The rigidity of Clifford torus S1 (...) x Sn-1 (...).

Autor(en): **Cheng, Qing-Ming**

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The rigidity of Clifford torus $S^1\left(\sqrt{\frac{1}{n}}\right) \times S^{n-1}\left(\sqrt{\frac{n-1}{n}}\right)$

QING-MING CHENG

Abstract. In this paper, we prove that if M is an n-dimensional closed minimal hypersurface with two distinct principal curvatures of a unit sphere $S^{n+1}(1)$, then $S = n$ and M is a Clifford torus if $n \le S \le n + [2n^2(n+4)/3(n(n+4)+4)]$, where S is the squared norm of the second fundamental form of M.

1. Introduction

Let M be an n-dimensional closed hypersurface in a unit sphere $S^{n+1}(1)$ of dimension $n+1$. Let S denote the squared norm of the second fundamental form of M. It is well-known that Chern, do Carmo and Kobayashi [2] and Lawson [3] obtained independently that Clifford tori are the only closed minimal hypersurfaces of the unit sphere with $S = n$. When the scalar curvature of M is constant, there are very nice results on the rigidity of the Clifford torus (see [5] and [6]). On the other hand, Otsuki[4] studied the converse problem for minimal hypersurfaces in $S^{n+1}(1)$. He proved that if M is a closed minimal hypersurface in $S^{n+1}(1)$ with two distinct principal curvatures and the multiplicities of them are at least two, then M is $S^m(\sqrt{m/n}) \times S^{n-m}(\sqrt{(n-m)/n})$ $(1 < m < n-1)$. But for the case in which one of the two principal curvatures is simple, he constructed infinitely many minimal hypersurfaces other than $S^1(\sqrt{1/n}) \times S^{n-1}(\sqrt{(n-1)/n})$ which are not congruent to each other in $S^{n+1}(1)$. When professor K. Shiohama visited China in 1993, he proposed the following interesting problem:

PROBLEM. *Let M be a closed minimal hypersurface in $S^{n+1}(1)$ with two distinct principal curvatures λ_1 and λ_2 and one of them be simple (we assume λ_1). Is there a constant $\epsilon = \epsilon(n)$ such that if $|\lambda_1 - \lambda_{10}| < \epsilon$ and $|\lambda_2 - \lambda_{20}| < \epsilon/(n-1)$ then M is $S^1(\sqrt{1/n}) \times S^{n-1}(\sqrt{(n-1)/n})$, where λ_{i0} are the corresponding principal curvatures of $S^1(\sqrt{1/n}) \times S^{n-1}(\sqrt{(n-1)/n})$.*

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This problem is equivalent to whether there is constant $\delta = \delta(n) > 0$ such that if $n - \delta \leq S \leq n + \delta$, then M is $S^1(\sqrt{1/n}) \times S^{n-1}(\sqrt{(n-1)/n})$.

In this paper, we consider the problem and give a partial answer.

THEOREM. *Let M be an n-dimensional closed minimal hypersurface of a unit sphere $S^{n+1}(1)$ with two distinct principal curvatures and one of them be simple. If*

$$n \leq S \leq n + \frac{2n^2(n+4)}{3[n(n+4)+4]}$$

then $S = n$ and M is $S^1(\sqrt{1/n}) \times S^{n-1}(\sqrt{(n-1)/n})$.

COROLLARY. *Let M be an n-dimensional closed minimal hypersurface of a unit sphere $S^{n+1}(1)$ with two distinct principal curvatures. If*

$$n \leq S \leq n + \frac{2n^2(n+4)}{3[n(n+4)+4]}$$

then $S = n$ and M is a Clifford torus.

Proof of Corollary. This is obvious from the result due to Otsuki and Theorem.

2. Local formulae

Let M be an n-dimensional closed minimal hypersurface in a unit sphere $S^{n+1}(1)$. We choose a local orthonormal frame field $\{e_1, \ldots, e_{n+1}\}$ in $S^{n+1}(1)$, restricted to M, so that $e_1, \ldots, e_n$ are tangent to M. Let $\omega_1, \ldots, \omega_n$ denote the dual coframe field on M. The connection form ω_{ij} are characterized by the structure equations

$$d\omega_i + \sum_j \omega_{ij} \wedge \omega_j = 0, \qquad \omega_{ij} + \omega_{ji} = 0,$$

$$d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj} = \Omega_{ij}, \tag{2.1}$$

$$\Omega_{ij} = \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l,$$

where Ω_{ij} (resp. R_{ijkl}) denotes the curvature form (resp. the components of the curvature tensor) of M. The second fundamental form α of M is given by

$$\alpha = \sum_{i,j} h_{ij}\omega_i\omega_j e_{n+1} \quad \text{and} \quad \sum_i h_{ii} = 0. \tag{2.2}$$

Since α is a symmetric tensor, $h_{ij} = h_{ji}$. The Gauss equation, Codazzi equation and Ricci formulas for the second fundamental form and its covariant derivatives are given by

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{2.3}$$

$$h_{ijk} = h_{ikj} = h_{jik}, \tag{2.4}$$

$$h_{ijkl} - h_{ijkl} = \sum_m h_{im}R_{mjkl} + \sum_m h_{mj}R_{mikl}, \tag{2.5}$$

$$h_{ijklm} - h_{ijkml} = \sum_r h_{rjk}R_{rilm} + \sum_r h_{irk}R_{rjlm} + \sum_r h_{ijr}R_{rklm}, \tag{2.6}$$

where h_{ijk}, h_{ijkl} and h_{ijklm} are the coefficients of the first, the second and the third covariant derivatives of the second fundamental form of M, respectively. The components of the Ricci curvature and the scalar curvature are given by

$$R_{ij} = (n-1)\delta_{ij} - \sum_k h_{ik}h_{jk}, \tag{2.7}$$

$$R = n(n-1) - \sum_{i,j} h_{ij}^2. \tag{2.8}$$

Now we compute some local formulae. For any fixed point p in M, we can choose a local frame field $e_1, \ldots, e_n$ such that

$$h_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ \lambda_i & \text{if } i = j. \end{cases} \tag{2.9}$$

The following formulas can be found in [1]. Let

$$S := \sum_{i,j} h_{ij}^2 = \sum_i \lambda_i^2.$$

$$\frac{1}{2} \Delta S = \sum_{i,j,k} h_{ijk}^2 - S(S - n). \tag{2.10}$$

$$\frac{1}{2} \Delta \sum_{i,j,k} h_{ijk}^2 = \sum_{i,j,k,l} h_{ijkl}^2 + (2n + 3 - S) \sum_{i,j,k} h_{ijk}^2 + 3(2B - A) - \frac{3}{2} |\nabla S|^2, \tag{2.11}$$

where $A = \Sigma_{i,j,k} \lambda_i^2 h_{ijk}^2$ and $B = \Sigma_{i,j,k} \lambda_i \lambda_j h_{ijk}^2$.

$$\frac{1}{3} \Delta f_3 = (n - S) f_3 + 2 \sum_{i,j,k} \lambda_i h_{ijk}^2, \tag{2.12}$$

where $f_3 = \Sigma_i \lambda_i^3$.

3. Proofs of theorems

At first we give an algebra Lemma which will play a crucial role in the proof of our theorems.

LEMMA. *Let a_{ij} and b_i $(i, j = 1, \ldots, n)$ be real numbers satisfying $\Sigma_i b_i = 0$ and $\Sigma_i b_i^2 = b > 0$, $\Sigma_{i,j} b_i a_{ij} = b(n - b)$ and $\Sigma_{i,j} b_j a_{ij} = 0$. Then*

$$\sum_i a_{ii}^2 + 3 \sum_{i \neq j} a_{ij}^2 + 3 \sum_{i,j} (b_j^2 b_i - b_i^2 b_j) a_{ij} - 3b(n - b)$$
$$\geq \frac{3b(n - b)^2}{2(n + 4)} - \frac{3}{2} \left[b \sum_i b_i^4 - \left(\sum_i b_i^3 \right)^2 - 2b^2 + nb \right].$$

Proof. We consider $F = \Sigma_i a_{ii}^2 + 3 \Sigma_{i \neq j} a_{ij}^2 + 3 \Sigma_{i,j} (b_j^2 b_i - b_i^2 b_j) a_{ij} - 3b(n - b)$ as a function of a_{ij}. Solve the following problem for the conditional extremum:

$$f = \sum_i a_{ii}^2 + 3 \sum_{i \neq j} a_{ij}^2 + 3 \sum_{i,j} (b_j^2 b_i - b_i^2 b_j) a_{ij}$$
$$- 3b(n - b) + \lambda \left(\sum_{i,j} b_i a_{ij} - b(n - b) \right) + \mu \sum_{i,j} b_j a_{ij}, \tag{3.1}$$

where λ and μ are the Lagrange multipliers. It is obvious that the critical point of f is the minimum point of f. Taking derivatives of f with respect to a_{ij}, we get

$$f_{a_{ij}} = 6a_{ij} + 3(b_j^2 b_i - b_i^2 b_j) + \lambda b_i + \mu b_j = 0, \qquad \text{for } i \neq j, \tag{3.2}$$

$$f_{a_{ii}} = 2a_{ii} + \lambda b_i + \mu b_i = 0, \qquad \text{for } i = j. \tag{3.3}$$

Hence

$$\sum_i a_{ii} f_{a_{ii}} = 2\sum_i a_{ii}^2 + \lambda \sum_i b_i a_{ii} + \mu \sum_i b_i a_{ii} = 0, \tag{3.4}$$

$$\sum_{i \neq j} a_{ij} f_{a_{ij}} = 6 \sum_{i \neq j} a_{ij}^2 + 3 \sum_{i,j} (b_j^2 b_i - b_i^2 b_j) a_{ij} + \lambda \sum_{i \neq j} b_i a_{ij} + \mu \sum_{i \neq j} b_j a_{ij} = 0, \tag{3.5}$$

$$\sum_i b_i f_{a_{ii}} = 2 \sum_i a_{ii} b_i + \lambda \sum_i b_i^2 + \mu \sum_i b_i^2 = 0, \tag{3.6}$$

$$\sum_{i \neq j} b_i f_{a_{ij}} = 6 \sum_{i \neq j} b_i a_{ij} + 3b^2 + \lambda \sum_{i \neq j} b_i^2 + \mu \sum_{i \neq j} b_i b_j = 0, \tag{3.7}$$

$$\sum_{i \neq j} b_j f_{a_{ij}} = 6 \sum_{i \neq j} b_j a_{ij} - 3b^2 + \lambda \sum_{i \neq j} b_i b_j + \mu \sum_{i \neq j} b_j^2 = 0. \tag{3.8}$$

$(3.4) + (3.5)$ implies

$$2\left(\sum_i a_{ii}^2 + 3 \sum_{i \neq j} a_{ij}^2 + 3 \sum_{i,j} (b_j^2 b_i - b_i^2 b_j) a_{ij} - 3b(n-b)\right)$$
$$- 3 \sum_{i,j} (b_j^2 b_i - b_i^2 b_j) a_{ij} + \lambda \sum_{i,j} b_i a_{ij} + \mu \sum_{i,j} b_j a_{ij} = -6b(n-b).$$

Thus

$$2f_{\min} = 2\left(\sum_i a_{ii}^2 + 3 \sum_{i \neq j} a_{ij}^2 + 3 \sum_{i,j} (b_j^2 b_i - b_i^2 b_j) a_{ij} - 3b(n-b)\right)$$
$$= 3 \sum_{i,j} (b_j^2 b_i - b_i^2 b_j) a_{ij} - 6b(n-b) - \lambda b(n-b). \tag{3.9}$$

According to (3.6), we get

$$2\sum_i a_{ii}b_i + (\lambda + \mu)b = 0. \tag{3.10}$$

(3.6) + (3.7) + (3.8) yield

$$-4\sum_i a_{ii}b_i + 3b^2 + 6b(n - b) + nb\lambda = 0, \tag{3.11}$$

$$-4\sum_i a_{ii}b_i - 3b^2 + nb\mu = 0. \tag{3.12}$$

Solving the system of the linear equations (3.10), (3.11) and (3.12) with unknown λ, μ and $\Sigma_i a_{ii}b_i$, we obtain

$$\lambda - \mu = -6$$

and

$$\lambda + \mu = -\frac{6(n-b)}{(n+4)}.$$

$$-\lambda = 3 + \frac{3(n-b)}{(n+4)}. \tag{3.13}$$

From (3.2) we have

$$-6a_{ij} = 3(b_j^2 b_i - b_i^2 b_j) + \lambda b_i + \mu b_j.$$

Hence

$$\begin{aligned}
&\sum_{i,j} 3(b_j^2 b_i - b_i^2 b_j)a_{ij} \\
&= -\frac{3}{2}\sum_{i,j}(b_j^2 b_i - b_i^2 b_j)^2 - \frac{1}{2}\sum_{i,j}(b_j^2 b_i - b_i^2 b_j)(\lambda b_i + \mu b_j) \\
&= -3\left[b\sum_i b_i^4 - \left(\sum_i b_i^3\right)^2\right] - \frac{1}{2}b^2(\lambda - \mu) \\
&= -3\left[b\sum_i b_i^4 - \left(\sum_i b_i^3\right)^2\right] + 3b^2.
\end{aligned}$$

From (3.9) and (3.13) and the above equality, we conclude

$$f_{\min} = \frac{3b(n-b)^2}{2(n+4)} - \frac{3}{2}\left[b\sum_i b_i^4 - \left(\sum_i b_i^3\right)^2 - 2b^2 + nb\right]. \tag{3.14}$$

Thus we complete the proof of Lemma.

Proof of theorem.

$$\begin{aligned}
\frac{1}{3}\sum_{i,j} h_{ij}(f_3)_{ij} &= \frac{1}{3}\sum_i \lambda_i (f_3)_{ii} \\
&= \sum_i \lambda_i\left(\sum_j \lambda_j^2 h_{jjii} + 2\sum_{j,k}\lambda_k h_{jki}^2\right) \\
&= \sum_{i,j}\lambda_i\lambda_j^2 h_{jjii} + 2\sum_{i,j,k}\lambda_i\lambda_j h_{ijk}^2 \\
&= \sum_{i,j}\lambda_i\lambda_j^2(h_{iijj} + (\lambda_j - \lambda_i)(1+\lambda_i\lambda_j)) + 2\sum_{i,j,k}\lambda_i\lambda_j h_{ijk}^2 \\
&= \sum_i \frac{\lambda_i^2 S_{ii}}{2} + \sum_{i,j}\lambda_i\lambda_j^2(\lambda_j - \lambda_i)(1+\lambda_i\lambda_j) + 2B - A \\
&= \frac{1}{2}\sum_{i,j,k} h_{ik}h_{kj}S_{ij} + \left[S\sum_i \lambda_i^4 - S^2 - \left(\sum_i \lambda_i^3\right)^2\right] + 2B - A.
\end{aligned}$$

Hence

$$\begin{aligned}
0 = \frac{1}{3}\int_M \sum_{i,j} h_{jji}(f_3)_i\, dM &= \frac{1}{3}\int_M \sum_{i,j} h_{ij}(f_3)_{ij}\, dM \\
&= \int_M \left[\frac{1}{3}\sum_{i,j,k} h_{ik}h_{kj}S_{ij} + (2B - A) + Sf_4 - f_3^2 - S^2\right] dM,
\end{aligned}$$

where f_4 is defined by $f_4 = \Sigma_i \lambda_i^4$, i.e.,

$$\int_M (A - 2B)\, dM = \int_M \left[Sf_4 - S^2 - f_3^2 - \frac{1}{4}|\nabla S|^2\right] dM. \tag{3.15}$$

$$
\begin{aligned}
\sum_{i,j,k,l} h_{ijkl}^2 &\geq \sum_i h_{iiii}^2 + 3\sum_{i\neq j} h_{ijij}^2 \\
&= \sum_i h_{iiii}^2 + \frac{3}{4}\sum_{i\neq j}(h_{ijij}+h_{jiji})^2 + \frac{3}{4}\sum_{i\neq j}(h_{ijij}-h_{jiji})^2 \\
&= \sum_i h_{iiii}^2 + 3\sum_{i\neq j} h_{iijj}h_{jjii} + \frac{3}{2}\sum_{i\neq j}(h_{ijij}-h_{jiji})^2 \\
&= \sum_i h_{iiii}^2 + 3\sum_{i\neq j} h_{iijj}h_{jjii} + 3\left[nS - 2S^2 + S\sum_i \lambda_i^4 - \left(\sum_i \lambda_i^3\right)^2\right].
\end{aligned} \tag{3.16}
$$

Since

$$
\sum_p h_{ijpp} = -(S-n)h_{ij},
$$

$\Sigma_i \lambda_i = 0$ and $\Sigma_i \lambda_i^2 = S > 0$, we have

$$
\sum_{i,j} h_{iijj}\lambda_i = S(n-S) \quad \text{and} \quad \sum_{i,j} h_{iijj}\lambda_j = 0. \tag{3.17}
$$

From Ricci formulas, we have

$$
h_{iijj} - h_{jjii} = (\lambda_i - \lambda_j)(1 + \lambda_i\lambda_j).
$$

Note that in view of (3.17) h_{iijj} and λ_i satisfy the conditions of Lemma. Hence we have

$$
\begin{aligned}
&\sum_i h_{iiii}^2 + 3\sum_{i\neq j} h_{iijj}h_{jjii} \\
&\quad = \sum_i h_{iiii}^2 + 3\sum_{i\neq j} h_{iijj}^2 + 3\sum_{i,j}(\lambda_j^2\lambda_i - \lambda_i^2\lambda_j)h_{jiijj} - 3S(n-S) \\
&\quad \geq \frac{3S(S-n)^2}{2(n+4)} - \frac{3}{2}(Sf_4 - f_3^2 - 2S^2 + nS).
\end{aligned} \tag{3.18}
$$

Since M has only two distinct principal curvatures λ_1 and λ_2 and λ_1 is simple, we have $\lambda_1 = -(n-1)\lambda_2$ and

$$
\frac{3}{2}(Sf_4 - f_3^2 - 2S^2 + nS) = \frac{3}{2n}S(S-n)^2.
$$

Thus we get, from (3.16) and the above inequality,

$$\sum_{i,j,k,l} h_{ijkl}^2 \geq 3(Sf_4 - f_3^2 - 2S^2 + nS) - \frac{6S(n-S)^2}{n(n+4)}. \tag{3.19}$$

According to (2.10) and (2.11), we obtain

$$\int_M \sum_{i,j,k} h_{ijk}^2 \, dM = \int_M [S(S-n)] \, dM, \tag{3.20}$$

$$\int_M \sum_{i,j,k,l} h_{ijkl}^2 \, dM = \int_M \left[-(2n+3-S) \sum_{i,j,k} h_{ijk}^2 - 3(2B-A) + \frac{3}{2} |\nabla S|^2 \right] dM. \tag{3.21}$$

From (3.15), (3.19), (3.20) and (3.21), we infer

$$\int_M \left\{ (2n-S) \sum_{i,j,k} h_{ijk}^2 - \frac{3}{4} |\nabla S|^2 - \frac{6S(n-S)^2}{n(n+4)} \right\} dM \leq 0. \tag{3.22}$$

From (2.10), we get

$$-\int_M \frac{1}{2} |\nabla S|^2 = \int_M \left[S \sum_{i,j,k} h_{ijk}^2 + (n-S)S^2 \right] dM. \tag{3.23}$$

(3.22) and (3.23) yield

$$\int_M \left\{ \left(2n + \frac{S}{2} \right) \sum_{i,j,k} h_{ijk}^2 - \frac{3}{2} S^2(S-n) - \frac{6}{n(n+4)} S(S-n)^2 \right\} dM \leq 0.$$

Using again (3.20) and the inequality

$$n \leq S \leq n + \frac{2n^2(n+4)}{3[(n(n+4)+4]},$$

we have

$$0 \geq \int_M \left\{\left(2n+\frac{n}{2}\right)\sum_{i,j,k} h_{ijk}^2+\frac{1}{2}(S-n)\sum_{i,j,k} h_{ijk}^2 -\frac{3}{2}S^2(S-n)-\frac{6}{n(n+4)}S(S-n)^2\right\}dM$$

$$=\int_M \left\{\frac{1}{2}(S-n)\sum_{i,j,k} h_{ijk}^2+\left(\frac{5}{2}n-\frac{3}{2}S-\frac{6(S-n)}{n(n+4)}\right)S(S-n)\right\}dM \geq 0.$$

Hence

$$\int_M \frac{1}{2}(S-n)\sum_{i,j,k} h_{ijk}^2\, dM = 0.$$

Since S and $\Sigma_{i,j,k} h_{ijk}^2$ are continuous functions, we have $S=n$. Thus from the assumption of Theorem, M is $S^1(\sqrt{1/n})\times S^{n-1}(\sqrt{(n-1)/n})$ according to a result due to Chern, do Carmo and Kobayashi [2] or Lawson [3]. We complete the proof of Theorem.

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REFERENCES

[1] CHENG, Q. M., *The classification of complete hypersurfaces with constant mean curvature of space formof dimension 4*, Mem. Fac. Sci. Kyushu Univ. *47* (1993), 79–102.

[2] CHERN, S. S., DO CARMO, M. and KOBAYASHI, S., *Minimal submanifolds of a sphere with second fundamental form of constant length, Functional analysis and related fields*, edited by F. Browder, Springer-Verlag, Berlin 1970, 59–75.

[3] LAWSON, H. B., *Local rigidity theorems for minimal hypersurfaces*, Ann. of Math. *89* (1969), 179–185.

[4] OTSUKI, T., *Minimal hypersurfaces ina Riemannian manifold of constant curvature*, Amer. J. Math. *92* (1970), 145–173.

[5] YANG, H. C. and CHENG, Q. M., *A note on the pinching constant of minimal hypersurfaces with constant scalar curvature in the unit sphere*, Chinese Science Bull. *36* (1991), 1–6.

[6] YANG, H. C. and CHENG, Q. M., *An estimate of the pinching constant of minimal hypersurfaces with constant scalar curvature in the unit spheres*, Manuscripta Math. *82* (1994), 89–100.

Department of Mathematics
Saga University
Saga 840
Japan.
e-mail:cheng@math.ms.saga-u.ac.jp

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