

**Zeitschrift:** Commentarii Mathematici Helvetici  
**Herausgeber:** Schweizerische Mathematische Gesellschaft  
**Band:** 71 (1996)

**Artikel:** A Kleinian group with contractible quotient not simply connected at infinity.  
**Autor:** Cooper, Daryl / Long, Darren  
**DOI:** <https://doi.org/10.5169/seals-53834>

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# A Kleinian group with contractible quotient not simply connected at infinity

DARYL COOPER\* AND DARREN LONG\*\*

*Abstract.* We give an example of a co-compact Kleinian group  $\Gamma$  which contains a subgroup  $\Gamma_0$  having the property that  $\mathbb{H}^3/\Gamma_0$  is contractible but not simply connected at infinity.

## 1. Introduction

The purpose of this article is to prove the following theorem:

**THEOREM 1.1.** *There is a hyperbolic 3-orbifold  $\tilde{X}$  homeomorphic to a contractible 3-manifold without boundary that is not simply connected at infinity. The singular locus of the orbifold  $\tilde{X}$  is a circle at which the cone angle is  $\pi$ . Furthermore  $\tilde{X}$  is an orbifold covering of a closed hyperbolic orbifold  $X$  which is homeomorphic to  $S^3$  and the singular locus of  $X$  is a link of two components at which the cone angle is  $\pi$ .*

We recall that a hyperbolic 3-orbifold is the quotient of  $\mathbb{H}^3$  by a discrete group of hyperbolic isometries. The theorem may thus be reformulated as:

**REFORMULATION.** *There is a co-compact Kleinian group  $\Gamma$  which contains an infinitely generated subgroup  $\Gamma_0$  having the property that  $\mathbb{H}^3/\Gamma_0$  is contractible but not simply connected at infinity. There are two conjugacy classes of torsion element in  $\Gamma$  and each has order two.*

This result is perhaps somewhat surprising. Of course Thurston [Th2] has shown that many closed 3-manifolds have hyperbolic structures. Furthermore, the fact that there is a universal hyperbolic link [Th3, HLM] implies that every closed orientable 3-manifold has a hyperbolic orbifold structure. However such general

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\* Research partially supported by NSF.

\*\* Research partially supported by NSF and Sloan Foundation.

results do not seem to predict the existence of an example of this type. The orbifold  $\tilde{X}$  is an irregular orbifold covering of a closed hyperbolic orbifold  $X$  which is  $S^3$  with a singular locus the link of two components shown in Fig. 1. The cone angle around each component is  $\pi$ . It is an unresolved question whether a closed 3-manifold can be covered by a contractible manifold other than Euclidean space. However, it has been shown that many contractible manifolds cannot do this [My, Wr]. Our examples shows that this can almost happen in the sense that the closed orbifold  $X$  has such an orbifold cover. Perhaps the most surprising feature of our example is that we could prove that it exists at all. It will be seen in the construction that several fortuitous accidents combine to enable the construction to succeed. For a more general definition of orbifold, see [Mo]. The authors thank the referee for finding errors in the original proof of 1.2(2) and for other helpful comments.

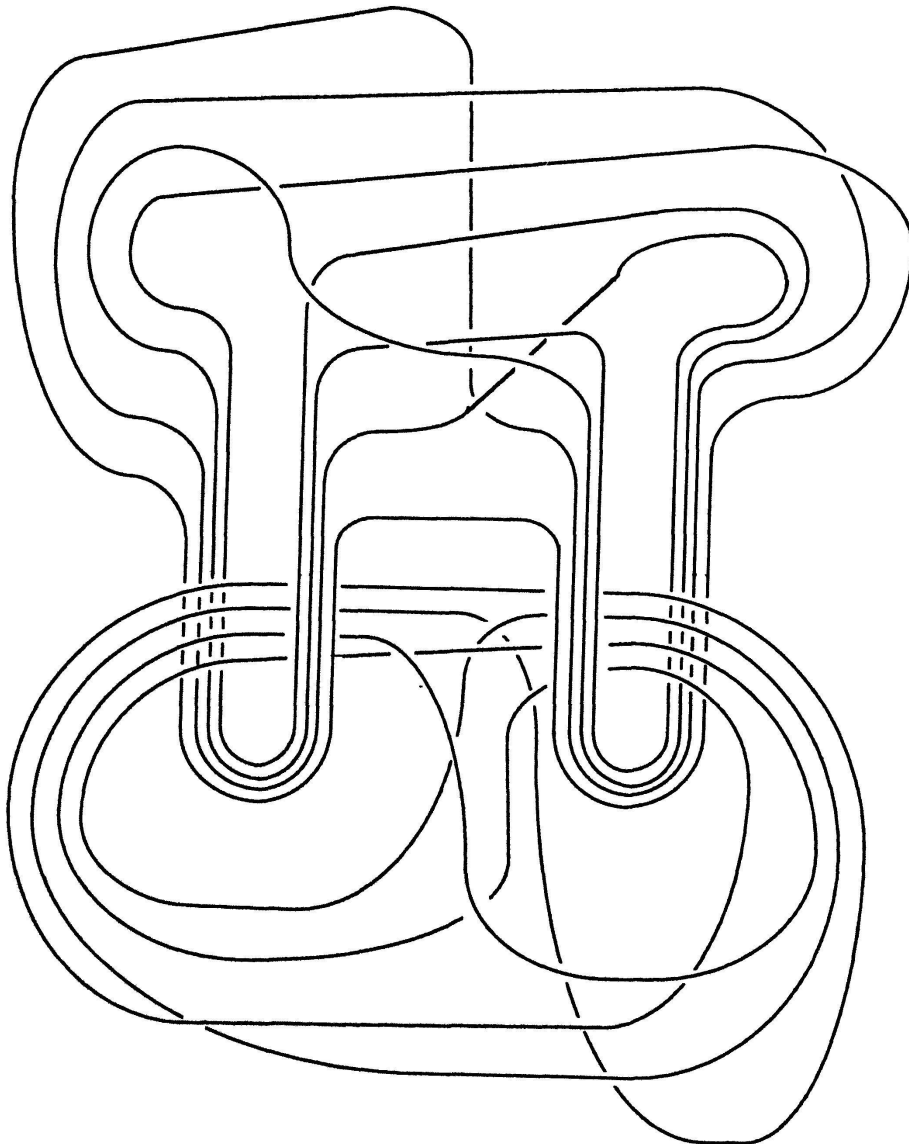


Figure 1

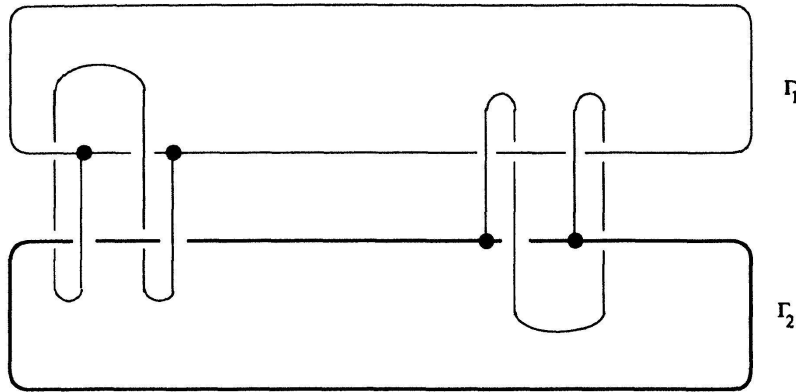


Figure 2

Let  $\Gamma_1$  and  $\Gamma_2$  be the pair of graphs embedded in  $S^3$  shown in Fig. 2. Each graph is homeomorphic to the graph shown in Fig. 3, which we call a theta-curve. We will denote by  $M$  the compact 3-manifold  $S^3 - \text{int}(N_1 \cup N_2)$  where  $N_i$  is a regular neighborhood of  $\Gamma_i$ . Thus  $\partial M$  consists of two genus 2 surfaces  $\partial_i M = \partial N_i$ , for  $i = 1, 2$ . The proof of the theorem depends on the following technical result the proof of which is deferred to section 2.

**PROPOSITION 1.2.**

- (1)  $M$  has incompressible boundary.
- (2)  $\pi_1(M)$  contains no  $\mathbb{Z} \times \mathbb{Z}$  subgroup.
- (3) Every properly embedded annulus  $A$  in  $M$  is isotopic rel  $\partial A$  into  $\partial M$ .
- (4)  $M$  contains no essential 2-sphere.

There is an involution  $\tau$  of  $S^3$  given by rotation around the circle  $C$  shown in Fig. 4 which exchanges  $\Gamma_1$  and  $\Gamma_2$ . The restriction of this to  $M$  gives an involution, also called  $\tau$ , of  $M$  which exchanges the boundary components of  $M$ .

Let  $\phi: \partial_1 M \rightarrow \partial_1 M$  be a diffeomorphism with  $\phi^2$  the central element in the mapping class group of  $\partial_1 M$  and such that  $\phi$  exchanges the un-oriented meridians of  $\Gamma_1$  with the un-oriented longitudes. To be precise we require that  $\phi(l_i^1) = m_i^1$  and  $\phi(m_i^1) = l_i^{1-1}$  for  $i = 1, 2$ , where  $m_1^1, m_2^1$  are the meridians of  $\Gamma_1$  and  $l_1^1, l_2^1$  are the longitudes of  $\Gamma_1$  shown in Fig. 5. Similarly we define meridians  $m_1^2, m_2^2$  and the longitudes  $l_1^2, l_2^2$  of  $\Gamma_2$  to be the images under  $\tau$  of the corresponding loops for  $\Gamma_1$ .

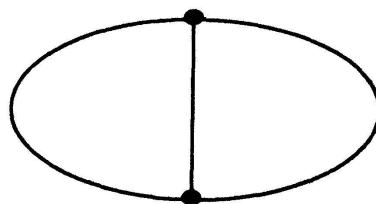


Figure 3



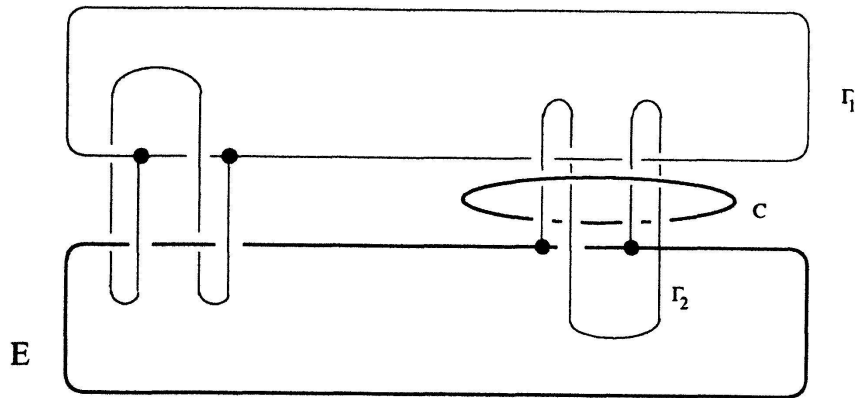


Figure 4

To see that such  $\phi$  exists, consider the genus 2 surface  $\partial_1 M$  as the union of two punctured tori. A punctured torus is a punctured square with opposite sides identified. A quarter rotation of the square gives an order 4 symmetry of the punctured torus, see Fig. 6. Then  $\phi$  is the map of  $\partial_1 M$  which restricts to the above map on each punctured torus.

Take 2 copies of  $M$  which are denoted by  $M$  and  $h(M)$  where  $h: M \rightarrow h(M)$  is a diffeomorphism. Define an involution  $\bar{\tau}$  on the disjoint union of  $M$  and  $h(M)$  by  $\bar{\tau}|_M = \tau$  and  $\bar{\tau}|_{h(M)} = h\tau h^{-1}$ . Now construct a closed 3-manifold  $N$  by identifying the boundary of  $M$  with the boundary of  $h(M)$  as follows. Identify  $\partial_1 M$  with  $h(\partial_1 M)$  via  $\phi_1 = h\phi$ . Identify  $\partial_2 M$  with  $h(\partial_2 M)$  via  $\phi_2 = \bar{\tau}h\phi\bar{\tau}$ . Then the involution  $\bar{\tau}$  passes to the quotient to give a well defined involution, also denoted  $\bar{\tau}$ , of  $N$ . See Fig. 7.

Then proposition 1.2 implies that  $N$  is Haken. Suppose that  $\pi_1 N$  contains a  $\mathbb{Z} \times \mathbb{Z}$  subgroup. The Torus theorem implies that  $N$  contains an essential torus  $T$ ,

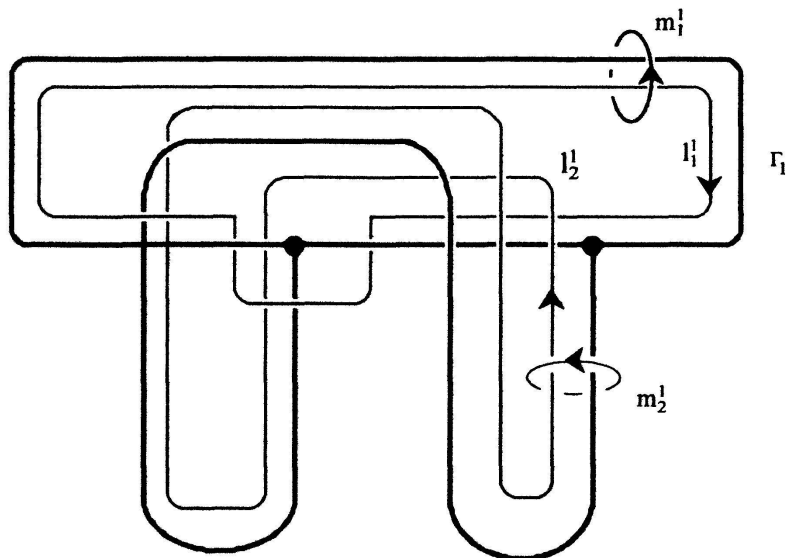


Figure 5

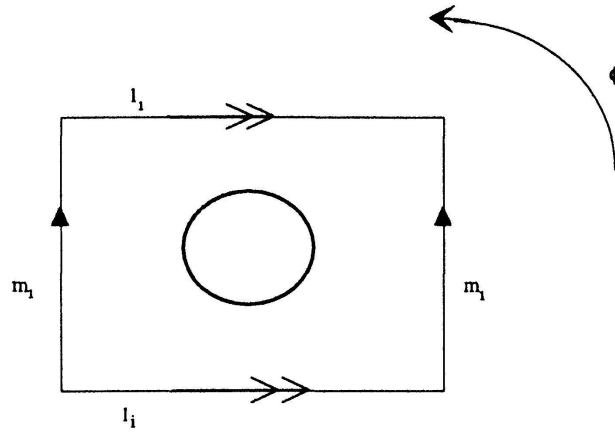


Figure 6

by 1.2(2)  $T$  cannot be isotoped into either copy of  $M$ . Thus  $T \cap M$  contains an essential non-boundary parallel annulus which is impossible by 1.2(3). Thus  $N$  contains no  $\mathbb{Z} \times \mathbb{Z}$  subgroup. Thus Thurston's uniformization theorem implies that  $N$  has a hyperbolic structure. It follows from Mostow rigidity that  $\bar{\tau}$  is homotopic to an isometry of  $N$ . A complete proof of Thurston's Uniformization theorem has been published by McMullen [McM1, McM2]. In fact it can be shown that  $N$  does not fiber over the circle, and so the particular case of the uniformization theorem which we appeal to is Haken manifolds that don't fiber.

If we knew that  $\bar{\tau}$  was *conjugate* to an isometry by a diffeomorphism isotopic to the identity then we could conclude that  $N/\bar{\tau}$  was a hyperbolic orbifold. Instead we argue as follows. The involution,  $\bar{\tau}$  of  $N$  has 1 dimensional fixed locus  $C \cup h(C)$ , and

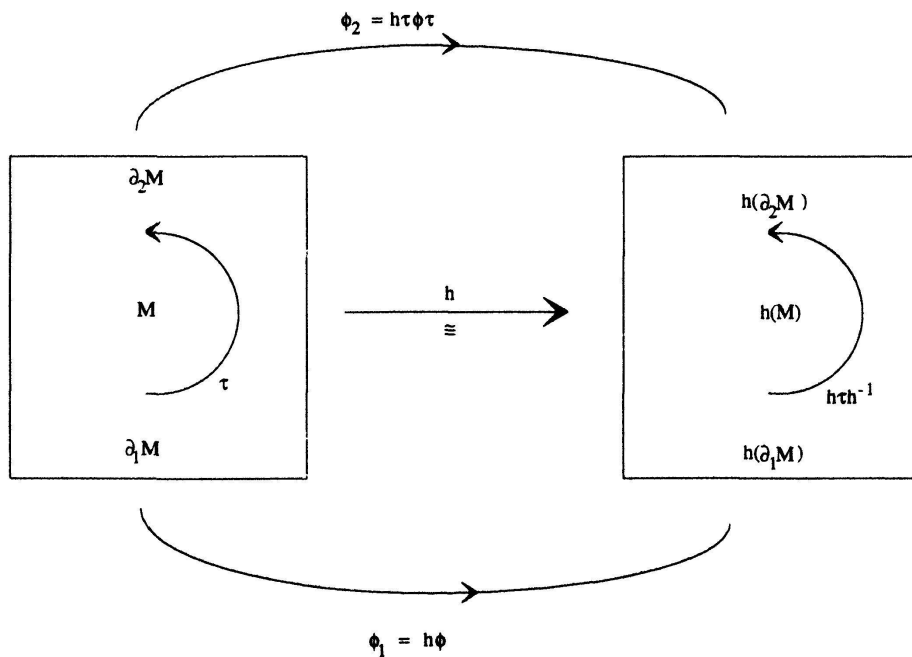


Figure 7

so by Thurston's Orbifold Theorem [Th, Ho], the quotient has a geometric decomposition. However since the 2-fold orbifold (branched) cover gives  $N$  back, the quotient  $N/\bar{\tau}$  must in fact be a hyperbolic orbifold. Set  $X = N/\bar{\tau}$ , a closed, orientable, hyperbolic orbifold.

The referee has pointed out that we may avoid appealing to the Orbifold Theorem as follows. By a result of Tollefson [To] two involutions of a Haken 3-manifold that are homotopic are in fact conjugate by a diffeomorphism isotopic to the identity provided that the manifold is not a Seifert fiber space and  $H_1(M)$  is infinite. We may apply this to the manifold  $N$  and to  $\bar{\tau}$  and the isometry provided by Mostow rigidity.

Now  $X = (M/\tau) \bigcup_{\bar{\phi}_1} h(M/\tau)$  identified along  $\partial(M/\tau)$  by the map

$$\bar{\phi}_1: \partial(M/\tau) \rightarrow \partial(h(M/\tau))$$

which is covered by  $\phi_1$ . Let  $\pi: N \rightarrow N/\tau$  be the projection; we will also use  $\pi$  for the restriction  $\pi: M \rightarrow M/\tau$ . Now  $N/\tau$  is  $S^3$ , and Fig. 8 shows  $\pi(\Gamma_1) = \pi(\Gamma_2)$  and  $\pi(C)$ . The graph  $\pi(\Gamma_1)$  is easily seen to be isotopic in  $S^3$  to an un-knotted theta curve, thus  $\pi(M) = S^3 - N(\pi\Gamma_1)$  is a genus 2 handlebody  $H$ . The branch locus  $\pi(C)$  is shown in a standard handlebody in Fig. 9. The following result is crucial to our construction, and appears to be a fortuitous accident:

**LEMMA 1.3.**  $\pi(l_1^1)$  and  $\pi(l_2^1)$  bound discs in  $H$ .

*Proof.* We sketch two proofs. First the curves  $\pi(l_1^1)$  and  $\pi(l_2^1)$  are shown in  $H = S^3 - N(\pi\Gamma_1)$  in Fig. 10. A little manipulation shows that these curves are unlinked from  $\pi(\Gamma_1)$  and are unknotted. The second proof is to calculate the (free) homotopy classes of  $l_1^1, l_2^1$ . One then adds the relations which identify an element of

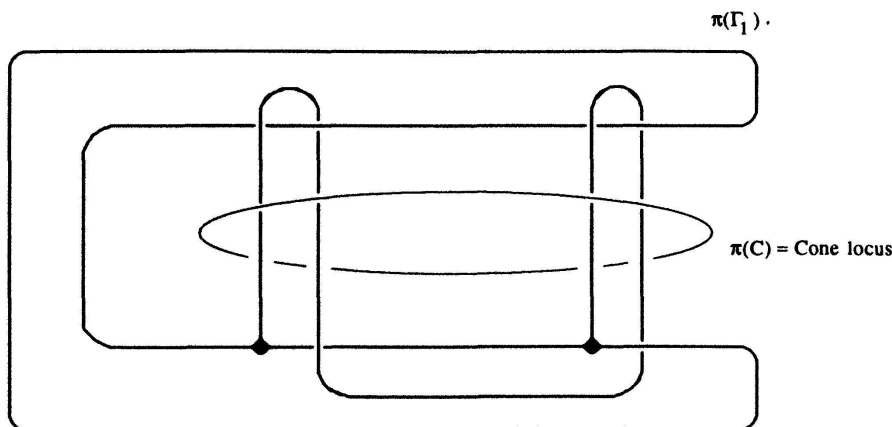


Figure 8

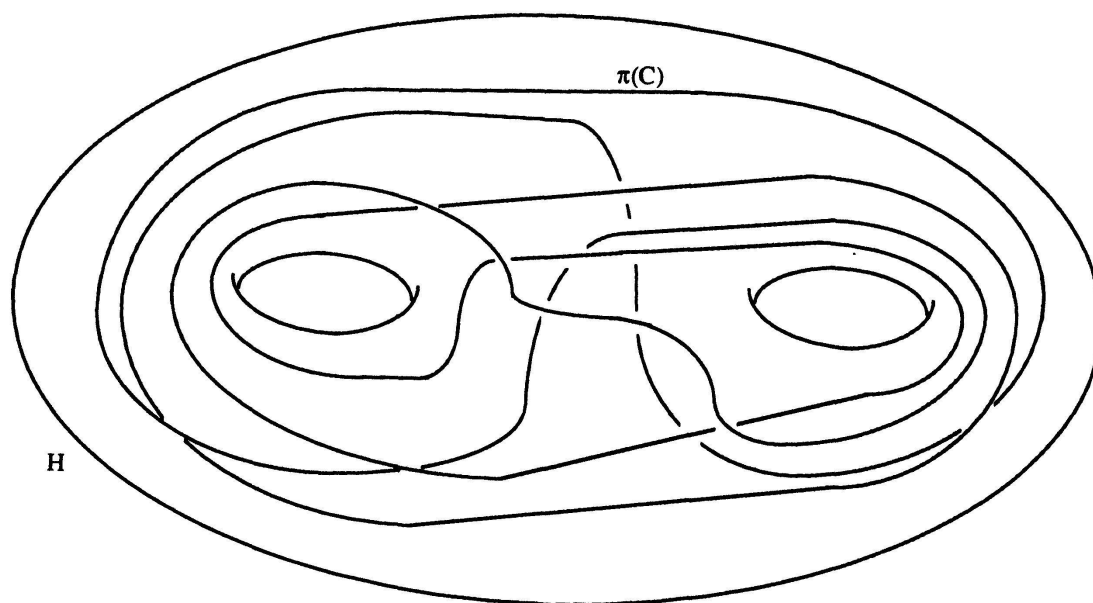


Figure 9

$\pi_1(M)$  with its image under  $\tau_*$  and checks that  $l_1^1, l_2^1$  are killed by this. This calculation is shown in Fig. 11 where we have made the identifications induced by  $\tau_*$  writing down the Wirtinger presentation of  $\pi_1(M)$ . Thus  $\pi(l_1^1), \pi(l_2^1)$  are simple closed curves in the boundary of the handlebody  $H$  which are inessential in  $H$  and thus bound discs in  $H$ .  $\square$

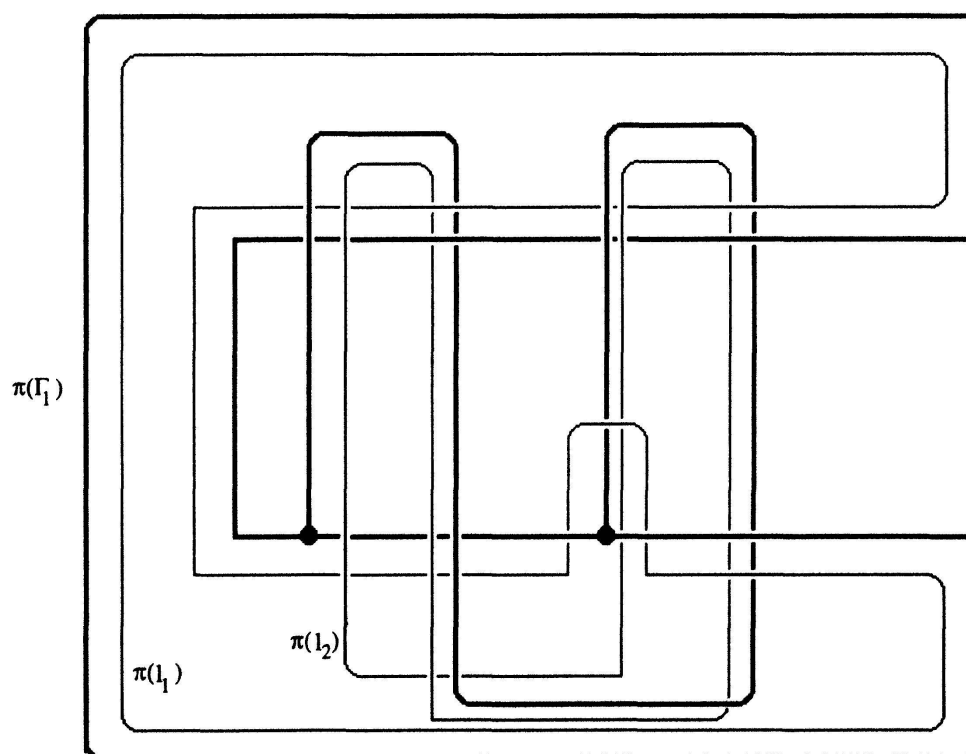
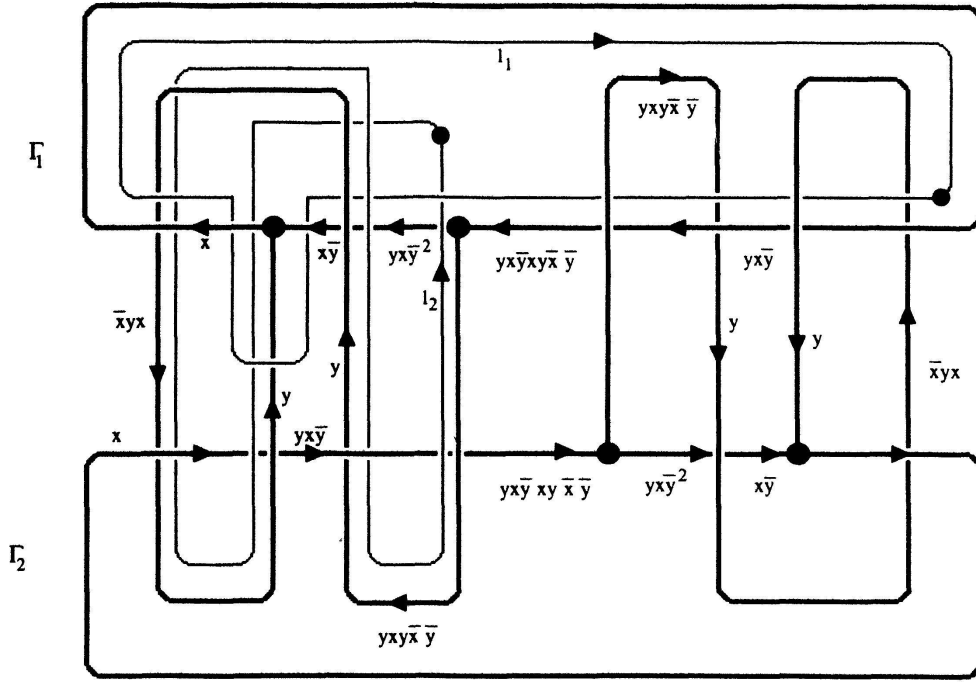


Figure 10



$$l_1 = \bar{y}(yxy\bar{x}\bar{y})y(x\bar{y})\bar{x} = 1$$

$$l_2 = yxx\bar{x}\bar{y}(y\bar{x}\bar{y}) = 1$$

Figure 11

The curves  $\pi(m_1^1)$ ,  $\pi(m_2^1)$  are longitudes of  $H$ , and it follows from (1.3) that  $X$  is topologically  $S^3$  since the handlebodies  $M/\tau$  and  $h(M/\tau)$  are glued together by identifying meridians to longitudes via  $\phi_1$ . As a hyperbolic orbifold,  $H$  contains a singular locus, a topological circle, with cone angle  $\pi$ , shown in Fig. 8 and also in Fig. 9. Thus  $X$  has singular locus a link of 2 components  $C_1 \cup C_2$  each with a cone angle of  $\pi$ , this link is shown in Fig. 1. The linking number of  $C_1$  with  $C_2$  is zero, in fact since  $C_1$  bounds a Seifert surface in  $H$ , we see that  $C_1 \cup C_2$  is a boundary link in  $S^3$ . Thus there is a homomorphism from  $\pi_1(S^3 - (C_1 \cup C_2))$  onto the free group of rank 2. This in turn maps onto  $\mathbb{Z}_2 * \mathbb{Z}_2$  where the meridians of  $C_1$  and  $C_2$  map to the generators of order 2 in  $\mathbb{Z}_2 * \mathbb{Z}_2$ . This determines a homomorphism  $G \rightarrow \mathbb{Z}_2 * \mathbb{Z}_2$  where  $G$  is the orbifold fundamental group of  $X$ . Now let  $\tilde{X}$  be the irregular orbifold covering space of  $X$  corresponding to the subgroup  $\langle \alpha_1 \rangle$  of order 2 in  $\mathbb{Z}_2 * \mathbb{Z}_2$  generated by the meridian  $\alpha_1$  of  $C_1$ . Thus  $\tilde{X}$  is a hyperbolic orbifold.

**LEMMA 1.4.** *Denoting the normal closure by  $\langle \cdot \rangle_N$  we have:*

- (1)  $l_1^1$  and  $l_2^1$  are trivial in  $\pi_1 M / \langle m_1^2, m_2^2 \rangle_N$ .
- (2)  $l_1^2$  and  $l_2^2$  are trivial in  $\pi_1 M / \langle m_1^1, m_2^1 \rangle_N$ .

*Proof.* Referring to Figs. 2 and 5, the manifold obtained from  $M$  by filling in

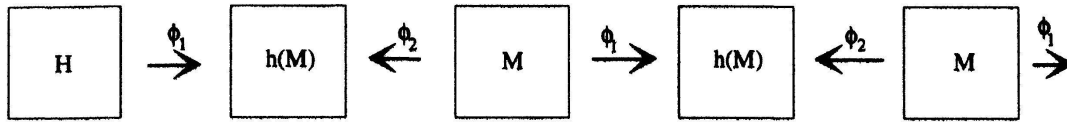


Figure 12

$N(\Gamma_2)$  is seen to be a handlebody in which  $l_1^1, l_2^1$  bound discs. From this it follows that after attaching 2-handles to  $\partial_1 M$  along meridians  $m_1^2, m_2^2$  that  $l_1^1, l_2^1$  bound discs, this proves (1). Applying the involution  $\tau$  of  $M$  proves (2).  $\square$

*Proof of Theorem.* The orbifold  $\tilde{X}$  is obtained by glueing copies of  $M$  to a single copy of  $H$  using  $\phi_1$  and  $\phi_2$  to do the glueing, as shown in Fig. 12. We calculate the topological (not orbifold) fundamental group  $\pi_1(\tilde{X})$  by applying Van Kampen's theorem to this decomposition to show that  $X$  is simply connected. For each positive integer  $n$  let  $M_n$  be a copy of  $M$  and let  $H_n$  denote the union of  $H$  and the first  $n$  copies of  $M$  with boundaries identified appropriately. Then  $\tilde{X}$  is the union of the increasing family of submanifolds  $H_n$ . The boundary  $\partial H_n$  is a component of  $M_n$ , a genus two surface with copies  $l_1^n, l_2^n$  of  $l_1, l_2$  marked on it.

Note that  $H$  is attached to  $M_1$  by the map  $\phi_1$  which identifies the longitudes  $\pi l_1^2, \pi l_2^2$  in  $H$  with  $m_1^1, m_2^1$  in  $M$ , but  $\pi l_1^2, \pi l_2^2$  are trivial in  $\pi_1(H)$  by the lemma 1.3, and so  $m_1^1, m_2^1$  are trivial in  $\pi_1(H \cup_{\phi_1} M)$ . By lemma 1.4,  $l_1^2, l_2^2$  are trivial in  $\pi_1(H \cup_{\phi_1} M)$ , and these are identified by  $\phi_2^{-1}$  to  $m_1^1, m_2^1$  in the second copy of  $M$  in  $\tilde{X}$ . Thus these loops are trivial in  $\pi_1(H \cup_{\phi_1} M \cup_{\phi_2^{-1}} M)$ . Continuing in this way, we see that  $\pi_1(\tilde{X})$  is trivial. A detailed argument will now be given.

We claim that  $H_n$  is a handlebody and that  $l_1^n, l_2^n$  bound discs in  $H_n$ . Indeed Lemma (1.3) implies this for the case that  $n = 0$ . Suppose inductively this is true for  $H_n$  then since  $l_1^n, l_2^n$  bound discs in  $H_n$  it follows that  $H_{n+1}$  is obtained from  $M_{n+1}$  by attaching 2-handles to  $M_{n+1}$  along the curves  $m_1^{n+1}, m_2^{n+1}$  in  $\partial M_{n+1}$  to which  $l_1^n, l_2^n$  are identified. One then caps off the resulting two-sphere boundary component with a 3-handle to obtain  $H_{n+1}$ . This proves the claim.

Thus there is a homeomorphism  $\theta: H_{n+1} \rightarrow S^3 - \text{int}[N(\Gamma_1)]$  taking  $H_n$  to  $N(\Gamma_2)$  and taking  $M_{n+1}$  onto  $S^3 - \text{int}[N(\Gamma_1) \cup N(\Gamma_2)]$ . We show below that the map induced by inclusion

$$(i_n)_*: \pi_1(H_n) \rightarrow \pi_1(H_{n+1})$$

has infinite cyclic image contained in the commutator subgroup of  $\pi_1(H_{n+1})$ . It follows from this that  $(i_{n+1} \circ i_n)_* = 0$  and thus that  $\tilde{X}$  is simply connected.

Since  $H_n$  is a handlebody in which  $l_1^n, l_2^n$  bound discs it follows that  $\pi_1(H_n)$  is freely generated by the copies  $m_1^n, m_2^n$  of  $m_1, m_2$  on  $\partial H_n$ . These are identified to copies of  $l_1, l_2$  on  $\partial M_{n+1}$ . Now  $\theta(m_1^n), \theta(m_2^n)$  are  $l_1^2, l_2^2$  (recall the identification of

$\partial H_n$  with a component of  $\partial M_{n+1}$  swaps meridians and longitudes.) Referring to Figs. 2 and 5 (with  $\tau$  applied which relabels  $\Gamma_1$  as  $\Gamma_2$ ), one sees that the loops  $l_1^2, l_2^2$  in  $S^3 - N(\Gamma_1)$  are both homotopic rel basepoint to the loop  $E$  shown in Fig. 4. One also sees that  $E$  is homologically unlinked from  $\Gamma_1$  and thus lies in the commutator subgroup of  $\pi_1(S^3 - N(\Gamma_1))$ . This proves the claim and completes the proof that  $\tilde{X}$  is simply connected.

We next show that  $\pi_1(\tilde{X} - \text{int}(H))$  is not finitely generated. Now  $\tilde{X} - \text{int}(H)$  is obtained by glueing copies of  $M$  together using the maps  $\phi_1, \phi_2$ .  $M$  has incompressible boundary, and it is clear that  $\text{incl}_*: \pi_1(\partial_1 M) \rightarrow \pi_1(M)$  is not surjective, otherwise it would be an isomorphism. This proves the claim. If  $\tilde{X}$  is simply connected at infinity then there is an open set  $U$  disjoint from the compact set  $H$  and which has compact complement and such that  $\pi_1(U)$  maps to zero in  $\pi_1(\tilde{X} - H)$ . Thus  $\pi_1(\tilde{X} - H)$  is the image of  $\pi_1$  of some compact submanifold of  $\tilde{X} - \text{int}(H)$ , and is thus finitely generated, a contradiction.  $\square$

## 2. Proof of 1.2

We now turn to proving proposition 1.2 We will consider a particular 2-fold branched covering  $p: S^3 \rightarrow S^3$  branched over the circle  $E$  contained in  $\Gamma_2$  shown in Fig. 4. The restriction of  $p$  to  $\tilde{M} = p^{-1}(M)$  gives an unbranched 2-fold cover  $p: \tilde{M} \rightarrow M$ . Set  $\tilde{\Gamma}_i = p^{-1}(\Gamma_i)$  and  $\tilde{N}_i = p^{-1}(N_i)$  then  $\tilde{N}_i$  is a regular neighborhood of  $\tilde{\Gamma}_i$  and the graphs  $\tilde{\Gamma}_i$  embedded in  $S^3$  are shown in Fig. 13. Now  $\tilde{N}_2$  is a genus-3 handlebody and  $\tilde{N}_1$  is the disjoint union of genus-2 handlebodies. The two

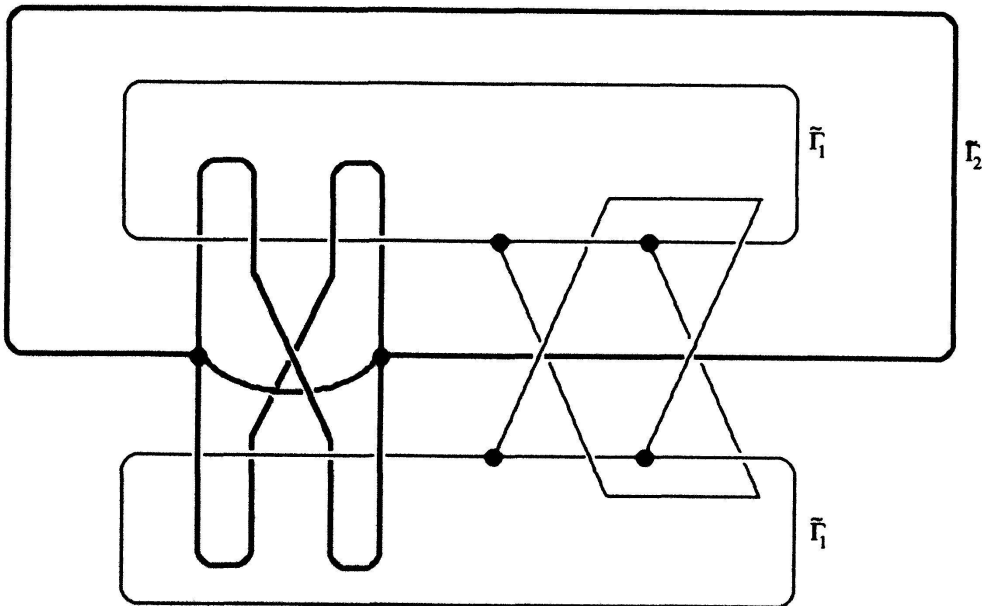


Figure 13

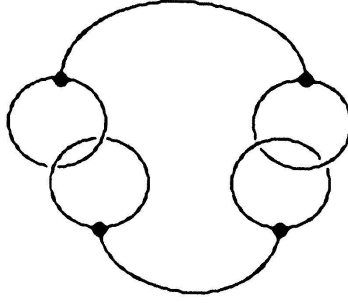


Figure 14

components of  $\partial_1 \tilde{M}$  will be denoted by  $G_1$  and  $G_2$ , each of which is a closed genus-2 surface. Note that  $\tilde{M} = S^3 - \text{int}(\tilde{N}_1 \cup \tilde{N}_2)$ .

LEMMA 2.1.  $\tilde{M} \cup \tilde{N}_2$  is diffeomorphic to  $G_1 \times I$ .

*Proof.* Slide  $\tilde{F}_1$  around to obtain the configuration in Fig. 14, which clearly gives  $G_1 \times I$ .  $\square$

From the lemma we see that  $\pi_1(G_1)$  injects into  $\pi_1(\tilde{M} \cup \tilde{N}_2)$  under the map induced by inclusion, and therefore also injects into  $\pi_1(\tilde{M})$ . Since  $\partial_1 M$  lifts to  $G_1$  in  $\tilde{M}$ , it follows that  $\pi_1(\partial_1 M)$  injects into  $\pi_1(M)$ . Thus  $\partial_1 M$  is incompressible, and by using the involution  $\tau$  of  $M$ , one sees that  $\partial_2 M$  is also incompressible, proving 1.2(1).

If  $M$  contains an essential 2-sphere  $S$  then  $S$  must separate  $\Gamma_1$  from  $\Gamma_2$  otherwise by the Schönflies theorem  $S$  would bound a ball. Now  $S$  lifts to a 2-sphere  $\tilde{S}$  in  $\tilde{M}$  which separates  $\tilde{F}_1$  from  $\tilde{F}_2$ . However inspection of Fig. 13 reveals that each component of  $\tilde{F}_1$  and  $\tilde{F}_2$  are algebraically linked in  $S^3$  thus  $\tilde{S}$  cannot separate them. This proves  $S$  cannot exist, establishing 1.2(4).

Consider the sphere  $S$  in  $S^3$  shown in Fig. 20, which meets  $(\Gamma_1 \cup \Gamma_2)$  in 4 points. Then  $S$  separates  $S^3$  into two closed balls  $B_1$  and  $B_2$  and  $S$  may be chosen so that  $\tau$  exchanges these balls. We may arrange that  $S$  meets  $N(\Gamma_1 \cup \Gamma_2)$  standardly in 4 discs, each of which contains one point of  $(\Gamma_1 \cup \Gamma_2)$ . Set  $S_- = M \cap S$ , a 4-punctured sphere,  $Q_i = M \cap B_i$  for  $i = 1, 2$ . Then  $S_- = \partial Q_1 \cap \partial Q_2$ .

LEMMA 2.2.  $S_-$  is incompressible in both  $Q_1$  and  $Q_2$ .

*Proof.* Suppose  $D$  is a properly embedded disc in  $Q_1$  with  $\partial D \subset S_-$ . Then  $D$  separates  $B_1$  into two balls and if  $D$  compresses  $S_-$  then  $\Gamma_1$  must lie on one side of  $D$  and  $\Gamma_2$  on the other side of  $D$ . Thus  $\pi_1(Q_1)$  splits as a free product. Now there is a loop  $\gamma$  in a neighborhood of  $\Gamma_2$  which is a commutator of meridians in  $\Gamma_1$  and  $\Gamma_2$ . Thus  $\gamma$  lies on the same side of  $D$  as  $\Gamma_2$  but such a commutator cannot be disjoint from  $D$ . Thus there is no compressing disc for  $S_-$ . Since  $S_-$  is incompressible in  $Q_1$ , applying the involution  $\tau$  we see that  $S_-$  is also incompressible in  $Q_2$ .  $\square$



LEMMA 2.3.  $Q_1$  is a genus-3 handlebody.

*Proof.*  $Q_1$  is the complement in  $S^3$  of an open regular neighborhood of the graph in  $S^3$  shown in Fig. 21. By sliding this graph, one obtains the graph in Fig. 22, the complement of which is clearly a genus-3 handlebody.  $\square$

Now suppose that  $M$  contains an essential torus  $T$ . Then we may assume  $T$  is transverse to  $S_-$  and has the least possible number of circles of intersection with

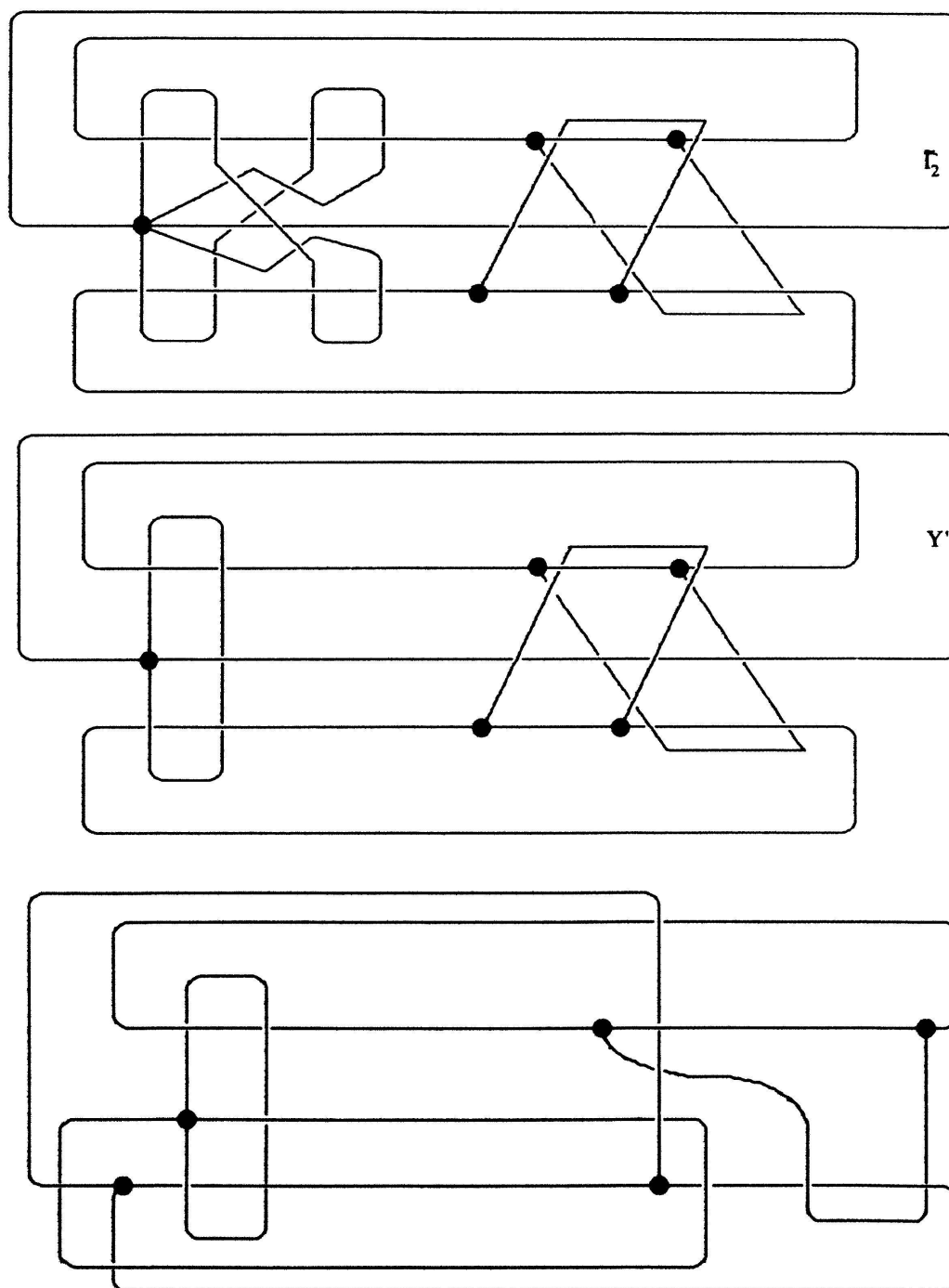


Figure 15(a)–(c)

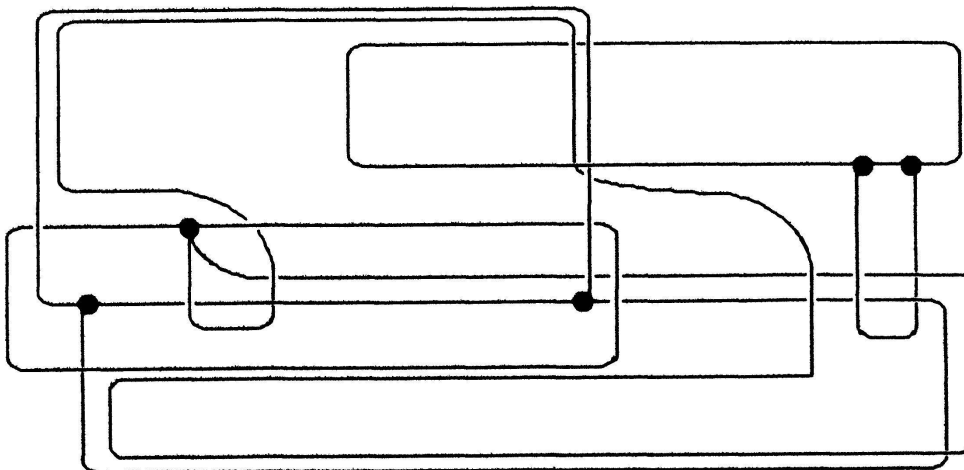
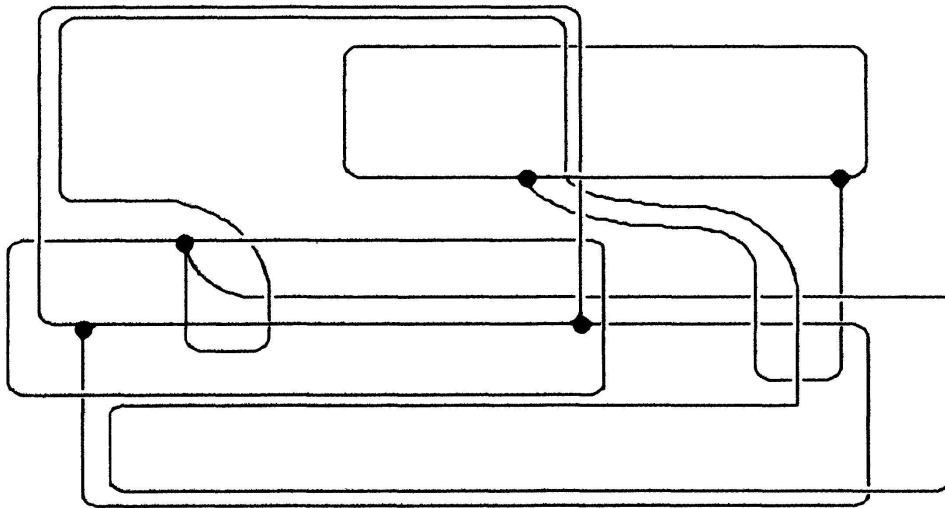
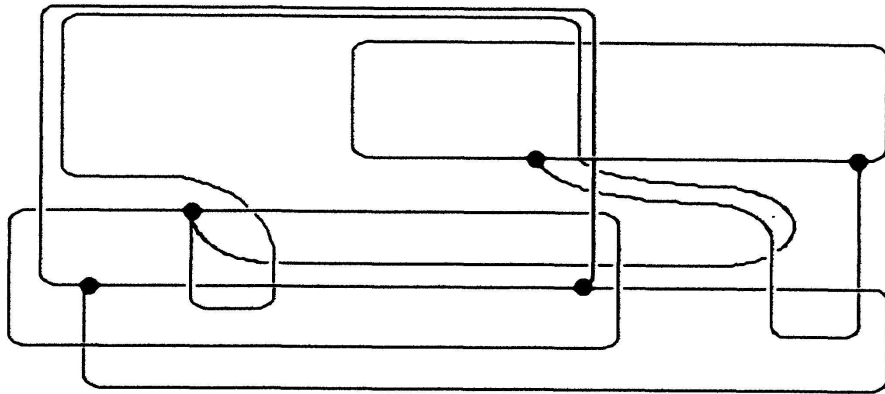


Figure 15 (d-f)

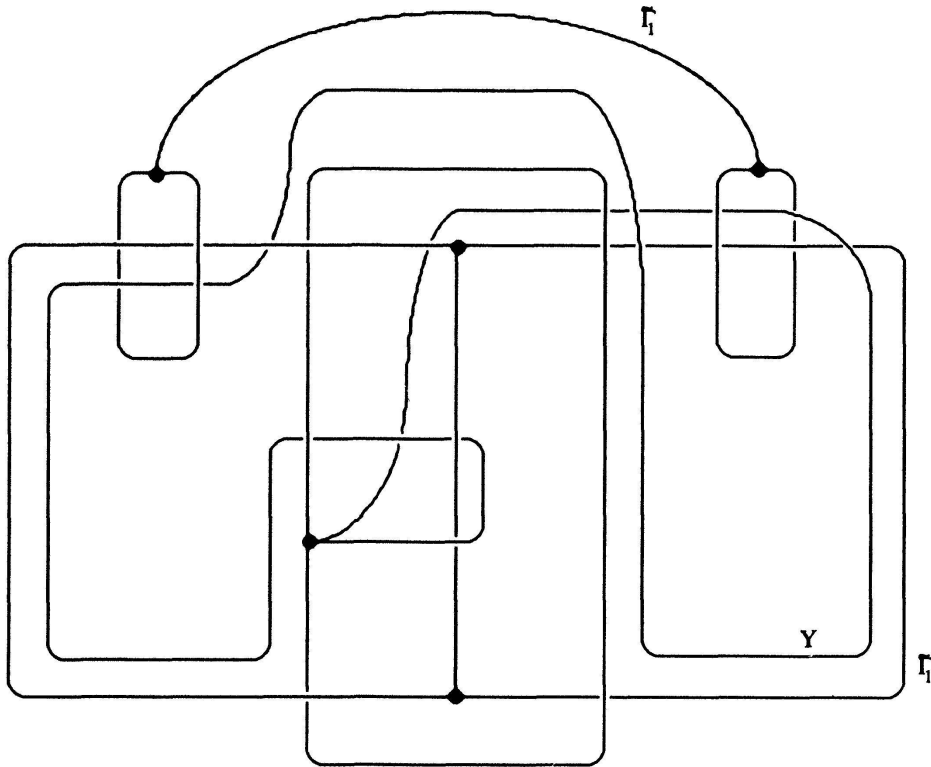


Figure 15(g)

$S_-$ . Since  $S_-$  is incompressible it follows that every circle of intersection is essential in  $T$ . Since a handlebody contains no essential torus, by (2.3)  $T$  must have non-empty intersection with  $S_-$ . Thus  $S_-$  separates  $T$  into components each of which is an annulus and none of these annuli can be isotoped rel boundary into  $S_-$ . Let  $A$  be such an annulus properly embedded in  $Q_1$  with boundary  $\partial A = \alpha_1 \cup \alpha_2$  two disjoint circles in the four punctured sphere  $S_-$ . These circles are essential in  $S_-$ . They cannot be isotopic in  $S_-$  because this would give a torus  $K$  consisting of the union of  $A$  and an annulus in  $S_-$ . But  $Q_1$  is a handlebody so  $K$  compresses and thus  $A$  can be isotoped into  $S_-$  a contradiction.

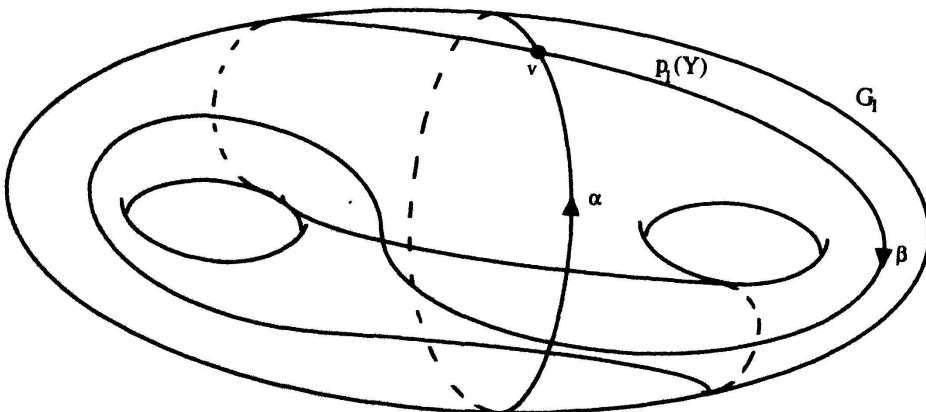


Figure 16

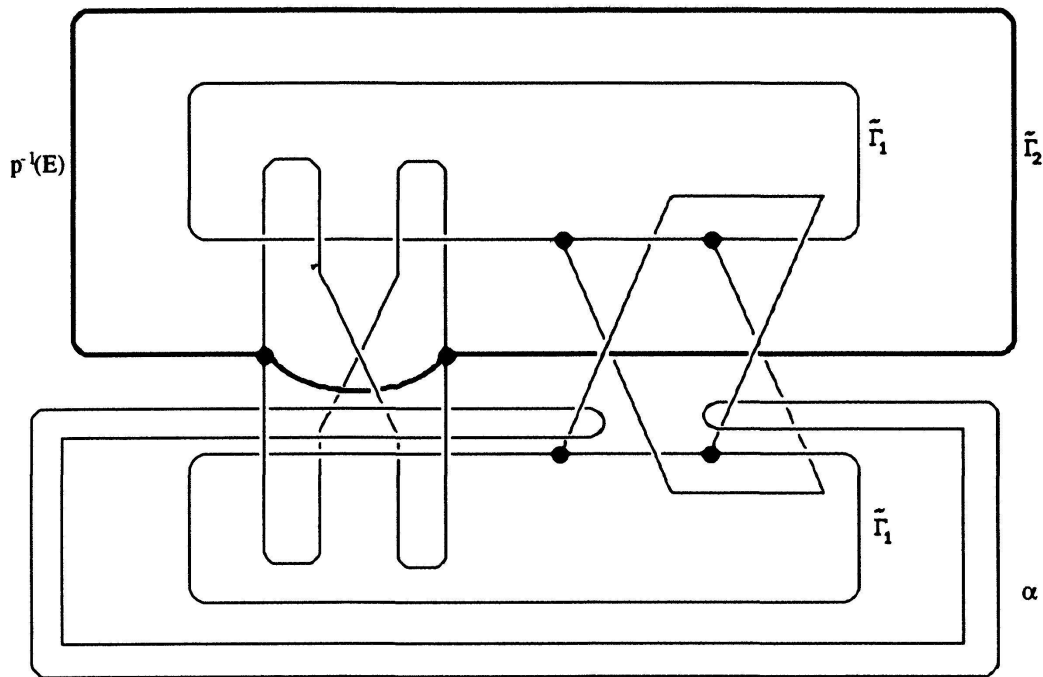


Figure 17

Now  $\alpha_1$  is a simple closed curve on the 4 punctured sphere  $S_-$  and if  $\alpha_1$  has 2 punctures on either side then since  $\alpha_1 = \alpha_2$  in  $H_1(Q_1)$  one sees that  $\alpha_2$  must also have 2 punctures on either side. But since  $\alpha_1$  and  $\alpha_2$  are disjoint this means that they are isotopic, a contradiction. It follows that  $\alpha_1$  has one puncture on one side and 3 punctures on the other side. Again considering  $H_1(Q_1)$  one sees that  $\alpha_2$  must also have one puncture on one side and that there are only two possibilities for  $\alpha_1, \alpha_2$  up to isotopy. Either they are the two meridians of  $\Gamma_1$  on  $S_-$  or they are the two meridians of  $\Gamma_2$  on  $S_-$ . Referring to Fig. 20 we see that the first case is possible, there is an annulus in a neighborhood of  $\Gamma_1$  in  $Q_1$ . However the second case is impossible. One way to see this is to observe that the annulus provides a free homotopy in  $Q_1$  between the two meridians of  $\Gamma_2$  on  $S_-$ . One calculates these two

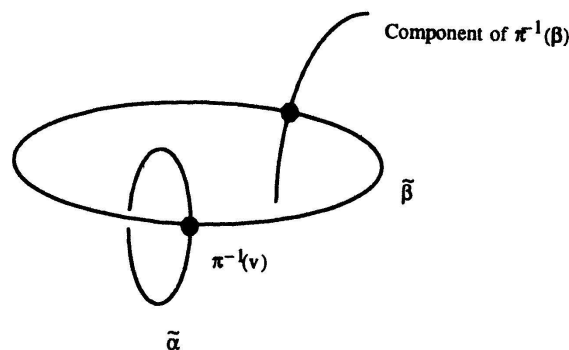


Figure 18

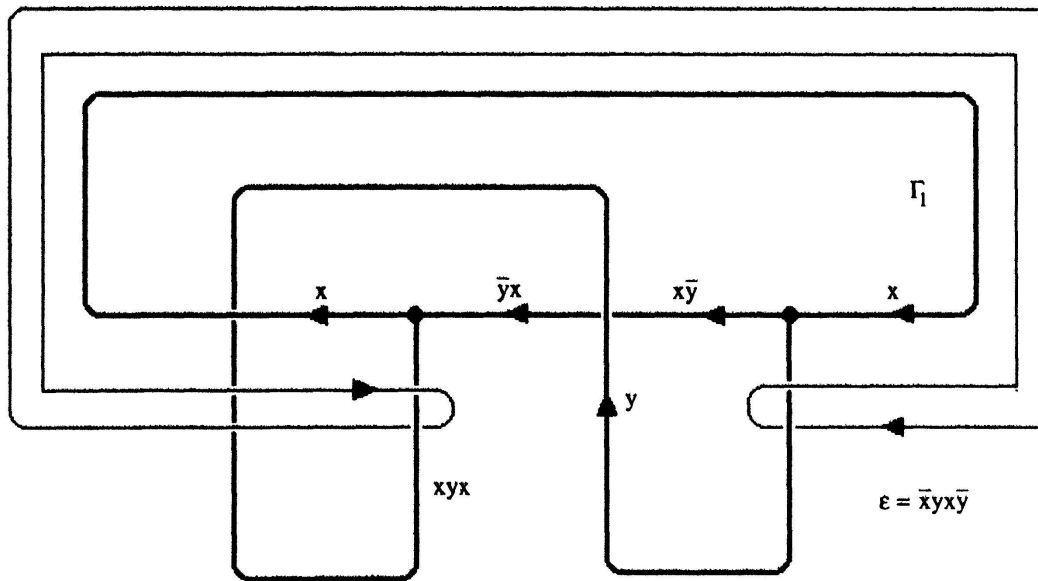


Figure 19

meridians using the Wirtinger presentation and since  $\pi_1 Q_1$  is a free group the fact that these two elements are not conjugate is visible.

It follows that every component of  $T \cap S_-$  is a meridian of  $\Gamma_1$  but using the involution  $\tau$  the above analysis applied to  $Q_2$  implies that these curves must also be meridians of  $\Gamma_2$  and so  $T \cap S_-$  is empty, a contradiction. This proves 1.2(2)

Suppose now that  $M$  contains a properly embedded non-boundary parallel annulus  $A$ . Using the involution  $\tau$  we may assume that  $A$  meets  $\partial_1 M$ . Then  $p^{-1}(A)$  consists of either one or two components each of which is a non-boundary parallel annulus properly embedded in  $\tilde{M}$ . Choose a component  $\tilde{A}$  of  $p^{-1}(A)$ , and note that

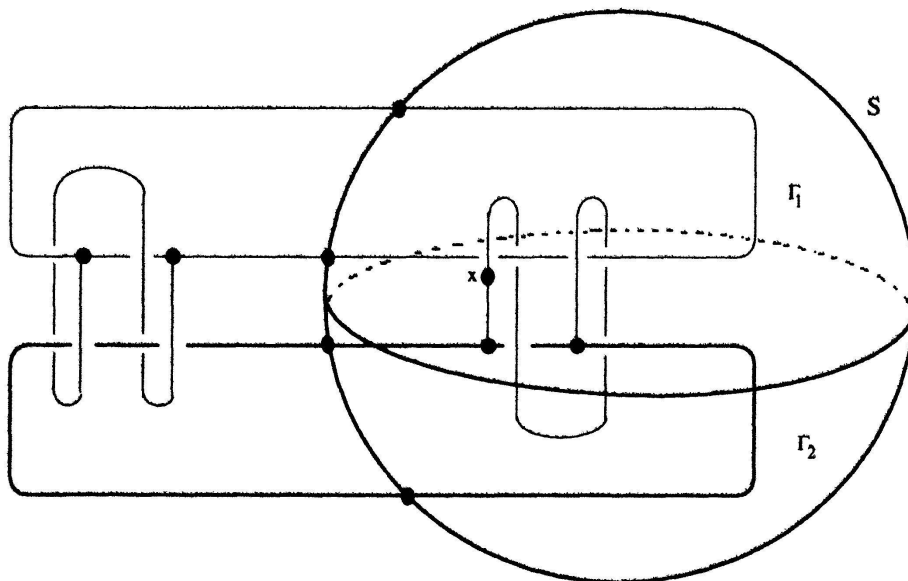


Figure 20

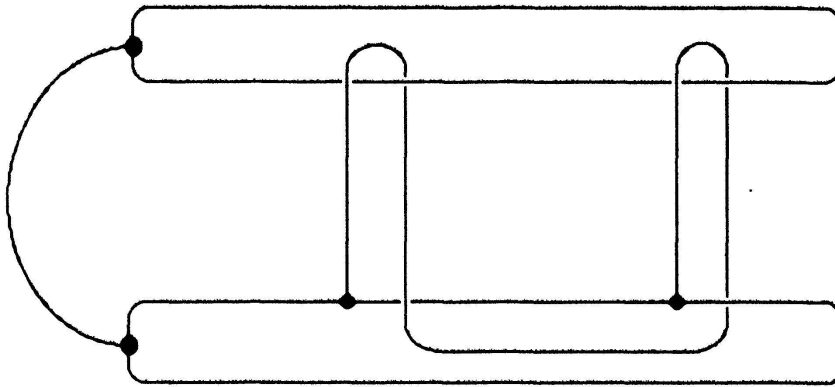


Figure 21

$\tilde{A}$  meets  $\partial_1 \tilde{M}$ . The covering  $p: \tilde{M} \rightarrow M$  is regular and so there is a covering transformation exchanging  $G_1$  and  $G_2$ . Thus we may assume that a boundary component of  $\tilde{A}$  lies in  $G_1$ . The boundary of  $\tilde{A}$  consists of 2 disjoint essential simple closed curves,  $\gamma, \delta$  in  $\partial \tilde{M}$  and we label them so that  $\gamma$  lies in  $G_1$ . We will now distinguish 3 cases, according to whether the second boundary component  $\delta$  of  $\tilde{A}$  lies in  $G_1, G_2$  or  $\partial_2 \tilde{M}$ .

First suppose that  $\delta$  is contained in  $G_2$ . By lemma 2.1,  $\tilde{M} \cup \tilde{N}_2 = G_1 \times I$  and we may do an ambient isotopy of  $G_1 \times I$  so that  $\tilde{A} = \gamma \times I$  is vertical in  $G_1 \times I$ , where  $\gamma$  is some essential simple closed curve in  $G_1$ . The image of  $\tilde{\Gamma}_2$  under this isotopy must be disjoint from  $\gamma \times I$ . Let  $Y$  be the graph in  $G_1 \times I$  shown in Fig. 15(g), and  $p_1: G_1 \times I \rightarrow G_1$  be projection onto the first factor.

LEMMA 2.4.  $P_{1*} \Pi_1(Y)$  is conjugate to  $P_{1*} \Pi_1(\tilde{\Gamma}_2)$  in  $\Pi_1(G_1)$ .

*Proof.* This is done in the sequence of figures 15(a) to 15(g). First,  $\tilde{\Gamma}_2$  is homotoped from the position in Fig. 13 to that in Fig. 15(a). Now observe that there are 2 distinct loops in  $\tilde{\Gamma}_2$  which are homotopic to each other in  $G_1 \times I$ . Let  $Y'$  be the graph in  $G_1 \times I$  shown in Fig. 15(b). Then  $p_{1*} \pi_1(Y') = p_{1*} \pi_1(\tilde{\Gamma}_2)$ . Perform

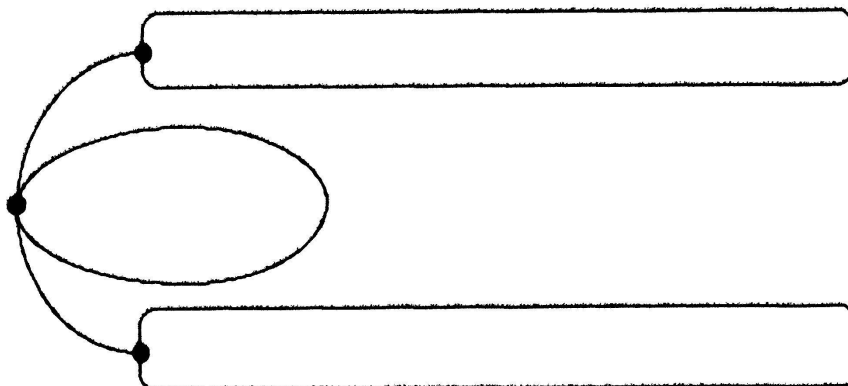


Figure 22

the sequence of homotopies of  $Y'$  in  $G_1 \times I$  shown in Figs. 15(c) to 15(g) to transform  $Y'$  into  $Y$ .  $\square$

The graph  $Y$  shown in Fig. 15(g) lies in a regular neighborhood of a component of  $\tilde{\Gamma}_1$ . The image of  $Y$  and  $G_1$  under the projection  $p_1$  is shown in Fig. 16. Topologically  $Y$  is a wedge of two circles, the projection of which are the two loops  $\alpha, \beta$  in  $G_1$  shown in Fig. 16. The vertex of  $Y$  projects to the point  $v$  in Fig. 16 on the intersection of  $\alpha$  and  $\beta$ . Thus  $p_1(\tilde{\Gamma}_2)$  contains 2 loops which are homotopic to the 2 loops  $\alpha$  and  $\beta$  in  $G_1$  shown in Fig. 16. The loops  $\alpha$  and  $\beta$  fill  $G_1$  and so cannot be homotoped to be disjoint from any essential closed curve such as  $\gamma$ . This contradicts the disjointness of  $\tilde{A}$  and  $\tilde{\Gamma}_2$ , proving that no annulus  $\tilde{A}$  can exist in this case.

The next case that we consider is that  $\delta$  is contained in  $G_1$ . Since  $\tilde{M} \cup \tilde{N}_2 = G_1 \times I$ , there is an annulus  $A'$  in  $G_1$  with the same boundary as  $\tilde{A}$ . It follows that the torus  $\tilde{A} \cup A'$  bounds a solid torus  $T$  in  $G_1 \times I$  on one side. We may perform an isotopy of  $G_1 \times I$  so that  $T = A' \times [0, 1/2]$ . If  $T$  contains  $\tilde{\Gamma}_2$  then  $\gamma \times I$  is an essential annulus disjoint from  $\tilde{\Gamma}$  which cannot exist by the previous case. Otherwise if  $T$  does not contain  $\tilde{\Gamma}_2$  then  $T$  is a solid torus in  $\tilde{M}$  and so  $\tilde{A}$  is boundary parallel in  $\tilde{M}$ . But this implies that  $A$  is boundary parallel in  $M$ , a contradiction.

The last case is that  $\delta$  is contained in  $\partial_2 \tilde{M}$ .

**LEMMA 2.5.**  *$\gamma$  is isotopic in  $G_1$  to the curve labelled  $\alpha$  in Fig. 16.*

*Proof.* We first observe that  $\delta$  is an essential  $G_1 \times I$  and that  $\delta$  can be homotoped in  $G_1 \times I$  into  $\tilde{\Gamma}_2$ , and thus homotoped into an essential loop in  $Y$ . It follows that  $p_1 \delta$  is freely homotopic into  $p_1(Y)$ . Let  $v$  be the point in  $G_1$ , shown in Fig. 16, which is the image under  $p_1$  of the vertex in the graph  $Y$ . We claim that the only non-trivial element of  $p_{1*} \pi_1(Y)$  which is homotopic to an essential simple closed curve is  $\alpha^{\pm 1}$ . To see this, let  $\pi: \tilde{G}_1 \rightarrow G_1$  be the covering of  $G_1$  corresponding to the subgroup  $p_{1*} \pi_1(Y)$  of  $\pi_1(G_1)$ . Then  $\tilde{G}_1$  is a punctured torus, on which there are unique lifts  $\tilde{\alpha}, \tilde{\beta}$  of  $\alpha, \beta$ . Now  $\tilde{\alpha}, \tilde{\beta}$  intersect in a single point lying over  $v$  as shown in Fig. 18. Also  $\gamma$  is homotopic to  $p_1 \delta$  and therefore lifts to a loop  $\tilde{\gamma}$  on  $\tilde{G}_1$ . If  $\tilde{\gamma}$  cannot be homotoped in  $\tilde{G}_1$  into  $\tilde{\alpha}$ , then  $\tilde{\gamma}$  runs around  $\tilde{\beta}$  and intersects other components of  $\pi^{-1}(\beta)$  because  $\beta$  has an essential self-intersection on  $G_1$ , and therefore  $\tilde{\gamma}$  intersects other components of  $\pi^{-1}(\gamma)$ . But this contradicts the simplicity of  $\gamma$  and proves the lemma.  $\square$

We have shown that  $\gamma$  is isotopic in  $G_1$  to  $\alpha$  and thus the boundary component of  $A$  on  $\partial_1 M$  is isotopic to  $\varepsilon = p(\alpha)$ . By tracing the loop  $\alpha$  back through the Figs. 15(g) to 15(a), we see that  $\alpha$  is homotopic in  $G_1 \times I$  to the loop  $p^{-1}(E)$  shown in Fig. 17. Thus  $\alpha$  is homotopic in  $G_1$  to the loop labelled  $\alpha$  in Fig. 17. Hence  $\varepsilon = p(\alpha)$

is homotopic in  $\partial M_1$  to the loop labelled  $\varepsilon$  in Fig. 19. Applying the involution  $\tau$  we see that the other boundary component of  $A$  must be isotopic in  $\partial_2 M$  to  $\tau\varepsilon$ . From Fig. 19 one sees that  $\varepsilon$  is contractible in  $M \cup N_1$  and hence that  $\tau\varepsilon$  is contractible in  $M \cup N_2$ . The annulus  $A$  provides a free homotopy from  $\varepsilon$  to  $\tau\varepsilon$ , and thus  $\varepsilon$  is contractible in  $M \cup N_2$  also. We compute the homotopy class  $[\varepsilon] \in \pi_1(M \cup N_2)$  from Fig. 19, and see that it is non-trivial. This contradicts the existence of the annulus  $A$  in this last case, and proves 1.2(3), completing the proof of the proposition.  $\square$

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University of California  
 Department of Mathematics  
 Santa Barbara, CA 93106-3080  
 USA

Received July 29, 1993; February 20, 1995