Fake spherical spaceforms of constant positive scalar curvature.

Comment. Math. Helvetici 71 (1996) 1–40 0010–2571/96/010001–40\$1.50 + 0.20/0

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If M^n is a compact riemannian manifold, the global topological properties of M^n often restrict the curvature properties of the riemannian metric. For example, the solution of the classical spaceform problem shows that M^n admits a metric with constant positive sectional curvature only if its universal covering manifold $\tilde{M}^n$ is the sphere S^n. In fact, a stronger conclusion is true: Up to a positive scale factor $\tilde{M}^n$ is isometric to the standardly embedded n-sphere in $\mathbb{R}^{n+1}$. This paper deals with a converse problem: If M^n is a smooth manifold such that $\tilde{M}^n$ is homeomorphic to S^n, what sorts of positive curvature properties can be realized by some riemannian metric on M^n? The weakest of these properties is positivity of the *scalar curvature* function $k : M^n \to \mathbb{R}$ which is essentially an iterated average value for the sectional curvature (see [LM, p. 60]). Results of N. Hitchin imply that metrics with positive scalar curvature need not exist. Specifically, this happens if $n = 8k + 1 \geq 9$ and the universal covering does not bound a spin manifold (see Hi, p. 42] or [LM, Thm. II.8.12, p. 162]); simply connected examples of this sort are well known (compare [Hi, p. 44] or [LM, Thm. II.8.13, p. 162]), and examples with nontrivial fundamental groups are given by taking connected sums of the simply connected examples with lens spaces whose (cyclic) fundamental groups have odd order. Our main results provide a converse to Hitchin's result and the preceding observations: If $n \geq 5$ and $\tilde{M}^n$ bounds a spin manifold, then M^n admits a riemannian metric with positive scalar curvature. Furthermore, if the fundamental group $\pi_1(M^n)$ has even order, then such a metric always exists (the fundamental group $\pi_1(M^n)$ must be finite because the universal covering is compact).

Complete riemannian manifolds with constant positive sectional curvature all have the form S^n/G, where G acts freely on S^n via some homomorphism $G \to O_{n+1}$, and are often called *linear spherical spaceforms.* A smooth manifold M^n will be called a *fake (smooth) spherical spaceform* if its universal covering is homeomorphic to S^n but M^n is not diffeomorphic to a linear spaceform. By the solution of the

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The first named author was partially supported by NSF Grants DMS 89-01583 and 91-01575, and the second named author was partially supported by NSF Grants DMS 86-02543, 89-02622, 91-02711, and the Mathematical Sciences Research Institute (Berkeley, CA).

Generalized Poincaré Conjecture one can replace "homeomorphic" by "homotopy equivalent" in dimensions $\neq 3$, but in general $\widetilde{M}^n$ need not be diffeomorphic to S^n. Such manifolds have been studied extensively by topologists over the past quarter centry (*cf.* [DM, Md1–2, MThW]), and questions about positive curvature can be viewed as a first step towards understanding the geometric properties of such manifolds (*e.g.*, see [Md2, Question, p. 98]).

With the preceding terminology our main results can be stated as follows:

THEOREM. *Let M^n be a fake spherical spaceform with $n \geq 5$ and let G be the fundamental group of M^n.*

(A) *If $n \not\equiv 1, 2 \bmod 8$, then M^n admits a riemannian metric with constant positive scalar curvature.*

(B) *If $n \equiv 1, 2 \bmod 8$ and G has even order, the same conclusion holds.*

(C) *If $n \equiv 1 \bmod 8$ and G has odd order, then M^n admits a riemannian metric with constant positive scalar curvature if and only if $\widetilde{M}^n$ does; more precisely, such a metric exists if $\alpha(\widetilde{M}^n) = 0$ in $KO^{-n}(\{pt\})$, where α is the characteristic number associated to the KO-theoretic Dirac orientation on MSpin, and no metric with positive scalar curvature exists if $\alpha(\widetilde{M}^n) \neq 0$.*

Remarks.

1. The theorem does not specifically mention the case where $\dim M^n \equiv 2 \bmod 8$ and G has odd order, but this is covered by the results of [GL1] because $G \cong \{1\}$ is the only possibility.

2. The number $\alpha(M^n)$ is the one considered in [Hi] (also see [LM]); it is denoted by π^0 in [ABP] and [Stg].

3. Three-dimensional manifolds with metrics of positive scalar curvature have been studied by R. Schoen and S. T. Yau [SY2] and also by R. Hamilton [Ha1]. Thurston's geometrization conjecture for 3-manifolds implies that all 3-manifolds with finite fundamental groups are diffeomorphic to linear spherical spaceforms.

4. Although fake spherical spaceforms are known to exist in dimension 4 (compare [CS, FS]), very little is known about their curvature properties (the results of [Ha2] are the best currently known).

5. The possibilities for G in the theorem were completely determined by I. Madsen, C. Thomas, and C. T. C. Wall [MThW]; specifically, for each prime p dividing the order of G, all subgroups of order p^2 and $2p$ are cyclic. In contrast, the fundamental groups of linear spaceforms satisfy an additional condition – for all pairs of primes p, q dividing the order of G every subgroup of order pq is cyclic (see [Wo]).

6. A closed smooth manifold Σ^n that is homotopy equivalent to S^n automatically bounds a spin manifold if $n \not\equiv 1, 2 \bmod 8$ or $n \leq 2$. If $n \geq 3$, then Σ^n bounds a spin manifold if and only if $\alpha(\Sigma^n) = 0$ (see [ABP]).

Outline of the proof

By the positive solution of the Yamabe problem (see [Shn]), it suffices to show that one can find metrics with (possibly variable) positive scalar curvature in the appropriate cases (a brief summary of this topic appears in [RS, Section 1]). The next steps are elaborations of results in our previous paper [KwS] (which in turn uses earlier work of Gromov-Lawson, Schoen-Yau, and J. Rosenberg [R1–3]). Specifically, by the methods of [KwS] it suffices to consider fake spherical spaceforms whose fundamental groups are 2-groups. The proof then splits into cases depending upon the dimension of M^n mod 4. For even values of n the only spherical spaceforms are fake real projective spaces, and the existence of metrics with positive scalar curvature follows from results of Rosenberg and S. Stolz (*i.e.*, [RS, Thm. 5.3(6)–(7)]), so therefore it suffices to consider cases where n is odd. If $n \equiv 1 \bmod 4$ the result is established by proving a special case of a general conjecture due to Rosenberg [R4]. We verify this using methods developed by Stolz to characterize the closed 1-connected manifolds with metrics of positive scalar curvature [Stz]. In the remaining cases the initial step is to notice that a fake spherical spaceform has the homotopy type of a linear spaceform if its fundamental group is a finite 2-group (*cf.* [DM], [Md1]). This suggests a more general problem:

PROPAGATION QUESTION. *If M^n has a riemannian metric with positive scalar curvature and $h: N^n \to M^n$ is a homotopy equivalence, does N^n also have such a metric?*

The results of [Hi] and [GL1] answer this completely when $M^n = S^n$; the answer is yes if and only if N^n bounds a spin manifold. More generally, the results of [GL] and [SY] on surgery and positive scalar curvature imply that the answer to the Propagation Question only depends upon the normal cobordism class of the homotopy equivalence. Thus it suffices to determine which bordism classes of degree 1 normal maps have representatives $h: P \to M$ where P has a metric with positive scalar curvature. This analysis has several parts. The classes of degree 1 normal maps are in 1-1 correspondence with the abelian groups of homotopy classes $[M^n, F/O]$ given by stable vector bundles over M^n with stable fiber homotopy trivializations. Since it suffices to consider cases where $\pi_1(M^n)$ is a finite 2-group, one can reduce further to examination of the localized normal invariant in the localized group $[M, F/O]_{(2)}$. The solution of the Adams Conjecture then yields a splitting $F/O_{(2)} \sim BSO_{(2)} \times \mathrm{Cok}\, J_{(2)}$ that passes to a (nonadditive!) splitting of $[M, F/O]_{(2)}$ into $\widetilde{KO}_+(M)_{(2)} \times [M, \mathrm{Cok}\, J_{(2)}]$, where KO_+ denotes the kernel of the Stiefel–Whitney class map

$$w_1 : \widetilde{KO}(M) \to H^1(M; \mathbb{Z}_2).$$

The first factor of the splitting is relatively easy to compute, but the second is highly nontrivial (the homotopy groups of Cok $J_{(2)}$ are the "bad" part of the 2-primary stable homotopy groups of spheres). Fortunately, the answer to the Propagation Question turns out to be independent of the second coordinate of the 2-localized normal invariant; the crucial point in proving this is a result of V. Snaith on triviality of $\widetilde{KO}(\mathrm{Cok}\, J_{(2)})$ [Sn1]. Analaysis of the $BSO_{(2)}$-component requires a variety of tricks from KO-theory, homotopy theory, and the representation theory of compact Lie groups. As in the work of Stolz [Stz1], fiber bundle constructions provide important examples of manifolds with metrics of positive scalar curvature. Another important theme in our work is the analysis of bordism classes for degree 1 normal maps in terms of normal maps with other degrees.

This paper is divided into six sections. The first section introduces some necessary terminology and contains some straightforward variants of some results in [Stz1–2]. In Section 2 we prove the main result when $n \equiv 1 \bmod 4$; the argument is similar to the proof of [RS, Thm. 5.3(4)]. Section 3 develops a theory of oriented normal maps whose degrees are arbitrary integers; this is similar to the nonoriented theories in [BrM] and [HM] for which the degree is a nonnegative integer, but the extra orientation data allow one to construct a well defined sum operation by disjoint union. The general setting for the Propagation Question is presented in Section 4, and the final two sections (5 and 6) deal with the remaining cases in which the dimension is congruent to 3 mod 4 and the fundamental group is either a cyclic or generalized quaternionic 2-group. Separate techniques are required for these two subcases; the generalized quaternionic case is treated in Section 5, and the cyclic case is treated in Section 6.

1. Stable splittings and reduction principles

If M^n is a closed spin manifold of dimension ≥ 5 and $G = \pi_1(M)$ is a finite 2-group, then the techniques of Stolz [Stz2] show that M^n has a metric with positive scalar curvature if a characteristic class in the connective KO-homology of $K(G, 1)$ is trivial. In this section we shall give analogous results for certain *semispin manifolds* M^n such that M^n is not a spin manifold but its universal covering is a spin manifold. The basic examples for our purposes are linear and fake spherical spaceforms whose dimensions are not congruent to 3 mod 4.

We begin by recalling some elementary facts about the fundamental groups and the first two Stiefel–Whitney classes of linear spherical spaceforms whose fundamental groups are nontrivial 2-groups.

OBSERVATION 1.1. *Let M^n be a linear spaceform whose fundamental group G is a finite 2-group, and let $k : M^n \to K(G, 1)$ be 2-connected.*

(i) *If* $n \equiv 3 \bmod 4$ *then* M^n *is a Spin manifold and* G *is either cyclic or generalized quaternionic.*
(ii) *If* $n \equiv 1 \bmod 4$ *then* G *is cyclic,* $w_1(M^n) = 0$, *and* $w_2(M^n) = \beta k^* \iota$, *where* $\iota \in H^1(K(G, 1); G)$ *corresponds to the identity and* β *is the Bockstein operator for the short exact sequence* $0 \to 2G \to G \to \mathbb{Z}_2 \to 0$.

CONSEQUENCE 1.2. *If* $n \equiv 1 \bmod 4$ *then* M^n *is a* $Spin^c$ *manifold, where* $Spin^c$ *is the homotopy pullback in the following diagram*:

$$\begin{array}{ccc} Spin^c & \longrightarrow & K(\mathbb{Z}, 1) \\ \downarrow & & \downarrow \\ SO & \xrightarrow{\text{2-conn}} & K(\mathbb{Z}, 1) \end{array}$$

In case (i) the classifying map $M^n \to BSpin$ associated to the *Spin* structure and the 2-connected map k combine to yield a 2-connected map from M^n to $BSpin \times K(G, 1)$, and the surgery invariance principle of [GL, SY] shows that if $n \geq 5$ then M^n has a metric with positive scalar curvature if and only if the bordism class $[M^n$, structure; $M^n \to K(G, 1)]$ in $\Omega_n^{Spin}(K(G, 1)$, has a representative $[N^n, \ldots]$ for which N^n has such a metric.

We need a similar principle when $n \equiv 1 \bmod 4$, but we cannot use $BSpin^c$ because the lifting $M^n \to BSpin^c$ of the normal bundle classifying map $M^n \to MSO$ is not 2-connected. The appropriate classifying spaces in this case are the spaces $Y(G, \beta)$ that are the colimits of the spaces $Y_n(G, \beta)$ constructed in [KwS, pp. 282–283]. Specifically, $Y(G, \beta)$ is the homotopy fiber of the maps $BSO \times K(G, 1) \to K(\mathbb{Z}_2, 2)$ corresponding to

$$w_2 \times 1 + 1 \times \beta(\iota) \in H^2(BSO \times K(G, 1); \mathbb{Z}_2).$$

PROPOSITION 1.3. *The space* $Y(G, \beta)$ *is homotopy equivalent to* $BSpin \times K(G, 1)$, *and the canonically associated Thom spectrum* $Th(G, \beta)$ *is stably homotopy equivalent to* $MSpin \wedge S^{-2}(K(G, 1)/C)$, *where* $C \cong S^1 \subseteq K(G, 1)$ *is the* 1*-skeleton in the standard cell decomposition of* $K(G, 1)$ *with one cell in each nonnegative dimension.*

Proof. (*Sketch*). The idea is standard and resembles the proofs that $BSpin^c \simeq BSpin \times K(\mathbb{Z}, 2)$ and $MSpin^c \simeq MSpin \wedge S^{-2}(K(\mathbb{Z}, 2))$ (compare [Stg]). A map $f_n : BSpin_n \times K(G, 1) \to Y_{n+2}(G, \beta)$ is defined by taking the direct sum of the standard n-plane vector bundle over $BSpin_n$ and the canonical complex line bundle over BG (recall that G is cyclic). The maps f_n pass to a homotopy equivalence on the stable level; the assertion about Thom spectra follows because the stable Thom spectrum on the left is just $MSpin \wedge S^{-2}(K(G, 1)/1$ – skeleton). □

PROPOSITION 1.4. *Let M^n be as in Proposition* 1.1 *with* $n \equiv 1 \bmod 4$. *Then M^n admits a $Y(G, \beta)$ structure, and M^n has a metric with positive scalar curvature if*

$$\alpha[M^n, \ldots] \in bo_n(S^{-2}(K(G, 1)/C))_{(2)}$$

is equal to $\alpha[N^n, \ldots]$ where N^n has a metric with positive scalar curvature.

Proof. (*Sketch*) The existence of the $Y(G, \beta)$ structure is established in [KwS, p. 283], and the balance of the argument is formally parallel to the reasoning in [RS, Section 5] for the nonorientable cases. □

2. The nonspin cases

In this section we shall consider fake spherical spaceforms M^n where $n \geq 5$ and $n \equiv 1 \bmod 4$. Observation 1.1 and elementary considerations show that M^n is a spin manifold if and only if $\pi_1(M^n)$ has odd order (see also [KwS, pp. 281–282]). Since the positive scalar curvature properties in the odd order case are completely determined by the results of [KwS, Section 1], we shall also assume that $\pi_1(M^n)$ has even order henceforth. In fact, the first steps in our approach are already contained in [KwS], and the result for $n \equiv 1 \bmod 4$ is essentially a combination of these steps and the verification of a conjecture due to Rosenberg in certain cases (see [RS, Thm. 5.3(5)]).

As noted in the proof of [KwS, Thm.2.1, Case 2, pp. 282–283], if M^n is a fake spherical spaceform of dimension $n = 4k + 1 \geq 5$ and $G = \pi_1(M^n)$ has even order, then the Sylow 2-subgroup is cyclic and the second Stiefel–Whitney class is nontrivial. The methods of [KwS] also yield the following reduction:

PROPOSITION 2.1. *Let M^n be a fake smooth spherical spaceform of dimension $2m + 1 \geq 5$, let N^n be the covering manifold associated to a Sylow 2-subgroup of $\pi_1(M^n)$, and assume that N^n has a riemannian metric with positive scalar curvature. Then M^n also has such a metric.*

Proof. Let p be a prime dividing the order of $G = \pi_1(M)$ and let N_p be the covering associated to a Sylow p-subgroup; then the techniques of [KwS] show that M^n has a metric with positive scalar curvature if and only if N_p does for all primes p dividing the order of G. If $p = 2$ this is given; on the other hand, if $p > 2$ then [KwS, Cor. 1.9] implies that N_p admits such a metric if and only if its universal covering $\tilde{N}_p$ does. But if p is odd then $\tilde{N}_p = \tilde{M} = \tilde{N}_2$, and this manifold admits a metric with positive scalar curvature because N_2 admits such a metric. □

From now on let G be a nontrivial cyclic 2-group, and let $Y(G, \beta)$ be the space considered in Section 1. This space is similar to the classifying space for $Spin^c$ structures on manifolds. Since these structures are given by liftings of the second Stiefel–Whitney class to $H^2(M^n; \mathbb{Z})$, it follows that every $Y(G, \beta)$ structure defines a $Spin^c$ structure (because β lifts canonically to $H^2(G; \mathbb{Z}) \approx G$). The following result shows that $Y(G, \beta)$ and $BSpin^c$ are clsoely related.

PROPOSITION 2.2. *The space $Y(G, \beta)$ is homotopy equivalent to the total space of the principal fibration*

$$\omega : S^1 \hookrightarrow Y(G, \beta) \to BSpin^c$$

with characteristic class

$$|G| \cdot \text{generator} \in H^2(BSpin^c; \mathbb{Z}) \approx \mathbb{Z}.$$

Proof. The spaces $BSpin^c$ and $Y(G, \beta)$ are homotopy fibers of maps from $BSO \times K(\mathbb{Z}, 2)$ and $BSO \times K(G, 1)$ to $K(\mathbb{Z}_2, 2)$; specifically, the restrictions of $BOS \times K(\mathbb{Z}, 2) \to K(\mathbb{Z}_2, 2)$ to the first and second factors are given by the second Stiefel–Whitney classes, and the restrictions of $BSO \times K(G, 1) \to K(\mathbb{Z}_2, 2)$ to the factors are given by the second Stiefel–Whitney class and the Bockstein β' respectively. This implies that

$$\begin{array}{ccc} Y(G, \beta) & \longrightarrow & BSpin^c \\ \downarrow & & \downarrow \\ BSO \times K(G, 1) & \xrightarrow{1 \times \beta'} & BSO \times K(\mathbb{Z}, 2) \end{array} \qquad (2.3)$$

is a pullback square. Since $\beta' : K(G, 1) \to K(\mathbb{Z}_2, 2)$ is a principal S^1-bundle classified by $|G|$ times the generator of the 2-dimensional integral cohomology, the conclusion of the proposition follows. □

As in [KwS, Section 2] we shall let $Th(G, \beta)$ denote the Thom spectrum associated to $Y(G, \beta)$; the usual transversality arguments imply that the stable homotopy groups $\pi_*^S(Th(G, \beta))$ are isomorphic to the bordism groups $\Omega_*(G, \beta)$ of manifolds with $(Y, (G, \beta) \to BO)$-structures on their stable normal bundles (more precisely, in the setting of [Stg, Chapter II] we take the structure associated to the maps $Y_n(G, \beta) \to BO_n$). It is fairly easy to show that the groups $\Omega_{2k+1}(G, \beta)$ are finite; this fact was noted in [KwS, Section 2], and it played a crucial role in the proof of [KwS, Thm. 2.1]. In this paper we shall need more precise information.

The first step is a direct consequence of Proposition 2.2; namely, the map $Y(G, \beta) \to BSpin^c$ defines a morphism of Thom spectra $\rho : (G, \beta) \to MSpin^c$. This is useful because the homotopy groups of $MSpin^c$ can be described quite well via the equivalence of the Thom spectra $MSpin^c \simeq S^{-2}K(\mathbb{Z}, 2) \wedge MSpin$ associated to the homotopy equivalence $BSpin^c \simeq K(\mathbb{Z}, 2) \times BSpin$ (compare [Stg, p. 354]). In particular, the bordism classes in $\Omega_*^{Spin^c} \cong \pi_*^S(MSpin^c)$ are detected by characteristic numbers over $\mathbb{Z}_2$ and the rationals [Stg, p. 337]. The following result establishes an even closer relationship between $\Omega_*(G, \beta)$ and $Spin^c$ bordism.

THEOREM 2.4. *The homomorphisms $\rho_* : (G, \beta) \to \Omega_*^{Spin^c}$ fit into a long exact sequence of graded Ω_*^{Spin} modules*

$$\cdots \longrightarrow \Omega_{k-1}^{Spin^c} \xrightarrow{\omega^!} \Omega_k(G, \beta) \xrightarrow{\rho_*} \Omega_k^{Spin^c} \longrightarrow \Omega_k^{Spin^c} \longrightarrow \cdots$$

where $\omega^!$ sends a $Spin^c$-manifold $(M, f : M \to BSpin^c)$ into the circle bundle that is the pullback of f and $Y(G, \beta) \to BSpin^c$. This sequence is canonically isomorphic to the twisted Gysin sequence in Ω_*^{Spin} homology associated to the map of Thom spectra $\rho' : K(G, 1)/C \to K(\mathbb{Z}, 2) \simeq K(\mathbb{Z}, 2)^\gamma$ associated to the Bockstein $\beta' : K(G, 1) \to K(\mathbb{Z}, 2)$ arising from the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{|G|} \mathbb{Z} \to G \to 0$$

and the universal complex line bundle γ over $K(\mathbb{Z}, 2)$.

Results of this type are fairly well known to workers in the area, but the complete derivations are not well documented in the literature; therefore we shall discuss the proof of Theorem 2.4 at the end of this section for the sake of completeness.

Bordism classes with positively curved representatives

As in [KwS, p. 283], let $\mathrm{Pos}_k(G, \beta) \subseteq \Omega_k(G, \beta)$ denote the bordism classes that can be represented by a manifold M^n (with appropriate extra data) such that M^n has a riemannian metric with positive scalar curvature. The usual arguments show that $\mathrm{Pos}_*(G, \beta)$ is a graded subgroup of $\Omega_*(G, \beta)$, and the main objective of this section is the following description of $\mathrm{Pos}_*(G, \beta)$.

THEOREM 2.5. *If $\alpha_{4m} : \Omega_{4m}(G, \beta) \to \mathbb{Q}$ is defined by the $\hat{A}$-genus, then* $\mathrm{Pos}_{4m}(G, \beta)$ *is equal to the kernel of α_{4m}. If $k \not\equiv 0$* mod 4, *then* $\mathrm{Pos}_k(G, \beta) = \Omega_k(G, \beta)$.

Proof of 2.5. The first step is to note that $\mathrm{Pos}_{4m}(G, \beta)$ is contained in the kernel of α_{4m}. Let $(M^{4m}; g : M \to K(G, 1), \ldots)$ represent $u \in \mathrm{Pos}_k(G, \beta)$, where M has a metric with positive scalar curvature, and let $M' \to M$ be the regular covering determined by g. It follows that M' is a spin manifold, and therefore the $\hat{A}$ genus of M' is zero. But this genus clearly satisfies $\hat{A}(M') = |G|\hat{A}(M)$, and therefore $\hat{A}(M) = 0$.

The second step in the proof is to show that $\omega^!(\Omega_{k-1}^{Spin^c})$ is contained in $\mathrm{Pos}_k(G, \beta)$ if $k \geq 5$. To see this, we first note that each element of $\Omega_{k-1}^{Spin^c}$ has a representative of the form $M^{k-1}; g : M \to K(\mathbb{Z}, 2) \cdots)$, where M is 1-connected and has a metric with positive scalar curvature. In fact, by surgery and taking connected sums with the nonspin sphere bundle over S^2 one can find 1-connected nonspin representatives of all bordism classes if $k > 5$, and by [GL2] all such manifolds have metrics with positive scalar curvature; the existence of similar representatives if $k = 5$ follows directly from the structure of $\Omega_4^{Spin^c}$ as described in [Gk, Thm. 3.1.4, pp. 205–206]. Given such a representative for a class u, it follows that $\omega^!(u)$ is represented by the circle bundle $g^*K(G, 1) \to M$ with its induced (G, β) structure, and standard results (*cf.* [Na], p. 250) now imply that $g^*K(G, 1)$ has a metric with positive scalar curvature so that $\omega^!(u) \in \mathrm{Pos}_k(G, \beta)$.

The third step in the proof is considerably deeper and requires the full strength of Stolz's methods [Stz2]:

STEP III. *Let bo be the connective KO-spectrum, and let $D : MSpin \to bo$ be the Dirac orientation as in [KwS] or [RS]. Then* $\mathrm{Pos}_k(G, \beta)$ *contains the kernel of the composite*

$$\tilde{D}_* : \Omega_k(G, \beta) \cong \tilde{\Omega}_{k+2}^{Spin}(K(G, 1)/C) \xrightarrow{D_*} \tilde{bo}_{k+2}(K(G, 1)/C).$$

Proof. Let PSp_3 be the projective symplectic group $Sp_3/\{\pm I\}$, and consider the natural action of PSp_3 on the quaternionic projective plane $\mathbb{K}\mathbb{P}^2$ by projective transformations. Recall that the approach of [Stz2] invovles a *Grothendieck bundle transfer* $f^\# : S^8(BPSp_{3+}) \wedge MSpin \to MSpin$ (in the sense of Boardman [Bd]) determined by the associated fiber bundle $\mathbb{K}\mathbb{P}^3 \to E \to BPSp_3$. The Diarac map yields a splitting $MSpin_{(2)} \simeq bo_{(2)} \vee \overline{MSpin}$ such that the localization of $f^\#$ at 2 factors through $\overline{MSpin}$ and the induced factorization $\bar{f}^\# : S^8(BPSp_{3+}) \wedge Spin_{(2)} \to \overline{MSpin}$ is a retraction in the homotopy category of spectra. A similar construction yields a Grothendieck bundle transfer

$$h^\# : S^8(BPSp_{3+}) \wedge Th(G, \beta) \to Th(G, \beta)$$

with a corresponding geometric interpretation; namely, if we view a class in

$$\pi_{k-8}(BPSp_{3+} \wedge Th(G, \beta))$$

via a representative object $(M^{k-8}; f: M \to K(G, 1), g: M \to BPSp_3, \dots)$ then the induced stable homotopy map $h^!$ takes this to the class of the $\mathbb{KP}^2$-bundle $g^*E \downarrow M^{k-8}$ together with extra data including $g^*E \to M \to K(G, 1)$. Since PSp_3 is the structure group for this bundle and PSp_3 acts isometrically on $\mathbb{KP}^3$ with respect to its canonical riemannian metric – which has positive scalar curvature – it follows as in [Stz1] that

$$\text{Image } h^! \subseteq \mathrm{Pos}_k(G, \beta). \tag{2.6}$$

Since G is a 2-group it follows that the localization map

$$\widetilde{bo}_*(K(G, 1)/C) \to \widetilde{bo}_*(K(G, 1)/C) \otimes \mathbb{Z}_{(2)}$$

is injective, and therefore it suffices to show that

(i) $\Omega_k(G, \beta) \to \Omega_k(G, \beta)_{(2)}$ is injective,
(ii) the image of $h^!_{(2)}$ equals the kernel of $\tilde{D}_{*(2)}$.

Assertion (i) amounts to saying that $\Omega_k(G, \beta)$ has no odd torsion; this can be checked directly by localizing at an arbitrary odd prime p (because $K(G, 1)_{(p)}$ is contractible). The proof of assertion (ii) requires the following elementary consequence of the construction of the Grothendieck bundle transfer:

(2.7). *Under the equivalence from $Th(G, \beta)$ to $MSpin \wedge S^{-2}(K(G, 1)/C)$ the bundle transfer $h^{\#}$ corresponds to the smash product of $f^{\#}$ and the identity on $S^{-2}(K(G, 1)/C)$.*

A proof of (2.7) appears at the end of this section.

By (2.6) and (2.7) it follows that $\mathrm{Pos}_k(G, \beta)$ contains the image of

$$f^! : \tilde{\Omega}^{Spin}_{k+6}((K(G, 1/C) \wedge (BPSp_{3+})) \to \tilde{\Omega}^{Spin}_{k+2}(K(G, 1)/C) \cong \Omega_k(G, \beta).$$

But Stolz's results imply that the image of $f^!$ is the kernel of $\tilde{D}_*$.

Conclusion of the proof. Consider the following commutative diagram, the rows of which arise from twisted Gysin sequences:

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \tilde{\Omega}^{Spin^c}_{k-1} & \xrightarrow{\omega^!} & \tilde{\Omega}_k(G, \beta) & \xrightarrow{p_*} & \tilde{\Omega}^{Spin^c}_{k} & \longrightarrow \cdots \\
& \downarrow\cong & & \downarrow\cong & & \downarrow\cong & \\
\cdots \longrightarrow & \tilde{\Omega}^{Spin}_{k+1}(K(\mathbb{Z}, 2)) & \xrightarrow{\omega_1} & \tilde{\Omega}^{Spin}_{k+2}(K(G, 1)/C) & \longrightarrow & \tilde{\Omega}^{Spin}_{k+2}(K(\mathbb{Z}, 2)) & \longrightarrow \cdots \\
& \downarrow D'_* & & \downarrow D_* & & \downarrow D''_* & \\
\cdots \longrightarrow & \widetilde{bo}_{k+1}(K(\mathbb{Z}, 2)) & \xrightarrow{\omega_1} & \widetilde{bo}_{k+2}(K(G, 1)/C) & \xrightarrow{J_*} & \widetilde{bo}_{k+2}(K(\mathbb{Z}, 2)) & \longrightarrow \cdots
\end{array}$$

Recall that the maps D'_*, D_*, D''_* are split surjective after localization at 2. If M^{4m} and other data represent $u \in \Omega_{4m}(G, \beta)$ and M^{4m} has a metric with positive scalar curavture, then $\hat{A}(M^{4m}) = 0$ by the first step of the proof. Suppose now that $\hat{A}(M^{4m}) = 0$ (or $\dim M \not\equiv 0 \bmod 4$), and let M plus the other necessary data represent $u \in \tilde{\Omega}_k(G, \beta)$. By 1.4 it suffices to show that the 2-localization of u lies in the subgroup $\mathrm{Pos}_k\,(G, \beta)_{(2)}$.

By the second and third steps of the proof we know that $\mathrm{Pos}_k\,(G, \beta)_{(2)}$ contains the image of $\omega^!_{(2)}$ and the kernel of $D_{(2)*}$. Since the composite of $D_{(2)*}$ with the projection $\widetilde{bo}_{k+2}(K(G, 1)/C)/\mathrm{Torsion}$ is detected by the $\hat{A}$ genus, it follows that $D_* u$ is torsion.

It is well known that $\widetilde{bo}_*(K(\mathbb{Z}, 2))_{(2)}$ is torsion free; an elementary proof of this fact can be obtained by the method indicated in [MhMi, §6] (alternate references for the change of rings results mentioned there are [Lv1, Ch. I] and [Lv2, §1]). Since $D_* u$ is torsion it follows that $J_{(2)*} D_{(2)*} u = 0$ and thus $D_{(2)*} u = \omega_1 z$ for some $z \in \widetilde{bo}_{k+1}(K(\mathbb{Z}, 2))_{(2)}$. A diagram chase now shows that $D_{(2)*} u = D_{(2)*} \omega^! v$ for some v, and therefore by the first two steps of the proof and Proposition 1.4 we conclude that $u \in \mathrm{Pos}_k\,(G, \beta)$. □

Implications for spherical spaceforms

Theorem 2.5 immediately yields the main result of this paper for $(4k+1)$-dimensional fake spherical spaceforms.

THEOREM 2.8. *If M^{4k+1} $(k \geq 1)$ is a fake spherical spaceform with an even order fundamental group, then M^{4k+1} admits a riemannian metric with positive scalar curavture.*

Proof. By Proposition 2.1 it suffices to consider examples whose fundamental groups are (nontrivial) 2-groups. On the other hand, if M^n is a closed smooth oriented manifold such that $n \geq 5$ is not divisible by 4, the fundamental group $G \cong \pi_1(M)$ is a nontrivial cyclic 2-group, the second Stiefel–Whitney class of M^n is nonzero, but the universal covering $\widetilde{M}^n$ is a spin manifold, then Theorem 2.5 and surgery invariance imply that M^n admits a riemannian metric with positive scalar curvature. Finally, if $M^{4k+1}(k \geq 1)$ is a fake spherical spaceform such that $\pi_1(M)$ is a nontrivial finite cyclic 2-group, then M satisfies the conditions in the preceding sentence and therefore has a riemannian metric with positive scalar curvature. □

QUESTION. *If M is a fake spherical spaceform as in Theorem 2.8, does M have infinitely many cobordism or concordance classes of metrics with positive scalar* *curvature*? – One can combine the results of [BG] and this paper to obtain conclusions of this type for fake spherical spaceforms such that $\dim M = 4k + 3 \geq 7$ or $|\pi_1(M^n)|$ is odd.

Addendum. Proofs of technical assertions

We shall now give some of the details that were deferred in the course of proving Theorem 2.5.

Proof of Theorem 2.4. (*Sketch*) Diagram (2.3), at the end of the proof of Proposition 2.2, induces a corresponding commutative diagram of Thom spectra:

$$\begin{array}{ccc} MSpin \wedge S^{-2}(K(G,1)/C) & \xrightarrow{\cong} & Th(G,\beta) \\ \downarrow{\scriptstyle 1 \wedge S^{-2}(\rho')} & & \downarrow{\scriptstyle \rho} \\ MSpin \wedge S^{-2}(K(\mathbb{Z},2)) & \xrightarrow{\cong} & MSpin^c \end{array}$$

The theorem will follow directly from general considerations involving *Gysin bordism sequences*. Here is a version that suffices for our purposes.

PROPOSITION 2.9 (Twisted Gysin sequences). *Let $\omega : S^1 \longrightarrow E \xrightarrow{\pi} B$ be a principal S^1-bundle over a finite CW complex B, and let ξ be a high dimensional vector bundle over B. Then the stable homotopy cofiber of the induced map $E^{\pi^*\xi} \longrightarrow B^{\xi}$ of Thom complexes is the Thom complex $B^{\xi\oplus\omega}$ (where ω is identified with its* 2*-plane bundle), and the connecting homomorphisms*

$$\partial : \pi_{k+n}(B^{\xi\oplus\omega}) \to \pi_{k+n-1}(E^{\pi^*\xi})$$

have a bordism theoretic interpretation by taking induced circle bundles.

EXPLANATIONS.

(1) We use the Atiyah notation X^{α} to denote the Thom complex/spectrum for a virtual vector bundle α over X for which the dimension of the bottom cell is $\dim \alpha$ [At].

(2) By transversality every class in $\pi_{k+n}(B^{\xi\oplus\omega})$ is represented by a submanifold M^{k-2} of $\mathbb{R}^{n+k}$ with an isomorphism φ of the normal bundle of M^{k-2} with a pullback $f^*(\xi \oplus \omega)$ for some map $f : M \to B$. Since the circle bundle f^*E is canonically embedded in the total space $f^*\omega$ with trivial normal bundle, it follows that we have an associated realization of f^*E as a submanifold of $\mathbb{R}^{n+k}$ and an identification φ^* of the normal bundle of f^*E with the pullback $\tilde{f}^*(\xi \oplus \mathbb{R})$, where $\tilde{f} : f^*E \to E$

is the bundle projection. The connecting homomorphism takes the class represented by the data $\{M, \varphi, f\}$ to the class represented by the data $\{f^*E, \varphi^*, \tilde{f}\}$.

Proof of Proposition 2.9. The bundle projection from E to B factors as a composite

$$E \xrightarrow{\subset} D(\omega) \xrightarrow{\simeq} B$$

where $D(\omega)$ is the associated D^2-bundle and $q : D(\omega) \to B$ is the vector bundle projection. This yields an inclusion of Thom complexes $E^{\pi^*E} \to D(\omega)^{q^*\xi}$, and the quotient complex $D(\omega)^{q^*\xi}/E^{\pi^*\xi}$ is equal to $B^{\omega\oplus\xi}$. This proves the assertion about the stable homotopy cofiber.

The preceding observations and a standard corollary of the Blakers–Massey Theorem (see [Wh], Thm. 7.12, p. 368) yield an exact sequence

$$\cdots \longrightarrow \pi_{n+k+1}(B^{\omega\oplus\xi}) \xrightarrow{\partial'} \pi_{n+k}(E^{\pi^*\xi}) \longrightarrow \pi_{n+k}(B^{\xi}) \longrightarrow \pi_{n+k}(B^{\omega\oplus\xi}) \xrightarrow{\partial'} \cdots$$

provided $k \ll n = \dim \xi$.

The only thing remaining to prove is the assertion regarding the boundary homomorphism ∂. To see this, consider the isomorphism $C_* : \pi_{n+k}(D(\omega)^{q^*\xi}, E^{\pi^*\xi}) \to \pi_{n+k}(B^{\omega\oplus\xi}, *)$ that is implicit in the exact sequence. By transversality the elements of the domain are represented by neatly embedded manifolds $(W, \partial W) \subseteq (D^{n+k}, S^{n+k-1})$ with $\{q^*\xi \downarrow D(\omega)\}$-structured normal bundles and refinements to $\{\pi^*\xi \downarrow E\}$-structured normal bundles on the boundaries; relative groups of this sort are defined in [Stg, Chapter II]. The usual transversality considerations also show that such representatives can be made transverse to B viewed as the zero section of $D(\omega) \downarrow B$. Thus geometrically the isomorphism is given by sending the transverse inverse image of the zero section $D(\omega) \subseteq D(\omega \oplus \xi)$ into the transverse inverse image of the zero section $B \subseteq D(\omega \oplus \xi)$. Similarly, if M is a submanifold of S^{n+k} with an $\{\omega \oplus \xi \downarrow B\}$-structured normal bundle and reference map $g : M \to B$, then the pair $(g^*D(\omega), g^*E)$ with appropriate extra data will represent the inverse image of the class determined by $(M; g$, other data). Since elementary considerations show that ∂C_* is given geometrically by sending the class of $(W, \partial W; \ldots)$ to the class determined by ∂W and its extra data, the asserted description of ∂ follows immediately.

COROLLARY 2.10. *The maps of bordism groups $\Omega_k(G, \beta) \to \Omega_k^{Spin^c}$ correspond to the maps of Spin bordism groups*

$$\Omega_{k+2}^{Spin}(K(G,1)/C) \to \Omega_{k+2}^{Spin}(K(\mathbb{Z},2))$$

under the canonical equivalences of Propositions 2.2 *and* 2.9 *and thus are embedded in the following long exact twisted Gysin sequences of Ω_*^{Spin} homology groups*:

$$\cdots \to \Omega_{k+1}^{Spin}(K(\mathbb{Z}, 2)) \xrightarrow{\omega!} \Omega_{k+2}^{Spin}(K(G, 1)/C) \xrightarrow{p_*} \Omega_{k+2}^{Spin}(K(\mathbb{Z}, 2)) \to \cdots$$

Proof. (*Sketch*) Choose finite subcomplexes of $K(\mathbb{Z}, 2)$ and *BSpin* such that the inclusions are highly connected, and construct the corresponding finite approximations to $K(G, 1)/C$ and *MSpin*; similarly, choose finite approximations to $BSpin^c$ whose inclusions are highly connected, and take corresponding approximations to $Y(G, \beta)$ and the associated Thom spectra. By Proposition 2.9 one has isomorphic twisted Gysin bordism sequences for the finite approximations to $Th(G, \beta) \to MSpin^c$ and

$$MSpin \wedge S^{-2}(K(G, 1)/C) \to MSpin \wedge S^{-2}K(\mathbb{Z}, 2)$$

through some large range of dimensions. The corollary follows directly from these and the standard transversality isomorphisms $\Omega_*^{Spin}(X) \cong \pi_*^S(MSpin \wedge X_+)$. □

Proof of (2.7). (*Sketch*) Let τ be the bundle of tangents along the fibers in the total space E of the $\mathbb{KP}^2$ bundle over $BPSp_3$; as noted in [Stz1] the bundle τ is a *Spin* bundle. If $U : S^8(BSp_{3+}) \to E^{8-\tau}$ is the Umkehr map in S-theory associated to the bundle projection, where $8 - \tau$ is the 0-dimensional virtual vector bundle that is stably inverse to τ, then the Grothendieck bundle transfer for $A = MSpin$ has the form

$$\begin{array}{ccc} S^8(BSp_{3+}) \wedge A & \xrightarrow{U \wedge 1_A} & E^{8-\tau} \wedge A \\ & & \downarrow{\scriptstyle \Gamma^\bullet \wedge 1_A} \\ & & MSpin \wedge A \\ & & \downarrow{\scriptstyle \oplus} \\ & & A \end{array}$$

where $\Gamma^\bullet$ is induced by the classifying map of $-\tau$ from E to *BSpin* and $\oplus$ is the E_∞ ring spectrum structure on *MSpin* given by direct sum of vector bundles (compare [May]). We have introduced the symbol A because we want a similar formula for $A = Th(G, \beta)$. This will hold if we have an analogous direct sum pairing $MSpin \wedge Th(G, \beta) \to Th(G, \beta)$. But this is elementary to construct because the direct sum of a Spin vector bundle and a $Y(G, \beta)$ vector bundle has a canonical $Y(G, \beta)$-structure; in fact, if $\oplus$ represents this module structure on spectra then the canonical splitting of $Th(G, \beta)$ is the composite of $\oplus$ with the obvious map

$$MSpin \wedge S^{-2}(K(G, 1)/C) \to MSpin \wedge Th(G, \beta).$$

Since the module structure for $Th(G, \beta)$ and the ring spectrum structure for $MSpin$ satisfy a mixed associative law up to homotopy (by the associativity properties of direct sums of vector bundles), the validity of (2.7) follows immediately. □

3. Oriented normal maps with signed degrees

In this section we shall establish some results on normal maps of degree $\neq 1$ that will be needed later. For our purposes it is necessary to consider normal maps whose degrees have definite signs, in contrast to the normal maps of [BrM] and [HM] which have unsigned degrees. This can be done quite simply for oriented manifolds by taking bundle data involving oriented vector bundles and orientation-preserving bundle maps (the trivial bundle is taken to have a standard orientation – for example, the one associated to the ordered basis of standard unit vectors on $\mathbb{R}^k$). Specifically, one can proceed as follows:

DEFINITION. Let M be a closed oriented manifold, let ξ be an oriented vector bundle over M, and let d be an integer. The set of normal bordism classes of oriented degree d normal maps into M with oriented bundle ξ is given by taking all pairs (f, b), where $f: N \to M$ has degree d and b represents orientation preserving bundle data, and factoring out the equivalence relation generated by

(i) normal bordisms (F, B) where F; $W \to B \times I$ is a degree d maps of triads, B is orientation preserving, and the stable tangent bundle of B is the pullback of ξ to $M \times I$,

(ii) orientation preserving vector bundle isomorphisms $\xi \approx \xi'$ covering the identity,

(iii) bundle data stabilization covering the identity for which ξ is replaced by $\xi \oplus \mathbb{R}$ and b by $b \oplus \mathbb{R}$.

This definition is virtually identical to the concepts of degree $\neq 1$ normal maps in earlier work of Agoston [Ag] and G. Anderson [AnG]. The normal bordism classes obtained in this fashion will be denoted by $\Omega(M, \xi, d)$.

In the setting above an oriented degree d normal map on a connected manifold M^n corresponds to a family of classes in the homotopy group of some Thom complex $\pi_{n+k}(M^\xi)$, where $k \gg n$ and the common Hurewicz image of the classes in $H_{n+k}(M^\xi) \cong \mathbb{Z}$ is d times the generator determined by the orientations of M and ξ; one needs a family of classes because the normal map is represented by a set of classes $\varphi^\bullet_* u$ where φ runs through all orientation preserving vector bundle automorphisms of ξ and $\varphi^\bullet_*$ is the automorphism associated to the one point compactification $\varphi^\bullet$, viewed as a self homeomorphism of M^ξ.

If ξ is the oriented bundle ν_M, then the following result illustrates the usefulness of normal maps with signed degrees.

PROPOSITION 3.1. *If* ν_M *is the oriented normal bundle of* M *in* $\mathbb{R}^{n+k}$ *(where* $k \gg n$ *as usual), then disjoint union with the identity on* M *defines an isomorphism* $\Omega(M, \nu_M, d) \to \Omega(M, \nu_M, d+1)$ *for all integers* d.

Proof. Let $\varepsilon_M^* : S^{n+k} \to Th(\nu_M)$ be a degree 1 collapse map defined by the Pontrjagin–Thom construction. if $\Delta : Th(\nu_M) \to S^{n+k}$ is a degree 1 collapsing map onto the one point compactification of a coordinate disk, then there is an associated splitting

$$\pi_{n+k}(Th(\nu_M)) \cong \mathbb{Z}\varepsilon_M^* \oplus \text{Kernel } \Delta_*.$$

The set $\Omega(M, \nu_M, d)$ is then equal to

$$\Delta_*^{-1}(\{d\})/\text{Aut}_+(\nu_M)$$

where $\text{Aut}_+(\nu_M)$ is the group of homotopy classes of orientation-preserving vector bundle automorphisms of M; as before, the action is given by $[\varphi] \cdot u = \varphi_*^\bullet(\text{u})$, and $\text{Aut}_+(\nu_M)$ sends each set $\Delta_*^{-1}(\{d\})$ into itself because $\Delta = \varphi^\bullet$ is homotopic to Δ by Hopf's Theorem). Thus one has an algebraic isomorphism from $\{d\} \times \text{Ker } \Delta_* / \text{Aut}_+(\nu_M)$ to $\{d+1\} \times \text{Ker } \Delta_* / \text{Aut}_+(\nu_M)$ induced by sending the class $[v]$ represented by v into $[v + \varepsilon_M^*]$. The geometric assertion in the proposition follows because if v is represented by $f : N \to M$ and a bundle map $B : \nu_N \to f^*\nu_M$, then $v + \varepsilon_M^*$ is represented by the disjoint union $f \sqcup id_M$ and the bundle map $B \sqcup$ identity (ν_M). □

We shall also need the following comparison principle for oriented normal maps into linear spherical spaceforms:

PROPOSITION 3.2. *Let* N^n *be an oriented linear spaceform whose fundamental group* G *is a finite 2-group and* $n \equiv 3 \bmod 4$, *let* V *be a free* $(n+1)$*-dimensional* G*-module so that* $N^n = S(V)/G$, *let* $k_V : N \to BG$ *be a 2-connected map classifying the orbit space bundle, and let* $\{M^n, W, k_W\}$ *be another set of such data. Then there is an odd degree map* $f : M^n \to N^n$ *such that the following hold*:

(i) *The maps* k_W *and* $k_V f$ *are homotopic.*

(ii) *Given a 2-connected degree one normal map* $g : P \to N$, *there is a 2-connected degree one normal map* $h : Q \to M$ *such that* fh *is normally cobordant to a disjoint union of* $d = \text{degree}(f)$ *copies of* g.

Remarks

1. The hypotheses imply that the map $f^* : \widetilde{KO}(N) \to \widetilde{KO}(M)$ is an isomorphisism, and thus for each stable vector bundle ζ over M we can find a stable vector bundle ξ over N such that $f^*\xi = \zeta$.

2. Given degree 1 bundle data on an oriented manifold M^n we can always extend it to oriented bundle data because the bundle ξ is orientable (it is stably fiber homotopy equivalent to ν_M and orientability is invariant under stable fiber homotopy equivalence) and we can simply choose orientations to make the bundle map $B : \nu_N \to \xi$ orientation preserving.

Proof of Proposition 3.2. The existence of f is a standard exercise in obstruction theory; specifically, one can find a G-equivariant map $F : S(V) \to S(W)$ and since $\mathbb{Z}_2 \subseteq G$ acts by the antipodal map it follows that F has odd degree. If $f = F/G$ then f also has odd degree and condition (i) follows from the equivariance of F.

Given an oriented vector bundle ω^k over N or M, define $\Delta : Th(\omega^k) \to S^{n+k}$ to be a degree one collapsing map as in the proof of 3.1. Let ξ be an oriented vector bundle over N such that some degree 1 normal map $(X \to N, \nu_X \to \xi)$ exists, and let d be an odd integer. Then it is elementary to verify that multiplication by d maps $\Delta_*^{-1}(\{1\})$ bijectively to $\Delta_*^{-1}(\{d\})$; specifically, the kernel of Δ_* is isomorphic to $\pi_{n+k}(N_0^\xi)$, where $N_0 = N -$ disk, and since $\tilde{H}^*(N_0)$ is 2-primary the same is true for $H_*(N_0^\xi)$ and $\pi_*(N_0^\xi)$, and thus multiplication by d is bijective on Kernal Δ_*.

Next let $(h : Q \to M, \tilde{h} : \nu_Q \to \zeta)$ represent a degree 1 normal map. By Remark 1 we may write $\zeta = f^*\xi$ for some vector bundle ξ over N, and we can choose an orientation of ξ consistent with the orientation ζ. If $\tilde{f} : \zeta \to \xi$ is the associated bundle map, then its one point compactification induces a homomorphism

$$(\tilde{f})_*^\bullet : \pi_{n+k}(M^\zeta) \to \pi_{n+k}(N^\xi)$$

sending representing classes for degree 1 normal maps into (M, ζ) to representing classes for degree $d(=\deg(f))$ normal maps into (N, ξ). By the reasoning in the first paragraph it will suffice to show the existence of some degree one normal map into (N, ξ). As usual, this holds if ξ is (stably) fiber homotopy equivalent to ν_N. If F/O is the classifying space for stable vector bundles with stable fiber homotopy trivializations, then it is a straightforward exercise to show that $f^* : [N, F/O] \to [M, F/O]$ is bojective at the prime 2 and $[N, F/O] \to [N, BO]$ (giving the underlying vector bundle) is trivial at odd primes; consequently, it suffices to show that $f^*\xi$ and $f^*\nu_N$ are stably fiber homotopy equivalent. But f is a 2-local homotopy equivalence, and this implies that $cf^*\nu_N = f^*(c\nu_N)$ is stably fiber homotopy equivalent to $c\nu_M$ for some odd integer c (this is implicit in [Shz2, §1]); since $\widetilde{KO}(M)$ is 2-primary it follows that $f^*\nu_N$ and ν_M are stably fiber homotopy equivalent. Finally, $f^*\xi = \zeta$, and ζ is stably fiber homotopy equivalent to ν_M because there is a degree one normal map into (M, ζ). Combining the preceding two sentences, we conclude that $f^*\xi$ and $f^*\nu_M$ are stably fiber homotopy equivalent as required. □

Here is another comparison result that will be needed:

PROPOSITION 3.3. *Let N^n be a $\mathbb{Z}_{2^r}$ lens space where $r \geq 1$ and n is odd (if $r = 1$ then $N^n = \mathbb{RP}^n$), let ξ be a vector bundle over N^n, and let d, q be positive odd integers. Then $\Omega(N, \xi, d)$ is nonempty if and only if $\Omega(N, \xi, qd)$ is nonempty, in which case q-fold disjoint union defines an isomorphism from $\Omega(N, \xi, d)$ to $\Omega(N, \xi, qd)$.*

Proof. If $\Omega(N, \xi, d) \neq \varnothing$ then a disjoint union of q copies of some representative

$$(f : M \to N, F : \nu_M \to \xi)$$

defines a class in $\Omega(N, \xi, qd)$. Conversely, if $\Omega(N, \xi, qd) \neq \varnothing$, then one can use the argument at the end of the proof of 3.2 to show that $\Omega(N, \xi, 1) \neq \varnothing$, and by taking d-fold disjoint unions we again obtain $\Omega(N, \xi, d) \neq \varnothing$.

As before we have $\Omega(N, \xi, d) \cong \Delta_*^{-1}(\{d\})/\mathrm{Aut}_+(\xi)$. The sets $\Delta_*^{-1}(\{d\})$ are cosets of the kernel of $\Delta*$; since this kernel is a finite 2-group it follows that multiplication by the odd integer q defines an isomorphism from $\Delta_*^{-1}(\{d\})$ to $\Delta_*^{-1}(\{dq\})$. This passes to an isomorphism from $\Delta_*^{-1}(\{d\})/\mathrm{Aut}_+(\xi)$ to $\Delta_*^{-1}(\{dq\})/\mathrm{Aut}_+(\xi)$ because the action of $\mathrm{Aut}_+(\xi)$ on $\pi_{n+k}(N^\xi)$ sends the sets $\Delta_*^{-1}(\{c\})$ to themselves and the linearity of the action implies that $\varphi*(qx) = q\varphi*(x)$ for all $\varphi \in \mathrm{Aut}_+(\xi)$ and $x \in \Delta_*^{-1}(\{d\})$. □

4. Homotopy propagation of positive scalar curvature metrics

In [R3] and [R4] Rosenberg has formulated some very striking conjectures for characterizing manifolds with finite fundamental groups that have metrics of positive scalar curvature. Since the results of Section 2 for fake spherical spaceforms were consequences of special cases of Rosenberg's conjecture, one natural approach to the remaining cases would be to proceed similarly. However, our current knowledge about the relevant cases is still fragmentary (see [RS, §5]). We shall view the scalar curvature properties of fake spherical spaceforms in dimensions $4k + 3 (k \geq 1)$ as essentially a special case of the following:

PROPAGATION QUESTION 4.1. *Let M^n and N^n be closed smooth manifolds that are homotopy equivalent, and suppose that N^n has a riemannian metric with positive scalar curvature. Does M^n also admit such a metric?*

The terminology is motivated directly by the results on propagating group actions through homotopy equivalences as in work of S. Cappell, S. Weinberger, and several others (*cf.* [CW]).

Example. If $N^n = S^n$, then M^n has a riemannian metric with positive scalar curvature if and only if M^n bounds a spin manifold (*cf.* [GL1–2]). More generally, if N^n is simply connected, then the results of [STz1] show that M^n has a riemannian metric with positive scalar curvature if and only if $\alpha(M^n) = 0$ in $\mathrm{KO}^{-n}(\{pt\})$.

The preceding discussion raises another question; namely, how can one reduce the proof of our main result for $(4k+3)$-diemensional fake spherical spaceforms to the propagation question? This requires two steps:

(1) By Propositon 2.1 it suffices to consider cases where the fundamental group is a 2-group.

(2) If M^n is a fake spherical spaceform whose fundamental group is a 2-group, then M^n is homotopy equivalent to a linear spaceform. In this case the fundamental group is either cyclic or generalized quaternionic; in the cyclic case it is well known that M^n is homotopy equivalent to a lens space, and in the generalized quaternionic case this is still true but requires additional work (*e.g.*, see [Md1]).

As noted in Section 1, every fake spherical spaceform of dimension $4k+3$ is a spin manifold. Therefore the standard bordism invariance property implies that the existence of a positive scalar curvature metric on a $(4k+3)$-dimensional fake spherical spaceform N (where $k \geq 1$) only depends upon the bordism class of N, some spin structure σ_N, and a 2-connected reference map $k_N : N \to BG$ in $\Omega_{4k+3}^{Spin}(BG)$. Of course, a similar assertion holds for every closed, connected spin manifold N^n where $n \geq 5$.

The first step in handling the homotopy propagation question for positive scalar curvature is to show that the answer only depends upon the normal cobordism class. Considerations of this type were implicit in [KwS, §2]. For the sake of completeness, here is a general statement.

PROPOSITION 4.2. *Let $N^n (n \geq 5)$ be a closed connected spin manifold, let $(f : M^n \to N^n, B : \nu_M \to \xi)$ be a degree one normal map, let σ_N be a spin structure on N^n, and let $k_N : N \to BG$ be a 2-connected reference map. Suppose that $f : M^n \to N^n$ is 2-connected and $(f : M^n \to N^n, \ldots)$ is normally cobordant to a normal map $(f' : M' \to N, \ldots)$ where M' has a riemannian metric with positive scalar curvature. Then M^n also has such a metric.*

Proof. (*Sketch*) A good reference map for M is $k_M = k_N \circ f$. Also, since ξ is fiber homotopy equivalent to ν_N and spin structures are invariants of fiber homotopy type, it follows that one can move the spin structure from ν_N to ξ by the fiber homotopy equivalence and from ξ to ν_M by B. In fact, with these conventions and surgery theory to kill extra low-dimensional homotopy one can construct a well defined map from $[N, F/O] \cong$ {bordism classes of degree 1 normal maps} to $\Omega_n^{Spin}(BG)$. Given this, the existence of a positive scalar curvature metric on M^n follows immediately from the bordism invariance property. □

The next step is to analyze the map

$$B : [N^n, F/O] \to \Omega_n^{Spin}(BG)$$

constructed in the proof of Proposition 4.2. Let $q : N \to F/O$ be given, and let $(f : M \to N, F : \nu_M \to \xi)$ be a degree one normal map associated to the homotopy class of q in the usual fashion (*cf.* [Wa]). By definition, the bordism class representing $X = M$ or N is given by the composite b_X of the maps in the following diagram:

$$S^{n+k} \xrightarrow{c_X} X^\nu \xrightarrow{\Delta_2^\bullet} X_+ \wedge X^\nu \xrightarrow{(k_X+)\wedge C(\nu_X)^\bullet} BG_+ \wedge MSpin.$$

Here c_X is the degree one collapse map, $\Delta_2^\bullet$ is the map of Thom spaces induced by the diagonal $\Delta_2 : X \to X \times X$ and the identification $\nu \cong \Delta_2^*$ (0-dimensional trivial bundle $\times\ \nu$), the map k_X is a 2-connected reference map, and $C(\nu_X)^\bullet$ is the map of Thom spectra associated to the classifying map $C(\nu_X) : X \to BSpin$ for ν_X. We need to relate these composites using the data associated to the normal map.

THEOREM 4.3. *Let $X = M$ or N, let $(f : M \to N, F : \nu_M \to \xi)$ be a degree one normal map where f is 2-connected, and let k_N, etc. be as above. Then b_X is given by the composite*

$$\begin{array}{c} S^{n+k} \xrightarrow{c_N} N^{\nu_N} \xrightarrow{(\Delta_3)^\bullet} N_+ \wedge N^\xi \wedge N^{\nu_N-\xi} \xrightarrow{E_X} N_+ \wedge N^\xi \wedge N^{\nu_N-\xi} \\ \Big\downarrow k_{N+} \wedge C(\xi)^\bullet \wedge C(\nu_N-\xi)^\bullet \\ BG_+ \wedge MSpin \wedge MSpin \\ \Big\downarrow 1\wedge\oplus \\ BG_+ \wedge MSpin \end{array}$$

where $(\Delta_3)^\bullet$ is the map of Thom complexes associated to the diagonal $\Delta_3 : N \to N \times N \times N$ and the identity $\nu_N = \Delta_3^*(0 \times \xi \times (\nu_N - \xi))$, the maps $C(\omega)^\bullet$ are induced by the classifying maps of the Spin vector bundles $\omega = \xi$ and $\nu_N - \xi$, the map $\oplus$ is defined by direct sum, and E_X is given as follows:

(i) *E_N is the identity.*

(ii) *E_M is the smash product of the identity on $N_+ \wedge N^\xi$ with the composite $j \circ \rho(f, F)$ where $j : S^0 \to N^{\nu-\xi}$ is induced by fiber inclusion and $\rho(f, F)$ is an S-map $N^{\nu-\xi} \to S^0$ that is S-dual to $F^\bullet \circ c_M$.*

EXPLANATION. We are viewing $\nu - \xi$ as a zero-dimensional virtual vector bundle, so that $N^{\nu-\xi}$ has a canonical map $j : S^0 \to N^{\nu-\xi}$ corresponding to inclusion of the Thom space over a point in N. The composite $\rho(f, F) \circ j$ is the identity on S^0 in S-theory.

Proof. First of all, the map $(\Delta_3)^\bullet$ factors through $(\Delta_2)^\bullet$ as a composite $(\Delta'_2 \wedge id) \circ (\Delta_2)^\bullet$ where

$$\Delta'_2 : N^{\nu_N} \to N^\xi \wedge N^{\nu_N - \xi}$$

is induced by the diagonal map from N into $N \times N$. Since $C(\nu_N)^\bullet$ is stably homotopic to the composite

$$\oplus \circ (C(\xi)^\bullet \wedge C(\nu_N - \xi)^\bullet) \wedge k_{N+}) \circ (\Delta_3)^\bullet \circ \Delta'_2$$

it follows that b_N is given by the composite

$$(id \wedge \oplus) \circ (C(\xi)^\bullet \wedge C(\nu_N - \xi)^\bullet \wedge k_{N+}) \circ (\Delta_3)^\bullet \circ c_N$$

as claimed in (i). To prove (ii), first notice that b_M is given by the composite

$$(id \wedge \oplus) \circ (C(\xi)^\bullet \wedge C(\nu_N - \xi)^\bullet \wedge k_{N+}) \circ j' \circ (\Delta_2^\xi)^\bullet \circ F^\bullet \circ c_M$$

where $F^\bullet$ is the map of Thom complexes induced by F, the map $(\Delta_2^\xi)^\bullet : N^\xi \to N_+ \wedge N^\xi$ is induced by the diagonal, and j' is given by the smash product of j with the identity on $N_+ \wedge N^\xi$ (under the usual identification $N_+ \wedge N^\xi \equiv N_+ \wedge N^\xi \wedge S^0$). Thus the proof of (ii) is reduced to checking that $j' \circ (id \wedge \rho(f, F)) \circ (\Delta_3)^\bullet \circ c_N$ is stably homotopic to $j' \circ (\Delta_2^\xi)^\bullet \circ F^\bullet \circ c_M$; of course, it also suffices to prove the corresponding result for the shorter composites with j' removed from the left ends, and thus it remains to compare

$$(id \wedge \rho(f, F)) \circ (\Delta_3)^\bullet \circ C_N \quad \text{and} \quad (\Delta_2^\xi)^\bullet \circ F^\bullet \circ c_M.$$

Consider the following diagram

$$\begin{tikzcd}
S^{n+k} \arrow[r, "c_M"] \arrow[d, "="] & N^\nu \arrow[r, "\Delta^{\xi, \nu-\xi}"] \arrow[d] & N^\xi \wedge N^{\nu-\xi} \arrow[r, "(\Delta_2^\xi)^\bullet \wedge 1"] \arrow[d, "1 \wedge \rho"] & N_+ \wedge N^\xi \wedge N^{\nu-\xi} \arrow[d, "1 \wedge \rho"] \\
S^{n+k} \arrow[r, "c_M"] & M^\nu \arrow[r, "F^\bullet"] & N^\xi \arrow[r, "(\Delta_2^\xi)^\bullet"] & N_+ \wedge N^\xi
\end{tikzcd}$$

in which $\Delta^{\xi, \nu-\xi}$ is induced by the diagonal map. The right hand square commutes by the elementary properties of smash products, the left hand square commutes in S-theory because ρ is S-dual to $F^\bullet$, and the composite along the top row is just $(\Delta_3)^\bullet \circ c_N$. Therefore $(1 \wedge \rho)(\Delta_3)^\bullet \circ c_N \simeq (\Delta_2^\xi)^\bullet \circ F^\bullet \circ c_M$ as required by the discussion in the preceding paragraph. □

Remark. The map $C(\nu_N - \xi)^\bullet$ can be factored (in the stable homotopy category) as a composite

$$N^{\nu_N - \xi} \xrightarrow{q^\bullet} (F/O)^\gamma \xrightarrow{s^\bullet} MSpin$$

where $q : N \to F/O$ classifies the degree one normal map, γ is the universal fiber in homotopically trivial vector bundle, $q^\bullet$ is the associated map of Thom spaces, and s is the canonical lifting of the classifying map $h_\gamma : F/O \to BSO$ to $BSpin$ (note that γ has a unique Spin structure because $H^1(F/O; \mathbb{Z}_2) = 0$).

We would like to apply Theorem 4.3, the preceding remark, and the homotopy-theoretic properties of F/O to obtain usable information comparing b_M to b_N. The first result is a localization formula.

PROPOSITION 4.4. *Suppose that N^n, M^n, etc. are given as in Theorem 4.3, let $q : N \to F/O$ classify (f, F) as in the preceding remark, let p be a prime, and suppose that the image of q is zero in the localization $[N, F/O]_{(p)} \cong [N, F/O_{(p)}]$. Then the images of b_M and b_N in $\Omega_n^{Spin}(BG)_{(p)}$ are equal.*

Proof. By the basic properties of localization at p in the stable category it suffices to show that the p-localization of $q^\bullet : N^{\nu - \xi} \to (F/O)^\gamma$ factors through the p-localization of the S-map $\rho : N^{\nu - \xi} \to S^0$. Let Y be the homotopy fiber of the localization map $F/O \to F/O_{(p)}$, and let $\gamma\langle p \rangle$ be the pullback of γ to Y; then $Y_{(p)}$ is contractible, and it is an elementary exercise to show that the canonical map of S^0 into the Thom spectrum $Y^{\gamma | Y}$ is a (p)-local stable equivalence. But the triviality of q under localization implies that q factors through some map $N \to Y$, and thus the assertion about the p-localization of $q^\bullet$ follows immediately. □

COROLLARY 4.5. *Suppose we are given $(f_i : M_i \to N,\ F_i : \nu_{Mi} \to \xi_i)$ as above where $i = 1$ or 2, let $q_i : N \to F/O$ classify (f_i, F_i), and let $b_i \in \Omega_n^{Spin}(BG_+)$ be the Spin bordism class associated to M_i with the induced Spin structure and reference map $k_N \circ f_i$. If p is a prime such that the difference class $[q_2] - [q_1]$ maps to zero in $[N, F/O]_{(p)}$, then the images of b_1 and b_2 in $\Omega_n^{Spin}(BG)_{(p)}$ are equal.*

Proof. Write $[q_0] = [q_2] - [q_1]$, so that $q_0 \oplus q_1$ represents q_2. It then follows that b_1 and b_2 are given by composites $\Phi(A, B)$ of the following type:

$$\begin{array}{ccc} S^{n+k} \xrightarrow{c_N} N^{\nu_N} \xrightarrow{(\Delta_4)^\bullet} & & N_+ \wedge N^\xi \wedge N^\alpha \wedge N^\beta \\ & & \downarrow{\scriptstyle 1 \wedge 1 \wedge A \wedge B} \\ & & N_+ \wedge N^\xi \wedge N^\alpha \wedge N^\beta \\ & & \downarrow{\scriptstyle k_{N+} \wedge C(\xi)^\bullet \wedge C(\alpha)^\bullet \wedge C(\beta)^\bullet} \\ & & BG_+ \wedge MSpin \wedge MSpin \wedge MSpin \\ & & \downarrow{\scriptstyle 1 \wedge \oplus} \\ & & BG_+ \wedge MSpin \end{array}$$

The basic idea is that α and a fiber trivialization represent q_1, while β and a fiber trivialization represent q_0. Let $E(\alpha) : N^\alpha \to N^\alpha$ and $E(\beta) : N^\beta \to N^\beta$ be given by composing the compactified fiber retraction $N^\alpha \to S^0$, $N^\beta \to S^0$ with the fiber inclusions of S^0 in N^α, N^β respectively. By Theorem 4.3 we have the following conclusions:

(0) If A and B are identity maps, then $\Phi(A, B) = b_N$.

(1) If $A = E(\alpha)$ and B is the identity, then $\Phi(A, B) = b_1$.

(2) If $A = E(\alpha)$ and $B = E(\beta)$, then $\Phi(A, B) = b_2$.

One can now apply the argument proving Proposition 4.4 to conclude that $C(\beta)^\bullet$ and $C(\beta)^\bullet{}_\circ E(\beta)$ become homotopic after localization at p, and by the preceding observations it follows that b_1 and b_2 become equal after localization at p. □

As before, let N^n be a closed connected Spin manifold with fundamental group G and 2-connected reference map $k_N : N \to BG$. If p is a prime, the previous considerations show that the map

$$V_p : [N, F/O] \to \Omega_n^{Spin}(BG) \to \Omega_n^{Spin}(BG)_{(p)}$$

factors through the image of $[N, F/O]$ in $[N, F/O_{(p)}] = [N, F/O]_{(p)}$. This allows us to study V_p by means of the *Adams Conjecture splittings*

$$F/O_{(p)} \simeq BSO_{(p)} \times \operatorname{Cok} J_{(p)}$$

(see [MdMi, Chapter 5, p. 106]). If we consider the corresponding splitting

$$[N, F/O]_{(p)} \cong [N, BSO]_{(p)} \times [N, \operatorname{Cok} J_{(p)}]$$

(which is not necessarily additive!)

the first factor looks manageable at least in some cases, but the second factor is highly complicated; in particular, $\pi_k(\operatorname{Cok} J_{(p)})$ is the "bad" summand of the

p-primary component of $\pi_{n+k}(S^n)$ for $n \gg k$. In order to work effectively we must show that the second factor is irrelevant for our purposes. Since a general discussion would be quite lengthy, we shall only prove a results that suffices for cases where N^n is a linear spherical spaceform, where $n = 4k+3$ with $k \geq 1$ and $G = \pi_1(N^n)$ is a (nontrivial) finite 2-group.

Preliminary remark. For each prime p the standard map $\operatorname{Cok} J_{(p)} \to F/O_{(p)}$ lifts to F/O. This is true because it lifts to $F_{(p)}$ by the basic construction of the Adams Conjecture splitting and the finiteness of the groups $\pi_k(F)$ shows that F is a weak product of its localizations $F_{(p)}$ over all primes p.

THEOREM 4.6. *Let N^n be a closed connected Spin manifold with $n \geq 5$ and fundamental group G, let $q_1 : N \to F/O$ classify the 2-connected degree one normal map $(f_1 : M_1 \to N, F_1 : \nu_{M_1} \to \xi)$, and let b_N, b_1 be the classes in $\Omega_n^{Spin}(BG)$ representing M_1 and N. Let $q_0 : N \to \operatorname{Cok} J_{(2)}$, let $i : \operatorname{Cok} J_{(2)} \to F/O$ be the map described in the preceding paragraph, let $q_2 = q_1 \oplus iq_0$, let (f_2, F_2) represent q_2 where f_2 is 2-connected, and let $b_2 \in \Omega_n^{Spin}(BG)$ represent the domain M_2 of f_2. If $D_{(2)} : \Omega_n^{Spin}(BG)_{(2)} \to bo_n(BG)_{(2)}$ is the 2-local Dirac map, then $D_{(2)}(b_1) = D_{(2)}(b_2)$.*

Before proving this result, we shall derive its application to positive scalar curvature.

THEOREM 4.7. *Suppose we are given the setting of Theorem 4.6, and assume further that G is a finite 2-group and M_1 has a riemannian metric with positive scalar curvature. Then M_2 also has a metric with positive scalar curvature.*

Proof that 4.6. *implies* 4.7. Since G is a 2-group and bo_* has no odd torsion, it follows that all torsion in $bo_*(BG)$ is 2-primary and hence the 2-localization map is injective. Therefore the unlocalized Dirac map satisfies $D(b_1) = D(b_2)$. But b_1 is represented by M_1, which has a riemannian metric of positive scalar curvature. Since $D(b_1) = D(b_2)$ and M_2 is connected, by [Stz2, Thm. 1.1] the difference $b_2 - b_1$ is represented by a manifold with a metric of positive scalar curvature. Hence the same also holds for $b_2 = (b_2 - b_1) + b_1$, and by the 2-connectedness of $M_2 \to BG$ it follows that M_2 also admits such a metric. □

Proof of Theorem 4.6. Let $\mu : bo \wedge bo \to bo$ be the map of spectra determined by the tensor product pairing $\otimes : BO \times BO \to BO$ (see [May, Section VIII.2] for a construction of this map); this makes bo into a ring spectrum such that the Dirac map D induces a weakly multiplicative map of spectra (where $\oplus$ induces the ring spectrum structure on $MSpin$). As in the proof of Corollary 4.5, the classes $D_{(2)}(b_1)$ and $D_{(2)}(b_2)$ are given by the following composites:

$$
\begin{array}{l}
S^{n+k} \xrightarrow{c_N} N^{\nu_N} \xrightarrow{(\Delta_4)^\bullet} \quad N_+ \wedge N^{\xi} \wedge N^{\alpha} \wedge N^{\beta} \\
\qquad\qquad\qquad\qquad\qquad \downarrow 1 \wedge 1 \wedge A \wedge B \\
\qquad\qquad\qquad\qquad\quad N_+ \wedge N^{\xi} \wedge N^{\alpha} \wedge N^{\beta} \\
\qquad\qquad\qquad\qquad\qquad \downarrow k_{N+} \wedge C(\xi)^\bullet \wedge C(\alpha)^\bullet \wedge C(\beta)^\bullet \\
\qquad\qquad\qquad BG_+ \wedge MSpin \wedge MSpin \wedge MSpin \\
\qquad\qquad\qquad\qquad\qquad \downarrow 1 \wedge D_{(2)} \wedge D_{(2)} \wedge D_{(2)} \\
\qquad\qquad\qquad BG_+ \wedge bo_{(2)} \wedge bo_{(2)} \wedge bo_{(2)} \\
\qquad\qquad\qquad\qquad\qquad \downarrow 1 \wedge \mu_{(2)} \\
\qquad\qquad\qquad\qquad\quad BG_+ \wedge bo_{(2)}
\end{array}
$$

More precisely, the proof of 4.5 shows that $D_{(2)}(b_1)$ and $D_{(2)}(b_2)$ are given by composites where $1 \wedge (\wedge^3 D_{(2)})$ and $1 \wedge \mu_{(2)}$ are replaced by $1 \wedge \oplus$ and $1 \wedge D_{(2)}$ respectively, but one obtains the same classes from either pair because $D_{(2)}$ is a map of ring spectra. Most of the argument proving Corollary 4.5 also applies in the present setting; the crucial difference is that the map q_o represents the trivial class after localization in the setting of 4.5, but here $q_o : N \to F/O$ is assumed to factor through $\mathrm{Cok}\, J_{(2)}$. However, in analogy with 4.5 it suffices to prove that the composites $D_{(2)} \circ C(\beta)^\bullet$ and $D_{(2)} \circ C(\beta)^\bullet \circ E(\beta)^\bullet$ are homotopic (*i.e.*, $D_{(2)} \circ C(\beta)^\bullet$ factors homotopically through $S^0 \to bo_{(2)}$. Since q_o factors through $\mathrm{Cok}\, J_{(2)}$ we can write $D_{(2)} \circ C(\beta)^\bullet$ as

$$
N^{\beta} \xrightarrow{q_3^\bullet} (\mathrm{Cok}\, J_{(2)})^{\gamma} \xrightarrow{i^{\gamma}} F/O^{\gamma} \xrightarrow{h\gamma} MSpin \xrightarrow{D_{(2)}} bo_{(2)}
$$

and therefore it suffices to show that $D \circ h^{\gamma} \circ i^{\gamma}$ factors homotopically through S^0. But in the stable category we have

$$
\{(\mathrm{Cok}\, J_{(2)})^{\gamma}, bo\} \cong \widetilde{KO}((\mathrm{Cok}\, J_{(2)})^{\gamma}),
$$

and the right hand side is isomorphic to $\widetilde{KO}(\mathrm{Cok}\, J_{(2)+})$ by the Thom isomorphism (since γ is a Spin bundle). Since $\widetilde{KO}(\mathrm{Cok}\, J_{(2)})$ is trivial (see [Sn1, Thms. 9.3 and 9.9]) it follows that $\widetilde{KO}((\mathrm{Cok}\, J_{(2)})^{\gamma}) \cong \widetilde{KO}(S^0)$, where the isomorphism is given by the fiber induced mapping $S^0 \to (\mathrm{Cok}\, J)^{\gamma}$. Therefore $D_{(2)} \circ C(\beta)^\bullet$ is stably homotopic to $D_{(2)} \circ C(\beta)^\bullet \circ E(\beta)$ as required. □

Remarks. By [ABP] the localized spectrum $MSpin_{(2)}$ is a wedge of suspensions of the Eilenberg–MacLane spectrum $H\mathbb{Z}_2$ and the 2-local connective K-spectra $bo_{(2)}$

and $bsp_{(2)}$; furthermore, the suspended $H\mathbb{Z}_2$ summands map into wedge summands of the Thom spectrum MO, which is itself a wedge of suspended copies of $H\mathbb{Z}_2$. This splitting of $MSpin_{(2)}$ and the argument for Theorem 4.6 lead to a proof that the images of b_1 and b_2 in $\Omega_n^{Spin}(BG)_{(2)}$ are equal; however, we are omitting this because the results of [Stz2] make Theorem 4.6 sufficient for our purposes. If p is an odd prime and $(M_1 \to N, \ldots)$, $(M_2 \to N, \ldots)$ describe 2-connected degree one normal maps that differ by an element of $[N, \operatorname{Cok} J_{(p)}]$, then it also seems likely that the p-localizations of the bordism classes b_1, b_2 are equal, but we have not attempted to check this.

We shall now define KO-theoretic invariants that often determine the obstruction to propagating a positive scalar curvature metric across a homotopy equivalence. As noted in [MdMi, Section V.C] there is a H-map $\beta_2 : F/O_{(2)} \to BSO^{\otimes}_{(2)}$ whose restriction to the Adams Conjecture summand $BSO_{(2)} \subseteq F/O_{(2)}$ is an H-space equivalence from $BSO^{\oplus}_{(2)}$ to $BSO^{\otimes}_{(2)}$; the inverse to this "exponential" equivalence [AS] will be denoted by $\mathscr{L}$. IF N^n is a closed connected Spin manifold and $f : M^n \to N^n$ is a homotopy equivalence, let $n(f) \in [N, F/O]$ be its normal invariant and let $\mathscr{L}\beta_2(f) \in [N, BSO]_{(2)}$ be the image of the 2-localization $\eta(f)_{(2)}$ under $\mathscr{L}\beta_2$. The following result shows that $\mathscr{L}\beta_2(f)$ gives a sufficient condition for M to have a metric of positive scalar curvature:

THEOREM 4.8. *Let N^n be a closed connected Spin manifold such that $n \geq 5$ and $G = \pi_1(N)$ is a finite 2-group. Let $f : M^n \to N^n$ be a homotopy equivalence and suppose that $\mathscr{L}\beta_2(f) = \mathscr{L}\beta_2(g)$ where $g : M' \to N$ is a degree one normal map such that M' admits a riemannian metric of positive scalar curvature. Then N also admits such a metric.*

Proof. The condition $\beta_2(f) = \beta_2(g)$ implies that $\eta(f) = \eta(g) + q_2 + q_1$ where $q_1 \in [N, F/O]$ has odd order and $q_2 \in [N, \operatorname{Cok} J_{(2)}]$. Let $(f_3 : M_3 \to M, \ldots)$ be a 2-connected degree one normal map representing $\eta(g) + q_2$, and let $b_3 \in \Omega_n^{Spin}(BG)$ be the associated bordism class. By the bordism invariance of positive scalar curvature and surgery in low dimensions, we may also assume that $M' \to N$ is 2-connected and has a metric of positive scalar curvature. By Theorem 4.7 it follows that M_3 also admits a metric of positive scalar curvature. □

5. Fake quaternionic spaceforms

The purpose of this section is to prove the main theorem when the dimension is $4k + 3 \geq 7$ and the fundamental group is a quaternionic 2-group.

THEOREM 5.1. *Let N^{4k+3} $(k \geq 1)$ be a fake spherical spaceform whose fundamental group G is a generalized quaternionic 2-group. Then N admits a riemannian metric with positive scalar curvature.*

The observations and results of Section 4 provide the first steps in the proof. Specifically, there is a homotopy equivalence $f: N \to M_V$, where M_V is the linear spaceform associated to the free G-representation V, and the existence of a metric with positive scalar curvature only depends upon an invariant $\mathscr{L}\beta(f) \in [N, BSO]$. Since $[N, BSO]$ is isomorphic to the kernel of the first Stiefel–Whitney class $w_1 : \widetilde{KO}(N) \to H^1(N; \mathbb{Z}_2)$ and $KO(N)$ is a quotient of the real representation ring [GKa], the value gorup for $\mathscr{L}\beta(f)$ is fairly tractable; recall that the map from $RO(G)$ to $KO(N)$ is given by sending a virtual G-representation $V - W$ to the virtual flat vector bundle:

$$(\tilde{N} \times_G V \downarrow N) - (\tilde{N} \times_G W \downarrow N).$$

More generally, if G is an arbitrary compact Lie group and $\operatorname{Hom}(G, \mathbb{Z}_2)$ is the abelian group of homomorphisms from G to $\mathbb{Z}_2$, then there is a unique homomorphism from $RO(G)$ to $\operatorname{Hom}(G, \mathbb{Z}_2)$ whose value on a representation is given by taking determinants. We shall denote the kernel of this representation by $RSO(G)$. With this notation the observations of the previous paragraph state that $[N, BSO]$ is a quotient of $RSO(Q(2^r))$. The following description of this group will be needed in our approach.

PROPOSITION 5.2. *Let $Q(2^r)$ be a generalized quaternionic 2-group where $r \geq 3$, and let $A : Q(2^r) \to \mathbb{Z}_2 \times \mathbb{Z}_2$ be the abelianization map. Then $RSO(Q(2^r))$ is the sum of the images of the map $A^* : RSO(\mathbb{Z}_2 \times \mathbb{Z}_2) \to RSO(Q(2^r))$ and the restriction map defined by the standard inclusion $Q(2^r) \subset S^3$.*

Proof. (*Sketch*) Let $D(2^{r-1})$ be the dihedral group $Q(2^r)/\mathbb{Z}_2$. The nontrivial irreducible real representations of $Q(2^r)$ separate naturally into three types:

(I) One-dimensional representations given by nontrivial homomorphisms into $\mathbb{Z}_2$; there are three of these up to equivalence.

(II) Two-dimensional representations defined by pulling back representations of the quotient group $D(2^{r-1})$; there are $2^{r-3} - 1$ of these up to equivalence (hence none if $r = 3$).

(III) Four-dimensional free representations that arise from different embeddings of $Q(2^r)$ into S^3; there are 2^{r-3} of these up to equivalence.

We shall dispose of the case $r = 3$ first because it is exceptional; namely, there are no Type II representations and there is a unique Type III representation up to equivalence. Thus a typical element of $RSO(Q(8))$ has the form $X + Y$ where X is a sum of Type I representations and Y is a sum of Type III representations. Since the latter is orientation preserving, it follows that the sum of Type I representations must also be orientation preserving and thus lie in the image of A^*. On the other hand, we also know that Y lies in the image of the restriction map from S^3, so this proves the proposition when $r = 3$.

Assume now that $r \geq 4$ and let Δ be the unique nontrivial 1-dimensional representation that is trivial on the unique cyclic subgroup of index 2; by construction Δ factors through a representation $\bar{\Delta}$ of the dihedral quotient group. If ρ is an irreducible representation of Type II then $\rho + \Delta$ is orientation preserving. Furthermore, if $\bar{\rho}$ is the irreducible dihedral representation that pulls back to ρ then $\bar{\rho} + \bar{\Delta}$ extends to an element of $RO(SO_3) = RSO(SO_3)$ by the standard description of irreducible representations of SO_3 in terms of weights (*cf.* [Hs, pp. 17–19]). Let Γ be a 1-dimensional representation of $Q(2^r)$ that is nontrivial on a generator of the index two subgroup; then the representations of Type I are given by Δ, Γ and their tensor product $\Delta\Gamma$.

Given an orientation preserving virtual representation ρ of $Q(2^r)$ write it as a sum of irreducible representations (that do not necessarily preserve orientations)

$$\rho = a_\Gamma \Gamma + a_\Delta \Delta + a_{\Gamma\Delta} \Gamma\Delta + \sum_i x_i \rho_i + \sum_j w_j \gamma_j$$

where each ρ_i is of Type II and each γ_i is of Type III. The right hand side can be rewritten in the form

$$a_\Gamma \Gamma + b_\Delta \Delta + a_{\Gamma\Delta} \Gamma\Delta + \sum_i x_i (\rho_i + \Delta) + \sum_j w_j \gamma_j$$

for a suitable choice of b_Δ, and by the previous paragraph the sums over i and j lie in the image of $RO(S^3) = RSO(S^3)$. Therefore $V = a_\Gamma \Gamma + b_\Delta \Delta + a_{\Gamma\Delta} \Gamma\Delta$ must also lie in $RSO(Q(2^r))$; the latter in turn implies that $a_\Gamma + a_{\Gamma\Delta}$, $a_\Gamma + b_\Delta$, and $b_\Delta + a_{\Gamma\Delta}$ are all even, which means that V is the sum of a multiple of $\Gamma + \Delta + \Gamma\Delta$ with even multiples of Γ, Δ and $\Gamma\Delta$. Since each of the representations $\Gamma + \Delta + \Gamma\Delta$, 2Γ, 2Δ and $2\Gamma\Delta$ lies in the image of A^* it follows that $V \in \text{Image } A^*$. This proves the conclusion of the proposition when $r \geq 4$. □

By the results of Section 4, the proof of Theorem 5.1 reduces to showing that each class in $[N, BSO]$ has the form $\mathcal{L}\beta_{(2)}(\eta)$, where η is the normal invariant of a degree one normal map $(f : M \to N, F : \nu_M \to \xi)$ such that M has a riemannian metric with positive scalar curvature. Separate considerations are needed for the images of $RO(S^3)$ and $RSO(\mathbb{Z}_2 \times \mathbb{Z}_2)$. The following result deals with the subgroup determined by representations of $\mathbb{Z}_2 \times \mathbb{Z}_2$:

THEOREM 5.3. *Suppose that $u \in [N, BSO]$ satisfies $u = \mathscr{L}\beta_{(2)}(\eta)$, where η is the normal invariant of a degree one normal map $(f : M \to N, \ldots)$ such that M has a metric of positive scalar curvature, and let $v \in [N, BSO]$ lie in the image of the composite*

$$A_N^* : RSO(\mathbb{Z}_2 \times \mathbb{Z}_2) \xrightarrow{A^*} RSO(G) \to [N, BSO].$$

Then $u + v = \mathscr{L}\beta_{(2)}(\eta')$, where η' is the normal invariant of a degree one normal map $(f' : M' \to N, \ldots)$ such that M' has a metric of positive scalar curvature.

In other words, the existence of a riemannian metric with positive scalar curvature only depends on the image of $\mathscr{L}\beta_2(\eta)$ in the quotient group of $[N, BSO]$ modulo the image of $RSO(\mathbb{Z}_2 \times \mathbb{Z}_2)$; by 5.2 this quotient is generated by the image of $RO(S^3)$.

Proof. The map A_N^* is given by the composite in the following diagram:

$$\begin{tikzcd}
RSO(\mathbb{Z}_2 \times \mathbb{Z}_2) \arrow[r, "V^b"] \arrow[d, "A^*"] & {[B(\mathbb{Z}_2 \times \mathbb{Z}_2), BSO]} \arrow[d, "A^*"] \\
RSO(G) \arrow[r, "V^b"] & {[BG, BSO]} \arrow[d, "k_N^*"] \\
 & {[N, BSO]}
\end{tikzcd}$$

Here $k_N : N \to BG$ is a 2-connected reference map, the maps A^* are induced by abelianization, and the maps V^b associate a flat oriented virtual vector bundle to each oriented virtual representation of the group in question.

Let $\partial : F/O \to BO$ be the homotopy fiber of the map $BO \to BF$, and consider the composite

$$\partial_{(2)*}\alpha_{2*}V^b_{(2)*} : RSO(\mathbb{Z}_2 \times \mathbb{Z}_2) \to \widetilde{KO}(B(\mathbb{Z}_2 \times \mathbb{Z}_2))_{(2)}.$$

Since $\partial_{(2)*}\alpha_{2*} = \psi^3 - 1$ where ψ^3 is the usual Adams operation, it follows that the composite sends the representation Ω into the flat bundle associated to the representation $\psi^3\Omega - \Omega$. But ψ^3 is the identity on one-dimensional representations and every irreducible representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$ is 1-dimensional, and therefore

$\psi^3-1=0$ on $RSO(\mathbb{Z}_2\times\mathbb{Z}_2)$. By exactness this means that the image of $\alpha_{(2)*}V^b_{(2)*}$ is contained in the image of $[B(\mathbb{Z}_2\times\mathbb{Z}_2), F]\cong\{B(\mathbb{Z}_2\times\mathbb{Z}_2), S^0\}$ (as usual, one must remember that the 1-1 correspondence does not send the direct sum on the left to the loop – or track – addition on the right). According to the Segal Conjecture, the abelian group $\{B(\mathbb{Z}_2\times\mathbb{Z}_2), S^0\}$ is isomorphic to the completion $IA(\mathbb{Z}_2\times\mathbb{Z}_2)^\wedge$ of the augmentation ideal in the Burnside ring $A(\mathbb{Z}_2\times\mathbb{Z}_2)$; this was shown more generally for elementary abelian 2-groups by G. Carlsson in [Ca1]. Furthermore, loop sum generators for $\{B(\mathbb{Z}_2\times\mathbb{Z}_2), S^0\}\cong IA(\mathbb{Z}_2\times\mathbb{Z}_2)^\wedge$ are given by the S-maps $B(\mathbb{Z}_2\times\mathbb{Z}_2)\to S^0$ of the form

$$B(\mathbb{Z}_2\times\mathbb{Z}_2)\subseteq B(\mathbb{Z}_2\times\mathbb{Z}_2)_+\xrightarrow{\text{transfer}}BC_+\xrightarrow{\text{aug}}S^0$$

where C runs through all proper subgroups of $\mathbb{Z}_2\times\mathbb{Z}_2$ and "aug" denotes the augmentation map collapsing BC to a point; this can be seen by combining the statement of the Strong Segal Conjecture in [Ca2, p. 190] with [tD, §7.6 and Thm. 8.5.1, p. 215] or by combining the construction of the map $A(G)^\wedge\to\{BG_+, S^0\}$ in [Lt, §0] with the definition of the stable homotopy transfer in [KP, §1]. By naturality it follows that the image of $\alpha_{2*}A^*_{N(2)}$ lies in the image of $\{N, S^0\}$, and since the image of $\{B(\mathbb{Z}_2\times\mathbb{Z}_2), S^0\}$ in $\{N, S^0\}$ is finite it follows that every class in the image of $\alpha_{2*}A^*_{N(2)}$ comes from a loop sum of reduced transfer maps $t_{G(i)}: N\to S^0$ where $G(i)$ is a sequence of proper subgroups (probably with repetitions) in G.

If $\Sigma\, t_{G(i)}\in\{N, S^0\}$ is a sum of reduced index two transfers, then one can realize this sum as the normal invariant of a stably tangential degree one normal map as follows: Each pair $(t_{G(i)}, 2)\in\{N_+, S^0\}\cong\{N, S^0\}\oplus\{\pi_0^S(S^0)\cong\mathbb{Z}\}$ is the normal invariant of the standard degree two of four (stably tangential) covering space projection $N_{G(i)}\to N$; this follows from standard duality considerations (*cf.* [BeS, §13]). If there are r_2 summands with $G(i)\cong\mathbb{Z}_2$ and r_4 summands with $G(i)=\{1\}$, this realizes $\{\Sigma_i\, t_{G(i)}, 2^r\}$ as the normal invariant of an oriented covering space projection of degree 2^r, where $r=r_2+2r_4$. Take the disjoint union of this with 2^r-1 copies of the identity map from the oriented manifold

$$-N=(N,\text{ negative of usual orientation})$$

to N with its usual orientation. By Proposition 3.1 and the standard rule $\partial(N\times[0,1])\cong N\sqcup -N$ it follows that this disjoint union is a stably tangential degree one normal map with normal invariant $\Sigma_i\, t_{G(i)}$. If $f: M\to N$ is the associated degree one map, then by construction f is a covering space projection (but M is in general disconnected); note that the degree of the mapping and the number of sheets in the covering are not necessarily equal because the degree takes orientations on different components into account.

We now claim the following: If $u \in$ image A_N^* represents the normal invariant of a degree one oriented covering space projection as above and $v \in [N, F/O]$ is represented by the degree one normal map $(g : P \to N, \ldots)$ then $w \oplus v$ is represented by the composite $(f^*P \to M \to N, \ldots)$ where $f^*P \to P$ is the covering space induced by $f : P \to N$ via g. The fastest way to see this is to consider the external direct sum $w \times f^*v$ in $[N \times M, F/O]$, which is represented by the product of $f : M \to N$ and $f^*g : f^*P \to M$. If $\Gamma : M \to N \times M$ is the graph of f, then $w \oplus v$ can be recovered by taking the transverse inverse image of $\Gamma(M)$ in $f^*P \times M$ (by construction $f^*g \times f$ is transverse to $\Gamma(M)$). Since this inverse image is precisely the graph of $f^*P \to M$, the claim follows.

To complete the proof, let u and v be as in the preceding paragraph and assume that the representative $(g : P \to N, \ldots)$ has a domain P with a metric of positive scalar curvature. Since f^*P is a covering space of P, it follows that f^*P also has such a metric. □

Simple spherical spaceforms

The next step in the proof of Theorem 5.1 involves quaternionic spherical spaceforms that fiber geometrically over quaternionic projective spaces. Specifically, we shall say that N is *simple* if it is given by the free linear $Q(2^r)$ action on $\tilde{N} = S^{4k+3}$ that extends to a free linear S^3 action. The following result implies Theorem 5.1 for fake spherical spaceforms that are homotopy equivalent to simple quaternionic linear spaceforms. In fact, it proves a little more:

THEOREM 5.4. *If M^{4k+3} $(k \geq 1)$ admits a 2-connected degree one normal map into a simple quaternionic spherical spaceform, then M^{4k+3} has a riemannian metric of positive scalar curvature.*

Proof. The argument uses an observation that also figures importantly in the work of Stolz [Stz1, §1]: If F is a riemannian manifold with a metric of positive scalar curvature such that the compact Lie group G acts by isometries, and $F \to E \to B$ is a compact smooth fiber bundle with structure group G, then E also has a riemannian metric of positive scalar curvature.

Consider the following commutative diagram:

$$\begin{array}{ccc}
RO(S^3) & \xrightarrow{\rho^*} & RSO(Q(2^r)) \\
{\scriptstyle V^b}\downarrow & & {\scriptstyle V^b}\downarrow \\
[\mathbb{K}\mathbb{P}^k, BSO] & \xrightarrow{\pi^*} & [S^{4k+3}/Q(2^r), BSO] \\
{\scriptstyle \alpha_{2*}}\downarrow & & {\scriptstyle \alpha_{2*}}\downarrow \\
[\mathbb{K}\mathbb{P}^k, F/O]_{(2)} & \xrightarrow{\pi^*} & [S^{4k+3}/Q(2^r), F/O]_{(2)} \\
{\scriptstyle \mathscr{S}(\beta_2)_*}\downarrow & & {\scriptstyle \mathscr{S}(\beta_2)_*}\downarrow \\
[\mathbb{K}\mathbb{P}^k, BSO]_{(2)} & \xrightarrow{\pi^*} & [S^{4k+3}/Q(2^r), BSO]_{(2)}
\end{array}$$

The map $\rho : Q(2^r) \to S^3$ is the inclusion homomorphism, and π is the projection of the fiber bundle

$$S^3/Q(2^r) \to S^{4k+3}/Q(2^r) \to \mathbb{KP}^k;$$

note that the structure group for this bundle is S^3, and the transitive action of S^3 on $S^3/Q(2^r)$ is by isometries of the constant curvature metric.

Let N be the simple spaceform $S^{4k+3}/Q(2^r)$. Then a chase of the diagram shows that every element z of $[N, BSO]_{(2)} \cong [N, BSO]$ can be written as a sum $A_N^* x + \mathscr{L}(\beta_2)^* \pi^* y$ for some $y \in [\mathbb{KP}^k, F/O]$. If we represent y be a degree 1 normal map $(f : B \to \mathbb{KP}^k, \ldots)$ then the induced map of total spaces $\tilde{f} : f^*B \to N, \ldots)$ represents $\pi^* y$; but now one can use the observation in the first paragraph of the proof to show that f^*B has a metric of positive scalar curvature. By Theorem 5.3 it follows that $z = A_N^* x + \mathscr{L}(\beta_2)_* \pi^* y$ has the form $\mathscr{L}\beta_2(\eta')$, where η' is the normal invariant of some degree one normal map $(f' : M' \to N, \ldots)$ such that M' has a metric of positive scalar curvature. If $(f'' : M'' \to N, \ldots)$ is a 2-connected map in the same normal bordism class, then surgery invariance implies that M'' also has a metric of positive scalar curvature. Since z was arbitrary, this completes the proof. □

One can extend Theorem 5.4 to nonsimple quaternionic spherical spaceforms by means of Proposition 3.2:

THEOREM 5.5. *The conclusion of Theorem* 5.4 *remains true in one considers* 2*-connected degree one normal maps into arbitrary linear spaceforms whose fundamental groups are quaternionic* 2*-groups* (*in dimensions* $4k + 3 \geq 7$).

Proof. Let G be a quaternionic 2-group, let N_0 be its simple spaceform in dimension $4k + 3$, let N be an arbitrary spaceform with the same fundamental group, and let $f : N_0 \to N$ be the odd degree map in Proposition 3.2; let d be the degree of f. If $(g : M \to N, \ldots)$ is a 2-connected degree one normal map, then by 3.2 there is a degree one normal map $g' : M' \to N_0$ such that fg' is normally cobordant to a sum of d copies of g. It follows that $d \cdot (k_N g : M \to BG, \ldots)$ and $(k_{N_0} fg' : M' \to BG, \ldots)$ determine the same element of $\Omega_{4k+3}^{Spin}(BG)$. By Theorem 5.4 we may assume without loss of generality that M' has a metric of positive scalar curvature.

Since G is a finite 2-group it follows that $\Omega_{4k+3}^{Spin}(BG)$ is also a finite 2-group. Choose an odd positive integer d' so that $d'd$ is congruent to 1 modulo the exponent of this group. Then $(k_N g : M \to BG, \ldots)$ and $d' \cdot (k_{N_0} fg' : M' \to BG, \ldots)$ determine the same element of $\Omega_{4k+3}^{Spin}(BG)$; since M' has a metric of positive scalar curvature and the map $k_N g$ is 2-connected, the existence of a metric with positive scalar curvature on M follows from surgery invariance. □

6. The remaining cases

We have now established the main theorem in all cases except the $(4k+3)$-dimensional case when the Sylow 2-subgroup is nontrivial and cyclic. As usual it suffices to dispose of the case where the fundamental group is a nontrivial 2-group (hence cyclic). Thus it remains to consider the propagation question for fake $(4k+3)$-dimensional lens spaces whose fundamental groups are isomorphic to $\mathbb{Z}_{2^r}$ for some $r \geq 1$. In analogy with Section 5 the discussion has two parts – a proof of the propagation result for fake spaceforms that are homotopy equivalent to certain simple lens spaces and an extension to the general case using Proposition 3.2. The argument presented here is somewhat different from the one outlined in [Shz4] and involves the subsequent work of Stolz [Stz1].

Simple lens spaces

A lens space L^{2r+1} with fundamental group $\mathbb{Z}_q$ is said to be *simple* if the associated free linear action of $\mathbb{Z}_q$ on S^{2r+1} extends to a free linear action of S^1. In this case one has a smooth fibering

$$S^1 \to L^{2r+1} \to \mathbb{C}\mathrm{P}^{2r}.$$

However, since positive scalar curvature is not a meaningful concept for 1-manifolds this situation is not completely analogous to the fibering over $\mathbb{K}\mathbb{P}^k$ in the preceding section. Despite this, one has a complete analog of Theorem 5.4.

THEOREM 6.1. *If M^{4k+3} $(k \geq 1)$ admits a 2-connected degree one normal map into a simple $\mathbb{Z}_{2^q}$ lens space (where $q \geq 1$), then M^{4k+3} has a riemannian metric of positive scalar curvature.*

Proof. The argument splits into two subcases depending upon the congruence class of $4k+3$ mod 8, but the first steps are the same for both cases. By the results of Section 4 we need to show that every class in $[L, BSO]_{(2)} \cong [L, BSO]$ has the form $\mathcal{L}(\beta_2)(\eta')$ where η' is the normal invariant of some degree one normal map $(f: M \to L, \ldots)$ such that M has a metric of positive scalar curvature. As in Section 4 we know that the map V^b from $RSO(\mathbb{Z}_{2^q})$ to $[K, BSO]$ is surjective (by [GKa] again).

Elementary considerations imply that the restriction homomorphism ρ^* from $RO(S^1)$ to $RSO(\mathbb{Z}_{2^q})$ is onto. If $S^1 \to L \xrightarrow{\pi} \mathbb{C}\mathrm{P}^{2k+1}$ is the fiber bundle discussed previously, then there is a commutative diagram

$$\begin{array}{ccc}
RO(S^1) & \xrightarrow{\rho^*} & RSO(\mathbb{Z}_{2q}) \\
{\scriptstyle V^b}\downarrow & & {\scriptstyle V^b}\downarrow \\
[\mathbb{CP}^{2k+1}, BSO] & \xrightarrow{\pi^*} & [L, BSO]_{(2)} \\
{\scriptstyle \alpha_{2*}}\downarrow & & {\scriptstyle \alpha_{2*}}\downarrow \\
[\mathbb{CP}^{2k+1}, F/O]_{(2)} & \xrightarrow{\pi^*} & [L, F/O]_{(2)} \\
{\scriptstyle \mathscr{L}(\beta_2)_*}\downarrow & & {\scriptstyle \mathscr{L}(\beta_2)_*}\downarrow \\
[\mathbb{CP}^{2k+1}, BSO]_{(2)} & \xrightarrow{\pi^*} & [L, BSO]_{(2)}
\end{array}$$

and a chase of this diagram shows that every class $u \in [L, BSO]_{(2)}$ has the form $\mathscr{L}(\beta_2)\pi^* v$ where $v \in [\mathbb{CP}^{2k+1}, F/O]$ is represented by a degree one normal map

$$(f: M \to \mathbb{CP}^{2k+1}, \ldots).$$

At this point we must consider the cases $4k + 3 \equiv 3$, $7 \bmod 8$ separately. We begin with the latter because it is easier.

Case 1. Suppose that $4k + 3 \equiv 7 \bmod 8$ (*i.e.*, k is odd). It will suffice to show that every degree one normal map $(f: M \to \mathbb{CP}^{2k+1}, \ldots)$ is normally cobordant to one $(q : M' \to \mathbb{CP}^{2k+1}, \ldots)$ such that M' has a metric of positive scalar curvature. For this will imply that the circle bundle g^*L^{4k+3} also has such a metric and thus that every class in the image of $\mathscr{L}(\beta_2)\pi^*$ has the form $\mathscr{L}(\beta_2)^*(\eta')$ where η' is represented by a degree one normal map whose domain has a metric of positive scalar curvature. But the vertical composites are surjective and ρ^* is also surjective, and therefore $\mathscr{L}(\beta_2)_*\pi^*$ is onto by a diagram chase.

Given a 2-connected degree one normal map $f: M \to \mathbb{CP}^{2k+1}$, by surgery invariance it is clear that M admits a metric of positive scalar curvature if M maps to zero in Ω^{Spin}_{4k+2}. This assertion can be verified as follows: Since $4k + 2 \equiv 6 \bmod 8$ and the forgetful map $\Omega^{Spin}_{8\ell+6} \to \Omega^{SO}_{8\ell+6}$ to oriented bordism in injective (*cf.* [ABP]), it suffices to show that M is an oriented boundary. But M has the same Stiefel–Whitney classes (and hence numbers) as $\mathbb{CP}^{2k+1}$ by the fiber homotopy invariance of these classes (the pullback of the tangent bundle of $\mathbb{CP}^{2k+1}$ is stably fiber homotopy equivalent to the tangent bundle of M). Since $\Omega^{SO}_{8\ell+6}$ is detected by Stiefel–Whitney numbers and $\mathbb{CP}^{2k+1}$ is an oriented boundary, the same is true for M. As noted before, it follows that M must be a Spin boundary and thus has a metric of positive scalar curvature.

Case 2. Suppose that $4k+3 \equiv 3 \bmod 8$ (i.e., k is even). Write $k = 2m$ so that the dimension becomes $8m+3$. As in Case 1 it suffices to consider degree one normal maps of the form $(f^*M \to L, \ldots)$ where $(f : M \to \mathbb{CP}^{4m+1}, \ldots)$ is a degree one normal map; more precisely, it suffices to show that each degree one normal map into $\mathbb{CP}^{4k+1}$ has a representative $(f : M \to \mathbb{CP}^{4k+1}, \ldots)$ such that f^*M has a metric of positive scalar curvature. By surgery we may assume that f is 2-connected.

In this case we cannot conclude that M automatically has a metric of positive scalar curvature. However, the work of Stolz [Stz1] yields a reasonable substitute. Namely, if M is 1-connected and Σ^* is a homotopy $(8m+2)$-sphere whose normal invariant is the generator $\mu_m \eta$ of

$$\mathbb{Z}_2 \cong \alpha_2(\pi_{8m+2}(BSO)) \subseteq \pi_{8m+2}(F/O)_{(2)}$$

then either M has a metric of positive scalar curvature or the connected sum $M \# \Sigma^*$ admits such a metric. If the first possibility holds, one can proceed as in Case 1 to show that the circle bundle f^*L has metric of positive scalar curvature. To deal with the second possibility let $f_1 : M \# \Sigma^* \to \mathbb{CP}^{4m+1}$ be the composite of f and the canonical degree 1 normal map $M \# \Sigma^* \to M$. The normal invariants of f and f_1 are related by

$$\begin{aligned} \eta(f_1) &= \eta(f) + h^*(\mu_m \eta) \\ &= \eta(f) + h^* \alpha_{2*} \sigma \end{aligned}$$

where $h : \mathbb{CP}^{4m+1} \to S^{8m+2}$ is the degree one collapse map and σ generates the group $\pi_{8m+2}(BSO) \cong \mathbb{Z}_2$. Let $\tilde{f}_1 : f_1^* L \to L$ be the associated degree 1 normal map of circle bundle total spaces and define $\tilde{f} : f^*L \to L$ similarly. By the reasoning of Case 1 we know that $f_1^* L$ has a metric of positive scalar curvature, and diagram chases show that $\tilde{f}_1$ and $\tilde{f}$ are both 2-connected. Since f_1 is normally cobordant to the disjoint union of f and the constant map $\Sigma^* \to \mathbb{CP}^{4m+1}$, it follows that $\tilde{f}_1$ is normally cobordant to the disjoint union of $\tilde{f}$ and $S^1 \times \Sigma^* \xrightarrow{\text{proj}} S_1 \xrightarrow{g} L$ where $g_* : \pi_1(S^1) \to \pi_1(L) \cong \mathbb{Z}_{2^r}$ is surjective. Comparing with the reference map $k_L : L \to B\mathbb{Z}_{2^r}$, we see that the difference between the classes representing $(k_L \tilde{f}_1 : f_1^* L \to B\mathbb{Z}_{2^r}, \ldots)$ and $(k_L \tilde{f} : f^*L \to B\mathbb{Z}_{2^r}, \ldots)$ in $\Omega_{8m+3}^{Spin}(B\mathbb{Z}_{2^r})$ has the form

$$(k_L : S^1 \to B\mathbb{Z}_{2^r}, \ldots) \times (\text{const.} : \Sigma^* \to \{\text{pt.}\}, \ldots).$$

There is a unique Spin structure on Σ^*, but S^1 has two Spin structures and thus it is necessary to see what happens to the bordism class if one changes the Spin structure on S^1. We claim that the bordism class does not depend upon the choice of Spin structures. This is true because there is an orientation-preserving diffeomorphism H from $S^1 \times \Sigma^*$ to $S^1 \times \Sigma^* \# \Sigma_1$, for some homotopy $(8m+3)$-sphere Σ_1, such that H commutes with projection onto S^1 up to homotopy and H sends one Spin structure to the other (*e.g.*, see [Shz1]); since Σ_1 bounds a Spin manifold (in fact, a parallelizable manifold) it follows that both Spin structures define the same Spin bordism class. If $\delta_{m,r} \in \Omega_{8m+3}^{Spin}(B\mathbb{Z}_{2^r})$ is the class described above, then it follows that for each $s < r$ the bordism transfer map $\Omega_{8m+3}^{Spin}(B\mathbb{Z}_{2^r}) \to \Omega_{8m+3}^{Spin}(B\mathbb{Z}_{2^s})$ sends $\delta_{m,r}$ to $\delta_{m,s}$.

It is well known that the homotopy $(8m+2)$-sphere Σ^* is spin cobordant to $P^m \cdot \eta^2$ where $P \in \Omega_8^{Spin}$ is a class with $\hat{A}$-genus equal to 1 and η^2 generates $\Omega_2^{Spin} \approx \pi_2 \approx \mathbb{Z}_2$ (*cf.* [ABP]). If we define $\delta_{0,r} \in \Omega_3^{Spin}(B\mathbb{Z}_{2^r})$ to be $(k_L, \ldots) \cdot \eta^2$, then by construction we have $2\delta_{0,r} = 0$, and since $\mu_m \eta = P^m \eta^2$ in Ω_*^{Spin} by the first sentence of this paragraph, the identity $\delta_{m,r} = P^m \delta_{0,r}$ follows immediately. The latter observation and the discussion in the preceding paragraphs yield the crucial reduction for Case 2:

PROPOSITION 6.2. *If $\delta_{0,r}$ lies in* $\mathrm{Pos}_3\,(B\mathbb{Z}_{2^r})$ *for each r then the conclusion of Theorem* 6.1 *holds.*

Proof. The preceding discussion shows that the Dirac invariant of a fake $(8m+3)$-dimensional $\mathbb{Z}_{2^r}$ lens space is either zero or $D(\delta_{m,r}) = D(P^m \delta_{0,r})$, so the conclusion of the main theorem is true if $\delta_{m,r} \in \mathrm{Pos}_{8m+3}\,(B\mathbb{Z}_{2^r})$ for each $m > 0$. On the other hand, since $\delta_{m,r} = P^m \delta_{0,r}$ the assertion of the previous sentence will hold if $\delta_{0,r}$ lies in $\mathrm{Pos}_3\,(B\mathbb{Z}_{2^r})$. □

The next step is to verify the hypothesis of Proposition 6.2.

PROPOSITION 6.3. *For each $r \geq 1$ we have $\delta_{0,r} \in \mathrm{Pos}_3\,(B\mathbb{Z}_{2^r})$.*

Proof. Let L be the simple $\mathbb{Z}_{2^r}$ lens space in dimension 3, let σ be some spin structure, and let $k : L \to B\mathbb{Z}_{2^r}$ be a polarization map. We shall first construct an odd degree normal map $g : L^* \to L$ such that L^* is a lens space and the 2-local normal invariant $\eta(g) \in [L, F/O]_{(2)}$ (in the sense of [Shz2] satisfies $\mathscr{L}\beta_2(\eta(g)) = V^b(\rho)$ for a nontrivial irreducible free representation ρ; standard results on the K-theory of classifying spaces imply that $[L, F/O] \approx [L, F/O]_{(2)} \cong [L, BSO]_{(2)} \approx \mathbb{Z}_2$, with the nontrivial element given by $V^b(\rho)$. The construction involves iterated branched coverings as in [Shz3]. Specifically, write $L = S(\rho \oplus \rho)/\mathbb{Z}_{2^r}$, where ρ is the standard irreducible free unitary representation, and using the invertibility of 3 in $\mathbb{Z}_{2^r}$ write $\rho = \psi_C^3 \rho_0$ for some other irreducible representation (specifically, $\rho_0 = \psi_C^X \rho_0$ where $3X \equiv 1 \mod 2^r$). Define a 3-sheeted equivariant branched covering $S(\rho \oplus \rho_0) \to S(\rho \oplus \psi_C^3 \rho_0)$ by sending $(v, z) \in S(\rho \oplus \rho_0)$ to

$$(|v|^2 + |z|^6)^{-2}(v, z^3) \in S(\rho \oplus \psi_{\mathbb{C}}^3 \rho_0).$$

This map passes to a degree 3 normal map f_0 of lens spaces that commutes with the canonical polarization maps from $L(\rho \oplus \rho) = S(\rho \oplus \rho)/\mathbb{Z}_{2^r}$ and $L(\rho \oplus \rho_0) = S(\rho \oplus \rho_0)/\mathbb{Z}_{2^r}$ to $B\mathbb{Z}_{2^r}$; furthermore, f_0 induces an isomorphism in 2-local homology (with twisted coefficients, in fact), and by [Shz3] the 2-local normal invariant $\eta(f_0)$ is equal to $\alpha_2 V^b(\rho_0)$.

To complete the proof we must relate f to a degree one normal map with the same normal invariant. By construction $\eta(f)$ is obtained by taking a stable homotopy class $y \in \{L^\omega, S^0\}$ on a finite Thom spectrum L^ω (where the virtual dimension of ω is zero) such that $y \mid S^0 = \text{degree}\,(f)$ and dividing by the degree. In the situation considered in the previous paragraph, by Proposition 3.3. it follows that

$$3([L, \sigma, k] + \delta_{0,r}) = [L^*, \sigma^*, k^*] \in \Omega_3^{Spin}(Bz_{2^r})$$

where L^* is the lens space described in the preceding paragraph and σ^* and k^* represent appropriate extra data. Since $\text{Pos}_3\,(B\mathbb{Z}_{2^r})$ is a subgroup it follows that it must contain $3\delta_{0,r}$ for all $r \geq 1$, and since this class has exponent 2 we obtain the desired relation $\delta_{0,r} \in \text{Pos}_3\,(B\mathbb{Z}_{2^r})$. □

The general case

As in Section 5 we have the following extension of Theorem 6.1; this will dispose of all cases in the main theorem that have not yet been treated.

THEOREM 6.4. *The conclusion of Theorem* 6.1 *remains true if one considers* 2-*connected degree one normal maps into arbitrary* $\mathbb{Z}_{2^r}$ *lens spaces* (*or real projective spaces if* $r = 1$) *in dimensions* $4k + 3 \geq 7$.

Proof. Modulo substituting lens spaces (*resp.*, simple lens spaces) for linear spaceforms (*resp.*, simple linear spaceforms) whose fundamental groups are quaternionic 2-groups, the proof of Theorem 5.5 goes through unchanged. □

Remark. If $r = 1$ the results of this section follow immediately from more general theorems of Rosenberg and Stolz (*i.e.*, [RS, Thm. 5.3(4)]).

Acknowledgements

We are extremely grateful to Stephan Stolz for explaining the ideas behind [RS, Thm. 5.3(4)] and [Stz1–2] before the latter were written and to Jonathan Rose berg for regular updates on his work and to Mark Mahowald for suggestions regarding *Spin*^{c} that were eventually decisive in some cases. The second author is also grateful to the Northwestern University Mathematics Department for providing generous access to its library and computing facilities during portions of the preparation of this paper and the Tulane University Mathematics Department for its hospitality in connection with numerous visits to discuss this and other joint work. Finally, both authors are grateful to the referee for pointing out certain misstatements in earlier drafats and suggesting various improvements to the exposition.

REFERENCES

[Ag] AGOSTON, M., *Browder–Novikov theory for maps of degree* $d > 1$, Topology *9* (1970), 251–265.

[AnG] ANDERSON, G. A., "*Surgery with Coefficients,*" Lecture Notes in Mathematics Vol. 591, Springer, Berlin-Heidelberg-New York, 1977.

[ABP] ANDERSON, D. W., BROWN E. H. and PETERSON, F. O., *Spin cobordism*, Bull. Amer. Math. Soc. *72* (1966), 256–260; [DETAILED VERSION] *The structure of the Spin cobordism ring*, Ann. of Math. *86* (1967), 271–298.

[At] ATIYAH, M., *Thom complexes*, Proc. London Math. Soc. (3) *11* (1961), 291–310.

[AS] ATIYAH, M. and SEGAL, G., *Exponential isomorphisms for λ-rings*, Quart. J. Math. Oxford (2) *22* (1971), 371–378.

[BeS] BECKER, J. C. and SCHULTZ, R. E., *Equivariant function spaces and stable homotopy theory*, Comment. Math. Helv. *49* (1974), 1–34.

[Bd] BOARDMAN, J. M., *Stable Homotopy Theory, Chapter V – Duality and Thom spectra*, mimeographed, University of Warwick, 1966.

[BG] BOTVINNIK, B. and GILKEY, P., *The Eta Invariant and Metrics of Positive Scalar Curvature*, preprint, Univ. of Oregon, 1993.

[BrM] BRUMFIEL, G. and MADSEN, I., *Evaluation of the transfer and universal surgery classes*, Invent. Math. *32* (1976), 133–169.

[CS] CAPPELL, S. and SHANESON, J., *Some new four-manifolds*, Ann. of Math. *104* (1976), 61–72.

[CW] CAPPELL, S. and WEINBERGER, S., *Homology propagation of gorup actions*, Comm. Pure Appl. Math. *40* (1987), 723–744.

[Ca1] CARLSSON, G., *G. B. Segal's Burnside ring conjecture for* $(\mathbf{Z}/2)^k$, Topology *22* (1983), 83–103.

[Ca2] CARLSSON, G., *Equivariant homotopy theory and Segal's Burnside ring conjecture*, Ann. of Math. *120* (1984), 189–224.

[DM] DAVIS, J. F., and MILGRAM, R. J., "*A survey of the Spherical Space Form Problem,*" Mathematical Reports Vol. 2 Part 2, Harwood Academic Publishers, London, 1985.

[tD] DIECK, T. TOM, "*Transformation Groups and Representation Theory,*" Lecture Notes in Mathematics Vol. 766, Springer, Berlin-Heidelberg-New York, 1979.

[FS] FINTUSHEL, R. and STERN, R., *An exotic free involution on* S^4, Ann. of Math. *113* (1981), 357–365.

[Gk] GILKEY, P., "*The Geometry of Spherical Space Form Groups,*" Series in Pure Mathematics Vol. 7, World Scientific, Singapore-Teaneck NJ-London, 1989.

[GKa] GILKEY, P. and KAROUBI, M., *K-theory for spherical space forms*, Topology Appl. *25* (1987), 179–184.

[GL1] GROMOV, M. and LAWSON, H. B., *Spin and scalar curvature in the presence of a fundamental group*, Ann. of Math. *111* (1980), 209–230.

[GL2] GROMOV, M. and LAWSON, H. B., *The classification of simply connected manifolds of positive scalar curvature*, Ann. of Math. *111* (1980), 423–434.

[HM] HAMBLETON, I. and MADSEN, I., *Local surgery obstructions and space forms*, Math. Zeitschrift *193* (1986), 191–214.

[Ha1] HAMILTON, R. S., *Three-manifolds with positive Ricci curvature*, J. Diff. Geom. *17* (1982), 255–306.

[Ha2] HAMILTON, R. S., *4-manifolds with positive curvature operator*, J. Diff. Geom. *24* (1986), 153–179.

[Hi] HITCHIN, N., *Harmonic spinors*, Adv. Math. *14* (1975), 1–55.

[HH] HOPKINS, M. J. and HOVEY, M. A., *Spin cobordism determines real K-theory*, Math. Zeit. *210* (1992), 181–196.

[Hs] HSIANG, W.-Y., "*Cohomology Theory of Topological Transformation Groups*," Ergeb. der Math. (2) Bd. 85, Springer, New York, 1975.

[KP] KAHN, D. S. and PRIDDY, S. B., *The transfer and stable homotopy theory*, Math. Proc. Camb. Phil. Soc. *83* (1978), 103–111.

[KwS] KWASIK, S. and SCHULTZ, R., *Positive scalar curvature and periodic fundamental groups*, Comment. Math. Helv. *65* (1990), 271–286.

[Lt] LAITINEN, E., *The Burnside ring and stable cohomotopy of a finite group*, Math. Scand. *44* (1979), 37–72.

[LM] LAWSON, H. B. and MICHELSON, M. L., "Spin Geometry," Princeton Math. Series Vol. 38, Princeton University Press, Princeton 1989.

[Lv1] LIULEVICIUS, A., *A theorem in homological algebra and stable homotopy of projective spaces*, Trans. Amer. Math. Soc. *109* (1963), 540–552.

[Lv2] LIULEVICIUS, A., *Notes on homotopy of Thom spectra*, Amer. J. Math. *86* (1964), 1–16.

[Md1] MADSEN, I., *Smooth spherical spaceforms*, in "Geometric Applications of Homotopy Theory I (Conference Proceedings, Evanston, 1977)," Lecture Notes in Mathematics Vol. 657, Springer, Berlin-Heidelberg-New York, 1978, pp. 303–352.

[Md2] MADSEN, I., *Smooth spherical spaceforms*, in "Eighteenth Scandinavian Congress of Mathematicians (Proceedings, 1980)," Progress in Mathematics Vol. 10, Birkhäuser, Basel-Boston-Stuttgart, 1981, pp. 82–103.

[MdMi] MADSEN, I. and MILGRAM, R. J., "*The classifying spaces for surgery and cobordism of manifolds*," Ann. of Math. Studies No. 92, Princeton University Press, Princeton, 1979.

[MTaW] MADSEN, I., TAYLOR, L. and WILLIAMS, B., *Tangential homotopy equivalences*, Comment. Math. Helv. *55* (1980), 445–484.

[MThW] MADSEN, I., THOMAS, C. B. and WALL, C. T. C., *The topological spherical space form problem II*, Topology *15* (1976), 375–382.

[Mh] MAHOWALD, M., *The image of J in the EHP sequence*, Ann. of Math. *116* (1982), 65–112.

[MhMi] MAHOWALD, M. and MILGRAM, J., *Operations which detect Sq^4 in connective K-theory and their applications*, Quart. J. Math. Oxford (2) *17* (1976), 415–432.

[May] MAY, J. P., (with contributions by F. Quinn, N. Ray, and J. Tørnehave), "*E_∞ Ring Spaces and E_∞ Ring Spectra*," Lecture Notes in Mathematics Vol. 577, Springer, Berlin-Heidelberg-New York, 1977.

[Na] NASH, J. C., *Positive Ricci curvature on fiber bundles*, J. Diff Geom. *14* (1979), 241–254.

[R1] ROSENBERG, J., *C*-algebras, positive scalar curvature, and the Novikov Conjecture*, Publ. Math. I.H.E.S. *58* (1983), 197–212.

[R2] ROSENBERG, J., *C*-algebras, positive scalar curvature, and the Novikov Conjecture – II*, in "Geometric Methods in Operator Algebras (Kyoto, 1983)," Pitman Res. Notes Math. Ser. 123, Longman Sci. Tech., Harlow, U.K., 341–374.

[R3] ROSENBERG, J., *C*-algebras, positive scalar curvature, and the Novikov Conjecture – III*, Topology *25* (1986), 319–336.

[R4] ROSENBERG, J., *The KO assembly map and positive scalar curvature*, in "Algebraic Topology Poznań 1989 (Conf. Proc.)," Lecture Notes in Mathematics Vol. 1474, Springer, Berlin-Heidelberg-New York, 1991, pp. 170–182.

[RS] ROSENBERG, J. and STOLZ, S., *Manifolds of positive scalar curvature*, in "Algebraic Topology and its Applications," M.S.R.I. Publications Vol. 27, Springer, Berlin-Heidelberg-New York-*etc.*, 1994, pp. 241–267.

[Shn] SCHOEN, R. M., *Conformal deformation of a Riemannian metric to a constant scalar curvature*, J. Diff. Geom. *20* (1984), 479–495.

[SY1] SCHOEN, R. and YAU, S.-T., *The structure of manifolds with positive scalar curvature*, Manuscr. Math. *28* (1979), 159–183.

[SY2] SCHOEN, R. and YAU, S.-T., *Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds of non-negative scalar curvature*, Ann. of Math. *110* (1979), 127–142.

[Shz1] SCHULTZ, R., *Smooth structures on $S^p \times S^q$*, Ann. of Math. *90* (1969), 187–198.

[Shz2] SCHULTZ, R., *Differentiable group actions on homotopy spheres I: Differential structure and the knot invariant*, Invent. Math. *31* (1975), 105–128.

[Shz3] SCHULTZ, R., *Finding framed $\mathbf{Z}_p$-actions on exotic spheres*, in "Algebraic Topology, Aarhus 1978 (Proceedings)," Lecture Notes in Math. Vol. 763, Springer, Berlin-Heidelberg-New York, 1979, pp. 591–603.

[Shz4] SCHULTZ, R., *Homotopy propagation of positive scalar curvature metrics*, pre-preprint, Purdue University, 1988.

[Sn1] SNAITH, V., *Dyer–Lashof operations in K-Theory*, in "Topics in K-Theory (Two Independent Contributions)," Lecture Note in Mathematics Vol. 496, Springer, Berlin-Heidelberg-New York, 1975, pp. 103–294.

[Sn2] SNAITH, V., *Algebraic Cobordism and K-Theory*, Memoirs Amer. Math. Soc. *221* (1979).

[Stz1] STOLZ, S., *Simply connected manifolds of positive scalar curvature*, Ann. of Math. *136* (1992), 511–540.

[Stz2] STOLZ, S., *Splitting certain MSpin-module spectra*, Topology *33* (1994), 159–180.

[Stg] STONG, R., "*Lectures on Cobordism Theory*," Princeton Mathematical Notes No. 7, Princeton Universtiy Press, Princeton, 1969.

[Wa] WALL, C. T. C., "*Surgery on Compact Manifolds*," London Math. Soc. Monographs, No. 1, Academic Press, London-New York, 1970.

[Wh] WHITEHEAD, G., "*Elements of Homotopy Theory*," Graduate Texts in Mathematics Vol. 61, Springer, Berlin-Heidelberg-New York, 1978.

[Wo] WOLF, J. A., "*Spaces of Constant Curvature* (Fourth Edition)," Publish or Perish, Berkeley, 1977.

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Received June 16, 1992; February 20, 1995