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## Unknotting tunnels in two-bridge knot and link complements

COLIN C. ADAMS AND ALAN W. REID

*Abstract.* We give a complete classification of the unknotting tunnels in 2-bridge link complements, proving that only the upper and lower tunnels are unknotting tunnels. Moreover, we show that the only strongly parabolic tunnels in 2-cusped hyperbolic 3-manifolds are exactly the upper and lower tunnels in 2-bridge knot and link complements. From this, it follows that the upper and lower tunnels in 2-bridge knot and link complements must be isotopic to geodesics of length at most  $\ln(4)$ , where length is measured relative to maximal cusps. Moreover, the four dual unknotting tunnels in a 2-bridge knot complement, which together with the upper and lower tunnels form the set of all known unknotting tunnels for these knots, must each be homotopic to a geodesic of length at most  $6\ln(2)$ .

### Section 1. Introduction

Given a compact 3-manifold with one or two torus boundary components, we say that a properly embedded arc is an *unknotting tunnel* if the complement of a regular neighborhood of the arc is a genus two handlebody. Note that if a manifold has an unknotting tunnel, its fundamental group can be given by a presentation with two generators and one relator. Given an arc in the 3-sphere that begins and ends on a given knot or link and such that its interior avoids the knot or link, we call the arc an *unknotting tunnel for the knot or link* if the restriction of the arc to the exterior of the knot or link is an unknotting tunnel.

Recently, there has been considerable effort expended on the determination of the unknotting tunnels for various classes of knots and links. In particular, in [5], Boileau, Rost and Zieschang completely classified the unknotting tunnels for torus knots. In [3], Bleiler and Moriah distinguished two types of unknotting tunnels for two-bridge knots, called the upper and lower tunnels and determined when they were equivalent. In [6], Kobayashi discovered additional unknotting tunnels for two-bridge knots and classified them up to homeomorphism. In [8], a complete determination of non-simple knots with unknotting tunnels was given, together with a classification of their tunnels. Moreover, Morimoto and Sakuma determined exactly which of the known unknotting tunnels for two-bridge knots are isotopic.

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In this paper, we use geometric techniques to give a complete classification up to isotopy of the unknotting tunnels for two-bridge links, showing that the upper and lower tunnels are all of the tunnels that occur. Note that in our usage of the word link, we do not include knots. It was recently announced by Martin Kuhn that he also has such a classification utilizing algebraic arguments.

Building on the work in [1] and [2], we also investigate parabolic tunnels. In [2], it was shown that a hyperbolic 3-manifold generated by two parabolic elements must be a 2-bridge knot or link. In [1], it was shown that an unknotting tunnel in a hyperbolic 3-manifold with two cusps must be isotopic to a vertical geodesic with length at most  $\ln(4)$  relative to a pair of canonical cusps.

A *parabolic tunnel* is an unknotting tunnel in the compact core of a cusped finite volume hyperbolic 3-manifold  $M$  with one or two cusps such that the fundamental group of the complementary handlebody is generated by two elements which correspond to parabolic isometries in the fundamental group of  $M$ . We say that a parabolic tunnel is *strongly parabolic* if the fundamental group of the complementary handlebody is generated by two elements, each of which can be freely homotoped to the boundary of  $M$  without passing through the tunnel. These loops will again correspond to parabolic isometries in the fundamental group of  $M$ .

We prove that strongly parabolic tunnels in hyperbolic 3-manifolds are exactly the upper and lower tunnels of 2-bridge knot and link complements, up to isotopy. We use this to show that the upper and lower tunnels must each be isotopic to a geodesic arc that has length at most  $\ln(4)$  with respect to the canonical cusps for the manifold.

In the case of a 2-bridge knot, there are four (not always distinct) additional unknotting tunnels other than the upper and lower tunnel that are known to occur, called dual tunnels (cf. [6] and [8]). We prove that such a dual tunnel must be homotopic to a geodesic with length at most  $6\ln(2)$  relative to a maximal cusp.

All of the manifolds in this paper are orientable. We give orientations to tunnels and utilize  $\alpha^{-1}$  to denote the tunnel  $\alpha$  with the opposite orientation. Given a horoball in the upper-half-plane model of hyperbolic 3-space, its *center* is its point of tangency with the boundary plane and its hyperbolic radius is infinite.

## Section 2. Unknotting tunnels in 2-bridge link complements

A 2-bridge knot or link can be drawn in a 4-plat projection, which is a projection with two maxima at the top and two minima at the bottom and a braid descending from the four strands that come out of the two maxima down to the four strands that come out of the two minima. The upper tunnel is a horizontal arc that begins on the one maximum and ends on the other and intersects the knot or

link only in its endpoints. The lower tunnel is a similar arc that connects the minima. One can easily check that both of these tunnels are unknotting tunnels, and an explicit argument appears at the beginning of the proof of Lemma 3.1. We will prove that in the case of a 2-bridge link, these are the only unknotting tunnels. By a 2-braid, we mean a knot or link that can be put in the form of a closed-braid with braid index two. In particular, this makes it a  $(p, 2)$ -torus knot or link.

**THEOREM 2.1.** *Up to isotopy, a two-bridge link with two components that is not a 2-braid has two unknotting tunnels, which are the upper and lower tunnels. Up to isotopy, a two-bridge link with two components that is a 2-braid has one unknotting tunnel.*

*Proof.* Let  $\alpha$  be an unknotting tunnel in a two-bridge link complement, where the link complement is not a 2-braid. By [7], the link complement is hyperbolic. Clearly, the link must have exactly two components, and  $\alpha$  must begin on one link component and end on the other. The complement of a regular neighborhood of  $\alpha$  in the link exterior is a handlebody. There is an involution of the handlebody sending each of the curves in a spine for a handlebody to its inverse. The attaching curve for the neighborhood of  $\alpha$  on the surface of the handlebody will be sent to itself with the same orientation by the involution, and therefore, the involution extends to the neighborhood of the tunnel, fixing the tunnel pointwise and giving an involution of the link complement. Since the manifold is Haken and hyperbolic, the axes of the involution will correspond to elliptic axes of order two in the universal cover  $H^3$ . The unknotting tunnel lifts to a set of such axes in  $H^3$ . Hence an unknotting tunnel is isotopic to a geodesic with both ends going out the cusps, and any unknotting tunnel must occur as a subset of the axes that correspond to a strong involution. Note that the axis of a strong involution on a 2-component link is comprised of four arcs, each of which has its endpoints on the link.

Every two-bridge knot or link has a tri-symmetric projection (cf. [3]) as in Figure 1. This follows from the fact that any rational number can be represented as a continued fraction with all coefficients even, the number of coefficients being even for a knot and odd for a link. (See [9].) The knot or link projection coming from that continued fraction representation given by  $[2b_0, 2b_1, \dots, 2b_n]$  can then be rearranged into the tri-symmetric projection.

A 2-component 2-bridge link has orientation preserving symmetry group  $Z_2 + Z_2$ . (See [4].) The symmetry group is evident in the tri-symmetric projection. In particular, one sees that there is a single strong involution corresponding to a  $180^\circ$  rotation about the horizontal line in the figure, together with two other obvious involutions that are not strong, corresponding to  $180^\circ$  rotations about the vertical line and about the line that is perpendicular to the page and that passes through the figure at the center.

Hence, the only possible unknotting tunnels are the four arcs that make up the axis corresponding to the single strong involution, each arc of which begins and

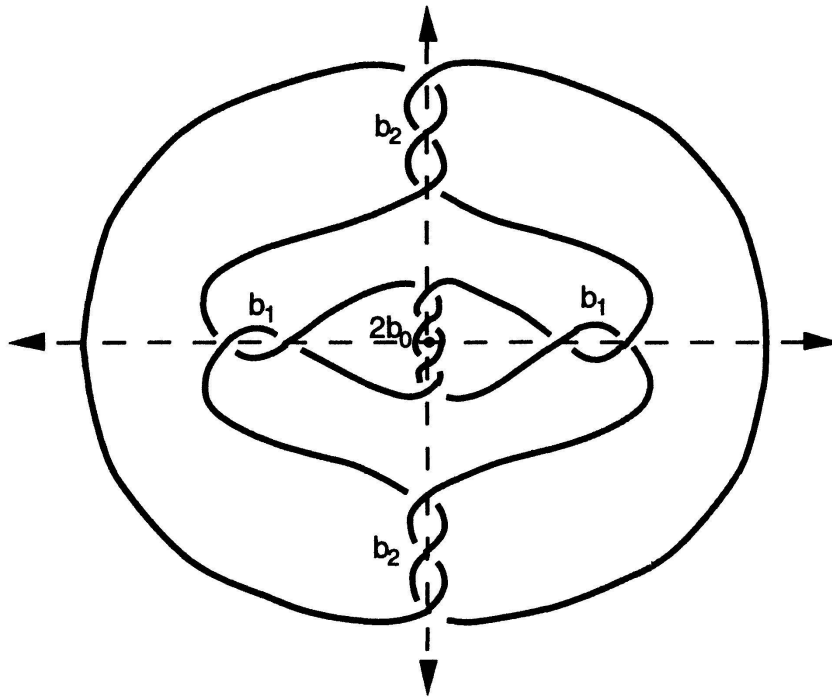


Figure 1

ends on the link. The arc at the center and the outermost arc (which in the figure appears as two infinite rays) are exactly the upper and lower tunnels, both of which are unknotting tunnels. We would like to eliminate the remaining two arcs as possibilities for unknotting tunnels.

Let  $m_1$  and  $m_2$  be the two axes of the strong involution that are not the upper or lower tunnel. Suppose for the sake of contradiction that one of them is an unknotting tunnel. Denote it by  $\alpha$  and denote the other remaining axis by  $\beta$ . Let  $M = S^3 - N(L)$ . The strong involution corresponding to  $\alpha$  can be extended to all of  $S^3$ . Taking the quotient of  $S^3$  by this involution yields  $S^3$ . Let  $p$  be the quotient map. The neighborhood of each component of  $L$  is taken to a ball by  $p$ . Since the complement of  $N(\alpha \cup L)$  is a handlebody, the complement of  $p(N(\alpha \cup L))$  in  $p(S^3)$  must be a ball  $W$ , and the three axes of the strong involution in the handlebody must project to three unknotted unlinked arcs in the ball. Hence the image of  $\beta$  in  $W$  must be unknotted. However, in order for  $p(\beta)$  to be unknotted in  $W$ , it must be the case that the knot  $p(\alpha \cup L \cup \beta)$  is unknotted in  $S^3$ . However, if  $L$  has tri-symmetric representation coming from the continued fraction  $[2b_0, 2b_1, \dots, 2b_n]$ , (so there are  $2b_0$  crossings at the center,  $b_1$  crossings to the left and right of that,  $b_2$  crossings above and below that, etc.,  $n$  even), then  $p(\alpha \cup L \cup \beta)$  is the two-bridge knot given by  $[b_0, 4b_1, b_2, 4b_3, \dots, b_n]$  and is therefore nontrivial (see Figure 2). Hence,  $\alpha$  cannot be an unknotting tunnel.

Note that in the case of a 2-braid link, it is known that there is exactly one unknotting tunnel, corresponding to both the upper and lower tunnels, as in this

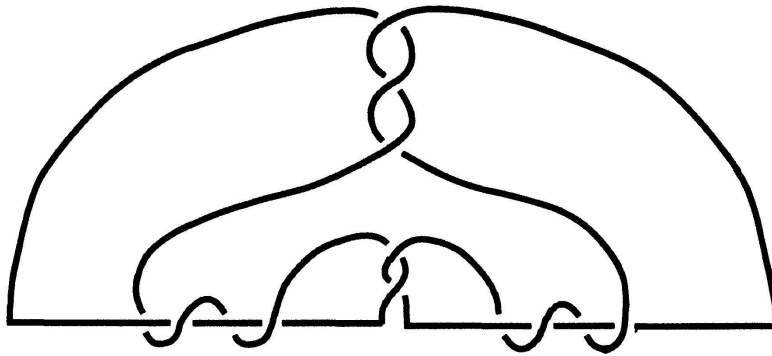


Figure 2

case, they are isotopic. For any other 2-bridge link, the upper and lower tunnels are known to be distinct up to isotopy (cf. Thm. 5.2 of [8] for example).  $\square$

In [3] and/or [8], the authors determine exactly when the upper and lower tunnels are equivalent, in the sense that there is a homeomorphism of  $S^3$  taking the link back to itself and the one unknotting tunnel to the other.

### 3. Strongly parabolic tunnels

**LEMMA 3.1.** *The upper and lower tunnels in a 2-bridge knot or link complement that is not a 2-braid are strongly parabolic.*

*Proof.* From [7], using the work of Thurston, a two-bridge knot or link is hyperbolic if and only if it is not a 2-braid. One example will suffice to prove the lemma. In Figure 3, the two loops shown at the right are freely isotopic to the boundary of a regular neighborhood of the knot. The tunnel  $\alpha$  at left is an upper tunnel (however, we have turned the knot on its side to save space). Hence, a regular neighborhood of the resulting graph can be unknotted. In particular, we can shrink the tunnel to a vertex (leaving the neighborhood of the graph unaffected), and then untwist the first sequence of crossings on the left. We can then untwist the

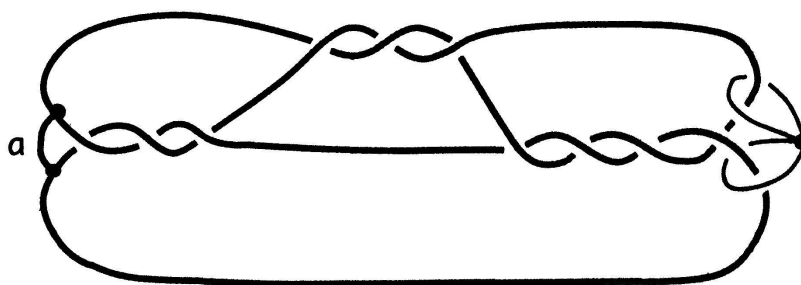


Figure 3

remaining sequences of crossings, one sequence at a time, moving left to right, by untwisting either the two uppermost strands or the two middle strands. At all stages of the unknotting of the graph, the two loops at right are unaffected. After unknotting, they form a pair of generators for the complementary handlebody. Hence, before unknotting, they form a pair of parabolic generators for the link complement. Thus, this unknotting tunnel is strongly parabolic.  $\square$

It was proved in [2] that a parabolic tunnel can only occur in a two-bridge knot or link complement. We will prove:

**THEOREM 3.2.** *An unknotting tunnel  $\alpha$  in the compact core  $M'$  of a finite volume hyperbolic 3-manifold  $M$  is strongly parabolic if and only if  $\alpha$  is either the upper or lower tunnel and  $M$  is a 2-bridge knot or link complement that is not a 2-braid. Such a strongly parabolic tunnel is isotopic to a geodesic arc with length at most  $\ln(4)$  relative to the canonical cusps.*

*Proof.* The existence of a strongly parabolic tunnel implies that the fundamental group of  $M$  is generated by a pair of parabolic elements. By Theorem 3.3 of [2], this implies that  $M$  is the complement of a 2-bridge knot or link in the 3-sphere that is not a 2-braid.

If  $M$  is the complement of a 2-bridge link that is not a 2-braid, Corollary 4.8 of [1] immediately implies that  $\alpha$  is isotopic to a geodesic arc with length at most  $\ln(4)$  relative to the canonical cusps. Moreover we have already proved that  $\alpha$  must be an upper or lower tunnel. Hence we can restrict  $M$  to be the exterior of a 2-bridge knot that is not a 2-braid.

Let  $c$  and  $d$  be two parabolic elements of the fundamental group of  $M$  that generate the fundamental group of the handlebody  $H$ , and such that representative loops in  $H$  can be homotoped to the boundary of  $M$  through  $H$ . Then, we will show that there exists a choice of representative loops  $c'$  and  $d'$  such that a regular neighborhood of their union is a handlebody, call it  $H'$ , and  $\partial H'$  is isotopic to  $\partial H$  through  $H$ . We argue as follows. If  $F$  is a free group generated by elements  $a$  and  $b$ , any automorphism of  $F$  is a product of automorphisms of two types, the first of which fixes one generator and sends the other generator to its inverse, and the second of which fixes one generator and sends the other to a product of the two generators. Hence, we can move from any one pair of generators to any other by a sequence of such operations. Let  $e$  and  $f$  be the two loops that form the spine of  $H$ . Applying an appropriate sequence of these operations, we can obtain  $c'$  and  $d'$ . However, if at any stage, we have a pair of loops that represent a pair of generators of the fundamental group of  $H$ , such that the boundary of a regular neighborhood of the pairs is isotopic through  $H$  to  $\partial H$ , then the same holds true after we apply

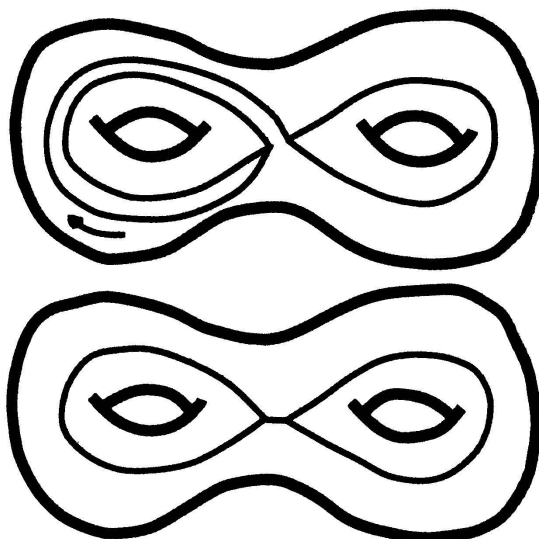


Figure 4

one of these two operations to those elements of the fundamental group. The first operation does not change  $H'$ , while an unwinding of part of  $H'$  as in Figure 4 shows that the second operation preserves the fact that  $\partial H'$  is isotopic to  $\partial H$  through  $H$ .

Hence,  $\alpha$  serves as a tunnel for the handlebody  $H'$ . However, since  $c'$  and  $d'$  are parabolic, there exist three nontrivial involutions of  $M$ , one switching  $c'$  and  $d'$ , one switching  $c'$  and  $c'^{-1}$  and  $d'$  and  $d'^{-1}$  and one switching  $c'$  and  $d'^{-1}$ . This follows from the fact that if we have a fundamental group generated by two parabolic isometries with distinct fixed points, be they  $c'$  and  $d'$  or  $c'$  and  $d'^{-1}$ , we can choose a geodesic such that conjugation by the order two elliptic isometry about that geodesic switches the two parabolics. This elliptic isometry projects to an involution on the corresponding manifold. Similarly, conjugation by the order two elliptic isometry about the geodesic that has endpoints at the fixed points of the two parabolics sends the group to itself by sending each of the generating parabolics to its inverse. This elliptic isometry also projects to an involution of the manifold.

Each of these three involutions sends  $H'$  to itself and extends to the manifold, hence it must send a neighborhood of  $\alpha$  to itself. Since it must send the two disks on  $\partial N(\alpha) \cup \partial M$  back to themselves, it must send  $\alpha$  to  $\alpha$  or to  $\alpha^{-1}$ . At least one of the three must send the attaching curve for the 2-handle back to itself with the same orientation, since any one involution is the product of the other two. Hence, one of the three fixes  $\alpha$ , up to isotopy. In particular, this means that  $\alpha$  is a subset of the axis of a strong involution.

However, if we place the knot  $K$  into a tri-symmetric projection, then there are two obvious axes for strong involutions, and since there are exactly two strong involutions for a 2-bridge knot (cf. [4]), these are all of the possibilities. Each of these axes in  $S^3$  is cut into two arcs by the knot. Two of the resulting four arcs are

the upper and lower tunnels. We will show that the remaining two arcs cannot be unknotting tunnels, thereby proving that a strongly parabolic tunnel must be either the upper or lower tunnel.

Let  $m$  be one of these two arcs and let  $g$  be the strong involution axis within which it lies. Suppose that  $m$  is an unknotting tunnel. Then the quotient of  $S^3$  by the strong involution yields  $S^3$ . Let  $p$  be the quotient map. The axis  $g$  intersects each of the two handlebodies  $N(K \cup m)$  and  $S^3 - N(K \cup m)$  in an arc, and each handlebody is preserved by the strong involution. Hence,  $p(N(K \cup m))$  and  $p(S^3 - N(K \cup m))$  are each solid tori. This implies that  $p(K \cup m)$  must be unknotted in  $S^3$ . However, if  $K$  is given by  $[2b_0, 2b_1, \dots, 2b_n]$  with  $n$  odd, then  $p(K \cup m)$  will be given by  $[b_0, 4b_1, b_2, 4b_3, \dots, 4b_n]$  or  $[4b_0, b_1, 4b_2, b_3, \dots, b_n]$  depending on the strong involution. Neither of the resulting knots are trivial, therefore showing that the only strongly parabolic tunnels are the upper and lower tunnels.

We now return to the three involutions, one of which fixes  $\alpha$ . It can be lifted to an elliptic isometry  $x$  that rotates about a geodesic lift of  $\alpha$ , which for convenience we can choose to be a geodesic with endpoints at 0 and  $\infty$ . The other two isometries then can be lifted to rotations, call them  $y$  and  $z$ , about a pair of geodesics that intersect this lift of  $\alpha$ , such that all three are perpendicular. Hence the other two isometries must both send  $\alpha$  to  $\alpha^{-1}$ .

Let  $g$  be an isometry in the fundamental group that takes the horoball centered at 0 to the horoball centered at  $\infty$ . Then  $g \circ y$  and  $g \circ z$  are lifts of the two involutions, each of which fixes  $\infty$ . At least one of the pair must be a parabolic isometry, as  $x$  is an elliptic isometry fixing  $\infty$ , and if all three were elliptic fixing  $\infty$ , we could not have the product of two giving the third.

Let  $r$  be this parabolic isometry. Since it projects to an involution of the manifold,  $r^2$  must be a parabolic isometry in  $\pi_1(M)$  so that it projects to the identity isometry of  $M$ . Since  $r$  must send the lift of  $\alpha$  with endpoints at 0 and  $\infty$  to a lift of  $\alpha^{-1}$ , with one endpoint at  $\infty$ , and since  $r$  is realized as a Euclidean translation, the two endpoints of  $\alpha$  on the cusp boundary must be equally spaced on the cusp boundary. That is to say, there are two shortest paths on the cusp boundary to get from one to the other. Since every parabolic isometry that projects to an isometry of the manifold moves points in a canonical cusp a distance of at least one, the two endpoints of  $\alpha$  occurring in the cusp must be a distance apart on the cusp boundary of at least 1. The argument given in the proof of Corollary 4.8 of [1] applies in this new situation to give a bound on the length of that part of  $\alpha$  that is outside the canonical cusp, as follows.

The lift of the canonical cusp and the unknotting tunnel must give a connected graph in  $H^3$ , thinking of the horoballs as vertices. The fact that the boundary of the handlebody that is complementary to a neighborhood of  $\alpha$  is compressible implies that the graph is not a tree. Choosing a cycle in the graph and picking the point at

$\infty$  appropriately, we know that there is a smallest horoball  $H$  in the cycle. Since it is attached by lifts of  $\alpha$  to two other horoballs, each of which is at least as large as it is, and since we can choose  $H$  such that one of its neighbors is larger than it is, we know that the two endpoints of the lifts of  $\alpha$  are a distance less than  $2e^{-x/2}$  apart on the surface of  $H$ , where  $x$  is the length of that part of  $\alpha$  outside of the canonical cusp. However, then  $1 < 2e^{-x/2}$ , giving us  $x < \ln(4)$ .  $\square$

#### Section 4. Dual tunnels

In [6], it was shown that there are up to four other unknotting tunnels in addition to the upper and lower tunnel for a 2-bridge knot. In [8], these tunnels were described as dual to the upper and lower tunnels. We now show that these additional unknotting tunnels are homotopic to geodesics, which have a universal bound on their length.

**COROLLARY 4.1.** *Let  $K$  be a two-bridge knot that is not a 2-braid. Then the dual tunnels to the upper and lower tunnels are themselves homotopic to geodesics of length less than  $6\ln(2)$ .*

*Proof.* Let  $t$  and  $t'$  be the upper or lower tunnels in either order, already isotoped to be geodesics in the hyperbolic structure on  $S^3 - K$ . Choose  $N(K)$  to be a neighborhood of  $K$  that is a cusp in the hyperbolic structure on  $S^3 - K$ . If  $d$  is a dual tunnel to  $t'$ , then by definition of  $d$  for a two-bridge knot, there is a disk  $D$  in  $S^3 - N(K)$  with embedded interior such that its boundary is made up of six arcs, the first of which runs the length of  $t$ , the next of which follows a meridian around the boundary of the neighborhood of the knot, then next of which runs the length of  $t$  in the opposite direction, the next of which runs along  $\partial N(K)$ , the next of which runs the length of  $d$ , and the last of which runs along  $\partial N(K)$  to where we started. The disk  $D$  lifts to a disk  $D'$  in  $H^3$ . The two arcs on  $\partial D$  that run along  $t$  lift to arcs within a pair of geodesics in  $H^3$ , where those geodesics share an endpoint at the end corresponding to the meridian. The three arcs on  $\partial D$  that lie on  $\partial N(K)$  lift to paths in three horospheres centered at the three endpoints of the geodesics. The arc corresponding to  $d$  must lift to an arc that is homotopic to the geodesic  $d'$  connecting the two remaining endpoints of the first two edges.

We will now obtain an upper bound on the length of the geodesic  $d'$  relative to a maximal cusp. By Theorem 2.2 and Corollary 3.5 of [2], the meridian curve can be isotoped to a loop in the boundary of the maximal cusp of length strictly less than 2. We have already seen above that  $t$  must be a geodesic of length less than  $\ln(4)$ . Choosing the ideal triangle in  $H^3$  that corresponds to  $D'$  so that the vertex

associated to the meridian is at  $\{\infty\}$  in the upper half space model, we can normalize so that the horoball  $H_\infty$  centered at  $\{\infty\}$  has boundary plane at Euclidean height 1. Then the horoballs at the other two vertices, call them  $H_a$  and  $H_b$ , each have a Euclidean diameter strictly greater than  $1/4$ , and their centers are a Euclidean distance apart that is strictly less than 2. By reflecting in a geodesic with one endpoint at the center of  $H_a$  and its highest point directly above the center of  $H_b$ , we will fix  $H_a$  and send  $H_b$  to a horoball centered at  $\infty$  with Euclidean height at most 16. Hence the geodesic with endpoints at the centers of  $H_a$  and  $H_b$  will be sent to a vertical geodesic with length relative to the maximal cusp strictly less than  $6\ln(2)$ .  $\square$

Although the expectation is that the four dual tunnels will all be isotopic to geodesics, we have no proof of this. However, in the case that the upper or lower tunnel has length 0, we can show that two of the four dual tunnels are isotopic to geodesics, which we do below. In fact, it is conjectured that for a two-bridge knot, either the upper or lower tunnel always does have length 0. This has been checked for two-bridge knots of low crossing number using the computer program SNAPPEA (cf. [10]).

**LEMMA 4.2.** *Let  $t$  and  $t'$  be the upper and lower tunnels in either order and let  $d$  be a dual tunnel to  $t'$ . If  $t$  has length 0, then  $d$  is isotopic to a geodesic.*

*Proof.* As in the preceding proof, since  $d$  is dual to  $t'$ ,  $d$  is isotopic through a disk  $D$  in  $S^3 - N(K)$  to a curve  $c$  that runs along  $t$ , then along a meridian around the boundary of the cusp and then back along  $t$ . Since we can assume  $t$  is geodesic by Theorem 3.1, we can lift  $c$  to a curve in  $H^3$  such that the two arcs of  $c$  that run along  $t$  lift to geodesic arcs on geodesics that share an endpoint at infinity. These two geodesics define an ideal triangle in  $H^3$ . Let  $f'$  be the third edge and  $f$  its projection to the manifold. If the projection of the ideal triangle to  $M$  is embedded, then we can isotope  $c$  through this embedded triangle to  $f$ , thereby showing that  $d$  is isotopic to a geodesic. In fact, it's enough to show that no other lift of  $f$  intersects the original ideal triangle in  $H^3$ . Place the vertex of the triangle corresponding to the two lifts of  $t$  at  $\{\infty\}$  and take the horoball centered at that point, denoted  $H_\infty$ , to be the set of all points in and above the plane of Euclidean height 1. Then the two horoballs at the other ends of the lifts of  $t$  must each be tangent to this one and therefore have a Euclidean diameter of 1. Since the parabolic isometry corresponds to a meridian of a 2-bridge knot, we know that its Euclidean translation distance  $y$  satisfies  $1 \leq y < 2$  from [2]. Let  $f''$  be another lift of  $f$ . Then there is a second ideal triangle and set of three horoballs at the vertices of this ideal triangle that correspond to  $f''$ . We wish to show that  $f''$  cannot intersect the ideal triangle corresponding to  $f'$ . However, the horoball  $H_\infty$  prevents any of the three horoballs corresponding to  $f''$  from having Euclidean diameter greater than 1. Looking down from  $\infty$ , if the centers of the two

horoballs at the end of  $f'$  form two opposite corners of a square and the centers of the two horoballs at the end of  $f''$  form the remaining two corners, then  $f'$  and  $f''$  will intersect. This situation can occur only when  $y = 2$ . Note that in this case the third horoball corresponding to  $f''$  is tangent to all four of the balls centered at the vertices of the square. If  $y < 2$ , then there will not be room for the horoballs at the end of  $f''$ , and  $f''$  will not reach high enough to intersect the ideal triangle of  $f'$ .

In fact, even the intersection when  $y = 2$  cannot occur, as if it did, there would have to be a covering translation sending the one lift  $f'$  of  $f$  to the other  $f''$ . But any such covering translation would be forced to fix the point of intersection between them, as that point is equidistant from the two horoballs at the end of each geodesic. This contradicts the fact that a covering translation can fix no points of  $H^3$ .  $\square$

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