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# Fenchel type theorems for submanifolds of $S^n$

REMI LANGEVIN and HAROLD ROSENBERG

*We dedicate this paper to the memory of Nicolaas Kuiper*

The total curvature of compact hypersurfaces $M$ of $\mathbf{R}^n$ ($\int_M |K|$) is related to the topology of $M$ and to the manner in which $M$ is embedded in $\mathbf{R}^n$. K is the Gauss-Kronecker curvature of M, i.e., the determinant of the second fundamental form.

For curves $C$ in $\mathbf{R}^3$, the theorems of Fenchel and Fary-Milnor, state the total curvature of $C$ is at least $2\pi$ (with equality precisely for convex planar curves) and if $C$ is knotted in $\mathbf{R}^3$ then $\int_C |k| > 4\pi$, [Fe], [Fa], [$M_1$], [$M_2$].

Chern and Lashof observed the total curvature of $M^k \subset \mathbf{R}^n$ is

$$c \int_{P^{n-1}} |\mu|(M, l),$$

where $c$ is a constant depending only on $n$ and $k$, $P^{n-1}$ is the projective space of lines $l$ through the origin in $\mathbf{R}^n$ and $|\mu|(M, l)$ is the number of critical points of the projection of $M$ to $l$. Since this projection is a Morse function for almost all $l$, they obtained $c\beta$ as a minoration of the total curvature, $\beta$ the sum of the betti numbers of M[C-L].

In particular for surfaces in $\mathbf{R}^3$ one has

$$\int_M |K| \geq 2\pi(2g + 2),$$

$g$ the genus of $M$. If a torus is knotted in $\mathbf{R}^3$, then the total curvature is at least twice as large, i.e., $16\pi$ [L-R]. Results of this type for knotted surfaces of higher genus in $\mathbf{R}^3$ have been obtained by Kuiper and Meeks [K-M].

In this paper we establish results of this nature for submanifolds of $S^n$. For surfaces in $S^3$, it is not sufficient to consider $\int_M |K|$, where $K$ is the extrinsic curvature of $M$ (consider the boundary of a small tubular neighborhood of a geodesic. Any two points of $M$ differ by an isometry of $S^3$ so the intrinsic curvature of $M$ is constant; it is zero by Gauss-Bonnet. So $|K| = 1$ and $\int_M |K|$ is the area of $M$). In fact, for curves $C$ in $S^2$, it's easy to see that $\int_C (|k_g|+1) \geq 2\pi$, and equality holds precisely when $C$ is a geodesic; $k_g$ the geodesic curvature of $C$. However for surfaces $M$ in $S^3$, it is still not enough to consider $\int_M (|K|+1)$. One must add to $|K|+1$, a function $h_1(x)$ = the average of the absolute values of the normal curvatures to $M$ at $x$. Then one has the desired results:

$$C(M) = \int_M (c_2|K| + c_1 h_1(x) + c_0) \geq 2\pi(2g+2),$$

for certain constants $c_0$, $c_1$, $c_2$, and $g$ the genus of $M$. Moreover, if $M$ is knotted in $S^3$, then $C(M) \geq 2\pi(2g+4)$.

The function $\int_M h_1$ has an interesting geometric interpretation. It is the total number of folds of $M$. We call this the 1-length of $M$. It is a one dimensional measure of $M$; for $M$ in $\mathbf{R}^3$ and $tM$ the homothety of $M$ by $t$, one has $L_1(tM) = tL_1(M)$. In general, for $M$ a $p$ dimensional submanifold of $\mathbf{R}^n$ or $S^n$, we introduce $i$-length of $M$ for every $i \leq p$. We then study the behaviour of $i$-length through projections and intersections obtaining local and cinematic-type formulae.

Notice that $h_1(x)$ is not (except if $M$ is convex) the first symmetric function of curvature $\sigma_1$ of $M$ at $x$. Chern and Slavsky have studied $\int_M \sigma_1$, for $M$ in $\mathbf{R}^n$ and proved cinematic formulae for these functions [Ch], [Sl].

The 2-length of $M \subset S^3$, $L_2(M)$, is the area of $M$, $L_0(M)$ is the total curvature of $M$. We define $L_1(M)$ as follows. Let $\Sigma$ be a geodesic 2-sphere of $S^3$ with $x$ a conjugate point of $\Sigma$ (i.e., $\operatorname{dist}(x, \Sigma) = \pi/2$). Let $p: S^3 - \{x, -x\} \to \Sigma$ be the projection along the geodesics starting at $x$. Denote by $\gamma_\Sigma$ the critical values of $p/M$. Define

$$L_1(M) = \frac{1}{\pi^2} \int_{G(4,3)} |\gamma_\Sigma| \, d\Sigma,$$

where $G(4, 3)$ is the Grassmann manifold of 3-planes through the origin of $\mathbf{R}^4$, identified with the space of geodesic 2-spheres of $S^3$.

We prove $L_1(M) = \pi^2 \int_M h_1$. Also we establish

$$L_0(M) = \frac{1}{2\mathrm{Vol}\, G(4, 2)} \int_{G(4,2)} |\gamma_l| \, dl,$$

where $l \in G(4, 2)$ is a geodesic of $S^3$, and $|\gamma_l|$ is the number of critical points of the projection of $M$ to $l$ (along the geodesic spheres orthogonal to $l$).

Now one uses the cinematic formulae to relate $L_0(M) + L_1(M) + L_2(M)$ to the critical points of a Morse function on $M$. For this, we construct an "adapted" singular foliation of $S^3$.

The theory is much simpler for curves on $S^2$; we indicate the argument here.

Let $l \in G(3,2)$ denote a geodesic of $S^2$ and for each $y \in P^2$ ($y = a$ pair of antipodal points of $S^2$), let $\mathscr{F}(y)$ be the foliation of $S^2$ (singular at $y$) by geodesics passing through $y$.

We have

$$\int_C |k_g| = \frac{1}{2} \int_{P^2} |\mu|(C, \mathscr{F}(y))\, dy,$$

where $|\mu|(C, \mathscr{F}(y))$ denotes the number of contact points of $C$ and $\mathscr{F}(y)$. Also

$$|C| = \frac{1}{2} \int_{l \in G(3,2)} \#(C \cap l)\, dl = \frac{1}{2\pi} \int_y \left( \int_{l \in \mathscr{F}(y)} \#(C \cap l) \right) dy,$$

where $|C|$ denotes the length of $C$. Hence

$$\int_C (|k_g| + 1) = \frac{1}{2} \int_y \left[ |\mu|(C, \mathscr{F}(y)) + \frac{1}{\pi} \int_{l \in \mathscr{F}(y)} \#(C \cap l)\, dl \right] dy.$$

Now for $y \in P^2$, if $C$ intersects every $l \in \mathscr{F}(y)$, then $C$ intersects every such $l$ in at least two points and

$$\int_{l \in \mathscr{F}(y)} \#(C \cap l) \geq 2\pi$$

If $C$ is disjoint from $l \in \mathscr{F}(y)$, then a moments thought shows there are at least two points of contact of $C$ with $\mathscr{F}(y)$. Thus $|\mu|(C, \mathscr{F}(y)) \geq 2$; so $\int_C (|k_g| + 1) \geq 2\pi$. This illustrates the integral geometric technique but for curves the result is not interesting since the last inequality is just an application of Fenchel's theorem for curves in $\mathbb{R}^3 = (k = \sqrt{k_g^2 + 1}$ is the curvature of $C$ in $\mathbf{R}^3$).

For surfaces in $S^3$ the argument requires the introduction of a foliation adapted to a flag of geodesic spheres.

We remark that this notion of length has been applied in oceanography [J-L].

## I. The length functions for submanifolds of $\mathbf{R}^n$ and their cinematic formulae

Let $M$ be a $p$-dimensional submanifold of $\mathbf{R}^n$ and let $h$ be a $i+1$ dimensional linear subspace of $\mathbf{R}^n$ (we will denote by $G(n, i+1)$ the Grassmann manifold of all such $h$). The critical points of the orthogonal projection $p_h$ of $M$ to $h$ will be denoted by $\Gamma_h(M)$ (or $\Gamma_h$ if there is no ambiguity) and we denote the set of critical values of $p_h$ by $\gamma_h$, or $\gamma(M, h)$.

When $p \geq i$, for almost every $h \in G(n, i+1)$, $\Gamma_h$ is almost everywhere an $i$-dimensional submanifold of $M$ and for almost every $x \in \Gamma_h$, $T_x(\Gamma_h)$ and $h^\perp$ are transverse in $T_x(M)$, so $\gamma_h$ is a hypersurface of $h$ in a neighborhood of $p_h(x)$.

We define the $i$-length functional as:

$$L_i(M) = c \int_{G(n,i+1)} |\gamma_h| \, dh,$$

where $|\gamma_h|$ denotes the volume of $\gamma_h$ (when $i = 0$, $\gamma_h$ is a finite set and $|\gamma_h|$ is the number of points in $\gamma_h$), and the constant $c$ is chosen so that if $M$ is the boundary of an $\varepsilon$-tubular neighborhood of an $i$-dimensional submanifold $C$ of an affine $p+1$ dimensional subspace of $\mathbf{R}^n$, then $\lim_{\varepsilon \to 0} L_i(M) = |C|$.

If $tM$ denotes a homothety of $M$ by $t > 0$, then clearly

$$L_i(tM) = t^i L_i(M).$$

The constant $c$ occurring in the definition of $L_0$ is $1/2|\mathbf{P}_{n-1}|$, since a sphere of any dimension $\geq 1$ satisfies $|\gamma_l| = 2$ for every line $l \in G(n, 1)$. We will see shortly that $L_0(M)$ is the total curvature of $M$.

Here are some examples of 1-lengths of surfaces in $\mathbf{R}^3$:

$$L_1(M) = \frac{1}{\pi^2} \int_{G(3,2)} |\gamma_h| \, dh.$$

If $M$ is a round cylinder of height $\lambda$, then $\gamma_h$ is (for almost all $h$) two parallel segments of length $\lambda |\cos \theta|$ where $\theta$ is the angle between the axis of $M$ and the plane $h$. Hence $L_1(M) = \lambda$. If $M$ is a sphere of radius $R$, $\gamma_h$ is a circle of radius $R$ and $L_1(M) = 4R$.

### *I.1. The local formulae*

We define extrinsic curvature functions $h_i$ on $M^p \subset \mathbf{R}^n$, and we prove $L_i(M) = c \int_M h_i(x) \, dx$, where $c = c(n, p, i)$.

Let us begin by $L_0$ and $L_1$ of a surface $M$ in $\mathbf{R}^3$. We know that

$$L_0(M) = \frac{1}{4\pi} \int_{\mathbf{P}_2} |\gamma_l| \, dl,$$

where $|\gamma_l|$ is the number of critical points of the projection of $M$ to $l$.

Let $\phi$: $M \to E$ be the map $\phi(x) = (l(x), p_{l(x)}(x))$, where $l(x)$ is the line through the origin parallel to the normal line to $M$ at $x$, $p_{l(x)}(x)$ is the orthogonal projection of $x$ to $l(x)$, and $E$ is the tautological line bundle over $P_2$. Let $N = \phi(M)$ and $H$ be the horizontal plane field of the Riemannian fibration $\pi$: $E \to P_2$.

Clearly $\pi\phi$ is the Gauss map of $M$ with $|\mathrm{Jac}(\pi\phi)| = |K(x)|$, $K$ the Gauss curvature of $M$ at $x$; so

$$|K(x)| = |\mathrm{Jac}\, \phi(x)|\, |\mathrm{Jac}\, p_{H(x)}|,$$

where $\mathrm{Jac}\, p_{H(x)}$ is the Jacobian of the orthogonal projection (in $E$) of $T_{\phi(x)}N$ to $H_{\phi(x)} = H(x)$.

Hence

$$\int_{\mathbf{P}_2} |\gamma_l|\, dl = \int_N |\mathrm{Jac}(p_H)| = \int_M |\mathrm{Jac}(\phi)|\, |\mathrm{Jac}\, p_H|\, dx = \int_M |K(x)|\, dx.$$

The first equality is a special case of the coarea formula and the second is a change of variables. Hence

$$L_0(M) = \frac{1}{4\pi} \int_M |K(x)|\, dx.$$

This formula for the total curvature of $M$ is the basis of the Chern-Lashof theorem and easily generalises to $\mathbf{R}^n$ [C-L].

For future calculations it is useful to introduce the following notation. Let $p$: $E \to B$ be a Riemannian fibration and $N \subset E$ a submanifold transverse to the fibers $F(y) = p^{-1}(y)$, $y \in B$. Let $H$ be the horizontal plane field of the fibration. At $x \in N$, $T_x(N)$ is the orthogonal sum $T_x(N \cap F_x) + V(x)$ where $V(x)$ is a subspace transverse to the fibers of dimension that of $H(x)$. Denote by $\mathrm{Jac}\, p_{H(x)}$ the Jacobian of the orthogonal projection of $V(x)$ to $H(x)$. Then the coarea formula yields:

$$\int_N |\mathrm{Jac}\, p_{H(x)}|\, dx = \int_B |F(y) \cap N|\, dy,$$

and more generally, if $\phi$: $M \to E$ is an immersion transverse to the fibers, $N = \phi(M)$, then

$$\int_M |\mathrm{Jac}\, \phi|\, |\mathrm{Jac}\, p_{H(x)}| = \int_N |\mathrm{Jac}\, p_{H(x)}|\, dx = \int_B |F(y) \cap N|\, dy.$$

Now we derive the local formula for a surface $M$ in $\mathbf{R}^3$. Let $l$ be a line in the tangent space to $x \in M$, and let $|k(x, l)|$ be the module of the normal curvature of $M$ at $x$ in the direction $l$; i.e., $k(x, l)$ is the curvature of the plane curve $M \cap (v_x \oplus l)$, $v_x$ the normal line to $M$ at $x$.

We define

$$h_1(x) = \frac{1}{\mathrm{Vol}(\mathbf{P}_1)} \int_{\mathbf{P}_1(T_x(M))} |k(x, l)|\, dl.$$

When $M$ is convex at $x$, $h_1(x)$ is the mean curvature of $M$ at $x$.

PROPOSITION I.2. *For $M$ a surface in* $\mathbf{R}^3$,

$$L_1(M) = \frac{1}{\pi} \int_M h_1(x)\, dx.$$

*Proof.* Let $\pi: E = E(3, 2) \to G(3, 2) = G$ be the tautological line bundle, $E = \{h \in G, x \in h\}$.

Let $\phi: P_1(M) \to E$ be the map

$$\phi(x, l) = (h = l^{\perp}, p_h(x)),$$

and let $\phi(P_1(M)) = N$. We know that

$$\int_G |\gamma_h|\, dh = \int_{P_1(M)} |\mathrm{Jac}\, \phi|\, |\mathrm{Jac}\, p_H|,$$

so we compute the Jacobians.

Let $l$ be a line through $x$ in $T_x(M)$, $v_x$ denote the line normal to $M$ at $x$, $h = l^{\perp}$ the subspace of $\mathbf{R}^3$ orthogonal to $l$ and $W$ the orthogonal to $v_x$ in $h$; cf. Figure 1.

We choose a basis of $T_{(x,l)}(P_1(M))$ as follows:

- $U_f$ is a unit vector tangent to the circle fiber of $\mathbf{P}_1(M)$ at $x$,
- $U_\Gamma$ is a horizontal lift of a unit vector tangent to $\Gamma_h$ at $x$,
- $U_l$ is a horizontal lift of a unit vector tangent to $(l \oplus v_x) \cap M$ at $x$.

Also, let $U_\gamma$ be a horizontal lift (in $E$) of a unit vector tangent to $\gamma_h$ at $y$.

The volume of the parallelepiped generated by the first three vectors is $|\cos\theta|$ where $\theta$ is the angle between $T_x\Gamma_h$ and $h$.

The image $d\phi(U_\Gamma)$ is the vector $\pm\cos(\theta)U_\gamma$. The vector $d\phi(U_f)$ and $d\phi(U_l)$ are projected by the differential $d\pi$ of the projection $\pi: E(3, 2) \to G(3, 2)$ on two orthogonal vectors of $T_{\pi\phi(x)}G(3, 2)$; the first unitary and the second of norm $|k(x, l)|$.

Hence

$$|\mathrm{Jac}\,\phi(x)|\,|\mathrm{Jac}\,p_H| = |k(x, l)|,$$

and I.2 follows by integrating over the fibers of $\mathbf{P}_1(M)$.

*Remark.* A different proof of this can be found in [L-S] based on a Meusnier formula.

Now we define the functions $h_i(x)$ when $M \subset \mathbf{R}^n$ is a hypersurface. Let $l = l^i$ be an $i$-dimensional subspace of $T_x(M)$, and let $\nu(x)$ be the normal line to $M$ at $x$. Denote by $|K|(x, l)$ the absolute value of the Gauss-Kronecker curvature at $x$ of the hypersurface $M \cap (l \oplus \nu(x))$ of $l \oplus \nu(x)$. Then we define

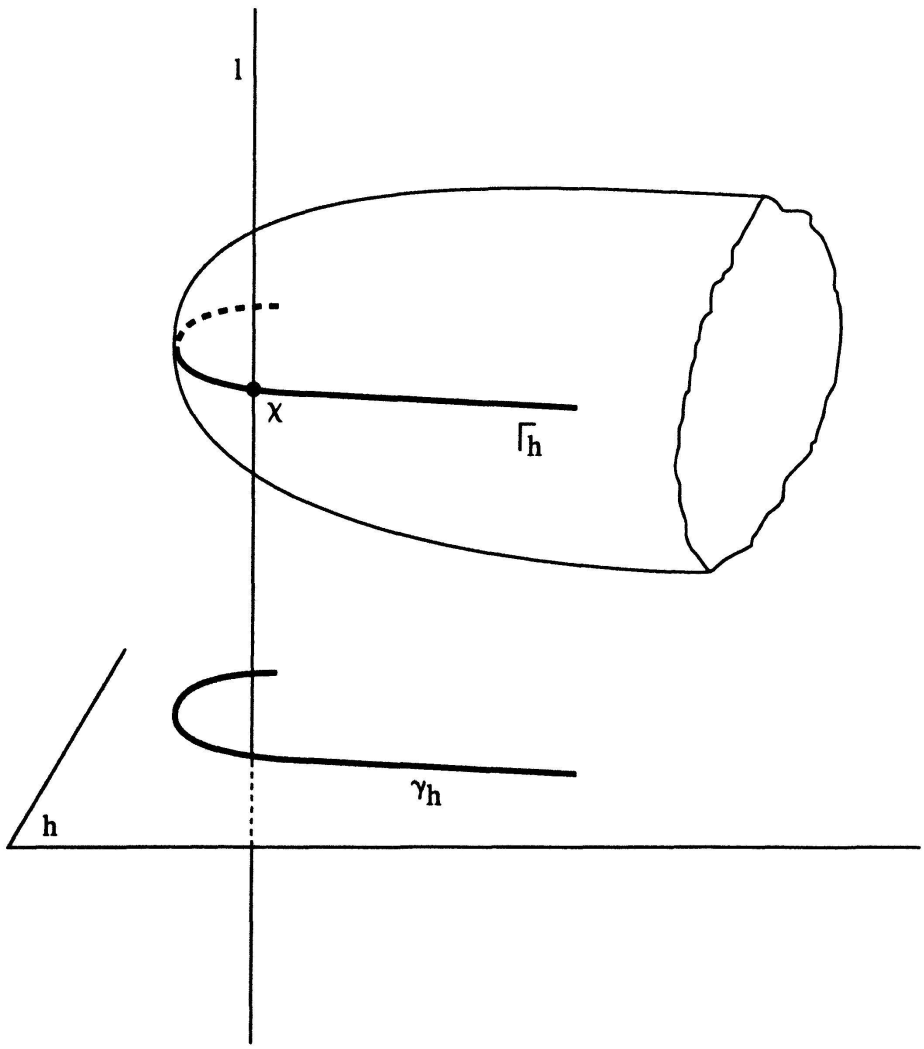


Figure 1

$$h_i(x) = \frac{1}{\operatorname{Vol} G(n-1, i)} \int_{G(T_x M, i)} |K|(x, l)\, dl,$$

where $G(T_x M, i)$ is the $i$-dimensional subspaces of $T_x(M)$.

Now I.2 generalizes to $\mathbf{R}^n$.

PROPOSITION I.3. *The functions $h_{n-i}(x)$ localize the functions $L_i(M)$; more precisely,*

$$\int_M h_{n-i}(x) = cL_i(M),$$

*where the constant $c$ depends only on the dimensions.*

*Proof.* Let $G$ be the bundle over $M$ whose fibers are the spaces $G(T_x M, l)$, $l$ an $n-1-i$ dimensional subspace of $T_x M$, and let $E = E(n, i+1) \to G(n, i+1)$ be the tautological bundle.

Define $\phi\colon G \to E$ by

$$\phi(x, l) = (h = l^{\perp}, p_h(x)).$$

Notice that the dimension of $G(M, n-1-i)$ is equal to the dimension of $N = \bigcup_{h \in G(n, i+1)} \gamma_h$, which is $in + n + i^2 - i - 1$.

Now the proof proceeds as in I.2; we leave the details to the reader.

## *I.4. The cinematic formulae*

We will show that the $p$-length of a submanifold $M \subset \mathbf{R}^n$ is equal to the $(p-i)$-length of the sections of $M$ by affine subspaces of codimension $i$ (up to a constant only depending on dimensions; we will denote such constants by $c$ here).

The idea is to use the Cauchy formula and a projection in cascade.

Let $D$ denote the flag of all pairs $(h, L)$ where $g \in G_{n,p+1}$ and $L$ is an affine subspace of $h$ of codimension $i$.

When $L$ is transverse to $\gamma_h$, the points of $\gamma_h \cap L$ are the critical points of the projection of $M \cap (L \oplus h^{\perp})$ to the vector subspace $l$ determined by $L$. Let $H = L \oplus h^{\perp}$; $H$ is an affine subspace of codimension $i$ in $\mathbf{R}^n$.

Since $\gamma(M \cap H, l) = \gamma_h \cap L$, we have

$$|\gamma_h| = c \int_{L \in A(h, p+1-i)} |\gamma(M \cap H, l)|.$$

Hence

$$L_p(M) = c \int_{G(n,p+1)} \left( \int_{A(h,p+1-i)} |\gamma(M \cap H, l)| \right).$$

Notice that $D$ can be thought of as $\{H \in A(n, n-i), l \in G(H, p+1-i)\}$, hence $D$ is a Riemannian fibration over $A(n, n-i)$ with fiber $G(H, p+1-i)$.

Now

$$c \cdot L_{p-i}(M \cap H) = \int_{G(H,p+1-i)} |\gamma(M \cap H, l)|,$$

hence one has the cinematic formula:

$$L_p(M) = c \int_{A(n,n-i)} L_{p-i}(M \cap H).$$

## II. Surfaces in $S^3$

In this section we will define the length functionals of surfaces in $S^3$ and establish the local and cinematic-type formulae. There are technical difficulties that arise here (in contrast to $\mathbf{R}^3$) due to the fact that the distortion of the projection in $S^3$ to a geodesic sphere depends on the point.

We begin with $L_2(M)$ (=the area of $M$) and the spherical Cauchy-Crofton formula [Sa].

THEOREM II.1. *For $M$ a compact surface in $S^3$,*

$$L_2(M) = \frac{1}{\pi} \int_{G(4,2)} |M \cap l| \, dl,$$

*where $l$ is a great circle of $S^3$ (which we can think of as a 2-plane through the origin of $\mathbf{R}^4$), $|M \cap l|$ is the number of points of $M \cap l$.*

*Proof.* Consider the map $\phi\colon P(TS^3/M) \to G(4,2)$, $\phi(x, L) = l$ where $l$ is the great circle whose tangent at $x$ is $L$

Write the tangent space to $G(4,2)$ at $l_0$ as an orthogonal sum:

$$T_{l_0}G(4,2) = T_{l_0}\{l/x \in l\} \oplus T_{l_0}\{l \perp \Sigma_{l_0,x}\},$$

where $\Sigma_{l,x}$ is the geodesic 2-sphere at $x$ orthogonal to $l$.

Write $T_{(x,L)}(PTS^3/M) = V \oplus H$ where $V$ is the tangent space to the fiber and $H = V^{\perp}$. Then

$$d\phi = \begin{pmatrix} Id & * \\ \bigcirc & p_{L^{\perp}} \end{pmatrix},$$

where $p_{L^{\perp}}$ is the orthogonal projection of $T_xM$ to $T_x(\Sigma_{l,x}) = L^{\perp}$. Then

$$\int_{L \in P_x(TS^3/M)} |\mathrm{Jac}\, d\phi| = \int_{P_2} |\cos \measuredangle (L^{\perp}, T_xM)| = \pi.$$

Since

$$\int_{G(4,2)} |\phi^{-1}(l)| = \int_{G(4,2)} |l \cap M|,$$

we have

$$\int_{G(4,2)} |l \cap M| = \pi |M|.$$

Now we discuss $L_1(M)$. Let $a = (x, -x) \in G(4, 1)$, be a pair of antipodal points of $S^3$ which are not on $M$. This point $a$ determines a projection $p_{\Sigma}: M \to \Sigma$ where $\Sigma$ is the geodesic 2-sphere of $S^3$ conjugate to a (i.e. dist $(x, \Sigma) = \pi/2$). By definition $p_{\Sigma}(y)$ is the point of $\Sigma$ which is the intersection with $\Sigma$ of the geodesic of $S^3$ through $a$ and $y$. Let $\Gamma_{\Sigma}$ be the critical points of $p_{\Sigma}$ and $\gamma_{\Sigma}$ the critical values.

DEFINITION. $L_1(M) = (1/2\pi^2) \int_{G(4,3)} |\gamma_{\Sigma}|\, d\Sigma$.

The constant is chosen so that the 1-length of an $\varepsilon$ tubular neighborhood of a curve $C$ tends to the length of $C$ as $\varepsilon \to 0$. This choice will be justified once we have established the cinematic formulae for $L_1$.

Now just as in $\mathbf{R}^3$ we define an extrinsic function $h_1$ on $M$. Let $k(x, l)$ be the geodesic curvature at $x$ of the curve $\Sigma_l \cap M$ in $\Sigma_l$, where $\Sigma_l$ is the geodesic 2-sphere at $x$ tangent to $l$ and $\nu_x = T_x(M)^{\perp}$. Then define

$$h_1(x) = \frac{1}{\pi} \int_{P_1(T_xM)} |k(x, l)|\, dl.$$

THEOREM II.2. *For $M$ a compact surface in $S^3$,*

$$L_1(M)=\frac{1}{\pi}\int_M h_1.$$

*Proof.* For $x\in M$, let $\Sigma_x$ be the geodesic 2-sphere tangent to $M$ at $x$. Let $P$ be the bundle over $M$ with fiber the projective space $P_2$:

$$P=\{(x,a)/a=(y,-y),y\in\Sigma_x\}.$$

Denote by $\Sigma_a^*$ the geodesic 2-sphere conjugate to the pair $a=(y,-y)$, and let $E=E(4,3)\to G(4,3)=G$ be the tautological bundle:

$$E=\{(\Sigma,y)/\Sigma\ a\text{ geodesic 2-sphere}, y\in\Sigma\}.$$

Then define $\phi: P\to E$ by:

$$\phi(x,a)=(\Sigma_a^*, p_{\Sigma_a^*}(x)).$$

By construction $N=\phi(P)$ is the union of the critical values $\gamma_\Sigma$; $N=\bigcup_\Sigma\gamma_\Sigma$ (cf. Figure 2; the polar curve $\Gamma_\Sigma$ is the set of critical points of the orthogonal projection on $\Sigma$, and the critical values $\Gamma_\Sigma$ is in $p_\Sigma(\Gamma_\Sigma)$).

Then

$$\int_{G_{4,3}}|\gamma_\Sigma|\,d\Sigma=\int_P|\text{Jac}\,\phi|\,|\text{Jac}\,p_H|,$$

so we must calculate the Jacobians.

To do this we decompose $T_{(x,a)}P$ and $TN$.

As $y$ varies on $\Sigma$, $\Sigma_y^*$ spans a sphere $S(\Sigma)$ contained in $G$.

Let $F$ be the 3-dimensional orthogonal complement of $T\gamma_\Sigma$ in $TN$, at the point $u=(\Sigma_a^*, p_{\Sigma_a^*(x)})$. Write $F=F_1\oplus F_2$ (at $x$), where $F_1$ is the lift of $T_{\Sigma_a^*}(S(\Sigma))$ to $F$ and $F_2$ is the orthogonal complement of $F_1$ in $F$. So $TN=F_1\oplus F_2\oplus T\gamma_\Sigma$, at $x$. Let $H_1$ be the horizontal lift to $H(E)$ of $T_{\Sigma_a^*}(S(\Sigma))$, and let $H_2$ be $H_1^\perp$ in $H(E)$.

Now define a splitting of $T_{(x,y)}P$, non orthogonal in general, as follows. Write $T_xM=T_x\Gamma_{\Sigma_y^*}+L$, where $L$ is the line tangent to the circle $l$ joining $x$ to $y$ (this is not orthogonal in general). Let $h_1$ and $h_2$ be the horizontal lifts to $P$ of $T_x\Gamma_{\Sigma_y^*}$ and $L$ respectively.

We shall see that the matrix of $p_H\circ d\phi$ is then:

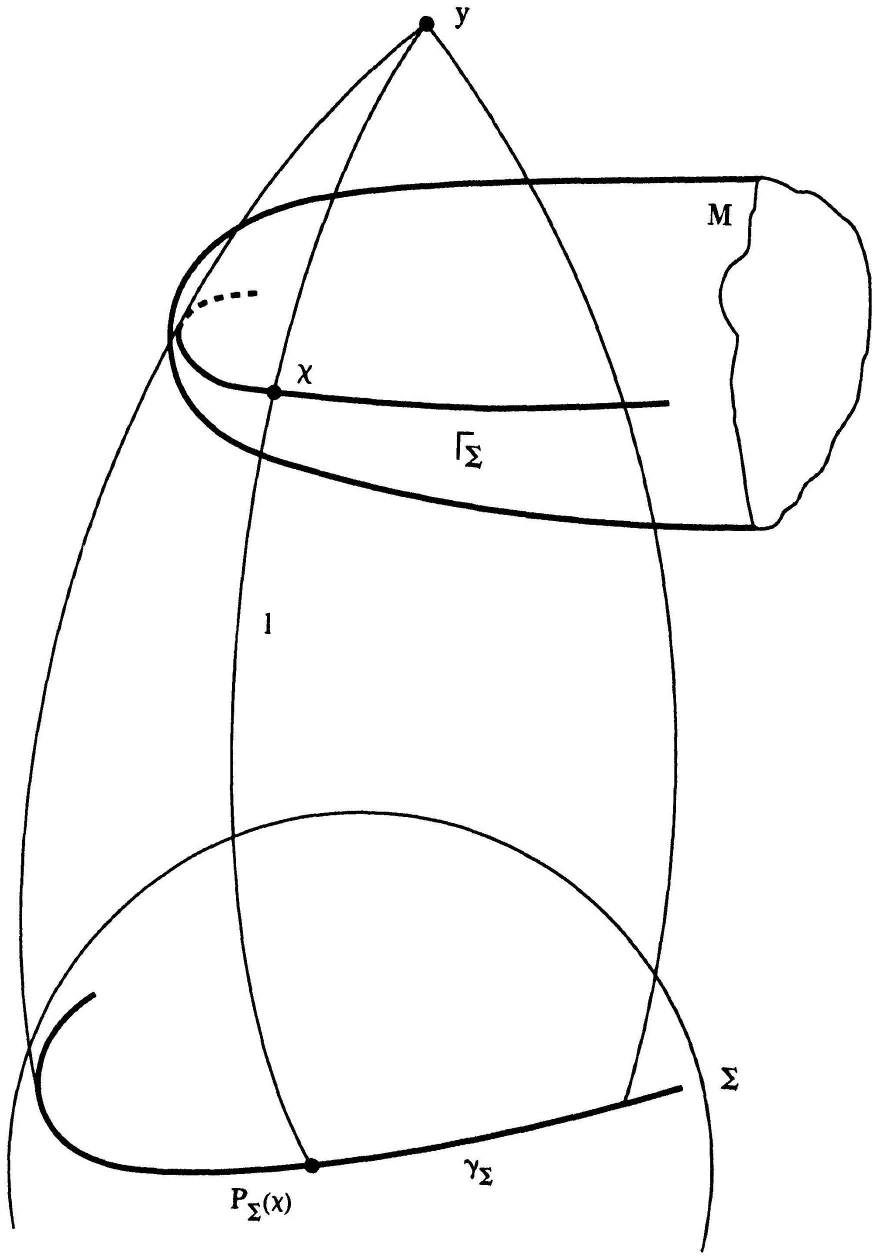


Figure 2

$$\begin{pmatrix} \alpha & * & * \\ 0 & Id & * \\ 0 & 0 & k(x, L)|\sin\theta| \end{pmatrix}$$

here $\alpha$ is the Jacobian of the projection of $\Gamma_\Sigma$ on $\gamma_\Sigma$ and $\theta$ is the arclength on $l$ between $x$ and $y$. This matrix is computed with respect to the basis vectors $\{h_1, T_{(x,y)}\Sigma_x, h_2\}$ of the domain and the basis vectors $\{T\gamma_\Sigma, H_1, H_2\}$ of the range. We calculate the matrix of $p_H \circ d\phi$ on $H_1 \oplus H_2$; identifying $H_1 \oplus H_2$ with $TG$.

By definition of $\Gamma_\Sigma$, $d\phi(h_1) \subset T\gamma_\Sigma$.

The coefficient $\alpha$ satisfies: $\alpha|\sin\theta| = \alpha_0$, where $\alpha_0$ is the Jacobian of the projection of $\Gamma_{\Sigma_0}$ on $\gamma_{\Sigma_0}$, when the geodesic sphere $\Sigma_0$ is orthogonal to $l$ at $x$. This follows from lemma II.3, which we prove shortly.

By definition of $T_{\Sigma_a^*}(S(\Sigma))$, $d\phi(T_{(x,y)}\Sigma_x)$ is of the form:

$$\begin{pmatrix} * \\ Id \\ 0 \end{pmatrix}.$$

It remains to determine the component of $d(p \circ \phi)(h_2)$ on $H_2$. For that, we follow a point on the circle tangent at $\xi$, where $\xi$ is a point moving on the curve $C$ of intersection of $M$ with the geodesic sphere at $x$ containing $l$ and the normal geodesic circle to $M$ at $x$ (cf. Figure 3). Figure 3 shows the analogous map for a curve on $S^2$: the length of the arc of the evolute (image of the arc $dl$ between $x$ and

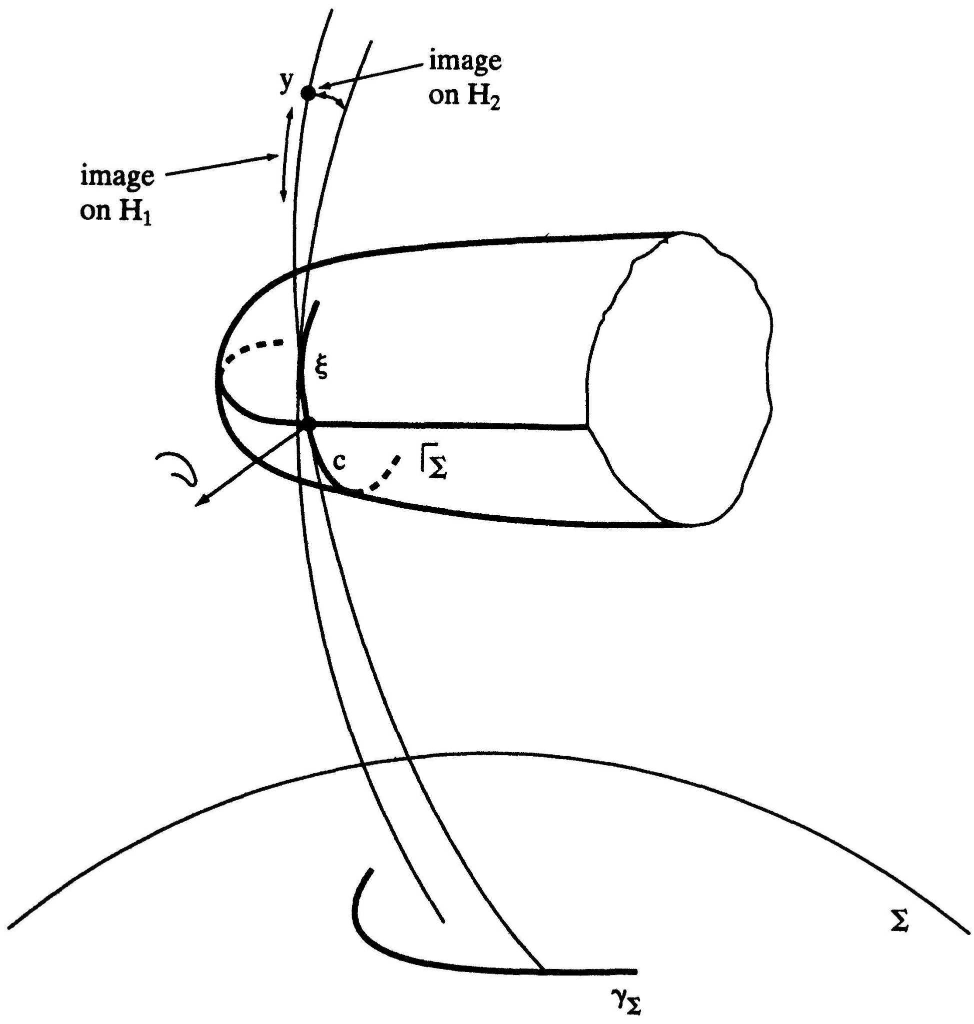


Figure 3

$\xi$) is $k(x)|\sin\theta|$, up to first order, where $\theta$ is the arc length along $l$ between $x$ and $y$ (since $k(x)=d\varphi/ds$).

The same analysis applies in $S^3$; one gets $k(x, l)|\sin\theta|$.

The decomposition of $TP$ is not orthogonal; the volume of the parallelepiped generated by $h_1$, $T_{(x,y)}\Sigma_x$ and $h_2$ is $\alpha_0$.

The volume density on $P(\Sigma_x)$ is $|\sin\theta\, d\theta\wedge d\varphi|$ where $(\theta, \varphi)$ are polar coordinates at $x$ on the space $P(\Sigma_x)$ of pairs of antipodal points on $\Sigma_x$.

Hence

$$\int_P |\mathrm{Jac}\,\phi|\,|\mathrm{Jac}\,p_H| = \int_M \int_{P(\Sigma_x)} \frac{\alpha_0|k(x, l)|\,|\sin\theta|\,|d\theta\wedge d\varphi|}{\alpha_0}$$

$$= 2\pi \int_M h_1(x)\,dx.$$

To complete the proof of theorem II.2 we now prove Lemma II.3.

LEMMA II.3. *Let $C(t)$ be a curve on a surface $M$ embedded in $\mathbf{R}^3$. Assume $\dot{C}(t)$ is not in the kernel of $\gamma$ at $C(t)$, $\gamma$ the Gauss map of $M$. Then the characteristic line of the envelope of the family of tangent planes to $M$ along $C(t)$ is $d\gamma(\dot{C})^{\perp}$.*

*Proof.* The equations of the envelope are:

$$\langle X-x, \gamma(x)\rangle = 0,$$

$$\langle X-x, d\gamma(\dot{C})\rangle = 0.$$

As an immediate corollary of this lemma we have: if $K(x)\neq 0$ (so $d\gamma(x)$ is non singular), all the curves $C$ through $x$ ($C$ on $M$), such that the characteristic line through $x$ of the envelope of the family of planes $T_{C(t)}M$ is a given line $D$, are tangent at $x$ to the line $\Delta$ such that $d\gamma(\Delta)=D$.

The analogous result in $S^3$, using envelopes of geodesic spheres tangent to $M$ along a curve, follows from the following remark concerning cones in $\mathbf{R}^4$, over $M\subset S^3$ and $C(t)$ a curve on $M$. Then the envelope of the family $T_{C(t)}(Z)$, contains the 2-plane $(d\gamma(\dot{C}(t)))^{\perp}$, (orthogonal in $T_{C(t)}Z$ to $d\gamma(\dot{C}(t))$) whenever $\dot{C}(t)$ is not contained in *Ker* $d\varphi$. This remark is clear since the equations of the 2-plane are as before:

$$\langle X-C(t), \gamma(C(t))\rangle = 0$$

$$\langle X-C(t), d\gamma(\dot{C}(t))\rangle = 0.$$

We finish this section with a discussion of $L_0(M)$. By definition:

$$L_0(M)=\frac{1}{2\ \mathrm{Vol}(G(4,2))}\int_{G(4,2)}|\gamma_l|\,dl,$$

where $|\gamma_l|$ is the number of critical points of the projection of $M$ to the geodesic $l$; the projection along the (singular) foliation $\mathscr{F}(l)$ of geodesic 2-spheres orthogonal to $l$. Notice that $|\gamma_l|$ is the number of points of contact of $M$ and $\mathscr{F}(l)$, for almost all $l$. The constant is chosen so that $L_0(\partial B(x,\varepsilon))=1$, for $\varepsilon\to 0$.

THEOREM II.4. *Let $M$ be a surface in $S^3$ and $K(x)$ be the extrinsic Gauss curvature of $M$ at $x$. Then*

$$L_0(M)=\frac{1}{4\pi}\int_M|K(x)|.$$

*Proof.* Let $E=E(4,2)\to G(4,2)=G$ be the tautological fibration and let $P(M)$ be the bundle over $M$ of the geodesic 2-spheres tangent to $M$. Define $\phi\colon P\to E$ by:

$$\phi(x,y)=(y,\ l\ \text{is orthogonal to}\ \Sigma_x\ \text{at}\ y).$$

Here $\Sigma_x$ is the geodesic sphere tangent to $M$ at $x$. Let $N=\phi(P)$ and $H$ be the horizontal field of the bundle $E\to G$.

Take a basis of $T_{(x,y)}P$ composed of a unitary frame tangent to $\Sigma_x$ at $y$ and two horizontal unit vectors that project to two unitary vectors tangent to the principal directions to $M$ at $x$. Then it is clear that the proof of II.4 follows from Lemma II.5 below.

First we define the 0-length of a curve $C$ on $S^2$:

$$L_0(C)=\frac{1}{4\pi}\int_{G(3,2)}|\gamma_l|\,dl.$$

Then we have:

LEMMA II.5. *Let $k_g$ be the geodesic curvature of a curve $C\subset S^2$. Then*

$$L_0(C)=\frac{1}{2\pi}\int_C|k_g|.$$

*Proof.* Let $E = E(3, 2) \to G(3, 2) = G$ be the tautological fibration and $P(C)$ the bundle over $C$ with fibers the geodesic circles of $S^2$ tangent to $C$. Define $\phi$: $P(C) \to E$ by

$$\phi(x, y) = (y, l \text{ is orthogonal to } \Sigma_x \text{ at } y).$$

Here $\Sigma_x$ is the geodesic circle tangent to $C$ at $x$. We have

$$|\mathrm{Jac}\, p_H| = |\cos d(x, y)|\, |k_g|,$$

so integrating on the fibers of $P(C)$ we have

$$\int_C |k_g| = C_0 \cdot L_0(C).$$

Since

$$\lim_{\varepsilon \to 0} \int_{\partial B(x,y)} |k_g| = 2\pi,$$

we see that $C_0 = 2\pi$.

Now we derive a cinematic-type formula satisfied by $L_1(M)$.

THEOREM II.6. *Let $M$ be a surface in $S^3$. Then*

$$L_1(M) = \frac{1}{\pi} \int_{G(4,3)} L_0(M \cap \Sigma).$$

*The constant is obtained by considering small spheres $S_t$. Then $L_1(S_t) \sim 4t$* and $\int_{G(4,2)} L_0(S_t \cap \Sigma) \sim 4\pi t$.

*Proof.* By definition,

$$L_1(M) = \frac{1}{2\pi^2} \int_{G(4,3)} |\gamma_\Sigma|.$$

The Cauchy-Crofton formula in $S^2$ says:

$$|\gamma_\Sigma| = \frac{1}{2} \int_{G(3,2)} |\gamma_\Sigma \cap l|.$$

The inverse image of the orthogonal projection onto $\Sigma$ of the great circle $l$ is a sphere $\Sigma_l$. The points of $\gamma_\Sigma \cap l$ are the critical points of the orthogonal projection of $\Sigma_l \cap M$ onto $l$. Hence

$$L_1(M) = \frac{1}{4\pi^2} \int_{G(4,3)} \int_{G(3,2)} |\gamma_\Sigma \cap l| = \frac{1}{4\pi^2} \int_{D(4,3,2)} |\mu|(\Sigma_l \cap M, P_l),$$

where $P_l$ is the (singular) foliation of $\Sigma_l$ by geodesics orthogonal to $l$. Here $D = D(4, 3, 2)$ is the space of flags $(\Sigma, l)$, $\Sigma \supset l$. The map $D \mapsto D$, $(\Sigma \supset l) \mapsto (l \subset \Sigma)$, is an isometry of $D$. Hence

$$L_1(M) = \frac{1}{4\pi^2} \int_{G(4,3)} 4\pi L_0(\Sigma \cap M) = \frac{1}{\pi} \int_{G(4,3)} L_0(\Sigma \cap M),$$

which completes the proof of II.6.

## III. The Fenchel theorem for surfaces in $S^3$

Let $D = D(4, 3, 2, 1)$ be the space of flags $\Delta = (y \subset l \subset \Sigma)$ where $y$ is a pair of antipodal points of a geodesic $l$ contained in a geodesic sphere $\Sigma$ of $S^3$. Given $\Delta$, let $\mathscr{F}(y)$ be the foliation (singular) of $\Sigma$ by the geodesics of $\Sigma$ passing through $y$ and let $\mathscr{F}(l)$ be the foliation of $S^3$ by the geodesic spheres of $S^3$ containing $l$.

For $M$ a compact surface in $S^3$ we define the geometry of $M$ with respect to $\Delta$, by

$$\mathrm{Geom}(M, \Delta) = \#(l \cap M) + |\mu|(M \cap \Sigma, \mathscr{F}(y)) + |\mu|(M, \mathscr{F}(l)),$$

where $|\mu|(M \cap \Sigma, \mathscr{F}(y))$ is the number of points of contact of $M \cap \Sigma$ and $\mathscr{F}(y)$, and $|\mu|(M, \mathscr{F}(l))$ the number of contact points of $M$ and $\mathscr{F}(l)$. If $M$ is transvere to $\Delta$ (i.e. $y \notin M$ and $l$ and $\Sigma$ are transverse to $M$) and if $M \cap \Sigma$ is in general position with respect to $\mathscr{F}(y)$, $M$ in general position with respect to $\Delta$, then $\mathrm{Geom}(M, \Delta)$ is well defined. This holds for almost every $\Delta \in D$.

Hence we can define the geometry of $M$:

$$\mathrm{Geom}(M) = \frac{1}{\mathrm{Vol}(D)} \int_D \mathrm{Geom}(M, \Delta).$$

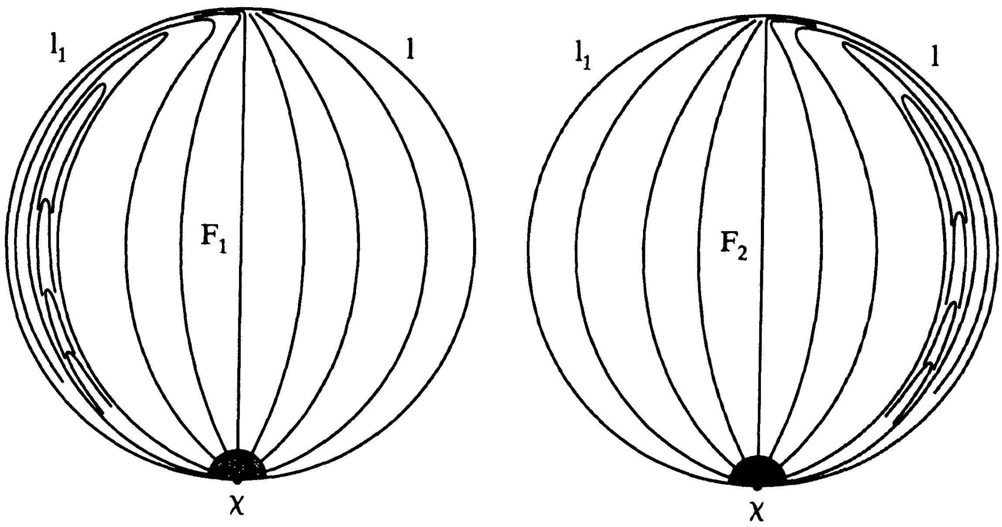


Figure 4

THEOREM III.1. Geom$(M) \geq 2g+2$, *g the genus of M, and if M is knotted in* $S^3$ Geom$(M) \geq 2g+4$. (*M oriented*).

*Proof.* It suffices to prove the inequalities for Geom$(M, \Delta)$ whenever $M$ is transverse to $\Delta$ and in general position with respect to $\mathscr{F}(y)$ and $\mathscr{F}(l)$. To do this we shall construct a foliation $\mathscr{F} = \mathscr{F}(t)$ of $S^3 - B(x, t)$ for $t > 0$ small, $x \in y$, $B(x, t)$ the $t$-ball of $S^3$ centered at $x$, satisfying:

- Geom$(M, \Delta) = |\mu|(M, \mathscr{F})$
- $\mathscr{F}$ is smoothly equivalent to a foliation of $\mathbf{R}^3$ by parallel planes,
- $M$ is in general position with respect to $\mathscr{F}$.

Then the standard Morse theory applies and the theorem follows.

Let $t > 0$ be chosen so that $B(x, t)$ is disjoint from $M$. Let $\Sigma_1$ be one of the hemispheres of $\Sigma$ bounded by $l$, $\Sigma = \Sigma_1 \cup \Sigma_2$, $\Sigma_1 \cap \Sigma_2 = l$. Let $\mathscr{F}_1$ be a one-dimensional foliation of $\Sigma_1 - B(x, t)$ as in Figure 4). Notice that $l$ is a leaf of $\mathscr{F}_1$ (actually $l - B(x, t)$). We require the leaves of $\mathscr{F}_1$ to be geodesics of $\Sigma_1$ through $y$, outisde of a small tubular neighborhood of $l$ in $\Sigma_1$.

This foliation of $\Sigma_1$ has a "Reeb-type" component near an arc $x = l_1$ of $l$ going from $-x$ to $\partial B(x, t)$ (the left side of $l$ in Figure 4). Notice that if $C$ is a curve on $\Sigma$, transverse to $l_1$, then the foliation $\mathscr{F}_1$ can be constructed so that $\#(C \cap l_1) =$ the number of contact points of $C$ and the Reeb-type component of $\mathscr{F}_1$. It suffices to construct $\mathscr{F}_1$ so the Reeb-type component is close enough to $l_1$.

Similarly, define a foliation $\mathscr{F}_2$ of $\Sigma_2 - B(x, t)$, with the Reeb type component of $\mathscr{F}_2$ close to the other arc of $l$, i.e. $l - l_1$; cf. Figure 4.

Now define $\mathscr{F}(\varepsilon)$; the trace of $\mathscr{F}(\varepsilon)$ on $\Sigma$ will be $\mathscr{F}_1 \cup \mathscr{F}_2$; $\varepsilon = t$.

Each leaf $\alpha$ of $\mathscr{F}_1$ bounds a 2-disk in $\Sigma_1$ (more precisely, each leaf of $\mathscr{F}_1$, together with an arc on $B(x, \varepsilon) \cap \Sigma_1$ joining the extremities of $\alpha$, bounds a disk in $\Sigma_1$). Let

$\alpha_1$ be a leaf of $\mathscr{F}_1$ as indicated in Figure 4, and consider the leaves of $\alpha$ of $\mathscr{F}_1$ inside the disk of $\Sigma_1$ bounded by $\alpha_1$. Let $D(\alpha)$ be the disk of $\Sigma_1$ bounded by $\alpha$. Let $F(\alpha)$ be a 2-disk in $S^3$ which is a thickened $D(\alpha)$; imagine $F(\alpha)$ as a thin pancake over $D(\alpha)$. $F(\alpha)$ is orthogonal to $\Sigma_1$ and $F(\alpha) \cap \Sigma_1 = \alpha$. In $S^3$, $\Sigma$ separates $S^3$ into two balls $B_1$ and $B_2$, and $F(\alpha)$ intersects each ball in a 2-disk close to $D(\alpha)$.

Choose the $D(\alpha)$, $\alpha$ inside $D(\alpha_1)$, so that the $\bigcup_\alpha F(\alpha)$ foliate a part of $S^3$, and all the $F(\alpha)$ are sufficiently flat so the foliated set is close to $D(\alpha)$. (One can do this by pushing one's thumb into $S^3 - B(x, \varepsilon)$, starting at $a \in \partial B(x, \varepsilon)$ to create the Reeb component. One keeps on pushing almost until $x$. The thumb starts out as a very thin thumb and then spreads out as a thin pancake till $\alpha_1$.)

Let $\Sigma(l)$ be the geodesic 2-sphere of $S^3$ containing $l$, which is orthogonal to $\Sigma$ along $l$ (in the ball $B_1$ for example, if one imagines $\Sigma_1$ as the upper hemisphere, then $\Sigma(l) \cap B_1$ is the equatorial plane). Now foliate the region of $S^3 - B(x, \varepsilon)$ between $F(\alpha_1)$ and $\Sigma(l) - B(x, l)$ by "blowing out" $F(\alpha_1)$ to $\Sigma(l)$. More precisely, the region in question is topologically $F(\alpha_1) \times [0, 1]$. One puts the product foliation in the region. However one does this so all the leaves outside a small tubular neighborhood of $\Sigma$, are leaves of $\mathscr{F}(l)$, i.e. they coincide with geodesic spheres containing $l$, outside of a tubular neighborhood of $\Sigma$.

This defines $\mathscr{F}(\varepsilon)$ on half of $S^3 - B(x, \varepsilon)$. To extend to the other half, one does the same thing we just did, blowing down to the foliation by thin pancakes close to the foliation $\mathscr{F}_2$ of $\Sigma_2$. In fact, if $\beta$ is the geodesic of $S^3$ through $y$ and orthogonal to $\Sigma$, then one extends $\mathscr{F}(\varepsilon)$ by rotating $\mathscr{F}(\varepsilon)$ by $\pi$ around $\beta$.

By construction, all the leaves of $\mathscr{F}(\varepsilon)$, outside a tubular neighborhood of $\Sigma$, are parts of the geodesic spheres of $\mathscr{F}(l)$. Now if $M$ is a surface in $S^3$, transverse to $\Sigma$, $y \notin M$ (i.e. $x \notin M$ and $-x \notin M$) and $M$ in general position with respect to $\mathscr{F}(y)$ and $\mathscr{F}(l)$, then constructing $\mathscr{F}(\varepsilon)$ so that the tubular neighborhoods of $l$ (to define $\mathscr{F}_1$) and of $\Sigma$, are small, one sees that $\mathrm{Geom}(M, \Delta) = |\mu|(M, \mathscr{F}(\varepsilon))$. A moments inspection shows $\mathscr{F}(\varepsilon)$ is equivalent to a parallel foliation of $\mathbf{R}^3$. This completes the proof of Theorem III.1.

THEOREM III.2. *Let $M$ be a compact surface in $S^3$. Then* $\mathrm{Geom}(M)$ *is a linear combination of $L_0(M)$, $L_1(M)$ and $L_2(M)$:*

$$\mathrm{Geom}(M) = \pi^3 L_2(M) + 4\pi^3 L_1(M) + 2\pi^2 \operatorname{Vol} G(4, 2) L_0(M).$$

*Proof.* We have

$$\int_D |l \cap M| = \pi^2 \int_{G(4,2)} |l \cap M| = \pi^3 L_2(M) \qquad \text{by II.1.}$$

Also

$$\int_D |\mu|(M\cap\Sigma, \mathscr{F}(y)) = \pi \int_{D(4,3,1)} |\mu|(M\cap\Sigma, \mathscr{F}(y))$$

$$= \pi \int_{G(4,3)} 4\pi L_0(M\cap\Sigma) = 4\pi^3 L_1(M) \qquad \text{by II.6.}$$

Finally

$$\int_D |\mu|(M, \mathscr{F}(l)) = \pi^2 \int_{G(4,2)} |\mu|(M, \mathscr{F}(l))$$

$$= 2\pi^2 \operatorname{Vol}(G(4,2))L_0(M) \qquad \text{by definition of } L_0(M).$$

COROLLARY III.3.

$$\operatorname{Geom}(M) = \int_M \pi^3 + 2\pi h_1(x) + \frac{\pi}{2}\operatorname{Vol} G(4,2)|K(x)|.$$

*Proof.* This follows immediately from Theorem III.2 and the local formulae.

## IV. Geometry of $M^{n-1} \subset S^n$

Let $D = D(n, n-1, \ldots, 1)$ be the space of flags $\Delta = (\Sigma^0 \subset \Sigma^1 \subset \cdots \subset \Sigma^n = S^n)$ each $\Sigma^i$ and $i$-dimensional geodesic sphere of $S^n$. Define $\mathscr{F}(i, i+2)$ to be the (singular) foliation of $\Sigma^{i+2}$ by geodesic $i+1$ spheres that contain $\Sigma^i$. Denote $M\cap\Sigma^{i+2}$ by $M_i$ when $M$ is in general position with respect to $\Delta$ (we subsequently assume this).

We define the geometry of $M$ with respect to $\Delta$.

$$\operatorname{Geom}(M, \Delta) = |M\cap\Sigma^1| + \sum_{i=2}^{n} |\mu|(M_i, \mathscr{F}(i-2, i)).$$

As in the proof of III.1 one has:

THEOREM IV.1. *Let $M^{n-1} \subset S^n$ be in general position with respect to the flag $\Delta$. Then there is an $\varepsilon > 0$ and foliation $\mathscr{F} = \mathscr{F}(\Delta)$ of $S^n - B(x, \varepsilon)$, $x \in \Sigma^0$, satisfying:*

– $\mathrm{Geom}(M, \Delta) = |\mu|(M, \mathscr{F})$, *and*
– $\mathscr{F}$ *is smoothly equivalent to a foliation of* $\mathbf{R}^n$ *by parallel hyperplanes.*

THEOREM IV.2. $\mathrm{Geom}(M)$ *is a linear combination of* $L_0(M)$, $L_1(M), \ldots,$ $L_{n-1}(M)$;

$$\mathrm{Geom}(M) = \int_D \mathrm{Geom}(M, \Delta) = \sum_{i=0}^{n-1} c_i L_i(M),$$

where $c_0, \ldots, c_{n-1}$ *are dimension constants.*

COROLLARY IV.3. *For* $M^{n-1} \subset S^n$, *one has*

$$\sum_{i=0}^{n-1} c_i L_i(M) \geq \beta(M),$$

$\beta(M)$ *the sum of the Betti numbers of* $M$.

## V. The geometry of submanifolds $M \subset S^n$ of arbitrary codimension

Similar results can be obtained in higher codimension. The construction of the foliation associated to a complete flag is unchanged. Therefore we can extend the results obtained in $\mathbf{R}^n$ (see [C-L], [Fe], [L-R]).

THEOREM V.1. *Let* $V$ *be a compact manifold immersed in* $S^n$. *Then*

$$\mathrm{Geom}(V) \geq \sum \beta_i,$$

*where the* $\beta_i$ *are the Betti numbers of* $V$.
*If* $V$ *is the sphere* $S^p$ *and is embedded, the condition*

$$\mathrm{Geom}(V) < 4$$

*implies that* $V$ *is an unknotted sphere* (*topologically and differentiably for* $p = 1$, *all* $n$; $p = 2$ $n = 4$; $p \geq 5$, $n = p + 2$).

The integral geometric construction requires one more step. For example, in the codimension 2 case ($V^{n-3} \subset S^{n-1}$), we need to consider the "quasi flag space" $D(n, n-2, n-1, n-2)$ of

$$\{h \subset k \supset l, \dim(h) = n-2, \dim(k) = n-1, \dim(l) = n-2\}.$$

Notice that the dimension of the fiber bundle $\mathfrak{D}$ on $V$

$$\mathfrak{D} = \{x \in V, h_x \subset k \supset l, \dim(k) = n-1, \dim(l) = n-2\},$$

where $h_x$ is the vector space spanned by the geodesic sphere tangent at $x$ to $V$, is $2(n-2)$, the same as that of the Grassmann manifold $G(n, n-2)$.

THEOREM V.2. *A curve $C$ embedded in $S^3$ satisfies*

$$\int_C |k_g| + 1 \geq 2\pi$$

$$\int_C |k_g| + 1 \geq 4\pi$$

*if $C$ is knotted, and more precisely*

$$\int_C |k_g| + 1 \geq 2\pi \cdot (\textit{bridge number of } C).$$

The first result was already proved by Banchoff [Ba]; the two others extend results of Fenchel, Fary and Milnor [Fe], [Fa], [$M_1$], [$M_2$]; and Sunday [Su].

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