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Autor: Clapp, Monica
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Critical point theory for perturbations of symmetric functionals

MÓNICA CLAPP

Abstract. Functionals which are invariant under the action of a compact transformation group G often have many critical values. Here we consider functionals which are not G-invariant and give conditions for them to have infinitely many critical values; including a mountain pass theorem. We apply it to prove the existence of infinitely many solutions of a nonlinear Dirichlet problem with perturbed G-symmetries.

1. Introduction

Let V be an infinite-dimensional Hilbert space with an orthogonal action of a compact Lie group G, let W be a finite-dimensional G-invariant linear subspace of V and $W^\perp$ be its orthogonal complement in V. Let $\Phi: V \to \mathbb{R}$ be a C^1-functional which satisfies the Palais-Smale condition and the following mountain pass conditions:

($\mathbf{MP}_1'$) There are constants $\alpha > 0$ and $\rho > 0$ such that $\Phi(x) \geq \alpha$ for all $x \in W^\perp$, $\|x\| = \rho$.

($\mathbf{MP}_2$) For every finite-dimensional linear subspace F to V there exists $R = R(F) > 0$ such that $\Phi(x) \leq 0$ for all $x \in F$, $\|x\| \geq R$.

If Φ is G-invariant, i.e. $\Phi(gx) = \Phi(x)$ for all $x \in V$, $g \in G$, these conditions, plus a condition on the action of G on V, guarantee the existence of an unbounded sequence of critical values. For the antipodal action of $G = \mathbb{Z}/2$ on V this is a classical result of Ambrosetti and Rabinowitz [1]. The condition on the G-action just says $\Phi(0) \leq 0$. For $G = \mathbb{S}^1$ this is due to Fadell, Husseini and Rabinowitz [15] and the condition on the G-action is that $\Phi(x) \leq 0$ for every fixed point $x \in V^G$ of this action. For more general group actions see [10], [8], [5] and [7].

This paper is concerned with the following question: If Φ is not G-invariant, when can one ensure the existence of an unbounded sequence of critical values? One would expect that, if Φ is not too far away from being G-invariant and if the mountain range is steep enough, then Φ should still have an unbounded sequence of critical values. More precisely, assume that $V^G \subset V_1 \subset \cdots \subset V_j \subset \cdots$ is a sequence of finite-dimensional G-invariant linear subspaces of V and that Φ satisfies the following conditions:

(DS) There are constants $\gamma > 0$, $\mu > 1$, such that for all $x \in V$, $g \in G$,

$$|\Phi(x) - \Phi(gx)| \leq \gamma(|\Phi(x)|^{1/\mu} + 1).$$

(**MP**$_1$) There are constants $\beta > 0$, $\theta > \mu/(\mu - 1)$, $j_0 \geq 1$, such that, for all $j \geq j_0$,

$$\sup_{\rho \geq 0} \inf\{\Phi(x) \colon x \in V_{j-1}^{\perp}, \|x\| = \rho\} \geq \beta j^{\theta}.$$

(**BU**$_1$) $G = \mathbb{Z}/p \times \cdots \times \mathbb{Z}/p$, p prime, or $G = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ and there exists $m > 0$ such that $\dim V_{j-1} - \dim V^G < mj$ for all $j \geq j_0$.

We will show that these conditions, together with the Palais-Smale condition and (MP$_2$) as above, guarantee the existence of an unbounded sequence of critical values of Φ.

(**BU**$_1$) can be replaced by another Borsuk-Ulam type condition introduced by Bartsch in [5] which applies to more general groups and is quite useful in applications, namely,

(**BU**$_2$) There exists a fixed *admissible* representation W of G such that for all $j \geq j_0$

$$V_j \cong W \oplus \cdots \oplus W \; (j \text{ summands}).$$

A representation W is admissible if, for example, G is a finite solvable group acting without fixed points on W. But the class of admissible representations is much larger than this (see [5], [6] and Section 5 below).

Critical point results for perturbations of symmetric functionals were first obtained by Bahri and Berestycki [2] and Struwe [25] who considered perturbations of even functionals on a sphere. The key tool was an invariant introduced by Krasnoselkii, without giving it a name, in his study of stable critical points of an even functional [17] Chapter VI.

Here we generalize this invariant to arbitrary compact Lie group actions and call it the *G-capacity* $\kappa(X)$ of the G-space X. It is dual to the G-genus [4] but it has the following rigidity property which neither the genus nor the equivalent Lusternik-Schnirelmann category have, namely,

(**Rigidity**) If $\kappa(X) = \kappa(Y) < \infty$ then no G-map $f \colon X \to Y$ which induces a homotopy equivalence $f^G \colon X^G \simeq Y^G$ on the fixed point sets can be nullhomotopic.

As was noted in [2] and [25] this property is crucial for obtaining critical point results for perturbations of symmetric functionals, because it provides information on the noncontractibility of the level sets between the minimax values defined in terms of the G-capacity. These minimax values are not critical values if Φ is not G-invariant, but the conditions given above will guarantee the existence of a critical value above each minimax value.

As an application, multiplicity results for nonlinear elliptic equations were obtained in [2], [25] and also by Rabinowitz [21], [22] and Dong and Li [14]. Here we extend these results as follows.

We look for solutions $u=(u_1, \ldots, u_m)\colon \bar{\Omega}\to R^m$ of the nonlinear Dirichlet problem

$$\begin{aligned} -\Delta u &= F_u(u)+f(x), && x\in\Omega \\ u &= 0, && x\in\partial\Omega, \end{aligned} \tag{D}$$

where Ω is a bounded domain in R^n with smooth boundary and $f=(f_1,\ldots,f_m)\in L^2(\Omega, R^m)$. We assume that F satisfies the following conditions:

(F1) $F\colon \mathbb{R}^m\to\mathbb{R}$ is a C^1-function.

(F2) There are constants $\alpha>0$ and $1<s<(n+2)/(n-2)$ if $n\geq 3$ such that

$$|F_u(u)|\leq\alpha(1+|u|^s).$$

(F3) There are constants $R>0$ and $\mu>2$ such that $0<\mu F(u)\leq u\cdot F_u(u)$ if $|u|\geq R$.

(F4) There exist a compact Lie group and an orthogonal action on $W=\mathbb{R}^m$ such that W is admissible and $F\colon W\to R$ is G-invariant.

If $f\equiv 0$ Bartsch showed [5] that, under these conditions, (D) has infinitely many weak solutions. Here we shall show that if s in (F2) is further restricted by

$$\theta\equiv\frac{(n+2)-(n-2)s}{n(s-1)}>\frac{\mu}{\mu-1}$$

then (D) possesses an unbounded sequence of weak solutions in $W_0^{1,2}(\Omega, \mathbb{R}^m)$.

This last condition coincides with the one given in [2], [25], [14], [21] and [22], for $m=1$ and $G=\mathbb{Z}/2$.

This paper is organized as follows: In Sections 2 and 3 we define and study the absolute and relative versions of the G-capacity respectively. In Section 4 we prove a general critical point theorem for perturbations of symmetric functions on a Banach manifold and apply it to obtain a critical point result for problems with constraints.

In Section 5 we prove the mountain pass theorems mentioned above and in Section 6 we apply them to obtain an unbounded sequence of weak solutions of (D). Finally in Section 7 we compute upper bounds for the G-capacity which are needed in our critical point results.

I am grateful to Antonio Ambrosetti for making me aware of some of these questions.

2. The G-capacity

Let G be a compact Lie group. A *G-space* is a topological space X with a continuous action of G. A *G-map* is a continuous map $f: X \to Y$ which preserves the G-action, i.e. $f(gx) = gf(x)$. We denote by $X^G \equiv \{x \in X: gx = x \text{ for all } g \in G\}$ the *fixed point set of X*.

The *join* $X * Y$ of two G-spaces X and Y is the quotient space of $X \times [0, 1] \times Y$ obtained by identifying $(x, 0, y)$ with $(x, 0, y')$ and $(x, 1, y)$ with $(x', 1, y)$ for all $x, x' \in X$, $y, y' \in Y$. It has a natural G-action given by $g(x, t, y) = (gx, t, gy)$.

We denote by

$$J_n G \equiv \underbrace{G * \cdots * G}_{n \text{ times}}, \qquad n \geq 1$$

the n-fold join of G, $J_0 G \equiv \emptyset$, and by

$$J_\infty G \equiv \bigcup_{n \geq 1} G^n$$

with the weak topology, the countable join of G. These are free G-spaces and $J_\infty G$ is contractible. If $G = \mathbb{Z}/2$ then $J_n G$ is (G-homeomorphic to) unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$ with the antipodal action, and if $G = \mathbb{S}^1$ then $J_n G$ is the unit sphere $\mathbb{S}^{2n-1}$ in $\mathbb{C}^n$ with the action given by multiplication on each coordinate.

Let Z be a topological space with the trivial G-action. Then $Z * J_n G$ is a G-space whose fixed-point set is exactly Z. Here $Z * J_0 G \equiv Z$.

DEFINITION 2.1. For every G-space X we define the *G-capacity* $\kappa(X)$ of X to be the greatest number $0 \leq n \leq \infty$ such that there exists a G-map $\sigma: Z * J_n G \to X$ whose restriction to the fixed point sets is a homotopy equivalence $\sigma^G: Z \simeq X^G$.

The G-capacity has the following easy property:

PROPOSITION 2.2. (Monotonicity) *If there exists a G-map $f: X \to Y$ with f^G: $X^G \simeq Y^G$ then $\kappa(X) \leq \kappa(Y)$.* □

The following proposition leads to a crucial property of the G-capacity.

PROPOSITION 2.3. *Let $f: X \to Y$ be a G-map. Then, f is nullhomotopic if and only if f can be extended to a G-map $\hat{f}: X * G \to Y$.*

Proof. Observe that X is contractible to a point in $X * X'$, via $(x, t) \mapsto (x, t, x_0')$. Therefore, if f has an extension as above, f is nullhomotopic. Now assume f is nullhomotopic and let $H: X \times [0, 1] \to Y$ be a homotopy with $H(x, 0) = f(x)$ and $H(x, 1) = y_0 \in Y$, for all $x \in X$. Then

$$\hat{f}(x, t, g) = gH(g^{-1}x, t), \qquad \text{for } (x, t, g) \in X \times [0, 1] \times G$$

gives the desired extension. □

An immediate consequence is the following:

COROLLARY 2.4. (Rigidity) *If $\kappa(X) = \kappa(Y) < \infty$ then every G-map $X \to Y$ which induces a homotopy equivalence $X^G \simeq Y^G$ is essential, i.e. it is not nullhomotopic.* □

We turn now to the question of computing $\kappa(X)$. Given an orthogonal representation V of G we denote by SV the unit sphere in V.

PROPOSITION 2.5. *If an orthogonal representation V of G is the orthogonal sum $V = W \oplus W'$ of two representations of G then $\kappa(SV) \geq \kappa(SW) + \kappa(SW')$.*

Proof. Given G-maps $\sigma: Z * J_m G \to SW$ and $\tau: Z' * J_n G \to SW'$ with $\sigma^G: Z \simeq SW^G$ and $\tau^G: Z' \simeq (SW')^G$, then the G-map $\rho: Z * J_m G * Z' * J_n G \to W \oplus W' - (0, 0)$ given by $\rho(x, t, y) = (1 - t)\sigma(x) + t\tau(y)$, composed with the radial retraction, is a G-map $Z * Z' * J_{m+n} G \to S(W \oplus W')$ which induces a homotopy equivalence $Z * Z' \simeq S(W \oplus W')^G$. □

For an arbitrary G-space X one gets a lower bound for $\kappa(X)$ as follows. We denote by $\mathbb{S}^j$ the unit sphere and by $\mathbb{B}^{j+1}$ the unit ball in euclidean $(j+1)$-space. Recall [23] that a space X is said to be m-connected if every map $f: \mathbb{S}^j \to X$, $0 \leq j \leq m$, has a continuous extension over $\mathbb{B}^{j+1}$. For example, the m-sphere $\mathbb{S}^m$ is $(m-1)$-connected. If X is $(m-1)$-connected and Y is $(n-1)$-connected then $X * Y$ is $(m+n)$-connected.

PROPOSITION 2.6. *If X is an $(m-1)$-connected G-space and X^G is a CW-complex, then $\kappa(X) \geq (m - \dim X^G)/(\dim G + 1)$.*

Proof. Let $\kappa(X) = n$ and $\sigma: Z * J_n G \to X$ be a G-map with $Z = X^G$. If $m - 1 \geq \dim X^G + n \dim G + n = \dim(Z * J_n G)$ then σ is nullhomotopic [23] 7.6.13. Proposition 2.3 gives a contradiction. □

This implies, in particular that $\kappa(SV) \geq (\dim V - \dim V^G)/(\dim G + 1)$ for every orthogonal representation V of G. Equality does not hold in general, not even for a finite group G, as the following example shows

EXAMPLE 2.7. There is a $\mathbb{Z}/4$-action on $\mathbb{S}^2$ such that $\kappa(\mathbb{S}^2) \geq 4$.

Proof. If $\mathbb{Z}/4 = \{\pm 1, \pm i\} \subset \mathbb{C}$ acts on $\mathbb{S}^3 \subset \mathbb{C}^2$ by scalar multiplication and on $\mathbb{S}^2 \subset \mathbb{R}^3$ by multiplication with ζ^2 on each coordinate, $\zeta \in \mathbb{Z}/4$, then the map $\mathbb{S}^3 \to \mathbb{R}^3 \backslash 0$ given by

$$(z_1, z_2) \mapsto ((z_1 + \bar{z}_2)(z_1 - \bar{z}_2), |z_1 + \bar{z}_2| - |z_1 - \bar{z}_2|)$$

composed with the radial retraction gives a $\mathbb{Z}/4$-map $\mathbb{S}^3 \to \mathbb{S}^2$ [4]. Monotonocity and the above proposition imply $\kappa(S^2) \geq 4$. □

For some groups G upper bounds for $\kappa(X)$ can be given in terms of the dimension of X. This will be done in Section 7. It will follow that

THEOREM 2.8. *If V is an orthogonal representation of G with* $\dim V^G < \infty$ *then*
(a) $\kappa(SV) = \dim V - \dim V^G$ *if G is a p-torus, i.e. $G = \mathbb{Z}/p \times \cdots \times \mathbb{Z}/p$, p a prime, and*
(b) $\kappa(SV) = \frac{1}{2}(\dim V - \dim V^G)$ *if G is a torus, i.e. $G = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$.*

Proof. That the given numbers are lower bounds for $\kappa(SV)$ follows from Proposition 2.6 if G is a p-torus, and from Proposition 2.5 if G is a torus, because every non-trivial irreducible representation of a torus is 2-dimensional [9] II.8.5. That they are also upper bounds follows immediately from Theorem 7.1. □

Some remarks are in order. For $G = \mathbb{Z}/2$ acting without fixed points on X, the G-capacity $\kappa(X)$ was introduced by Krasnoselskii [17] in his theory of stable critical points of an even functional, and further studied by Conner and Floyd [11] who called it the index of X. It is however not an index theory in the usual sense, since it does not satisfy the subadditivity property. The 2-sphere of Example 2.7 is the union of the $\mathbb{Z}/4$-subsets $X = \mathbb{S}^2 \backslash \{\pm(0, 0, 1)\}$ and $Y = \mathbb{S}^2 \backslash \mathbb{S}^1$. It is easy to see that $\kappa(X) = 2$ and $\kappa(Y) = 1$, so $\kappa(\mathbb{S}^2) > \kappa(X) + \kappa(Y)$.

Dual notions like the genus or the Lusternik-Schnirelmann category have the advantage of being subadditive [10], [6] which makes them quite useful for counting critical points. On the other hand, they do not satisfy the rigidity property: Conner and Floyd have given an example [11] 3.14 of a nullhomotopic $\mathbb{Z}/2$-map between spaces of the same genus (or co-index). Rigidity for $G = \mathbb{Z}/2$ was first proved by Krasnoselskii [17] Chapter VI.

It follows from Theorem 7.1 that, if G is a torus or a p-torus, $X^G = \emptyset$, then $\kappa(X) \leq \gamma(X) =$ the G-genus of X, which is defined to be the smallest number n such that there exist n proper closed subgroups $H_1, \ldots, H_n$ of G and a G-map $X \to G/H_1 * \cdots * G/H_n$ [6]. But this is not true for an arbitrary group, as Example 2.7 shows. Also, equality may not hold, even for $G = \mathbb{Z}/2$ [11] 3.15.

3. The relative G-capacity

By a *G-pair* (X, A) we mean a G-space X together with a G-invariant subspace A of X, and by a *G-map (of pairs)* $f: (X, A) \to (Y, B)$ a G-map $f: X \to Y$ which maps A into B.

Given a pair (Z, C) on which G acts trivially, we write

$$(Z, C) * J_n G \equiv (Z * J_n G, C * J_n G).$$

DEFINITION 3.1. For every G-pair (X, A) we define the *relative G-capacity* $\kappa(X, A)$ of (X, A) to be the smallest number $0 \leq n \leq \infty$ such that there exists a G-map $\sigma: (Z, C) * J_n G \to (X, A)$ whose restriction to the fixed point sets is a homotopy equivalence (of pairs) $\sigma^G: (Z, C) \simeq (X^G, A^G)$.

Then the relative G-capacity satisfies

PROPOSITION 3.2. (Monotonicity) *If there exists a G-map $f: (X, A) \to (Y, B)$ whose restriction to the fixed point sets is a homotopy equivalence $f^G: (X^G, A^G) \simeq (Y^G, B^G)$ then $\kappa(X, A) \leq \kappa(Y, B)$.* □

A map $f: (X, A) \to (Y, B)$ is said to be *nullhomotopic* if there is a homotopy (of pairs) $H: (X \times [0, 1], A \times [0, 1]) \to (Y, B)$ such that $H(x, 0) = f(x)$ and $H(x, 1) = b_0 \in B$, for all $x \in X$. A pair (X, A) is called *contractible* if the identity map of (X, A) is nullhomotopic.

The following proposition is proved just like Proposition 2.3.

PROPOSITION 3.3. *Let $f: (X, A) \to (Y, B)$ be a G-map. Then f is nullhomotopic if and only if it can be extended to a G-map $\hat{f}: (X * G, A * G) \to (Y, B)$.* □

COROLLARY 3.4. (Rigidity) *If $\kappa(X, A) = \kappa(Y, B) < \infty$ then every G-map $(X, A) \to (Y, B)$ which induces a homotopy equivalence $(X^G, A^G) \simeq (Y^G, B^G)$ is essential (i.e. it is not nullhomotopic).* □

Given an orthogonal representation V of G we denote by BV the closed unit ball in V.

PROPOSITION 3.5. *Let V be an orthogonal representation of G with* $\dim V^G < \infty$. *Then*

(a) $\kappa(BV, SV) = \dim V - \dim V^G$ *if G is a p-torus, and*

(b) $\kappa(BV, SV) = \frac{1}{2}(\dim V - \dim V^G)$ *if G is a torus.*

Proof. Every G-map $\sigma: SV^G * J_nG \to SV$ which is a homotopy equivalence on the fixed point sets has an extension $\hat{\sigma}: BV^G * J_nG \cong SV^G * J_nG * \{0\} \to SV * \{0\} \cong BV$, given by $\hat{\sigma}(x, t, 0) = (\sigma(x), t, 0)$, which is a homotopy equivalence on the fixed point pairs. On the other hand, every G-map $(BV^G, SV^G) * J_nG \to (BV, SV)$ which is a homotopy equivalence on the fixed point pairs induces a G-map of the quotient spaces $S(V^G \oplus \mathbb{R}) * J_nG \cong (BV^G * J_nG)/(SV^G * J_nG) \to BV/SV \cong S(V \oplus \mathbb{R})$ which is a homotopy equivalence on the fixed point sets. Here G acts trivially on $\mathbb{R}$. Now apply Theorem 2.8. □

We shall now use the relative G-capacity to prove a critical point theorem for perturbations of symmetric functions.

4. Critical points of perturbed symmetric functions

Let M be a complete $C^{1,1}$-Finsler manifold [20], [26] and $\Phi: M \to \mathbb{R}$ be a C^1-function. Φ is said to satisfy the *Palais-Smale condition* $(\mathbf{PS})_a$ *above* $a \in \mathbb{R}$ if

- Any sequence (x_n) in M such that $\Phi(x_n) \subset [a, b]$ for some $b \in \mathbb{R}$ and such that $\|d\Phi(x_n)\| \to 0$ as $n \to \infty$ has a convergent subsequence.

Given $c \in \mathbb{R}$, let

$$\Phi^c \equiv \{x \in M : \Phi(x) \leq c\}.$$

Recall that X is said to be *deformable into* Y *rel* Z *in* M if there is a homotopy $H: X \times [0, 1] \to M$ with $H(x, 0) = x$, $H(x, 1) \in Y$, $H(z, t) = z$ for all $x \in X$, $z \in Z$, $0 \leq t \leq 1$. It is well known that Φ has the following deformation property [20], [26] II.3.11.

PROPOSITION 4.1. (Deformation Lemma) *Assume Φ satisfies $(PS)_a$. If $d \geq a$ and if Φ has no critical values in $[d, \infty)$ then M is deformable into Φ^d rel Φ^d in M.* □

Let now G be a compact Lie group acting on M and let D be a fixed closed G-invariant subset of M. Let

$$\Delta_k = \{X \subset M: X \text{ is } G\text{-invariant}, X \supset M^G \text{ and } \kappa(X, X \cap D) \geq k\}.$$

Given a C^1-function $\Phi: M \to \mathbb{R}$ which is not necessarily G-invariant, define

$$d_k \equiv \inf_{x\in\Delta_k} \sup_{x\in X} \Phi(x).$$

Observe that $d_k \leq d_{k+1}$ for all k. If Φ satisfies (PS) and is G-invariant and if $d_k \in \mathbb{R}$, then the values d_k are critical values of Φ [10] Appendix A, but this need not be true in general. The following proposition gives conditions for the existence of a critical value above d_k.

PROPOSITION 4.2. *Assume that, for some $k \geq 1$, d_k has the following properties*:

(i) *Φ satisfies the Palais-Smale condition $(PS)_{d_k}$ above d_k.*

(ii) *$-\infty < d_k < a \leq b < d_{k+1}$ and the smallest G-invariant subset of M containing Φ^a lies below b, i.e.*

$$G[\Phi^a] \equiv \{gy: y\in M,\ \Phi(y) \leq a,\ g\in G\} \subset \Phi^b.$$

(iii) *$\Phi(x) \leq d_k$ for all $x \in D$.*

Then, if (M, D) is contractible, Φ has a critical value $c > d_k$.

Proof. Since $-\infty < d_k$, $\Delta_k \neq \emptyset$. Let $X \subset \Phi^a$ be G-invariant, $M^G \subset X$ and $\kappa(X, X\cap D) \geq k$. It follows from (ii) and (iii) that $\kappa(G[\Phi^a], D) \leq k$. Therefore, $\kappa(X, X\cap D) = \kappa(G[\Phi^a], D) = k$.

If Φ has no critical values in (d_k, ∞) then, by (i) and the Deformation Lemma 4.1, M is deformable into Φ^a rel Φ^a. So there is a (non-equivariant) map $r: (M, D) \to (\Phi^a, D)$ with $r(x) = x$ for all $x \in \Phi^a$. But (M, D) is contractible. Hence $(X, X\cap D) \subset (G[\Phi^a], D)$ is nullhomotopic. This contradicts the rigidity property 3.4. □

The following theorem provides conditions for (ii) to hold.

THEOREM 4.3. *Let $\Phi: M \to \mathbb{R}$ be a C^1-function, D be a closed G-invariant subset of M, $b > 0$ and $\mu: [0, \infty) \to [0, \infty)$ be a continuous non-decreasing function, $\mu \neq 0$, which satisfy the following properties.*

(P1) *Φ satisfies the Palais-Smale condition $(PS)_b$,*

(P2) *$|\Phi(x) - \Phi(gx)| \leq \mu(|\Phi(x)|)$ for all $x \in M$, $g \in G$,*

(P3) *$0 \leq d_k < \infty$ and $d_{k+1} > d_k + \mu(d_k)$ for infinitely many numbers k,*

(P4) *$\Phi(gx) \leq b$ if $\Phi(x) < -b$, for all $g \in G$, and*

(P5) *$\Phi(x) \leq b$ for all $x \in D$.*

Then, if (M, D) is contractible, Φ has an unbounded sequence of positive critical values.

Proof. First observe that, since $\mu \neq 0$ and μ is non-decreasing, property (P3) implies that $d_k \to \infty$ as $k \to \infty$. We now show that Φ has a critical value $c_k > d_k$ for infinitely many k's. Assume $d_k \geq b$ and $d_{k+1} > d_k + \mu(d_k)$. Let $\varepsilon > 0$ be such that $d_{k+1} - \varepsilon > d_k + \varepsilon + \mu(d_k + \varepsilon)$. Let $\Phi(x) \leq d_k + \varepsilon$. Then, by (P2), if $|\Phi(x)| \leq d_k + \varepsilon$,

$$\Phi(gx) \leq |\Phi(x)| + |\Phi(gx) - \Phi(x)| \leq d_k + \varepsilon + \mu(d_k + \varepsilon) < d_{k+1} - \varepsilon$$

and if $\Phi(x) < -(d_k + \varepsilon) < -b$, by (P4), $\Phi(gx) \leq b < d_{k+1} - \varepsilon$, for all $g \in G$. Therefore, $G[\Phi^{d_k+\varepsilon}]$ lies below $d_{k+1} - \varepsilon$. Now use Proposition 4.2. □

We apply this theorem to obtain a critical point result for a problem with constraints.

Let V be a G-Hilbert space, i.e. a Hilbert space with an orthogonal action of G. Let SV be the unit sphere in V and Φ: $SV \to \mathbb{R}$ be a C^1-function. As above, let

$$\Delta_k \equiv \{X \subset SV\colon X \text{ is } G\text{-invariant, } SV^G \subset X \text{ and } \kappa(X) \geq k\}$$

and let

$$d_k \equiv \inf_{X \in \Delta_k} \sup_{x \in X} \Phi(x).$$

We obtain lower bounds for these values as follows.

PROPOSITION 4.4. *Let W be a G-invariant linear subspace of V such that $W \supset V^G$ and $\kappa(SW) < k$. Then, if $\Delta_k \neq \emptyset$,*

$$d_k \geq \inf\{\Phi(x)\colon x \in SW^{\perp}\},$$

where $W^{\perp}$ is the orthogonal complement of W in V.

Proof. All we need to do is show that every G-invariant subset X of SV, such that $SV^G \subset X$ and $\kappa(X) \geq k$, intersects $SW^{\perp}$. If this were not so, the restriction to X of the obvious G-retraction $V \backslash W^{\perp} \to SW$ would given a G-map $X \to SW$ which is the identity on SV^G, hence $\kappa(X) \leq \kappa(SW) < k$. □

The following theorem gives sufficient conditions for the existence of infinitely many critical values of Φ.

THEOREM 4.5. *Let V be a G-Hilbert space and let $V^G \subset V_1 \subset V_2 \subset \cdots \subset V_j \subset \cdots$ be a sequence of finite-dimensional G-invariant linear subspaces of V. Let*

$\Phi: SV \to \mathbb{R}$ be a C^1-function which satisfies $(\mathbf{PS})_a$ *for some $a > 0$. Assume further*

(DS) *There are constants $\gamma > 0$, $\mu > 1$, such that for all $x \in SV$, $g \in G$,*

$$|\Phi(x) - \Phi(gx)| \leq \gamma(|\Phi(x)|^{1/\mu} + 1)$$

(S) *There are constants $\beta > 0$, $\theta > \mu/(\mu - 1)$, $j_0 \geq 1$, such that for all $j \geq j_0$,*

$$\inf\{\Phi(x): x \in SV_{j-1}^{\perp}\} \geq \beta j^{\theta}.$$

(BU) *There exists $m > 0$ such that $\kappa(SV_{j-1}) < mj$ for all $j \geq j_0$.*

Then Φ has an unbounded sequence of critical values.

Proof. We apply Theorem 4.3 above to the given Φ and to $D = \emptyset$, $\mu(t) = \gamma(t^{1/\mu} + 1)$, and $b > a$ large enough so that $t > \gamma(t^{1/\mu} + 1)$ if $t > b$. These data obviously satisfy (P1), (P2) and (P5). Assume $\Phi(x) < -b < 0$. Then, if $\Phi(gx) > 0$ for some $g \in G$, $\Phi(gx) < \Phi(gx) - \Phi(x) \leq \gamma(|\Phi(gx)|^{1/\mu} + 1)$. Hence, by our choice of b, $\Phi(gx) \leq b$. This proves (P4). Now we check (P3). Observe that (S) implies that dim $V = \infty$. So by 2.5, $\kappa(SV) = \infty$ and $\Delta_k \neq \emptyset$ for all k. In fact, $\kappa(SV_j) \to \infty$ as $j \to \infty$. Therefore $d_k < \infty$ for all k. Now, (S), (BU) and Proposition 4.4 imply that $d_{mj} \geq \beta j^{\theta}$ for all $j \geq j_0$. If for some $k_0 \geq 1$, $d_{k+1} \leq d_k + \gamma(d_k^{1/\mu} + 1)$ for all $k \geq k_0$, then there exists $\omega > 0$ such that $d_k < \omega k^{\mu/\mu - 1}$ for all $k \geq k_0$ [22] (10.53)–(10.57), [2] 5.3. But then $\beta j^{\theta} \leq d_{mj} \leq (\omega m^{\mu/\mu - 1}) j^{\mu/\mu - 1}$ for all sufficiently large j, which is impossible because $\theta > \mu/(\mu - 1)$. Finally, observe that, since dim $V = \infty$, SV is contractible. □

(DS) is a sublinearity condition, with respect to $|\Phi(x)|$, on the deviation of Φ from being G-invariant. (S) is a condition on the steepness of Φ on a sequence of orthogonal directions, which depends on the deviation from symmetry (DS). (BU) should be thought of as a Borsuk-Ulam condition which restricts the dimensions of these orthogonal directions. It is satisfied if, for example (Theorem 2.8),

$(\mathbf{BU}_1)$ G is a p-torus and dim $V_{j-1} < mj$ for all j and some fixed m.

If Φ is G-invariant one can guarantee the existence of an unbounded sequence of critical values under much weaker conditions: One needs only that Φ is bounded below on some infinite dimensional sphere $SW \subset SV$ and that the group G satisfies a weak Borsuk-Ulam condition, namley that there is no G-map from an infinite-dimensional into a finite dimensional G-representation sphere [8] 4.1, [6] 3.1. This is equivalent with G being an extension of a finite p-group by a torus [6] 3.5. One can in fact show that for these groups the G-capacity of a finite-dimensional sphere is finite (Theorem 7.3). But condition (BU) requires much more than that: One needs actually to be able to compute upper bounds for $\kappa(SV_j)$. For abelian p-groups the computation of upper bounds for similar invariants has been carried out by Stolz [24] for $p=2$ and by Meyer [18] for p prime, using quite sophisticated machinery from algebraic topology. In the following section we shall give an alternative Borsuk-Ulam condition (BU_2) which applies to more general groups than (BU_1).

Bahri and Berestycki [2] and Struwe [25], simultaneously, first noticed the relevance of the rigidity property for proving critical point results for perturbations of even functionals on a sphere. They applied them to prove multiplicity results for nonlinear elliptic equations of the type $-\Delta u=|u|^{s-1}u+f(x)$ in Ω, $u=0$ on $\partial\Omega$, $f\in L^2(\Omega)$, s small enough, by turning this into a variational problem on a sphere in $H_0^1(\Omega)$. We shall extend these results to more general symmetries in Section 6, via a mountain pass theorem.

5. Mountain pass theorems for perturbed symmetric functionals

Let G be a compact Lie group, V be a G-Hilbert space and $V^G\subset V_1\subset\cdots\subset V_j\subset\cdots$ be a sequence of finite-dimensional G-invariant linear subspaces of V.

For a given non-decreasing sequence of positive numbers (R_j) with $R_j\to\infty$ as $j\to\infty$ define

$$D\equiv\bigcup_{j>1}(V_j\backslash B_{R_j}V_j),$$

where $B_{R_j}V_j\equiv\{x\in V_j\colon \|x\|<R_j\}$, and

$$\Delta_k\equiv\{X\subset V\colon X \text{ is } G\text{-invariant, } X\supset V^G \text{ and } \kappa(X, X\cap D)\geq k\}.$$

Observe that D is a closed G-invariant subset of V. Let $\Phi\colon V\to\mathbb{R}$ be a (non-equivalent) C^1-functional and

$$d_k\equiv\inf_{X\in\Delta_k}\sup_{x\in X}\Phi(x).$$

We start by giving lower bounds for these values.

PROPOSITION 5.1. *If $\kappa(S(V_j\oplus\mathbb{R}))<k$, where G acts trivially on $\mathbb{R}$, then, if $\Delta_k\neq\emptyset$,*

$$d_k\geq\sup_{0\leq\rho<R_{j+1}}\inf\{\Phi(x)\colon x\in V_j^{\perp},\ \|x\|=\rho\}.$$

Proof. Let X be a G-invariant subset of V with $X \supset V^G$ and $\kappa(X, X \cap D) \geq k$. Assume that $X \cap S_\rho V_j^\perp = \emptyset$ for some $0 \leq \rho < R_{j+1} \equiv R$. There is a G-map

$$(V \backslash S_\rho V_j^\perp, D) \to (S_R V \cup B_R V_j, S_R V)$$

which induces a homotopy equivalence $(V^G, D^G) \simeq (B_R V^G, S_R V^G)$, obtained by projecting $V \backslash B_R V$ radiallly onto $S_R V$, $B_R V \backslash S_\rho V_j^\perp$ along straight lines onto $S_R V \cup B_R V_j$ and, finally, expanding $B_R V_j$ if necessary so that $B_R V_j \cap D$ gets mapped onto $S_R V_j$. By monotonicity,

$$\kappa(X, X \cap D) \leq \kappa(V \backslash S_\rho V_j^\perp, D) \leq \kappa(S_R V \cup B_R V_j, S_R V).$$

As in the proof of Proposition 3.5 one can easily show that $\kappa(S_R V \cup B_R V_j, S_R V) \leq \kappa(S(V_j \oplus \mathbb{R})) < k$. This gives a contradiction. $\square$

We are now ready to prove our first mountain pass theorem.

THEOREM 5.2. *Let V be a G-Hilbert space and $V^G \subset V_1 \subset \cdots \subset V_j \subset \cdots$ be a sequence of finite-dimensional G-invariant linear subspaces of V. Let $\Phi\colon V \to \mathbb{R}$ be a C^1-functional which satisfies* $(\mathbf{PS})_a$ *for some $a > 0$. Assume further*

(DS) *There are constants $\gamma > 0$, $\mu > 1$, such that for all $x \in V$, $g \in G$,*

$$|\Phi(x) - \Phi(gx)| \leq \gamma(|\Phi(x)|^{1/\mu} + 1).$$

$(\mathbf{MP}_1)$ *There are constants $\beta > 0$, $\theta > \mu/(\mu - 1)$, $j_0 \geq 1$, such that for all $j \geq j_0$,*

$$\sup_{\rho \geq 0} \inf\{\Phi(x)\colon x \in V_{j-1}^\perp, \|x\| = \rho\} \geq \beta j^\theta.$$

$(\mathbf{MP}_2)$ *For every $j \geq 1$ there exists $R_j > 0$ such that $\Phi(x) \leq 0$ for $x \in V_j$, $\|x\| \geq R_j$.*

(BU) *There exists $m > 0$ such that $\kappa(S(V_{j-1} \oplus \mathbb{R})) < mj$* for all $j \geq j_0$.

Then Φ has an unbounded sequence of positive critical values.

Proof. We may assume (R_j) is non-decreasing and $R_j \to \infty$ as $j \to \infty$. Let $D \equiv \bigcup_{j \geq 1} (V_j \backslash B_{R_j} V_j)$, Φ be the given function, $\mu(t) \equiv \gamma(t^{1/\mu} + 1)$, $t \geq 0$, and let $b \geq a$ be large enough so that $t > \gamma(t^{1/\mu} + 1)$ if $t > b$. This data obviously satisfy the hypotheses (P1), (P2) and (P5) of Theorem 4.3, and (P4) and (P3) can be proved just like in Theorem 4.5 (using Proposition 5.1). Now, D is a local ANR, hence it is an ANR. So it has the homotopy type of a CW-complex. It follows from (MP_1) and (MP_2) that dim $V_j \to \infty$ as $j \to \infty$ hence, D is weakly contractible. Therefore D is contractible [23] 7.6.24. And, since V is an AR, the pair (V, D) is contractible. By Theorem 4.3, Φ has an unbounded sequence of critical values. $\square$

Some remarks are in order. (MP_1) and (MP_2) are mountain pass conditions. (MP_2) is the usual one, whereas (MP_1) is considerably stronger than the correspondong assumption for the symmetric case, which appears as (MP_1') in the introduction [1], [22], [10]. (MP_1) is a condition on the steepness of the mountain range which depends on the deviation (DS) of Φ from being G-invariant. The Borsuk-Ulam condition (BU) is satisfied if, for example (Theorem 2.8),

(**BU**$_1$) is a torus or a p-torus and $\dim V_{j-1} - \dim V^G < mj$, for all $j > 1$ and some $m > 0$.

In the G-invariant case this theorem holds for much more general group actions [8], [7]. The difficulty here again lies in computing upper bounds for $\kappa(S(V_j \oplus \mathbb{R}))$, see remarks at the end of Section 4. We now prove a mountain pass theorem which involves a different BU-condition which is quite useful in applications (cf. Section 6). We start with some definitions.

A finite-dimensional orthogonal representation W of a compact Lie group G is said to be *admissible* [5] if there exist a closed subgroup H of G and a normal subgroup K of H of finite index in H with H/K solvable, $W^H = 0$ and $W^K \neq 0$. Recall that a finite group S is *solvable* iff there exists a sequence of subgroups of S, $\{e\} \equiv S_0 \lhd S_1 \lhd \cdots \lhd S_r = S$, each one being a normal subgroup of the following one, such that $S_i/S_{i-1} \cong \mathbb{Z}/p_i$ for some prime p_i. Such a sequence is called a *resolution of* S. In particular, every finite abelian group is solvable.

A typical example of an admissible representation is when the maximal torus T of G acts with fixed points on $W \backslash \{0\}$ whereas the action of its normalizer N is fixed point free, and the Weyl group $W = N/T$ is solvable. If G is a finite solvable group, then every representation without non-trivial fixed points is admissible. For an equivalent definition of admissibility in terms of a Borsuk-Ulam property see [5] and [6] 2.24.

We now state our second mountain pass theorem.

THEOREM 5.3. *Let V be a G-Hilbert space with $V^G = 0$ and let $V_1 \subset \cdots \subset V_j \subset \cdots$ be a sequence of finite-dimensional G-invariant linear subspaces of V. Let $\Phi\colon V \to \mathbb{R}$ be a C^1-functional which satisfies* (**PS**)$_a$ *for some $a > 0$. Assume that this data satisfy* (**DS**), (**MP**$_1$) *and* (**MP**$_2$) *as in Theorem 5.2, and the following Borsuk-Ulam condition.*

(**BU**$_2$) *There exists a fixed admissible representation W of G such that for all $j \geq j_0$*

$$V_j \cong \underbrace{W \oplus \cdots \oplus W}_{j \text{ times}}.$$

Then Φ has an unbounded sequence of critical values.

Proof. Let H be a closed subgroup of G and K be a normal subgroup of H of finite index in H with $S \equiv H/K$ solvable, $W^H = 0$ and $W^K \neq 0$. Then the S-spaces V^K, V_j^K and the restriction Φ^K: $V^K \to R$ of Φ to V^k again satisfy $(\mathrm{PS})_\alpha$, (DS), (MP_1), (MP_2) and (BU). This last condition now reads

($\mathbf{BU}_2$) There exists a non-trivial admissible representation $\hat{W} \equiv W^K$ of S such that $V_j^K \cong \hat{W} \oplus \cdots \oplus \hat{W}$ (j times) for all $j \geq j_0$.

Now take a resolution $\{e\} \equiv S_0 \lhd S_1 \cdots \lhd S_r = S$ of S with $S_i/S_{i-1} \cong \mathbb{Z}/p_i$. Choose $1 \leq i \leq r$ such that $W^{S_i} = 0$ but $\hat{W}^{S_{i-1}} \neq 0$. Then the $\mathbb{Z}/p_i$-spaces $(V^K)^{S_{i-1}}$, $(V_j^K)^{S_{i-1}}$ and the restriction of Φ to $(V^K)^{S_{i-1}}$ satisfy all hypotheses of Theorem 5.2, with (BU_1) instead of (BU). This proves this theorem. □

If Φ is G-invariant this result was proved by Bartsch [5] 2.5 under a much weaker assumption than (MP_1), namely

$$\sup_{\rho \geq 0} \inf\{\Phi(x) : x \in V_{j-1}^{\perp}, \|x\| = \rho\} \to \infty \qquad \text{as } j \to \infty.$$

See also [7] 3.2.

The following section contains an application of Theorem 5.3.

6. A nonlinear Dirichlet problem

We look for solutions $u = (u_1, \ldots, u_m)$: $\bar{\Omega} \to \mathbb{R}^m$ of the nonlinear Dirichlet problem.

$$\begin{aligned} -\Delta u &= F_u(u) + f(x), && x \in \Omega \\ u &= 0, && x \in \partial\Omega. \end{aligned} \tag{6.1}$$

Here Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary and $f = (f_1, \ldots, f_m) \in L^2(\Omega, \mathbb{R}^m)$. We assume that F satisfies the following conditions:

(F1) F: $\mathbb{R}^m \to \mathbb{R}$ is a C^1-function.

(F2) There are constants $\alpha > 0$ and $1 < s < (n+2)/(n-2)$ if $n \geq 3$ such that

$$|F_u(u)| \leq \alpha(1 + |u|^s).$$

If $n = 2$ this condition can be weakened, if $n = 1$ it can be omitted.

(F3) There are constants $R > 0$ and $\mu > 2$ such that $0 < \mu F(u) \leq u \cdot F_u(u)$ if $|u| \geq R$.

(F4) There exist a compact Lie group and an orthogonal action on $W=\mathbb{R}^m$ such that W is admissible and $F\colon W\to\mathbb{R}$ is G-invariant.

If $f\equiv 0$ Bartsch has shown that, under these conditions, (6.1) has infinitely many weak solutions [5]. Here we shall show that

THEOREM 6.2. *If F satisfies the hypotheses* **(F1)**, **(F2)**, **(F3)**, **(F4)** *and $f\in L^2(\Omega,\mathbb{R}^m)$ then* (6.1) *possesses an unbounded sequence of weak solutions in $W_0^{1,2}(\Omega,\mathbb{R}^m)$ provided that s in* (*F2*) *is further restricted by*

$$\theta=\frac{(n+2)-(n-2)s}{n(s-1)}>\frac{\mu}{\mu-1}. \tag{6.3}$$

It is easy to check that if s satisfies (6.3) then $s<(n+2)/(n-2)$. For $m=1$ and $G=\mathbb{Z}/2$ this theorem was proved by P. H. Rabinowitz [21], [22] Section 10, using a mountain pass argument. Slightly less general versions were previously proved by Bahri and Berestycki [2], Struwe [25] and Dong and Li [14] using other arguments. We shall apply our Mountain Pass Theorem 5.3 to prove Theorem 6.2.

Set $V=W_0^{1,2}(\Omega,\mathbb{R}^m)$ and consider the functional

$$\Psi(u)=\int_\Omega\left(\tfrac{1}{2}\|\nabla u\|^2-F(u)-f\cdot u\right)dx,$$

$u\in V$. By the Poincaré inequality we may take $\|u\|=(\int_\Omega\|\nabla u\|^2\,dx)^{1/2}$ as the norm in V. As a consequence of (F1) and (F2) $\Psi\in C^1(V,\mathbb{R})$ (cf. [22] Appendix B). Its derivative at $u\in V$ is

$$D\Psi(u)v=\int_\Omega(\nabla u\cdot\nabla v-F_u(u)\cdot v-f\cdot v)\,dx$$

and a critical point of Ψ is, by definition, a weak solution of (6.1).

The action of G on $W=\mathbb{R}^m$ induces an orthogonal action on V given by $(gu)(x)=g(u(x))$. Observe that Ψ is not G-invariant unless $f\equiv 0$.

In order to apply Theorem 5.3 we need that Ψ satisfies the deviation from symmetry condition (DS). Unfortunately it does not. We proceed as in [22] Section 10, and replace Ψ by a new functional Φ which does not satisfy (DS) and is such that large critical values of Φ are critical values of Ψ, i.e. weak solutions of (6.1).

Integrating condition (F3) shows that there are constants $\alpha_1,\alpha_2>0$ such that

$$F(u)\geq\alpha_1|u|^\mu-\alpha_2. \tag{6.4}$$

Following Rabinowitz [22] 10.11, one can easily show that

PROPOSITION 6.5. *There exists a constant $\gamma_1 > 0$ depending on $\|f\|_{L^2(\Omega,\mathbb{R}^m)}$ such that, if u is a critical point of Ψ, then*

$$\int_\Omega (F(u) + \alpha_2)\, dx \le \gamma_1(\Psi(u)^2 + 1)^{1/2}. \tag{6.6}$$

We modify Ψ as follows. Let $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ be such that $\chi(t) = 1$ for $t \le 1$, $\chi(t) = 0$ for $t \ge 2$ and $\chi'(t) \in (-2, 0)$ for $t \in (1, 2)$. Set

$$\eta(u) \equiv \chi\left(\frac{1}{2\gamma_1(\psi(u)^2 + 1)^{1/2}} \int_\Omega (F(u) + \alpha_2)\, dx\right)$$

and

$$\Phi(u) \equiv \int_\Omega (\tfrac{1}{2}\|\nabla u\|^2 - F(u) - \eta(u) f \cdot u)\, dx,$$

for $u \in V$. Note that, by (6.6), $\eta(u) = 1$ if u is a critical point of Ψ. Hence, $\Phi(u) = \Psi(u)$.

PROPOSITION 6.7. *If F satisfies* **(F1)–(F4)** *and $f \in L^2(\Omega, \mathbb{R}^m)$ then*

(a) *$\Phi \in C^1(V, \mathbb{R})$.*

(b) *There exists a constant $\gamma > 0$ depending on $\|f\|_{L^2(\Omega,\mathbb{R}^m)}$ such that for all $g \in G$, $u \in V$, $|\Phi(u) - \Phi(gu)| \le \gamma(|\Phi(u)|^{1/\mu} + 1)$.*

(c) *There is a constant $a_0 > 0$ such that, if $\Phi(u) \ge a_0$ and $D\Phi(u) = 0$, then $D\Psi(u) = 0$ and $\Psi(u) = \Phi(u)$.*

(d) *There is a constant $a \ge a_0$ such that Φ satisfies the Palais-Smale condition $(PS)_a$ above a.*

Proof. (a) It follows from (F1) and (F2), as in [22] Appendix B, that Ψ is C^1. Since χ is smooth the same is true for η. Hence, Φ is C^1.

(b) If $\eta(u) = 0$ then, by (F4), $\Phi(u) = \Phi(gu)$ for all $g \in G$. So assume $\eta(u) > 0$. Then, for all $g \in G$,

$$\left|\int_\Omega (f \cdot gu)\, dx\right| < \|f\|_{L^2} \|u\|_{L^2} \le \gamma_2 \|u\|_{L^\mu} \le \gamma_3\left(\int_\Omega (F(u) + \alpha_2)\, dx\right)^{1/\mu}$$

and by the Hölder and Schwarz inequalities and (6.4). And, since $\eta(u) > 0$,

$$\int_\Omega (F(u) + \alpha_2)\, dx \le 4\gamma_1(\Psi(u)^2 + 1)^{1/2} \le 4\gamma_1(|\Psi(u)| + 1)$$

hence,

$$\left|\int_\Omega (F \cdot gu)\, dx\right| \le \gamma_4(|\Psi(u)|^{1/\mu} + 1) \qquad \text{for all } g \in G, \tag{6.8}$$

where $\gamma_4 > 0$ depends on $\|f\|_{L^2}$. Now

$$\begin{aligned} |\Phi(u) - \Phi(gu)| &= \left|\int_\Omega (\eta(gu)f \cdot gu - \eta(u)f \cdot u)\, dx\right| \\ &\le \left|\int_\Omega (f \cdot gu)\, dx\right| + \left|\int_\Omega (f \cdot u)\, dx\right| \\ &\le 2M, \end{aligned} \tag{6.9}$$

where $M \equiv \max\{|\int_\Omega (f \cdot gu)\, dx|, |\int_\Omega (f \cdot u)\, dx|\}$. Observe that

$$|\Psi(u)| \le |\Phi(u)| + \left|\int_\Omega (f \cdot u)\, dx\right| \le |\Phi(u)| + M.$$

This, together with (6.8), gives

$$M \le \gamma_5(|\Phi(u)|^{1/\mu} + M^{1/\mu} + 1)$$

hence,

$$M \le \gamma_6(|\Phi(u)|^{1/\mu} + 1),$$

where $\gamma_6 > 0$ depends on $\|f\|_{L^2}$. This, together with (6.9), proves (b).

(c) and (d) are proved by the same arguments as in [22] 10.16, for the case $m = 1$. □

In order to prove Theorem 6.2 it is, therefore, enough to prove that Φ has an unbounded sequence of critical values.

We define a filtration $V_1 \subset V_2 \subset \cdots \subset V_j \subset \cdots$ of V as follows. Let $0 < \lambda_1 < \lambda_2 \le \cdots \le \lambda_j \le \cdots$ be the eigenvalues of the problem

$$\begin{aligned} -\Delta v &= \lambda v \qquad \text{in } \Omega \\ v &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

and $v^1, v^2, \ldots, v^j, \ldots$ be the corresponding eigenfunctions. They give a basis for $H_0^{1,2}(\Omega)$. For every $k \geq 1$ let

$$V^k \equiv \{u = (u_1, \ldots, u_m) \in V \colon u_i = t_i v^k, t_i \in \mathbb{R}, i = 1, \ldots, m\}$$

and for each $j \geq 1$ let

$$V_j \equiv \bigoplus_{1 \leq k \leq j} V^k.$$

Each V^k is a G-invariant linear subspace of V isomorphic to $W = \mathbb{R}^m$ as a G-space. So, by (F4), each V_j satisfies the Borsuk-Ulam condition (BU_2) of Theorem 5.3. In view of Proposition 6.7 we need only to check (MP_1) and (MP_2).

To prove (MP_2) we only need to show that $\Phi(tu) \leq 0$ for all $u \in V$, $u \neq 0$ and t large enough. But, using (6.4) and $|\eta(u)| < 1$ we obtain positive constants $\alpha_3, \alpha_4, \alpha_5, \alpha_6$ such that

$$\Phi(tu) \leq \alpha_3 t^2 - \alpha_4 t^\mu + \alpha_5 t + \alpha_6 \to -\infty \qquad \text{as } t \to \infty$$

because $\mu > 2$. Finally, we show that

PROPOSITION 6.10. *Therer are constants $\beta > 0$ and $j_0 \geq 1$ such that, for all $j \geq j_0$,*

$$\sup_{\rho \geq 0} \inf\{\Phi(u) \colon u \in V_{j-1}^{\perp}, \|u\| = \rho\} \geq \beta j^\theta$$

with

$$\theta \equiv \frac{(n+2) - (n-2)s}{n(s-1)}.$$

Proof. Let $u \in V_{j-1}^{\perp}$ with $\|u\| = \rho$. Then $\|u\|_{L^2} \leq \lambda_j^{-1/2} \rho$. By the Gagliardo-Nirenberg inequality [16] 1.9.3,

$$\|u\|_{L^{s+1}} \leq \beta_1 \|u\|_{L^2}^{\nu} \|u\|^{1-\nu} \leq \beta_1 \lambda_j^{-\nu/2} \rho,$$

where ν is defined by

$$(1-\nu)\left(\frac{1}{2} - \frac{1}{n}\right) + \frac{\nu}{2} = \frac{1}{s+1}, \text{ i.e. } \nu = \frac{n}{s+1} - \frac{n-2}{2},$$

and $\beta_1 > 0$. So, using (F2), we get that, for positive constants $\beta_2, \beta_3, \beta_4$,

$$\Phi(u) \geq \tfrac{1}{2}\|u\|^2 - \alpha\|u\|_{L^{s+1}}^{s+1} - \alpha - \|f\|_{L^2}\|u\|_{L^2}$$

$$\geq \rho^2(\tfrac{1}{2} - \beta_2\lambda_j^{-\nu(s+1)/2}\rho^{s-1}) - \beta_3\rho - a.$$

Taking $\rho = \rho_j = (4\beta_2)^{-1/(s-1)}\lambda_j^{\nu(s+1)/2(s-1)}$ we obtain

$$\Phi(u) \geq \tfrac{1}{4}\rho_j^2 - \beta_3\rho_j - a$$

$$\geq \tfrac{1}{8}\rho_j^2 - \beta_4.$$

Since $\lambda_j \geq \beta_5 j^{2/n}$ for j large enough and some $\beta_5 > 0$ [12], $\Phi(u) \geq \beta j^{\theta}$ for all $u \in V_{j-1}^{\perp}$ with $\|u\| = \rho_j$ and j sufficiently large, where β is some positive constant and

$$\theta = \frac{2\nu(s+1)}{n(s-1)} = \frac{(n+2)-(n-2)s}{n(s-1)}. \qquad \square$$

This together with (6.3), gives the mountain pass condition (MP_1) of Theorem 5.3 and completes the proof of Theorem 6.2.

7. Upper bounds for the G-capacity

Recall that a G-ANR is a metrizable G-space X such that every G-map $Z \to X$ from a closed G-subspace Z of a metrizable G-space Y can be extended to a G-map $U \to X$ on a G-neighborhood U of Z in Y [19].

If (X, A) is a G-pair, $H^*(X, A)$ will denote singular cohomology with $\mathbb{Z}/p$-coefficients if $G = \mathbb{Z}/p \times \cdots \times \mathbb{Z}/p$ is a p-torus, p prime, and with rational coefficients if $G = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ is a torus. The aim of this section is to prove the following:

THEOREM 7.1. *Let X be a compact G-ANR whose fixed point set is homeomorphic to the l-sphere $X^G \cong \mathbb{S}^l$. Assume $H^q(X, pt) = 0$ for $0 \leq q \leq l+1$ and $H^q(X, X^G) = 0$ for $q > n > l$. Then*

$$\kappa(X) \leq \begin{cases} n - l & \text{if } G \text{ is a } p\text{-torus} \\ \tfrac{1}{2}(n-l) & \text{if } G \text{ is a torus.} \end{cases}$$

For the proof we use Borel cohomology so let us recall what this is. For every compact Lie group G, the countable join $J_\infty G$ of G, which is usually denoted by EG, is a contractible G-space on which G acts freely. Its orbit space $BG = EG/G$ is known as the *classifying space of* G. Given a G-space X we write $X \times_G EG$ for the orbit space of $X \times EG$ with the diagonal G-action. The projection $X \times EG \to EG$ induces a map of the orbit spaces $p_X: X \times_G EG \to BG$ which is a fibration with fibre X, called the *Borel fibration*. Every G-map $f: X \to Y$ induces a fibre-preserving map $X \times_G EG \to Y \times_G EG$ over BG. By defining

$$H_G^*(X, A) \equiv H^*(X \times_G EG, A \times_G EG)$$

on every G-pair (X, A) one obtains a G-equivariant cohomology theory, called *Borel cohomology*. It satisfies the homotopy invariance, excision and exactness axioms in the G-equivariant setting [10] 4.1, [13] Chapter III. The cup-product endows the *coefficient ring* $H_G^*(pt) = H^*(BG)$ with a graded ring structure and induces an $H_G^*(pt)$-module structure on $H_G^*(X, A)$ by $\xi\omega \equiv \xi \smile p_X^*(\omega)$, for $\xi \in H_G^*(X, A)$, $\omega \in H_G^*(pt)$.

For torii and p-torii of rank r, i.e. with r factors $\mathbb{S}^1$ or $\mathbb{Z}/p$ respectively, the coefficient rings are

$$H^*(BG; \mathbb{Q}) = \mathbb{Q}[c_1, \ldots, c_r] \qquad \text{if } G \text{ is a torus}$$

$$H^*(BG; \mathbb{Z}/2) = \mathbb{Z}/2[w_1, \ldots, w_r] \qquad \text{if } G \text{ is a 2-torus}$$

$$H^*(BG; \mathbb{Z}/p) = \mathbb{Z}/p[c_1, \ldots, c_r] \otimes \Lambda[w_1, \ldots, w_r] \qquad \text{if } G \text{ is a } p\text{-torus}, p > 2.$$

So the first two are polynomial algebras in r generators and the third one is the tensor product of a polynomial algebra and an exterior algebra in r generators. Since every proper subgroup H of a p-torus G of rank r is a p-torus of rank $<r$, and since every proper subgroup H of a torus G of rank r is the product of a finite group with a torus of rank r, it follows that $\ker(H_G^*(pt) = H^*(BG) \to H^*(BH) = H_G^*(G/H))$ contains a polynomial element if $H \neq G$ [10] 5.7. We will use this fact to prove the following proposition.

PROPOSITION 7.2. *Let X be a compact G-ANR and let $A \subset X$ be a closed G-invariant subset which is also a G-ANR with $X^G \subset A$. Then, if G is a torus or a p-torus, there exists an element $\alpha \in H_G^*(pt)$, $\alpha \neq 0$, which annihilates $H_G^*(X, A)$, i.e. $\xi\alpha = 0$ for all $\xi \in H_G^*(X, A)$.*

Proof. Let U_0 be a G-invariant neighborhood of A which is G-deformable into A in X rel A and, for every G-orbit of $X \setminus A$, take a G-invariant neighborhood which is G-deformable into that orbit [9] 6.10. Extract a finite subcovering $X = U_0 \cup U_1 \cup \cdots \cup U_n$. Then $U_j, j = 1, \ldots, n$, is G-deformable into some homogeneous

G-space G/H_j with $H_j \neq G$. Choose a polynomial element $\alpha_j \in \ker(H_G^*(pt) \to H_G^*(G/H))$. Since $p_{U_j}^*(\alpha_j)=0$, $p_X^*(\alpha_j)$ has a preimage $\tilde{\alpha}_j \in H_G^*(X, U_j)$. This follows from the exact H_G^*-sequence of the pair (X, U_j). On the other hand, the inclusion $(U_0, A) \subset (X, A)$ induces the zero homomorphism in H_G^*, so from the exact H_G^*-sequence of the triple (X, U_0, A) we get that every $\xi \in H_G^*(X, A)$ has a preimage $\tilde{\xi} \in H_G^*(X, U_0)$. But the cup-product $\tilde{\xi} \smile \tilde{\alpha}_1 \smile \cdots \smile \tilde{\alpha}_n \in H_G^*(X, U_0 \cup U_1 \cup \cdots \cup U_n)=0$. Hence $\xi \cdot \alpha_1 \cdots \alpha_n = 0$ in $H_G^*(X, A)$. Now take $\alpha \equiv \alpha_1 \cdots \alpha_n$. □

Proof of Theorem 7.1. Assume first G is a p-torus and let $\sigma\colon \mathbb{S}^l * J_m G \to X$ be a G-map which induces a homotopy equivalence $\sigma\colon \mathbb{S}^l \simeq X^G$. Then σ induces a map of the relative Leray-Serre spectral sequences [27] XIII.7,

$$\sigma\colon E_r^{p,q} \to \tilde{E}_r^{p,q}$$

associated to the fibration pairs $(X \underset{G}{\times} EG, X^G \underset{G}{\times} EG)$ and $((\mathbb{S}^l * J_m G) \underset{G}{\times} EG, \mathbb{S}^l \underset{G}{\times} EG)$ over BG. Their E_2-terms are

$$E_2^{p,q} = H^p(BG; H^q(X, X^G))$$

$$\tilde{E}_2^{p,q} = H^p(BG; H^q(\mathbb{S}^l * J_m C, \mathbb{S}^l)).$$

Here the coefficients are local coefficients given by the action of G on (X, X^G) which induces an action of $G = \pi_0(G) = \pi_1(BG)$ on $H^q(X, X^G)$. Similarly for $(\mathbb{S}^l * J_m G, \mathbb{S}^l)$ [27] VI.1.12. Therefore,

$$E_2^{p,q} = 0 \qquad \text{for } q < l+1 \text{ and } q > n$$

$$\tilde{E}_2^{p,q} = 0 \qquad \text{for } q < l+1 \text{ and } l+1 < q < l+m.$$

The last assertion follows from the fact that $\mathbb{S}^l * J_m G$ is $(m+l-1)$-connected. Hence, $\tilde{E}_2^{p,l+1} = \tilde{E}_r^{p,l+1}$ for $r < m$, and $E_r^{p,l+1} = E_\infty^{p,l+1}$ for $r \geq n-l$.

Observe that, since $H^l(X^G) \cong H^{l+1}(X, X^G)$, G acts trivially on $H^{l+1}(X, X^G)$. Let $\alpha \in H^p(BG) \cong E_2^{p,l+1}$, $\alpha \neq 0$, annihilate $H_G^*(X, X^G)$. Then α must be zero in $E_{r_0}^{p,l+1}$ for some $r_0 \leq n-l$. Assume $m > n-l$. Then $\tilde{E}_2^{p,l+1} = \tilde{E}_{r_0}^{p,l+1}$ and $\sigma^*(\alpha)=0$ in $\tilde{E}_2^{p,l+1}$. But since $\mathbb{S}^l \simeq X^G$, $\sigma^*\colon E_2^{p,l+1} \cong \tilde{E}_2^{p,l+1}$. This gives a contradiction. Therefore $m \leq n-l$.

The proof for $G=$ torus is completely analogous. In this case $\mathbb{S}^l * J_m G$ is $(2m+l-1)$-connected. □

Using stable cohomotopy instead of singular cohomology one can prove, by completely analogous arguments as in [7] 6.1, that

THEOREM 7.3. *If G is a finite p-group, p prime, and X is a compact G-ANR whose fixed point set is a finite dimensional sphere then* $\kappa(X, X^G) < \infty$.

As we mentioned before, this is not enough to apply our critical point results. They require upper bounds for the G-capacity of finite-dimensional G-representation spheres. For abelian p-groups upper bounds for similar invariants have been computed by Stolz [24] for $p = 2$ and Meyer [18] for p prime using the Adams spectral sequence.

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Mónica Clapp
Instituto de Matemáticas
Universidad Nacional Autónoma de México
Circuito Exterior, Ciudad Universitaria
04510 México, D.F., México

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