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## On the existence of higher dimensional Enneper's surface

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Enneper's surface and the catenoid are the simplest minimal surfaces in  $\mathbf{R}^3$  that are complete, orientable and nonplanar. This is because a complete orientable minimal surface has the total curvature of  $-4k\pi$  for some nonnegative integer  $k$ , while  $k = 1$  for Enneper's surface and the catenoid. Enneper's surface has one end and is a minimal immersion of  $\mathbf{R}^2$  in  $\mathbf{R}^3$ , whereas the catenoid has two ends and is a surface of revolution.

Not only is  $\mathbf{R}^3$  but also in  $\mathbf{R}^n$ ,  $n \geq 4$ , the catenoid has been known to exist. It is a minimal hypersurface which is rotationally symmetric. The higher dimensional catenoid has been the only example that is a higher dimensional analogue of a 2-dimensional minimal surface. In this paper, however, we prove that there also exists an  $n$ -dimensional Enneper's surface  $\Sigma^n$  in  $\mathbf{R}^{n+1}$  for  $n = 3, 4, 5, 6$ , which is a minimal immersion of  $\mathbf{R}^n$  in  $\mathbf{R}^{n+1}$ .

For two-dimensional minimal surfaces in  $\mathbf{R}^3$  there is the Weierstrass representation. This representation makes it easy to write down an enormous number of complete minimal surfaces in  $\mathbf{R}^3$ . Moreover, one can construct arbitrarily many minimal submanifolds of even codimension in  $\mathbf{R}^{2n}$ , as every complex submanifold of  $\mathbf{R}^{2n}$  is minimal. But in higher dimension one does not even have a good way to construct examples of complete immersed minimal hypersurfaces. Among a few known examples are the higher dimensional catenoids, area minimizing cones and graphs constructed by Bombieri-De Giorgi-Giusti [BDG], minimal hypersurfaces in  $\mathbf{R}^4$  and  $\mathbf{R}^6$  passing through the Clifford tori in  $S^3$  and  $S^5$  [B], minimal hypersurfaces as leaves of a foliation arising from isoparametric hypersurfaces [FK], and  $F$ -invariant minimal hypersurfaces [W].

All the examples above have been found by solving ordinary differential equations which were induced from the partial differential equation of minimal hypersurfaces by exploiting certain symmetry conditions. Higher dimensional Enneper's surface  $\Sigma$ , by contrast, is constructed by solving the partial differential equation directly as follows. First construct a compact minimal hypersurface by finding Jenkins-Serrin's solution [JS] to the Dirichlet problem for the minimal

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surface equation with suitably prescribed boundary data. Second obtain a compact Enneper type surface by reflecting the minimal hypersurface across the totally geodesic part of its boundary. Third blow up the compact Enneper type surface by an appropriately chosen scale to obtain a complete minimal immersion of  $\mathbf{R}^n$  in  $\mathbf{R}^{n+1}$ . In this process we have used the curvature estimates of [SSY] and [SS], and for this reason we have the dimension restriction that  $n = 3, 4, 5, 6$ .

Our higher dimensional Enneper's surface  $\Sigma$  satisfies some properties which are analogous to those of classical Enneper's surface. Namely,  $\Sigma^n$  contains  $n$  mutually orthogonal  $(n-1)$ -planes. Asymptotically, i.e., viewed from infinity,  $\Sigma^n$  looks like an  $n$ -plane with multiplicity  $2^n - 1$ . On the other hand, a high dimensional analogue of the total curvature for  $\Sigma^n \int_{\Sigma} |A|^n$ ,  $A$  being the second fundamental form, becomes infinite. Moreover, the Gauss map is not well defined at the point at infinity of  $\Sigma$ . Several interesting features of higher dimensional Enneper's surface are remarked in Section 6.

We would like to thank Mike Anderson and Leon Simon for some useful discussions.

## 1. Definitions and notations

(1) Let  $O = (0, \dots, 0)$ ,  $p_1 = (1, 0, \dots, 0)$ ,  $p_2 = (0, 1, 0, \dots, 0)$ ,  $\dots$ ,  $p_n = (0, \dots, 0, 1, 0) \in \mathbf{R}^{n+1}$ . Define  $T$  to be the regular  $(n-1)$ -simplex with  $p_1, \dots, p_n$  as its vertices. Let  $p_\varepsilon = (-1, \dots, -1, \varepsilon) \in \mathbf{R}^{n+1}$ ,  $0 < \varepsilon < 1$ , and define  $\Gamma_\varepsilon = (O \times \partial T) \cup (p_\varepsilon \times \partial T) \subset \mathbf{R}^{n+1}$ . Here  $p \times S$  denotes the cone from  $p$  to  $S$ , the union of all line segments from  $p$  to the points of  $S$ .

(2) Define  $C$  as the  $n$ -dimensional catenoid which is rotationally symmetric about the  $x_{n+1}$ -axis.  $C$  satisfies the equation  $x_{n+1} = f(r)$ ,  $r = (x_1^2 + \dots + x_n^2)^{1/2}$ , where

$$f(r) = \int_1^r [t^{2(n-1)} - 1]^{-1/2} dt.$$

(3) For each  $r > 0$  we define

$$\mu_r: \mathbf{R}^{n+1} \rightarrow \mathbf{R}^{n+1}, \quad \mu_r(x) = rx,$$

and for each  $q \in \mathbf{R}^{n+1}$  define

$$\tau_q: \mathbf{R}^{n+1} \rightarrow \mathbf{R}^{n+1}, \quad \tau_q(x) = x - q.$$

(4) Let  $A_i$ ,  $1 \leq i \leq n+1$ , be the hyperplane  $\{(x_1, \dots, x_{n+1}): x_i = 0\}$  and let  $A_{n+2} = \{(x_1, \dots, x_{n+1}): x_1 + \dots + x_n = 0\}$ ,  $A_0 = \{(x_1, \dots, x_n, 0): x_1 \cdots x_n = 0\}$ ,

$A_{-1} = \{(x_1, \dots, x_n, 0) : \prod_{1 \leq i < j \leq n} (x_i^2 + x_j^2) = 0\}$ , i.e.,  $A_{-1}$  is the  $(n-2)$ -skeleton of  $A_0$ ,  $A^+ = \{(x_1, \dots, x_{n+1}) : x_1 \cdots x_{n+1} > 0\}$ ,  $A^* = \{(x_1, \dots, x_{n+1}) : x_i \geq 0, i = 1, \dots, n+1\}$ ,  $A_{n+1}^\# = A_{n+1} \cap A^*$ ,  $A_{n+1}^b$  = the closure of  $(A_{n+1} \sim A^*)$ ,  $A_{n+1}^\varepsilon = \{(x_1, \dots, x_{n+1}) : 0 \leq x_{n+1} \leq \varepsilon\}$ ,  $A_{n+1}^+ = \{(x_1, \dots, x_{n+1}) : x_{n+1} \geq 0\}$ . Let  $A_{i,\theta}$  be the hyperplane which passes through the origin, is disjoint from the interior of  $A^*$ , is perpendicular to  $A_{n+1}$ , and makes an angle of  $\theta$  with  $A_i$  and an angle of  $\varphi$ ,  $(n-1)\cos^2 \varphi + \cos^2 \theta = 1$ , with every  $A_j, j \neq i, n+1$ , and let  $A_{n+1,\theta}$  be the hyperplane in  $\mathbf{R}^{n+1}$  which contains the  $(n-1)$ -plane  $A_{n+1} \cap A_{n+2}$  and makes an angle of  $0 \leq \theta < \pi$  with  $A_{n+1}$ . Let  $\ell$  be the straight line  $\{(x_1, \dots, x_n, 0) : x_1 = \cdots = x_n\}$ .

(5) Define  $\pi_1, \pi_2$  as the projections from  $\mathbf{R}^{n+1}$  onto  $A_{n+1}, A_{n+2}$ , respectively. Define  $\rho_i, 1 \leq i \leq n$ , as the rotation by  $180^\circ$  about the  $(n-1)$ -plane  $A_{n+1} \cap A_i$  and  $\rho_{ij}, 1 \leq i \neq j \leq n$ , as the rotation by  $90^\circ$  about the  $(n-1)$ -plane  $A_i \cap A_j$  taking the positive  $x_i$ -axis to the positive  $x_j$ -axis.  $\xi_{n+1}$  is the reflection with respect to the hyperplane  $A_{n+1}$ .

(6) For  $1 \leq i \neq j \leq n$ , let  $\phi_{ij} = \xi_{n+1} \circ \rho_{ij}$  and let  $\xi_{ij}$  be the reflection with respect to the hyperplane  $x_i = x_j$ . Define  $G$  to be the subgroup of  $O(n+1)$  generated by  $\{\phi_{ij}, \xi_{ij}\}_{1 \leq i \neq j \leq n}$ .

(7) Let  $B_r(q)$  be the ball of radius  $r$  with center at  $q$  and  $\overset{\circ}{B}_r(q)$  its interior.  $Z_r$  is the cylinder defined by  $Z_r = \{(x_1, \dots, x_{n+1}) : x_1^2 + \cdots + x_n^2 \leq r^2\}$ .

## 2. Compact Enneper type surface

The first step towards the proof of the existence of higher dimensional Enneper's surface is to construct a compact minimal hypersurface which resembles the fundamental region of 2-dimensional Enneper's surface (Lemma 1). Then a compact Enneper type surface is obtained from this fundamental piece by  $180^\circ$  rotations (Lemma 2).

**LEMMA 1.** *For each  $\varepsilon > 0$  there exists a unique  $n$ -dimensional compact minimal hypersurface  $\Sigma_\varepsilon$  in  $\mathbf{R}^{n+1}$  bounded by  $\Gamma_\varepsilon$ .  $\Sigma_\varepsilon$  is area minimizing and stable.*

*Proof.* The projection  $\pi_2$  maps  $\Gamma_\varepsilon$  one-to-one onto  $\pi_2(\Gamma_\varepsilon)$  which is the boundary of a convex domain in  $A_{n+2}$ . By [GT, Theorem 16.8] the Dirichlet problem for the minimal surface equation is uniquely solvable. It is well known that a minimal graph over a convex domain is area minimizing. Hence it is stable.

**LEMMA 2.**  *$2^n$  congruent copies of  $\Sigma_\varepsilon$  can be pieced together to form a compact smooth minimal hypersurface  $\tilde{\Sigma}_\varepsilon$  which is invariant under the group  $G$ .*



*Proof.* Assume that for  $\alpha, \beta \in G$ ,

$$\alpha(\partial\Sigma_\varepsilon \cap \Lambda_{n+1}) = \beta(\partial\Sigma_\varepsilon \cap \Lambda_{n+1}). \quad (1)$$

Note that  $\phi_{ij}(\Lambda^+) = \Lambda^+$  and  $\xi_{ij}(\Lambda^+) = \Lambda^+$ . Hence

$$\beta^{-1}\alpha(\Lambda^+) = \Lambda^+. \quad (2)$$

It follows from (1) and (2) that

$$\beta^{-1}\alpha(\Lambda^*) = \Lambda^* \quad \text{and} \quad \alpha(\partial\Sigma_\varepsilon) = \beta(\partial\Sigma_\varepsilon).$$

Hence  $\beta^{-1}\alpha$  is the product of some  $\xi_{ij}$ 's. From the invariance of  $\Gamma_\varepsilon$  under  $\xi_{ij}$  and the uniqueness of  $\Sigma_\varepsilon$  spanning  $\Gamma_\varepsilon$ , one obtains  $\xi_{ij}(\Sigma_\varepsilon) = \Sigma_\varepsilon$ . Therefore

$$\alpha(\Sigma_\varepsilon) = \beta(\Sigma_\varepsilon) \quad (3)$$

Define

$$\tilde{\Sigma}_\varepsilon = \bigcup_{\alpha \in G} \alpha(\Sigma_\varepsilon).$$

Clearly  $\tilde{\Sigma}_\varepsilon$  is invariant under  $G$ . That (1) implies (3) shows that  $\tilde{\Sigma}_\varepsilon$  consists of  $2^n$  (=the number of the components of  $\Lambda^+$ ) copies of  $\Sigma_\varepsilon$ . Note now that

$$\rho_i(\Sigma_\varepsilon) = \phi_{ij}(\Sigma_\varepsilon) \quad \text{for every } 1 \leq i \neq j \leq n.$$

Then a standard theory of the elliptic partial differential equations states that  $\Sigma_\varepsilon \cup \phi_{ij}(\Sigma_\varepsilon)$  is an analytic extension of  $\Sigma_\varepsilon$  across  $\partial\Sigma_\varepsilon \cap \Lambda_{n+1} \cap \Lambda_i$ . Furthermore it follows that  $\tilde{\Sigma}_\varepsilon$  is an analytic extension of  $\Sigma_\varepsilon$  across  $\partial\Sigma_\varepsilon \cap \Lambda_{n+1}$ .

### 3. Curvature estimates

Extending a compact Enneper type surface to a complete hypersurface requires detailed estimates on the curvature of the surface. A lower bound of the curvature is obtained by the maximum principle (Lemma 4) and an upper bound is derived from stability (Lemma 5).

**LEMMA 3.** *Let  $\gamma(s) = (x(s), y(s))$ ,  $0 \leq s \leq a$ , be a  $C^2$  curve in  $\mathbf{R}^2$  parametrized by the arclength  $s$  satisfying  $\gamma(0) = (0, 0)$ ,  $\gamma'(0) = (1, 0)$ ,  $\gamma'(a) = (0, 1)$  and*

$0 \leq x(s) \leq b$ . Then there exists  $0 \leq s_0 \leq a$  such that the curvature of  $\gamma$  at  $\gamma(s_0)$  is not less than  $1/b$ .

*Proof.* Let  $\zeta_c$  be the quarter circle defined by  $\zeta_c(t) = (b \sin t, c - b \cos t)$ ,  $0 \leq t \leq \pi/2$ . If  $\bar{c} = \sup\{c < b : \zeta_c \cap \gamma = \emptyset\}$ , then  $\gamma$  lies on one side of  $\zeta_{\bar{c}}$  touching  $\zeta_{\bar{c}}$  at a point  $\gamma(s_0)$ ,  $0 \leq s_0 \leq a$ . Hence the curvature of  $\gamma$  at  $\gamma(s_0)$  is larger than or equal to that of  $\zeta_{\bar{c}}$  which is  $1/b$ .

LEMMA 4. For each  $\varepsilon > 0$  there exist  $q_\varepsilon \in \tilde{\Sigma}_\varepsilon$  and  $a(\varepsilon) > 0$  such that

$$\text{dist}(q_\varepsilon, \Lambda_0) \leq a(\varepsilon) + \varepsilon, \quad |A|(q_\varepsilon) \geq \frac{1}{na(\varepsilon)}, \quad \lim_{\varepsilon \rightarrow 0} a(\varepsilon) = 0, \quad (4)$$

where  $|A|$  is the length of the second fundamental form of  $\tilde{\Sigma}_\varepsilon$ .

*Proof.* Let  $a(\varepsilon) > 0$  be the smallest number such that for any  $r > a(\varepsilon)$ , the catenoid  $\tau_{q(r)}\mu_r(C)$ ,  $q(r) = (r, \dots, r, 0) \in \ell$ , is disjoint from  $\Gamma_\varepsilon \sim \Lambda_{n+1}$ . Then one can easily see that  $a(\varepsilon)$  converges to 0 as  $\varepsilon$  goes to 0. For any  $q(r)$  with  $r > a(\varepsilon)$ ,  $\tau_{q(r)}\mu_{a(\varepsilon)}(C)$  does not intersect  $\Gamma_\varepsilon$ . Also, for sufficiently large  $b > 0$ ,  $\tau_{q(b)}\mu_{a(\varepsilon)}(C)$  cannot intersect  $\Sigma_\varepsilon$ . It follows from the maximum principle that

$$\tau_{q(r)}\mu_{a(\varepsilon)}(C) \cap \Sigma_\varepsilon = \emptyset \quad \text{for } a(\varepsilon) < r \leq b.$$

Hence

$$\tau_{q(a(\varepsilon))}\mu_{a(\varepsilon)}(C) \cap (\Sigma_\varepsilon \sim \partial\Sigma_\varepsilon) = \emptyset.$$

Let

$$\bar{\ell} = \{q \in \Sigma_\varepsilon : \pi_1(q) \in \ell\}.$$

Since the plane curve  $\bar{\ell}$  is invariant under the reflections  $\xi_{ij}$ ,  $\bar{\ell}$  is a principal curve in  $\Sigma_\varepsilon$ , that is, every tangent vector of  $\bar{\ell}$  points along a principal direction of  $\Sigma_\varepsilon$ . Therefore

$$|A|(q) \geq \kappa(q), \quad \text{the curvature of } \bar{\ell} \text{ at } q \in \bar{\ell}. \quad (5)$$

The tangent cone of  $\Sigma_\varepsilon$  at the origin  $O$  is  $\Lambda_{n+1} \cap \Lambda^*$ . Hence a tangent vector of  $\bar{\ell}$  at  $O$  points along  $\ell \subset \Lambda_{n+1}$ . Moreover  $\bar{\ell}$  is tangent to  $p_\varepsilon \times T$  at  $p_\varepsilon$ . Hence the angle between two tangent vectors of  $\bar{\ell}$  at  $O$  and at  $p_\varepsilon$  is larger than  $90^\circ$ . Thus there exists  $q \in \bar{\ell}$  at which a tangent vector of  $\bar{\ell}$  is perpendicular to  $\ell$ . Since  $\bar{\ell} \sim \{p_\varepsilon\}$

is disjoint from  $\tau_{q(a(\varepsilon))}\mu_{a(\varepsilon)}(C)$  one can apply Lemma 3 and conclude that there exists a point  $q_\varepsilon \in \bar{\ell}$  at which

$$\kappa(q_\varepsilon) \geq \frac{1}{(\sqrt{n}-1)a(\varepsilon)}. \quad (6)$$

Combining (5) and (6), we get (4b). Finally we can compute

$$\text{dist}(q_\varepsilon, A_0) \leq \left[ \frac{1}{n} (\sqrt{n}-1)^2 a(\varepsilon)^2 + \varepsilon^2 \right]^{1/2},$$

which gives (4a).

DEFINITION. Fix  $0 < d < 1$  in such a way that for any  $\varepsilon$

$$\text{dist}(Z_d, \partial \tilde{\Sigma}_\varepsilon) \geq d. \quad (7)$$

Define  $\tilde{\Sigma}_{\varepsilon,c} = \{q \in \tilde{\Sigma}_\varepsilon \cap Z_d : \text{dist}(q, A_0) \leq c\}$ .

LEMMA 5. If  $n \leq 6$  and  $\tilde{\Sigma}_{\varepsilon,c}$  is stable, then there exists  $b > 0$  depending only on the dimension  $n$  such that for any interior point  $q$  of  $\tilde{\Sigma}_{\varepsilon,c}$

$$|A|(q) \leq \frac{b}{\text{dist}(q, \partial \tilde{\Sigma}_{\varepsilon,c})}. \quad (8)$$

*Proof.* Let  $\omega_{n+1}$  be the volume of a unit ball in  $\mathbf{R}^{n+1}$ . By Lemma 1,  $\alpha(\Sigma_\varepsilon) \cap \tilde{\Sigma}_{\varepsilon,c}$  is area minimizing for any  $\alpha \in G$ . So it is easy to show that if  $B_r(q)$  is disjoint from  $\partial \tilde{\Sigma}_{\varepsilon,c}$  then

$$\text{Vol}(\tilde{\Sigma}_{\varepsilon,c} \cap \alpha(\Sigma_\varepsilon) \cap B_r(q)) \leq \text{Vol}(\partial B_r(q)) = (n+1)\omega_{n+1}r^n, \quad \alpha \in G.$$

Summing up for all distinct  $\alpha(\Sigma_\varepsilon)$  gives

$$r^{-n} \text{Vol}(\tilde{\Sigma}_{\varepsilon,c} \cap B_r(q)) \leq 2^n(n+1)\omega_{n+1}.$$

Thus (8) follows from [SSY, Theorem 3] for  $n \leq 5$  and [SS, Theorem 3] for  $n = 6$ .

#### 4. Blowing up

We are now in a position to blow up a compact Enneper type surface to obtain a higher dimensional Enneper's surface. But in this process correct scaling is needed (Lemma 6). Blowing up by correct scaling gives us a complete analytic hypersurface (Lemma 8). It may happen that this hypersurface becomes the hyperplane. However, an eigenvalue estimate rules out this possibility (Lemma 7).

LEMMA 6. Suppose  $n \leq 6$ . For each  $\varepsilon$ , let

$$c(\varepsilon) = \max\{c : \tilde{\Sigma}_{\varepsilon,c} \text{ is stable}\}.$$

Then

$$\lim_{\varepsilon \rightarrow 0} c(\varepsilon) = 0. \quad (9)$$

*Proof.*  $\tilde{\Sigma}_{\varepsilon,c}$  is stable if and only if

$$\int_{\tilde{\Sigma}_{\varepsilon,c}} |\nabla f|^2 - |A|^2 f^2 \geq 0$$

for any smooth function  $f$  with compact support in  $\tilde{\Sigma}_{\varepsilon,c}$ . Hence  $\tilde{\Sigma}_{\varepsilon,c}$  is stable for sufficiently small  $c > 0$ . So  $c(\varepsilon) > 0$ . Suppose there exist  $\delta > 0$  and a sequence of positive numbers  $\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots$  converging to 0 such that  $c(\varepsilon_i) > \delta$  for all  $i = 1, 2, 3, \dots$ . Then (4a) and (4c) of Lemma 4 imply that  $q_{\varepsilon_i}$  lies in  $\tilde{\Sigma}_{\varepsilon_i, c(\varepsilon_i)}$  for sufficiently large  $i$ . And then from (4b), (4a), (8) we see that

$$\frac{1}{na(\varepsilon_i)} \leq |A|(q_{\varepsilon_i}) \leq \frac{b}{\delta - a(\varepsilon_i) - \varepsilon_i},$$

which contradicts (4c). Therefore we get (9).

LEMMA 7. For  $i = 1, \dots, n$ , let  $\Lambda_{r,i}^n = \{(x_1, \dots, x_n) \in \mathbf{R}^n : |x_i| \leq r\}$  and  $\Lambda_r^n = \bigcup_{1 \leq i \leq n} \Lambda_{r,i}^n$ . Then on a domain  $D \subset \Lambda_r^n$  the first nonzero eigenvalue  $\lambda_1(D)$  of the Laplacian satisfies

$$\lambda_1(D) \geq \frac{1}{4n^2 r^2}.$$

*Proof.* Define the projections  $\pi_i^n: \mathbf{R}^n \rightarrow \mathbf{R}^n$  by  $\pi_i^n(x_1, \dots, x_n) = (x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n)$ . Then for any  $D' \subset \subset D$  and any  $i$  we have

$$\text{Vol}(\partial D') \geq 2 \text{Vol}(\pi_i^n(\partial D')).$$

However,

$$\text{Vol}(D') \leq \sum_{1 \leq i \leq n} \text{Vol}(D' \cap \Lambda_{r,i}^n) \leq 2r \sum_{1 \leq i \leq n} \text{Vol}(\pi_i^n(\partial D')).$$

Hence from Cheeger's estimate [C] we see that

$$\begin{aligned} \lambda_1(D) &\geq \frac{1}{4} \left[ \inf_{D' \subset \subset D} \frac{\text{Vol}(\partial D')}{\text{Vol}(D')} \right]^2 \\ &\geq \frac{1}{4} \left[ \inf_{D' \subset \subset D} \frac{\frac{2}{n} \sum_{1 \leq i \leq n} \text{Vol}(\pi_i^n(\partial D'))}{2r \sum_{1 \leq i \leq n} \text{Vol}(\pi_i^n(\partial D'))} \right]^2 = \frac{1}{4n^2 r^2}. \end{aligned}$$

LEMMA 8. As  $\varepsilon \rightarrow 0$ ,  $\mu_{1/c(\varepsilon)}(\tilde{\Sigma}_\varepsilon)$  converges to a complete minimal hypersurface  $\Sigma$  in  $\mathbf{R}^{n+1}$ ,  $n = 3, 4, 5, 6$ .  $\Sigma$  is distinct from the hyperplane.

*Proof.* Since  $\Sigma_\varepsilon$  is area minimizing, one can apply the same argument as in the proof of Lemma 5 to show that

$$|A|(q) \leq \frac{b}{\text{dist}(q, \partial \alpha(\Sigma_\varepsilon))}, \quad q \in \alpha(\Sigma_\varepsilon), \quad \alpha \in G. \quad (10)$$

Take  $q \in \tilde{\Sigma}_\varepsilon \cap Z_d$ . Then

$$\text{dist}(q, \partial \tilde{\Sigma}_\varepsilon) \geq d, \quad (11)$$

and  $q$  must belong to  $\alpha(\Sigma_\varepsilon)$  for some  $\alpha \in G$ . Observe that  $\partial \alpha(\Sigma_\varepsilon) \subset \Lambda_0 \cup \partial \tilde{\Sigma}_\varepsilon$ . If  $\text{dist}(q, \Lambda_0) \leq c(\varepsilon)/2$ , then by Lemma 5

$$|A|(q) \leq \frac{2b}{c(\varepsilon)}. \quad (12)$$

If  $\text{dist}(q, A_0) > c(\varepsilon)/2$ , then (10) and (11) imply that

$$|A|(q) \leq \frac{b}{\min\{c(\varepsilon)/2, d\}}. \quad (13)$$

So it follows from (12) and (13) that for sufficiently small  $\varepsilon$

$$\sup|A| \leq \frac{2b}{c(\varepsilon)} \quad \text{on } \tilde{\Sigma}_\varepsilon \cap Z_d.$$

Hence on  $\mu_{1/c(\varepsilon)}(\tilde{\Sigma}_\varepsilon \cap Z_d)$  we have

$$\sup|A| \leq 2b.$$

Therefore  $\mu_{1/c(\varepsilon)}(\tilde{\Sigma}_\varepsilon \cap Z_d)$  converges as  $\varepsilon \rightarrow 0$  to an analytic minimal hypersurface  $\Sigma$  in the  $C^2$  topology. By (7) we see that the boundary of  $\mu_{1/c(\varepsilon)}(\tilde{\Sigma}_\varepsilon \cap Z_d)$  lies in  $\partial Z_{d/c(\varepsilon)}$ , which disappears as  $\varepsilon \rightarrow 0$ . Thus  $\Sigma$  is complete.

We now show that  $\Sigma$  cannot be the hyperplane. Since  $\tilde{\Sigma}_{\varepsilon, c(\varepsilon)}$  is stable and any subset of  $\tilde{\Sigma}_\varepsilon$  properly containing  $\tilde{\Sigma}_{\varepsilon, c(\varepsilon)}$  is unstable, the Jacobi operator  $\Delta + |A|^2$  on  $\tilde{\Sigma}_{\varepsilon, c(\varepsilon)}$  has an eigenfunction  $f_\varepsilon$  with the eigenvalue zero which is positive in the interior and zero on the boundary of  $\tilde{\Sigma}_{\varepsilon, c(\varepsilon)}$ . Consequently  $\bar{f}_\varepsilon = f_\varepsilon \circ \mu_{1/c(\varepsilon)}^{-1}$  is an eigenfunction of the Jacobi operator on  $\mu_{1/c(\varepsilon)}(\tilde{\Sigma}_{\varepsilon, c(\varepsilon)})$ . Let

$$\Sigma_J = \lim_{\varepsilon \rightarrow 0} \mu_{1/c(\varepsilon)}(\tilde{\Sigma}_{\varepsilon, c(\varepsilon)}) = \{q \in \Sigma : \text{dist}(q, A_0) \leq 1\}.$$

Suppose that  $\Sigma$  is the hyperplane.  $\Sigma$  must then coincide with  $A_{n+1}$ . Viewing  $A_{n+1}$  as  $\mathbf{R}^n$ , we see that  $\Sigma_J = A_1^n$ , as defined in the preceding lemma.  $\mu_{1/c(\varepsilon)}(\tilde{\Sigma}_{\varepsilon, c(\varepsilon)})$  is close to  $A(\varepsilon) = A_1^n \cap Z_{d/c(\varepsilon)}$  in the  $C^2$  topology. Hence one can push  $\bar{f}_\varepsilon$  forward to obtain a smooth function  $\tilde{f}_\varepsilon$  on  $A(\varepsilon)$  that vanishes on the boundary of  $A(\varepsilon)$  and satisfies

$$\Delta \tilde{f}_\varepsilon + q \tilde{f}_\varepsilon = 0 \quad \text{on } A(\varepsilon)$$

for a smooth function  $q$  with  $|q| \leq b(\varepsilon)$ , where  $b(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . Then

$$\lambda_1(A(\varepsilon)) \leq \frac{\int_{A(\varepsilon)} |\nabla \tilde{f}_\varepsilon|^2}{\int_{A(\varepsilon)} \tilde{f}_\varepsilon^2} = \frac{\int_{A(\varepsilon)} q \tilde{f}_\varepsilon^2}{\int_{A(\varepsilon)} \tilde{f}_\varepsilon^2} \leq b(\varepsilon),$$

which contradicts the preceding lemma. Therefore  $\Sigma$  is not the hyperplane.

## 5. Existence theorem

In conclusion we prove the following theorem on the existence of higher dimensional Enneper's surface.

**THEOREM.** *In  $\mathbf{R}^{n+1}$ ,  $n = 3, 4, 5, 6$ , there exists a complete minimal hypersurface  $\Sigma^n$  called higher dimensional Enneper's surface with the following properties.*

- (i)  $\Sigma$  is a minimal immersion of  $\mathbf{R}^n$  into  $\mathbf{R}^{n+1}$ .
- (ii) Asymptotically  $\Sigma$  is the hyperplane with multiplicity  $2^n - 1$ .
- (iii)  $\Sigma$  contains  $A_0$ , the union of  $n$  mutually orthogonal  $(n-1)$ -planes.
- (iv)  $\Sigma$  is invariant under  $G$ .
- (v)  $\int_{\Sigma} |A|^n = \infty$ ,  $|A|$  being the length of the second fundamental form of  $\Sigma$ .
- (vi) The Gauss map for  $\Sigma$  is not well defined at the point at infinity of  $\Sigma$ .
- (vii)  $\Sigma$  consists of  $2^n$  congruent embedded pieces. The union of two adjacent pieces is stable. More precisely, if  $\hat{\Sigma}$  is one of the pieces with  $\partial\hat{\Sigma} \subset A^*$ , then  $\hat{\Sigma} \cup \rho_i(\hat{\Sigma})$  is a stable subset of  $\Sigma$ .

*Proof.* (i) From the construction of  $\Sigma_\varepsilon$  in Lemma 1 it is clear that the interior of  $\Sigma_\varepsilon$  is diffeomorphic to the interior of  $O \times T$ . Let  $\hat{\Sigma} = \lim_{\varepsilon \rightarrow 0} \mu_{1/c(\varepsilon)}(\Sigma_\varepsilon)$ . Then one can see that  $\hat{\Sigma}$  is embedded and diffeomorphic to  $A_{n+1}^* (= \lim_{\varepsilon \rightarrow 0} \mu_{1/c(\varepsilon)}(O \times T))$ . Let  $\psi$  be a diffeomorphism of  $A_{n+1}^*$  onto  $\hat{\Sigma}$ . Note that

$$\Sigma = \bigcup_{\alpha \in G} \alpha(\hat{\Sigma}).$$

Define  $\tilde{\psi}: A_{n+1} \rightarrow \mathbf{R}^{n+1}$  by

$$\tilde{\psi}(x) = \alpha(\psi(y)) \quad \text{if } x = \alpha(y), \quad x \in A_{n+1}, \quad y \in A_{n+1}^*.$$

Then one easily verifies that  $\tilde{\psi}$  is an immersion of  $\mathbf{R}^n$  onto  $\Sigma \subset \mathbf{R}^{n+1}$ .

(ii) Since  $\Sigma_\varepsilon$  is area minimizing, we have for  $r \leq d$

$$\text{Vol}(\Sigma_\varepsilon \cap B_r(O)) \leq \text{Vol}(\Sigma_\varepsilon \cap Z_r) \leq \text{Vol}(A_{n+1}^b \cap Z_r) + \text{Vol}(\partial Z_r \cap A_{n+1}^\varepsilon).$$

Hence

$$\frac{1}{\omega_n r^n} \text{Vol}(\Sigma_\varepsilon \cap B_r(O)) \leq \frac{2^n - 1}{2^n} + \frac{n\varepsilon}{r}.$$

By the monotonicity of the volume ratio,

$$\frac{1}{\omega_n r^n} \text{Vol}(\Sigma_\varepsilon \cap B_r(O)) \leq \frac{2^n - 1}{2^n} + \frac{n\varepsilon}{d}, \quad 0 < r \leq d.$$

Because of the invariance of the volume ratio under scaling, we see that as  $\varepsilon \rightarrow 0$

$$\frac{1}{\omega_n r^n} \text{Vol}(\hat{\Sigma} \cap B_r(O)) \leq \frac{2^n - 1}{2^n}, \quad 0 < r < \infty. \quad (14)$$

Now define the tangent cone  $T_\infty$  of  $\hat{\Sigma}$  at infinity by the current limit

$$T_\infty = \lim_{\varepsilon \rightarrow 0} \mu_\varepsilon(\hat{\Sigma}).$$

Then (14) gives

$$\Theta^n(T_\infty, O) \leq \frac{2^n - 1}{2^n}. \quad (15)$$

$\text{spt } T_\infty$  lies in  $A_{n+1}^+$  because  $\hat{\Sigma} \subset A_{n+1}^+$ . Also

$$\text{spt}(\partial T_\infty) = \partial A_{n+1}^*.$$

If  $\text{spt } T_\infty \cap A_{n+1}^b \sim \partial A_{n+1}^b \neq \emptyset$ , then the maximum principle implies that  $\text{spt } T_\infty \supset A_{n+1}^b$ . It follows from (15) that  $\text{spt } T_\infty = A_{n+1}^b$ . So let us suppose that  $\text{spt } T_\infty \neq A_{n+1}^b$ . Then either

$$\text{spt } T_\infty \cap A_{n+1}^b \sim \partial A_{n+1}^b = \emptyset \quad \text{and} \quad \text{spt } T_\infty \sim A_{n+1} \neq \emptyset, \quad (16)$$

or

$$\text{spt } T_\infty = A_{n+1}^*. \quad (17)$$

In case of (16), there exists  $0 < \theta < \pi$  such that  $\text{spt } T_\infty$  is tangent to  $A_{n+1,\theta}$  and lies on one side of  $A_{n+1,\theta}$ . By the maximum principle one gets  $A_{n+1,\theta} \cap A_{n+1}^+ \subset \text{spt } T_\infty$ , and so  $\text{spt } T_\infty \cap A_{n+1}^b \supset A_{n+1} \cap A_{n+2}$ , which is a contradiction. In case of (17), assume that  $\hat{\Sigma} \cap A_{n+1,\theta} \sim \{O\} \neq \emptyset$  for some  $0 < \theta < \pi$ . (17) requires  $\hat{\Sigma} \cap A_{n+1,\theta}$  to be compact. Then one can find a hyperplane  $A$  parallel to  $A_{n+1,\theta}$  such that  $\hat{\Sigma}$  is tangent to  $A$  at an interior point of  $\hat{\Sigma}$  and lies on one side of  $A$ . This is impossible by the maximum principle. So  $\hat{\Sigma} = A_{n+1}^*$ . But then  $\Sigma = A_{n+1}$ , which contradicts



Lemma 8. Therefore  $\text{spt } T_\infty = A_{n+1}^b$  and it follows that the tangent cone of  $\Sigma$  at infinity is  $A_{n+1}$  with multiplicity  $2^n - 1$ .

(iii), (iv) These are obvious by Lemma 1 and Lemma 2.

(v) M. Anderson [A, Theorem 5.2] showed that if a complete  $n$ -dimensional minimally immersed submanifold  $S \subset \mathbf{R}^m$  has one end and satisfies  $\int_S |A|^n < \infty$ , then  $S$  is an  $n$ -plane. So our claim follows.

(vi) Since  $\Sigma \supset A_0$  and the  $(n-1)$ -planes of  $A_0$  intersect each other along  $A_{-1}$ , one can see that any vector  $v$  normal to  $\Sigma$  at  $q \in A_{-1}$  must be normal to every  $(n-1)$ -plane of  $A_0$  passing through  $q$ . It follows that  $v = (0, \dots, 0, a)$ ,  $a \neq 0$ . Hence the Gauss map for  $\Sigma$  maps  $A_{-1}$  to the north pole of  $S^n$ . Now let  $\hat{\Sigma}$  be the embedded surface as defined in the proof of part (i) above and let  $\hat{\ell} \subset \hat{\Sigma}$  be the plane curve which is invariant under the reflections  $\xi_{ij}$ ,  $1 \leq i \neq j \leq n$ . At the origin  $\hat{\ell}$  is tangent to the horizontal hyperplane  $A_{n+1}$ . But as  $\hat{\ell}$  goes toward the point at infinity, it is flipped over by  $180^\circ$  and becomes parallel to  $A_{n+1}$ . So the Gauss map maps  $\hat{\ell}$  onto a great semicircle connecting the north pole to the south pole in  $S^n$ . Therefore the Gauss map cannot take on a single value at the point at infinity.

(vii) It follows from Lemma 2 that  $2^n$  congruent copies of  $\hat{\Sigma}$  comprise  $\Sigma$ . Since  $\partial \Sigma_\varepsilon$  can be projected one-to-one into  $A_{i,\theta}$  for  $0 < \theta < \pi/2$ ,  $\Sigma_\varepsilon$  is a graph over  $A_{i,\theta}$ . Therefore, as a limiting case, the interior of  $\Sigma_\varepsilon$  is a graph over  $A_{i,0} = A_i$  although  $\Sigma_\varepsilon$  itself is not. Similarly one can show that the interior of  $\rho_i(\Sigma_\varepsilon)$  is a graph over  $A_i$ . Note that  $\Sigma_\varepsilon$  and  $\rho_i(\Sigma_\varepsilon)$  lie in the opposite sides of  $A_{n+1}$  and that  $(\Sigma_\varepsilon \cup \rho_i(\Sigma_\varepsilon)) \cap A_{n+1} \subset A_0$ . Hence the interior of  $\Sigma_\varepsilon \cup \rho_i(\Sigma_\varepsilon)$  is also a graph over  $A_i$ . Therefore  $\Sigma_\varepsilon \cup \rho_i(\Sigma_\varepsilon)$  is stable by [Ch, Corollary 3] and so is  $\hat{\Sigma} \cup \rho_i(\hat{\Sigma})$ .

## 6. Concluding remarks

(1) When  $n = 2$ , the same construction as described above gives rise to classical Enneper's surface. This can be verified by observing that  $\Sigma^2$  has one end and has the total curvature of  $-4\pi$  [O].

(2) We have seen that there exists  $n$ -dimensional Enneper's surface in  $\mathbf{R}^{n+1}$  for  $2 \leq n \leq 6$ . On the other hand, our method of construction breaks down for  $n \geq 7$  because the curvature estimates (8) and (10) are no longer valid. This is in sharp contrast with the following famous results: J. Simons [Si] has proved that there exists no  $n$ -dimensional entire nonlinear minimal graph in  $\mathbf{R}^{n+1}$  when  $2 \leq n \leq 7$ ; Bombieri-De Giorgi-Giusti [BDG] have shown that there exist  $n$ -dimensional entire nonlinear minimal graphs in  $\mathbf{R}^{n+1}$  if  $n \geq 8$ . Let us explain in a naive and heuristic way why these dichotomies occur: The curvature estimates imply that low dimensional stable minimal submanifolds are *rigid*; Invalidity of the curvature estimates in high dimensions indicates that high dimensional stable minimal submanifolds are *flexible*.

Roughly speaking, one can obtain  $\hat{\Sigma}$  from  $A_{n+1}^\# = \{(x_1, \dots, x_n, 0) : x_i \geq 0, i = 1, \dots, n\}$  by fixing the boundary of  $A_{n+1}^\#$  and bending the interior of  $A_{n+1}^\#$  by  $180^\circ$ . When the dimension is low, the minimal submanifold  $\hat{\Sigma}$  is so rigid that  $\hat{\Sigma}$  can withstand the extreme bending and thereby giving rise to the higher dimensional Enneper's surface. However, when the dimension is high,  $\hat{\Sigma}$  is so flexible that the  $180^\circ$  bending tears down and flattens  $\hat{\Sigma}$  and then  $\hat{\Sigma}$  becomes  $A_{n+1}^b$ . As for minimal graphs, one should note that graphs are obtained from the horizontal hyperplane by mild bendings of at most  $90^\circ$ . But low dimensional complete stable minimal submanifolds are too rigid to allow mild bendings and therefore hyperplanes are the only entire minimal graphs. Moreover, high dimensional stable minimal submanifolds are flexible enough to allow mild bendings to persist, thereby allowing nonlinear minimal graphs to exist.

In view of these interpretations let us make a guess as to  $n$ -dimensional Enneper's surface in  $\mathbf{R}^{n+1}$  for  $n \geq 7$ . If such Enneper's surface is to exist, its fundamental piece should be constructed by bending  $A_{n+1}^\#$  by less than  $180^\circ$ , and hence the support of its tangent cone at infinity should be distinct from  $A_{n+1}$ .

(3) L. Simon [S] proved that the curvature estimate of Lemma 5 also holds for  $n = 7$  in the nonparametric case. However, with the assumption that  $\tilde{\Sigma}_{\varepsilon, c}$  is a local graph instead of being a stable hypersurface, we had difficulty ruling out the possibility that  $\Sigma^7$  becomes the hyperplane or  $\Sigma^6 \times \mathbf{R}^1$  in the proof of Lemma 8.

(4) As for part (v) of the theorem, we recall that  $\int_\Sigma K = -4\pi$  for the two-dimensional Enneper's surface  $\Sigma \subset \mathbf{R}^3$ . In fact, the total curvature of  $\Sigma$  is concentrated near the origin since  $|K|$  takes on the maximum at the origin. On the other hand, for higher dimensional Enneper's surface we can argue that  $\int_\Sigma |A|^n$  is concentrated near  $A_{-1}$  as follows. By part (vii) of the theorem,  $\hat{\Sigma} \cup \rho_i(\hat{\Sigma})$  and  $\hat{\Sigma} \cup \rho_j(\hat{\Sigma})$ ,  $i \neq j$ , are stable. Hence for  $q \in \hat{\Sigma}$  one gets the curvature estimate

$$\begin{aligned} |A|(q) &\leq \min\{b/\text{dist}(q, \partial(\hat{\Sigma} \cup \rho_i(\hat{\Sigma}))), b/\text{dist}(q, \partial(\hat{\Sigma} \cup \rho_j(\hat{\Sigma})))\} \\ &\leq \sqrt{2}b/\text{dist}(q, \partial(\hat{\Sigma} \cup \rho_i(\hat{\Sigma})) \cap \partial(\hat{\Sigma} \cup \rho_j(\hat{\Sigma}))) \\ &= \sqrt{2}b/\text{dist}(q, A_{-1}). \end{aligned}$$

This estimate indicates that  $|A|(q)$  becomes large as  $q$  approaches  $A_{-1}$ , which also suggests that  $\int_\Sigma |A|^n$  becomes infinite since  $A_{-1}$  has infinite  $(n-2)$ -dimensional volume unless  $n = 2$ .

(5) From part (vii) of the theorem we see that the higher dimensional Enneper's surface  $\Sigma$  consists of  $2^{n-1}$  disjoint stable subsets. In light of [Ch, Theorem 1] it is tempting to conjecture regarding the Morse index of  $\Sigma$  that

$$\text{index}(\Sigma) = 2^{n-1} - 1.$$

(6) It is still interesting to show that  $\text{index}(\Sigma)$  is finite. This together with part (v) of the theorem would surprisingly contrast with Fischer-Colbrie's theorem that a complete minimal surface in  $\mathbf{R}^3$  has finite total curvature if and only if it has finite index [F].

(7) The higher dimensional catenoid  $C$  lies between two parallel hyperplanes. In the proof of Lemma 4,  $C$  was used as a barrier in applying the maximum principle to  $\Sigma_\varepsilon$ . For this reason it seems quite probable that higher dimensional Enneper's surface might also lie between two parallel hyperplanes.

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