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# Rationality of the moduli variety of curves of genus 3

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*Abstract.* We prove that the moduli variety of curves of genus 3 is rational.

## §0. Introduction

Let $g \geq 2$ be a natural number. Consider the moduli variety $M_g$ of curves of genus $g$. Recall that $M_g$ is an irreducible quasiprojective variety of dimension $\dim M_g = 3g - 3$ [5, 11]. For $g \geq 23$ the variety $M_g$ is not unirational [6]. If $g \leq 13$ then $M_g$ is unirational [1, 3, 13] and for $g = 2, 4, 5, 6$ it is known that $M_g$ is rational [4, 9, 14, 15]. The aim of this paper is to prove the following result.

MAIN THEOREM. *The moduli variety $M_3$ is rational.*

The group $SL_3$ acts canonically on the space $S^4\mathbb{C}^{3*}$ of ternary forms of degree 4. It is known [12] that

$$\mathbb{C}(M_3) \simeq \mathbb{C}(\mathbb{P}(S^4\mathbb{C}^{3*}))^{SL_3}. \tag{0.1}$$

As usual, $\mathbb{C}(X)$ denotes the field of rational functions on the variety $X$.

For $n \geq 0$ denote by $V(n)$ the space of forms of degree $n$ in the variables $z_1, z_2$. The group $SL_2$ acts canonically on $V(n)$ and $PSL_2$ on $V(2d)$. For $\lambda = (\lambda_0, \lambda_2, \lambda_4, \lambda_6) \in \mathbb{C}^4$ considers the homogeneous $PSL_2$-morphism of degree 2

$$\delta_\lambda : V(8) \oplus V(0) \oplus V(4) \to V(4),$$

$$f_8 + f_0 + f_4 \mapsto \lambda_6 \psi_6(f_8, f_8) + 2\lambda_4 \psi_4(f_8, f_4) + \lambda_2 \psi_2(f_4, f_4) + 2\lambda_0 f_4 f_0.$$

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Here $\psi_i$ denotes $i$th transvectant. Recall that $\psi_i$ is the bilinear $SL_2$-mapping

$$\psi_i\colon V(d_1) \times V(d_2) \to V(d_1 + d_2 - 2i),$$

$$\psi_i(h_1, h_2) = \frac{(d_1 - i)(d_2 - i)}{d_1!\, d_2!} \sum_{0 \le j \le i} (-1)^j \binom{j}{i} \frac{\partial^i h_1}{\partial z_1^{i-j}\, \partial z_2^j} \frac{\partial^i h_2}{\partial z_1^j\, \partial z_2^{i-j}},$$

where $i \le \min\{d_1, d_2\}$. Consider $\delta_\lambda^{-1}(0)$ for $\lambda_0 \ne 0$. It is obvious that the element $1 \in V(0) = \mathbb{C}$ belongs to $\delta_\lambda^{-1}(0)$ and that the tangent space to $\delta_\lambda^{-1}(0)$ at the point 1 coincides with $V(8) \oplus V(0)$. It follows that 1 is a regular point of the subvariety $\delta_\lambda^{-1}(0)$. Therefore, a unique 10-dimensional irreducible component $U_\lambda$ of the subvariety $\delta_\lambda^{-1}(0)$ contains 1. It is shown in [10] that we have the following isomorphism of fields

$$\mathbb{C}(\mathbb{P}(S^4\mathbb{C}^{3*}))^{SL_3} \simeq \mathbb{C}(U_{(-\frac{7}{72}, \frac{11}{54}, \frac{1}{1680}, -\frac{6}{1225})})^{\mathrm{PSL}_2 \times \mathbb{C}^*}. \tag{0.2}$$

THEOREM 0.1. *For all* $\lambda \in (\mathbb{C}\backslash 0)^4$ *the field* $\mathbb{C}(U_\lambda)^{\mathrm{PSL}_2 \times \mathbb{C}^*} \simeq \mathbb{C}(\mathbb{P}U_\lambda)^{\mathrm{PSL}_2}$ *is rational.*

(For a closed homogeneous subvariety $U$ of a vector space $V$ we denote by $\mathbb{P}U$ the corresponding closed subvariety of the projective space $\mathbb{P}V$.)

Clearly, our Main Theorem is a consequence of (0.1), (0.2) and Theorem 0.1. We will prove Theorem 0.1 in 1–6.

This paper is organized as follows. In §1 we reduce Theorem 0.1 to the special case where $\lambda = (1, 6\varepsilon, 1, 6)$, $\varepsilon \ne 0$. Then we fix a basis $e_1, \dots, e_9, a_0, a_1, \dots, a_5$ of the space $V(8) \oplus V(0) \oplus V(4)$ and describe the mapping $\delta_\lambda$ explicitly in terms of coordinates. In §2 we recall some facts about $(G, G')$-sections. In §3 we construct a $(\mathrm{PSL}_2, N(H))$-section $\mathbb{P}X_\lambda^0$ of the variety $\mathbb{P}U_\lambda$ where $H$ is a finite subgroup defined in §2, and obtain isomorphisms

$$\mathbb{C}(\mathbb{P}U_\lambda)^{\mathrm{PSL}_2} \simeq \mathbb{C}(\mathbb{P}X_\lambda^0)^{N(H)} \simeq \mathbb{C}(\mathbb{P}X_\lambda)^{N(H)}$$

where $X_\lambda = \overline{X_\lambda^0}$. In §4 we construct a 6-dimensional variety $Y_\lambda$ and a regular action of $N(H)$ on $Y_\lambda$ such that

$$\mathbb{C}(\mathbb{P}X_\lambda)^{N(H)} \simeq \mathbb{C}(Y_\lambda)^{N(H)}$$

where the subgroup $H \subset N(H)$ acts trivially on $Y_\lambda$. In §§5 and 6 we construct a birational $N(H)$-isomorphism of $Y_\lambda$ with $R \times N$, where $R$ is a 3-dimensional linear space, $N$ is isomorphic to $\mathbb{P}^3$, and the action of $N(H)$ on $R \times N$ is the direct product of a linear representation on $R$ and a projective representation on $N$. Thus

$$\mathbb{C}(Y_\lambda)^{N(H)} \simeq \mathbb{C}(R \times N)^{N(H)}.$$

This finishes the proof since the field $\mathbb{C}(R \times N)^{N(H)}$ is rational by the "Noname Lemma" and Castlenuovo's Theorem (see [2], [7]).

## §1. Reduction to a special case

We first note that it is sufficient to prove Theorem 0.1 for $\lambda = (1, 6\varepsilon, 1, 6)$ where $\varepsilon \neq 0$. Indeed, suppose that $6\mu_8^2 = \lambda_6$, $\mu_4\mu_8 = \lambda_4$, $6\varepsilon\mu_4^2 = \lambda_2$, $\mu_0\mu_4 = \lambda_0$. Then

$$\mathbb{P}U_{(\lambda_0,\lambda_2,\lambda_4,\lambda_6)} \to \mathbb{P}U_{(1,6\varepsilon,1,6)}: \overline{(f_0 + f_4 + f_8)} \mapsto \overline{\mu_0 f_0 + \mu_4 f_4 + \mu_8 f_8} \tag{1.1}$$

is a $\mathrm{PSL}_2$-isomorphism and so

$$\mathbb{C}(\mathbb{P}U_{(\lambda_0,\lambda_2,\lambda_4,\lambda_6)})^{\mathrm{PSL}_2} \simeq \mathbb{C}(\mathbb{P}U_{(1,6\varepsilon,1,6)})^{\mathrm{PSL}_2}.$$

Thus it remains to prove Theorem 0.1 for $\lambda = (1, 6\varepsilon, 1, 6)$ where $\varepsilon \neq 0$.

For further use we want to explicitly calculate the map $\delta_\lambda$ for $\lambda = (1, 6\varepsilon, 1, 6)$. Fix the following basis in the space $V(8) \oplus V(0) \oplus V(4)$:

$e_1 = 28(z_1^6 z_2^2 - z_1^2 z_2^6)$, $e_2 = 56(z_1^7 z_2 + z_1^5 z_2^3 - z_1^3 z_2^5 - z_1 z_2^7)$,

$e_3 = 56(z_1^7 z_2 - z_1^5 z_2^3 - z_1^3 z_2^5 + z_1 z_2^7)$, $e_4 = z_1^8 - z_2^8$,

$e_5 = 8(z_1^7 z_2 - 7z_1^5 z_2^3 + 7z_1^3 z_2^5 - z_1 z_2^7)$, $e_6 = 8(z_1^7 z_2 + 7z_1^5 z_2^3 + 7z_1^3 z_2^5 + z_1 z_2^7)$,

$e_7 = z_1^8 + z_2^8$, $e_8 = 28(z_1^6 z_2^2 + z_1^2 z_2^6)$,

$e_9 = 70 z_1^4 z_2^4$, $a_0 = 1$,

$a_1 = z_1^4 + z_2^4$, $a_2 = 6z_1^2 z_2^2$,

$a_3 = z_1^4 - z_2^4$, $a_4 = 4(z_1^3 z_2 - z_1 z_2^3)$,

$a_5 = 4(z_1^3 z_2 + z_1 z_2^3)$.

Let $(x, s) = (x_1, \ldots, x_9, s_0, s_1, \ldots, s_5)$ be the corresponding coordinates. We find

$$\delta_\lambda(x, s) = Q_1(x, s)(z_1^4 + z_2^4) + Q_2(x, s) 6z_1^2 z_2^2 + Q_3(x, s)(z_1^4 - z_2^4)$$
$$+ Q_4(x, s) 4(z_1^3 z_2 - z_1 z_2^3) + Q_5(x, s) 4(z_1^3 z_2 + z_1 z_2^3)$$

where

$$
\begin{aligned}
Q_1(x,s) &= q_1(x) + 2x_7s_1 + 12x_8s_2 + 2x_9s_1 + \varepsilon(12s_1s_2) + 2s_0s_1 \\
&\quad + 48x_2s_4 - 48x_3s_5 - 2x_4s_3 + 16x_5s_4 \\
&\quad - 16x_6s_5 + \varepsilon(-12s_4^2 - 12s_5^2), \\
Q_2(x,s) &= q_2(x) + 4x_8s_1 + 12x_9s_2 + \varepsilon(2s_1^2 - 6s_2^2) + 2s_0s_2 \\
&\quad - 4x_1s_3 + 16x_2s_4 + 16x_3s_5 - 16x_5s_4 \\
&\quad - 16x_6s_5 + \varepsilon(-2s_3^2 - 4s_4^2 + 4s_5^2), \\
Q_3(x,s) &= q_3(x) + 2x_4s_1 + 12x_1s_2 + 64x_2s_5 + 64x_3s_4 \\
&\quad - 2x_7s_3 + 2x_9s_3 + \varepsilon(12s_2s_3 - 24s_4s_5) + 2s_0s_3, \\
Q_4(x,s) &= q_4(x) + 4x_5s_1 + 12x_2s_1 - 12x_5s_2 + 12x_2s_2 \\
&\quad - 8x_1s_5 - 16x_3s_3 + 8x_8s_4 - 8x_9s_4 \\
&\quad + \varepsilon(-6s_1s_4 - 6s_2s_4 + 6s_3s_5) + 2s_0s_4, \\
Q_5(x,s) &= q_5(x) + 4x_6s_1 + 12x_3s_1 + 12x_6s_2 - 12x_3s_2 \\
&\quad + 8x_1s_4 - 16x_2s_3 - 8x_8s_5 - 8x_9s_5 \\
&\quad + \varepsilon(6s_1s_5 - 6s_2s_5 - 6s_3s_4) + 2s_0s_5,
\end{aligned}
\tag{1.2}
$$

and

$$
\begin{aligned}
q_1(x) &= -192x_6^2 - 192x_3x_6 + 384x_3^2 - 192x_5^2 - 192x_2x_5 + 384x_2^2 \\
&\quad - 12x_1x_4 + 12x_7x_8 + 180x_8x_9, \\
q_2(x) &= 64x_6^2 - 192x_3x_6 - 128x_3^2 - 64x_5^2 + 192x_2x_5 + 128x_2^2 \\
&\quad - 2x_4^2 + 16x_1^2 + 2x_7^2 - 16x_8^2 - 50x_9^2, \\
q_3(x) &= 96x_5x_6 - 672x_3x_5 - 672x_2x_6 + 1248x_2x_3 \\
&\quad - 12x_1x_7 + 12x_4x_8 + 180x_1x_9, \\
q_4(x) &= 6x_4x_6 + 42x_3x_4 + 84x_1x_6 + 156x_1x_3 \\
&\quad - 6x_5x_7 - 42x_2x_7 + 24x_5x_8 - 264x_2x_8 + 30x_5x_9 - 30x_2x_9, \\
q_5(x) &= -6x_4x_5 - 42x_2x_4 + 84x_1x_5 + 156x_1x_2 \\
&\quad + 6x_6x_7 + 42x_3x_7 + 24x_6x_8 - 264x_3x_8 - 20x_6x_9 + 30x_3x_9.
\end{aligned}
$$

## §2. $(G, G')$-sections

In this part we recall some facts about $(G, G')$-sections. Let $G$ be a linear algebraic group, $X$ an irreducible quasiprojective variety with a regular action of $G$, and let $G' \subset G$ a subgroup of $G$.

DEFINITION 2.1. An irreducible subvariety $X' \subset X$ is called $(G, G')$-section of $X$ iff

(1) $\overline{G \cdot X'} = X$,

(2) $G' \cdot X' = X'$,

(3) $(G \cdot x') \cap X' = G' \cdot x'$ for all $x' \in X'$.

If $X'$ is $(G, G')$-section of $X$ then the map $f \mapsto f|_{X'}$ clearly induces an isomorphism $\mathbb{C}(X)^G \xrightarrow{\simeq} \mathbb{C}(X')^{G'}$.

Let $X'$ be $(G, G')$-section of $X$, $Y$ an irreducible quasiprojective variety, with a regular action of $G$, $F$: $Y \to X$ a dominant $G$-morphism, and $Y' \subset Y$ an irreducible component of $F^{-1}(X')$. Then one easily proves the following result.

PROPOSITION 2.2. *Suppose that $G' \cdot Y' = Y'$ and $F(Y')$ is dense in $X'$. Then $Y'$ is $(G, G')$-section of $Y$.*

EXAMPLE 2.3. Let $G$ be a reductive linear algebraic group, $G{:}X$ a linear representation, and let $H \subset G$ be the stationary subgroup of general position of the representation $G{:}X$. There exists an open nonempty $G$-invariant subset $X^0$ such that $G_x$ is conjugate to $H$ for all $x \in X^0$. Moreover,

$$(X^H)^0 = (X^H) \cap X^0 = \{x \in X^H \mid G_x = H\}$$

is $(G, N(H))$-section of $X$ where $N(H)$ is the normalizer of the subgroup $H$ in $G$.

EXAMPLE 2.4. Consider the linear representation of $PSL_2$ on $V(4)$. It is known that the stationary subgroup of general position of this representation is $H = \{e, \omega, \rho, \omega\rho\}$ where

$$e = \overline{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}, \qquad \omega = \overline{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}, \qquad \rho\overline{\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}}.$$

It can easily be checked that $N(H) = \langle \tau, \sigma \rangle$ where

$$\tau = \overline{\begin{pmatrix} \theta^{-1} & 0 \\ 0 & \theta \end{pmatrix}}, \qquad \sigma = \overline{\frac{1}{\sqrt{2}}\begin{pmatrix} \theta^3 & \theta^7 \\ \theta^5 & \theta^5 \end{pmatrix}}, \qquad \theta = \exp(2\pi i/8).$$

We have $N(H) \simeq S_4$ and $N(H)/H \simeq S_3$. It follows from Example 2.3 that

$$(V(4)^H)^0 = \{f \in V(4)^H \mid (\mathrm{PSL}_2)_f = H\}$$

is a $(\mathrm{PSL}_2, N(H))$-section of $V(4)$.

## §3. A special section

In this part we construct a $(\mathrm{PSL}_2, N(H))$-section $\mathbb{P}X_\lambda^0$ of the variety $\mathbb{P}U_\lambda$ (see the definition of $N(H)$ in §2).

For convenience we first write down explicitly the actions of $H$ and $N(H)$ on the space $V(8) \oplus V(0) \oplus V(4)$:

$$\begin{aligned}
\omega \cdot (x, s) &= (-x_1, x_2, -x_3, -x_4, x_5, -x_6, x_7, x_8, x_9, s_0, s_1, s_2, -s_3, s_4, -s_5),\\
\rho \cdot (x, s) &= (x_1, -x_2, -x_3, x_4, -x_5, -x_6, x_7, x_8, x_9, s_0, s_1, s_2, s_3, -s_4, -s_5),\\
\tau \cdot (x, s) &= (-x_1, -ix_3, -ix_2, x_4, -ix_6, -ix_5, x_7, -x_8, x_9, s_0, -s_1, s_2,\\
&\qquad -s_3, is_5, is_4), \qquad (3.1)\\
\sigma \cdot (x, s) &= \Big(4x_3, -\frac{i}{4}x_1, ix_2, -8x_6, -\frac{i}{8}x_4, -ix_5,\\
&\qquad \frac{1}{8}x_7 + \frac{7}{2}x_8 + \frac{35}{8}x_9, -\frac{1}{8}x_7 - \frac{1}{2}x_8 + \frac{5}{8}x_9, \frac{1}{8}x_7 - \frac{1}{2}x_8 + \frac{3}{8}x_9,\\
&\qquad s_0, -\frac{1}{2}s_1 - \frac{3}{2}s_2, \frac{1}{2}s_1 - \frac{1}{2}s_2, 2s_5, \frac{i}{2}s_3, -is_4\Big).
\end{aligned}$$

From this we get

$$(V(8) \oplus V(0) \oplus V(4))^H = \langle e_7, e_8, e_9, a_0, a_1, a_2\rangle$$

and

$$(V(8) \oplus V(0) \oplus V(4))^{N(H)} = \langle 5e_7 + e_9, a_0\rangle.$$

The decomposition of the $N(H)$-module $V(8) \oplus V(0) \oplus V(4)$ is as follows:

$$\begin{aligned}
V(8) \oplus V(0) \oplus V(4) = {} & \langle e_1, e_2, e_3\rangle \oplus \langle e_4, e_5, e_6\rangle \oplus \langle e_8, 7e_7 - e_9\rangle\\
& \oplus \langle 5e_7 + e_9\rangle \oplus \langle a_0\rangle \oplus \langle a_1, a_2\rangle \oplus \langle a_3, a_4, a_5\rangle.
\end{aligned}$$

Let $p$: $V(8) \oplus V(0) \oplus V(4) \to V(4)$ be the projection $f_8 + f_0 + f_4 \mapsto f_4$. First we construct a $(\mathrm{PSL}_2, N(H))$-section $X_\lambda^0$ of the variety $U_\lambda$ by applying Proposition 2.2 to the $\mathrm{PSL}_2$-morphism $p|_{U_\lambda}$ and a $(\mathrm{PSL}_2, N(H))$-section $(V(4)^H)^0$ of $V(4)$ (see Example 2.4).

LEMMA 3.1. $5e_7 + e_9 \in U_\lambda$.

*Proof.* Consider the plane $\langle a_0, 5e_7 + e_9 \rangle \subset V(8) \oplus V(0) \oplus V(4)$. We have $N(H)$. $\delta_\lambda(x, s) = \delta_\lambda(N(H) \cdot (x, s)) = \delta_\lambda(x, s)$ for all $(x, s) \in \langle a_0, 5e_7 + e_9 \rangle$ (see (3.1)). Therefore, $\delta_\lambda(\langle a_0, 5e_7 + e_9 \rangle) \subset V(4)^{N(H)} = \{0\}$ and $\langle a_0, 5e_7 + e_9 \rangle \subset \delta_\lambda^{-1}(0)$. Note also that $a_0 \in U_\lambda$ and that $a_0$ is a regular point of $\delta_\lambda^{-1}(0)$. It follows that $\langle a_0, 5e_7 + e_9 \rangle \subset U_\lambda$ and hence $5e_7 + e_0 \in U_\lambda$. □

Consider $\tilde{X}_\lambda = p^{-1}(V(4)^H) \cap \delta_\lambda^{-1}(0)$. From §1 and (3.1) above we obtain the following equations for $\tilde{X}_\lambda \subset V(8) \oplus V(0) \oplus V(4)$:

$$\begin{aligned} & s_3 = s_4 = s_5 = 0, \\ & q_1(x) + 2x_7 s_1 + 12x_8 s_2 + 2x_9 s_1 + \varepsilon(12 s_1 s_2) + 2 s_0 s_1 = 0, \\ & q_2(x) + 4x_8 s_1 + 12x_9 s_2 + \varepsilon(2s_1^2 - 6s_2^2) + 2 s_0 s_2 = 0, \\ & q_3(x) + 2x_4 s_1 + 12x_1 s_2 = 0, \\ & q_4(x) + 4x_5 s_1 + 12x_2 s_1 - 12x_5 s_2 + 12x_2 s_2 = 0, \\ & q_5(x) + 4x_6 s_1 + 12x_3 s_1 + 12x_6 s_2 - 12x_3 s_2 = 0. \end{aligned} \tag{3.2}$$

LEMMA 3.2

(1) $5e_7 + e_9$ *is a regular point of the subvariety* $\tilde{X}_\lambda$, $\dim T_{5e_7+e_9}(\tilde{X}_\lambda) = 7$.

(2) *Exactly one irreducible component, denoted by* $X_\lambda$, *of the subvariety* $\tilde{X}_\lambda$ *contains* $5e_7 + e_9$ *and* $\dim X_\lambda = 7$.

(3) $N(H) \cdot X_\lambda = X_\lambda$.

*Proof.* The proof of (1) is by direct calculations and statement (2) is a consequence of (1).

For (3) we remark that $N(H) \cdot \tilde{X}_\lambda = \tilde{X}_\lambda$ (see above), that $N(H) \cdot (5e_7 + e_9) = 5e_7 + e_9$, and that $5e_7 + e_9$ is a regular point of the subvariety $\tilde{X}_\lambda$. Hence we see that $N(H) \cdot X_\lambda = X_\lambda$. □

It follows from Lemma 3.2 that $X_\lambda$ is an irreducible component of the subvariety $p^{-1}(V(4)^H) \cap U_\lambda$. We set

$$X_\lambda^0 = \{(x, s) \in X_\lambda \mid p(x, s) \in (V(4)^H)^0\} = X_\lambda \cap p^{-1}((V(4)^H)^0).$$

Since $N(H) \cdot X_\lambda = X_\lambda$, $N(H) \cdot (V(4)^H)^0 = (V(4)^H)^0$, we see that $N(H) \cdot X_\lambda^0 = X_\lambda^0$. It follows from Lemma 3.2 that $X_\lambda^0$ is a nonempty open subset of $X_\lambda$ and that $p(X_\lambda^0)$ is dense in $(V(4)^H)^0$. This and Proposition 2.2 imply that $X_\lambda^0$ is a $(\mathrm{PSL}_2, N(H))$-section of $U_\lambda$.

Now, consider the subsets $\mathbb{P}X_\lambda^0 \subset \mathbb{P}X_\lambda \subset \mathbb{P}U_\lambda$. It follows from the previous paragraph that $\mathbb{P}X_\lambda^0$ is a $(\mathrm{PSL}_2, N(H))$-section of $\mathbb{P}U_\lambda$. Hence

$$\mathbb{C}(\mathbb{P}U_\lambda)^{\mathrm{PSL}_2} \simeq \mathbb{C}(\mathbb{P}X_\lambda^0)^{N(H)} \simeq \mathbb{C}(\mathbb{P}X_\lambda)^{N(H)}.$$

Our goal now is to prove the rationality of $\mathbb{C}(\mathbb{P}X_\lambda)^{N(H)}$. Note that $\mathbb{P}X_\lambda$ is uniquely defined by the following conditions (see Lemma 3.2):

(1) $\overline{5e_7 + e_9} \in \mathbb{P}X_\lambda$,
(2) $\mathbb{P}X_\lambda$ is an irreducible component of $\mathbb{P}\tilde{X}_\lambda$,
(3) The subvariety $\mathbb{P}\tilde{X}_\lambda \subset \mathbb{P}(V(8) \oplus V(0) \oplus V(4))$ is defined by the equations (3.2).

## §4. Some special representations

In this part we define a linear representation of $N(H)$ on $R$, a projective representation of $N(H)$ on $\mathbb{P}^8$, and a 6-dimensional irreducible $N(H)$-invariant closed subvariety $Y_\lambda \subset R \times \mathbb{P}^8$ such that $\mathbb{C}(\mathbb{P}X_\lambda)^{N(H)} \simeq \mathbb{C}(Y_\lambda)^{N(H)}$ where $H$ acts trivially on $Y_\lambda$.

Define a linear representation of $N(H)$ on $R = \mathbb{C}^3$ in the following way:

$$\tau \cdot (r_1, r_2, r_3) = (-r_1, r_3, r_2), \qquad \sigma \cdot (r_1, r_2, r_3) = (-2r_3, r_1/2, -r_2).$$

Let $\bar{y} = (y_1 : y_2 : y_3 : y_7 : y_8 : \cdots : y_{12})$ be homogeneous coordinates in $\mathbb{P}^8$. Define a projective representation of $N(H)$ on $\mathbb{P}^8$ in the following way:

$$\tau \cdot \bar{y} = (y_1 : -y_3 : -y_2 : y_7 : -y_8 : y_9 : y_{10} : -y_{11} : y_{12}),$$

$$\sigma \cdot \bar{y} = \Big(\frac{1}{16}y_3 : -16y_1 : -y_2 : \frac{1}{8}y_7 + \frac{7}{2}y_8 + \frac{35}{8}y_9 : -\frac{1}{8}y_7 - \frac{1}{2}y_8 + \frac{5}{8}y_9 : \frac{1}{8}y_7$$
$$-\frac{1}{2}y_8 + \frac{3}{8}y_9 : y_{10} : -\frac{1}{2}y_{11} - \frac{3}{2}y_{12} : \frac{1}{2}y_{11} - \frac{1}{2}y_{12}\Big).$$

Clearly, the subgroup $H \subset N(H)$ acts trivially on $R \times \mathbb{P}^8$. Define the open $N(H)$-invariant subset $\mathbb{P}^{8'} = \{\bar{y} \in \mathbb{P}^8 \mid y_1 y_2 y_3 \neq 0\}$ and set

$$M' = \{(x, s) \in V(8) \oplus V(0) \oplus V(4) \mid s_3 = s_4 = s_5 = 0,\ x_1 x_2 x_3 \neq 0\}.$$

We see that $N(H) \cdot M' = M'$, and that $M = \overline{M'}$ is a linear subspace of $V(8) \oplus V(0) \oplus V(4)$. Define the morphism $\pi$: $\mathbb{P}M' \to R \times \mathbb{P}^{8'}$ by

$$(x, s) \mapsto \left(\left(\frac{x_4}{x_1}, \frac{x_5}{x_2}, \frac{x_6}{x_3}\right), \left(\frac{x_2 x_3}{x_1} : \frac{x_3 x_1}{x_2} : \frac{x_1 x_2}{x_3}\right) : x_7 : x_8 : x_9 : s_0 : s_1 : s_2\right).$$

It can easily be checked that $\pi$ is an $N(H)$-morphism and that the fibers of $\pi$ are $H$-orbits. Note that $\mathbb{P}X_\lambda \subset \mathbb{P}\tilde{X}_\lambda \subset \mathbb{P}M$. Put

$$X'_\lambda = X_\lambda \cap M', \qquad \tilde{X}'_\lambda = \tilde{X}_\lambda \cap M'.$$

LEMMA 4.1. *$X'_\lambda \neq \varnothing$. More precisely,*

$$x^0 = 13i(5e_7 + e_9) + 5(4e_1 - ie_2 + e_3) \in X'_\lambda.$$

*Proof.* Consider the subgroup $\langle \sigma \rangle = \{\sigma, \sigma^2, \sigma^3 = 1\} \subset N(H)$. We have

$$V(8)^{\langle \sigma \rangle} = \langle 5e_7 + e_9, 8e_4 - ie_5 - e_6, 4e_1 - ie_2 + e_3 \rangle,$$

$$V(4)^{\langle \sigma \rangle} = \langle 2(z_1^4 - z_2^4) + 4(z_1^3 z_2 + z_1 z_2^3) + 4i(z_1^3 z_2 - z_1 z_2^3) \rangle.$$

It follows from above that

$$\delta_\lambda(\alpha_1(5e_7 + e_9) + \alpha_2(8e_4 - ie_5 - e_6) + \alpha_3(4e_1 - ie_2 + e_3))$$
$$= q(\alpha_1, \alpha_2, \alpha_3)(2(z_1^4 - z_2^4) + 4(z_1^3 z_2 + z_1 z_2^3) + 4i(z_1^3 z_2 - z_1 z_2^3)). \tag{4.1}$$

Direct calculations give us

$$q(\alpha_1, \alpha_2, \alpha_3) = 48(5\alpha_1\alpha_3 + i\alpha_2^2 - 13i\alpha_3^2). \tag{4.2}$$

From (4.1) and (4.2) it follows that $x^0$, $5e_7 + e_9 \in V(8)^{\langle \sigma \rangle} \cap \tilde{X}_\lambda$ and that $V(8)^{\langle \sigma \rangle} \cap \tilde{X}_\lambda$ is irreducible. On the other hand $5e_7 + e_9$ is a regular point of $X_\lambda$ (Lemma 3.2). Hence $V(8)^{\langle \sigma \rangle} \cap \tilde{X}_\lambda \subset X_\lambda$ and so $x^0 \in X_\lambda$. □

From Lemma 4.1 it follows that $X'_\lambda$ is an open nonempty $N(H)$-invariant subset of $X_\lambda$. Thus we get an isomorphism

$$\mathbb{C}(\mathbb{P}X_\lambda)^{N(H)} \simeq \mathbb{C}(\mathbb{P}X'_\lambda)^{N(H)}. \tag{4.3}$$

Notice that $\mathbb{P}X'_\lambda$ is an irreducible component of $\mathbb{P}\tilde{X}'_\lambda$ and that $\overline{x^0} \in \mathbb{P}X_\lambda$.

We have an isomorphism

$$\mathbb{C}(\mathbb{P}X'_\lambda)^{N(H)} \simeq \mathbb{C}(\pi(\mathbb{P}X'_\lambda))^{N(H)}. \tag{4.4}$$

Notice that $\pi(\mathbb{P}X'_\lambda)$ is an irreducible component of $\pi(\mathbb{P}\tilde{X}'_\lambda)$, and

$$\pi(x^0) = ((0, 0, 0), (-5/4 : 20 : -20 : 65 : 0 : 13 : 0 : 0 : 0)) \in \pi(\mathbb{P}X'_\lambda).$$

It is not hard to obtain from (3.2) that the equations of the subvariety $\pi(\mathbb{P}\tilde{X}'_\lambda) \subset R \times \mathbb{P}^{8'}$ are given by

$$\begin{aligned}
0 &= (-192r_3^2 - 192r_3 + 384)y_1y_2 + (-192r_2^2 - 192r_2 + 384)y_1y_3 \\
&\quad + (-12r_1)y_2y_3 + 12y_7y_8 + 180y_8y_9 + 2y_7y_{11} + 12y_8y_{12} \\
&\quad + 2y_9y_{11} + \varepsilon(12y_{11}y_{12}) + 2y_{10}y_{11}, \\
0 &= (64r_3^2 - 192r_3 - 128)y_1y_2 + (-64r_2^2 + 192r_2 + 128)y_1y_3 \\
&\quad + (-2r_1^2 + 16)y_2y_3 + 2y_7^2 - 16y_8^2 - 50y_9^2 + 4y_8y_{11} + 12y_9y_{12} \\
&\quad + \varepsilon(2y_{11}^2 - 6y_{12}^2) + 2y_{10}y_{12}, \\
0 &= (96r_2r_3 - 672r_2 - 672r_3 + 1248)y_1 \\
&\quad - 12y_7 + (12r_1)y_8 + 180y_9 + 2r_1y_{11} + 12y_{12}, \\
0 &= (6r_1r_3 + 42r_1 + 84r_3 + 156)y_2 + (-6r_2 - 42)y_7 + (24r_2 - 264)y_8 \\
&\quad + (30r_2 - 30)y_9 + (4r_2 + 12)y_{11} + (-12r_2 + 12)y_{12}, \\
0 &= (-6r_1r_2 - 42r_1 + 84r_2 + 156)y_3 + (6r_1 + 42)y_7 + (24r_3 - 264)y_8 \\
&\quad + (-30r_3 + 30)y_9 + (4r_3 + 12)y_{11} + (12r_3 - 12)y_{12}.
\end{aligned} \tag{4.5}$$

Denote by $\tilde{Y}_\lambda \subset R \times \mathbb{P}^8$ the subvariety defined by the equations (4.5). The closure $Y_\lambda$ of $\pi(\mathbb{P}\tilde{X}'_\lambda)$ in $R \times \mathbb{P}^8$ is a union of some irreducible components of $\tilde{Y}_\lambda$. We see that $N(H) \cdot \tilde{Y}_\lambda = \tilde{Y}_\lambda$, $N(H) \cdot Y_\lambda = Y_\lambda$, and $Y_\lambda$ is an irreducible component of the subvariety $\tilde{Y}_\lambda$. Thus

$$\mathbb{C}(\pi(\mathbb{P}X'_\lambda))^{N(H)} \simeq \mathbb{C}(Y_\lambda)^{N(H)}. \tag{4.6}$$

From (4.3), (4.4), and (4.6) we obtain an isomorphism

$$\mathbb{C}(\mathbb{P}X_\lambda)^{N(H)} \simeq \mathbb{C}(Y_\lambda)^{N(H)}.$$

Our goal now is to prove the rationality of $\mathbb{C}(Y_\lambda)^{N(H)}$. Note that the following conditions hold for $Y_\lambda$:

(1) $\overline{\pi(x^0)} \in Y_\lambda$,
(2) $Y_\lambda$ is an irreducible component of $\tilde{Y}_\lambda$,
(3) the equations of the subvariety $\tilde{Y}_\lambda \subset R \times \mathbb{P}^8$ are (4.5).

## §5. Proof of rationality

In this section we prove the rationality of $\mathbb{C}(Y_\lambda)^{N(H)}$. Define

$$\eta\colon \tilde{Y}_\lambda \to R, \qquad (r, \bar{y}) \mapsto r,$$

$$\beta\colon \tilde{Y}_\lambda \to \mathbb{P}^8, \qquad (r, \bar{y}) \mapsto \bar{y}.$$

We have $\eta(\overline{\pi(x^0)}) = 0$. It follows from (4.5) that $\beta(\eta^{-1}(r))$ is an intersection of 2 quadrices and 3 hyperplanes in $\mathbb{P}^8$.

LEMMA 5.1. *$\eta^{-1}(0)$ is irreducible and 3-dimensional.*

*Proof.* The variety $\beta(\eta^{-1}(0))$ is the intersection of a 5-dimensional linear subspace $L_0$ of $\mathbb{P}^8$ and 2 quadrices. Consider the restriction of these 2 quadrices to $L_0$. One can calculate that

(1) some linear combination of these restrictions of the quadrices has maximal rank,
(2) the rank of all nontrivial linear combinations of these restrictions of the quadrices is $\geq 3$.

From (1) it follows that $\beta(\eta^{-1}(0))$ has no irreducible component of degree 1. From (2) it follows that $\dim(\beta(\eta^{-1}(0)) = 3$ and that $\beta(\eta^{-1}(0))$ has no irreducible component of degree 2. Therefore, $\eta^{-1}(0) \simeq \beta(\eta^{-1}(0))$ is irreducible and 3-dimensional.

Set

$$R' = \{r \in R \mid \eta^{-1}(r) \text{ is irreducible and 3-dimensional}\}.$$

From Lemma 5.1 it follows that $R'$ is an open nonempty $N(H)$-invariant subset of $R$, that $0 \in R'$, and that $\eta^{-1}(R')$ is an open nonempty $N(H)$-invariant subset of $Y_\lambda$. Hence

$$\mathbb{C}(Y_\lambda)^{N(H)} \simeq \mathbb{C}(\eta^{-1}(R'))^{N(H)}.$$

Let us prove now the rationality of $\mathbb{C}(\eta^{-1}(R'))^{N(H)}$. Consider the bundle

$$\eta|_{\eta^{-1}(R')}: \eta^{-1}(R') \to R'.$$

This bundle has the $N(H)$-section

$$r \mapsto (r, u'(r)), \qquad u'(r) = (0:0:0:0:0:0:1:0:0).$$

LEMMA 5.2. *There exists an open nonempty $N(H)$-invariant subset $R'' \subset R$ such that*

(1) $R'' \ni 0$,

(2) *the bundle $\eta|_{\eta^{-1}(R'')}: \eta^{-1}(R'') \to R''$ has the $N(H)$-section*

$$r \mapsto (r, u''(r)) = (r, u''_1(r) : \cdots : u''_9(r))$$

*where $u''_7(r) = u''_8(r) = u''_9(r) = 0$ for $r \in R''$,*

(3) $u''(0) = (-5/4 : 20 : -20 : 65 : 0 : 13 : 0 : 0 : 0)$.

The proof will be given in §6.

By (4.5) and Lemma 5.2 it follows that

$$\langle u'(r), u''(r)\rangle \subset \beta(\eta^{-1}(r)) \qquad \text{for } r \in R' \cap R''.$$

Set

$$N = \{\bar{y} \in \mathbb{P}^8 \mid y_1 = y_2 = y_3 = y_7 + 7y_9 = y_{10} = 0\},$$

$$N(r) = \langle u'(r), u''(r), (1:0:0:0:\cdots), (0:1:0:0:\cdots),$$
$$(0:0:1:0:\cdots)\rangle \subset \mathbb{P}^8, \qquad r \in R' \cap R''.$$

We have $N(H) \cdot N = N$ and $g \cdot N(r) = N(g \cdot r)$ for $g \in N(H)$.

LEMMA 5.3. *There exists an open $N(H)$-invariant subset $R''' \subset (R' \cap R'')$ containing* 0 *such that*
(1) $\dim N(r) = 4$ *and*
(2) $N(r) \cap N = \varnothing$
*for all $r \in R'''$.*

*Proof.* From Lemma 5.2 we get $\dim N(0) = 4$ and $N(0) \cap N = \varnothing$, and the lemma follows. □

For $r \in R'''$ let

$$\gamma_r\colon \mathbb{P}^8 \to N$$

be the projection of $\mathbb{P}^8$ to $N$ from $N(r)$.

LEMMA 5.4. *There exists an open $N(H)$-invariant subset $R'''' \subset R'''$ containing* 0 *such that $\gamma_r(\beta(\eta^{-1}(r))) = N$ for $r \in R''''$.*

*Proof.* It can easily be checked that $\gamma_0(\beta(\eta^{-1}(0))) = N$. From this the lemma follows. □

Clearly, we have an isomorphism

$$\mathbb{C}(\eta^{-1}(R'))^{N(H)} \simeq \mathbb{C}(\eta^{-1}(R''''))^{N(H)}.$$

It remains to prove the rationality of $\mathbb{C}(\eta^{-1}(R''''))^{N(H)}$. First recall the following fact.

LEMMA 5.5. *Let $X \subset \mathbb{P}^n$ be an intersection of a 5-dimensional linear subspace and two quadrices, and let $M_1, M_2 \subset \mathbb{P}^n$ be linear subspaces. Suppose that $X$ is irreducible*, $\dim X = 3$, $M_1 \cap M_2 = \varnothing$, $\dim M_1 = n - 4$, $\dim M_2 = 3$, *$M_1 \cap X$ contains a line, and $p_2(X) = M_2$, where $p_2$ is the projection of $\mathbb{P}^n$ to $M_2$ from $M_1$; then $p_2|_X$ is a birational isomorphism of $X$ and $M_2$.*

*Proof.* Let $L \subset M_1 \cap X$ be the line. For a point $u \in M_2$ in general position the intersection of any of the quadrices and the plane $\langle L, u\rangle$ splits into two lines where the line $L$ is one of the component. Therefore, $X \cap \langle L, u\rangle$ is the union of $L$ and some point $u'$ where $(p_2|_X)^{-1}(u) = \{u'\}$. It follows that $p_2|_X$ is a birational isomorphism. □

From Lemmas 5.4 and 5.5 it follows that

$$\gamma_r|_{\beta(\eta^{-1}(r))}\colon \beta(\eta^{-1}(r)) \to N$$

is a birational isomorphism for all $r \in R''''$. Therefore,

$$\Gamma\colon \eta^{-1}(R'''') \to R'''' \times N, \qquad (r, \bar{y}) \mapsto (r, \gamma_r(\bar{y}))$$

is a birational $N(H)$-isomorphism which defines an isomorphism of fields

$$\mathbb{C}(\eta^{-1}(R''''))^{N(H)} \simeq \mathbb{C}(R'''' \times N)^{N(H)}.$$

The rationality of the field

$$\mathbb{C}(R'''' \times N)^{N(H)} \simeq \mathbb{C}(R \times N)^{N(H)}$$

is now a consequence of "Noname Lemma" and Castelnuovo's Theorem [2], [7].

## §6. Proof of Lemma 5.2

In this section we give a proof of Lemma 5.2.

Let $X_1 \subset \mathbb{P}(V(8) \oplus V(0) \oplus V(4))$ be the projectivization of $\overline{\mathrm{PSL}_2 \cdot \langle z_1^8, z_1^7 z_2, z_1^6 z_2^2 \rangle}$ and let $X_2 \subset \mathbb{P}(V(8) \oplus V(0) \oplus V(4))$ be the projectivization of $\overline{\mathrm{PSL}_2 \cdot \langle 5e_7 + e_9 \rangle}$. It is obvious that $X_1$ and $X_2$ are irreducible, $\dim X_1 = \dim X_2 = 3$, and that $f \in X_1$ iff $f$ has a root of multiplicity $\geq 6$ (as an element of $V(8)$). It is also clear that $\delta_\lambda(\langle z_1^8, z_1^7 z_2, z_1^6 z_2^2 \rangle) = 0$ and that the differential $d(\delta_\lambda|_{V(8)})|_{z_1^6 z_2^2}$ is surjective. This implies that $X_1$ is an irreducible component of $\mathbb{P}(\delta_\lambda^{-1}(0) \cap V(8))$. Note also that

$$\deg X_1 = 16$$

(see [16]).

Since $\delta_\lambda(5e_7 + e_9) = 0$ and the differential $d(\delta_\lambda|_{V(8)})|_{5e_7 + e_9}$ is surjective, we see that $X_2$ is an irreducible component of $\mathbb{P}(\delta_\lambda^{-1}(0) \cap V(8))$. Since the stabilizer of $5e_7 + e_9$ in $\mathrm{PSL}_2$ coincides with $N(H)$ and $5e_7 + e_9$ has distinct roots, we have

$$\deg X_2 = \frac{8 \cdot 7 \cdot 6}{|N(H)|} = 14.$$

From the considerations above we obtain the following result.

LEMMA 6.1. $\mathbb{P}(\delta_\lambda^{-1}(0) \cap V(8)) = X_1 \cup X_2$.

For $r \in R$ define

$$L(r) = \overline{\{(x, s) \mid x_4 = r_1 x_1, x_5 = r_2 x_2, x_6 = r_3 x_3\}}.$$

We shall describe $L(r) \cap X_1$ and $L(r) \cap X_2$. Set

$$L_0 = \overline{\{(x, s) \mid x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = 0\}},$$

$$L_1(r) = \overline{\{(x, s) \mid x_1 \neq 0, x_4 = r_1 x_1, x_2 = x_3 = x_5 = x_6 = 0\}},$$

$$L_2(r) = \overline{\{(x, s) \mid x_2 \neq 0, x_5 = r_2 x_2, x_1 = x_3 = x_4 = x_6 = 0\}},$$

$$L_3(r) = \overline{\{(x, s) \mid x_3 \neq 0, x_6 = r_3 x_3, x_1 = x_2 = x_4 = x_5 = 0\}},$$

$$\tilde{L}_1(r) = \overline{\{(x, s) \mid x_2 x_3 \neq 0, x_5 = r_2 x_2, x_6 = r_3 x_3, x_1 = x_4 = 0\}},$$

$$\tilde{L}_2(r) = \overline{\{(x, s) \mid x_1 x_3 \neq 0, x_4 = r_1 x_1, x_6 = r_3 x_3, x_2 = x_5 = 0\}},$$

$$\tilde{L}_3(r) = \overline{\{(x, s) \mid x_1 x_2 \neq 0, x_4 = r_1 x_1, x_5 = r_2 x_2, x_3 = x_6 = 0\}},$$

$$L^0(r) = \overline{\{(x, s) \mid x_1 x_2 x_3 \neq 0, x_4 = r_1 x_1, x_5 = r_2 x_2, x_6 = r_3 x_3\}}.$$

The linear subspace $L(r)$ is the disjoint union of the subsets $L_0$, $L^0(r)$, $L_i(r)$, $\tilde{L}_i(r)$, $i = 1, 2, 3$. For $g \in N(H)$, $r \in R$ we have

$$g \cdot L(r) = L(g \cdot r), \qquad g \cdot L^0(r) = L^0(g \cdot r), \qquad g \cdot L_0 = L_0,$$

$$g \cdot L_j(r) = L_{\kappa(g)(j)}(g \cdot r),$$

where $\kappa \colon N(H) \to S_3$ is the homomorphism given by

$$\kappa(\tau) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \qquad \kappa(\sigma) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

LEMMA 6.2. *There exist an open nonempty $N(H)$-invariant subset $R'' \subset R$ containing* 0 *such that $L(r) \cap \mathbb{P}(\delta_\lambda^{-1}(0) \cap V(8))$ consists of* 32 *points of multiplicity* 1 *for all $r \in R''$ which satisfy the following conditions*:

(1) $\tilde{L}_j(r) \cap X_l = \varnothing$, $1 \le j \le 3$, $1 \le l \le 2$;
(2) $L_0 \cap X_1 = \varnothing$, $|L_0 \cap X_2| = 4$;
(3) $|L_j(r) \cap X_l| = 2$, $1 \le j \le 3$, $1 \le l \le 2$;
(4) $|L^0(r) \cap X_1| = 12$, $|L^0(r) \cap X_2| = 4$.

*Proof.* Set

$$R^0 = \{r \in R \mid 96r_2r_3 - 672r_2 - 672r_3 + 1248 \neq 0,$$
$$6r_1r_3 + 42r_1 + 84r_3 + 156 \neq 0, \ -6r_1r_2 - 42r_1 + 84r_2 + 156 \neq 0\}.$$

From (1.2) it follows that $\tilde{L}_j(r) \cap \mathbb{P}(\delta_\lambda^{-1}(0) \cap V(8)) = \varnothing$ for $r \in R^0$, $1 \leq j \leq 3$. It is sufficient to prove that

(a) $|L^0(0) \cap \mathbb{P}(\delta_\lambda^{-1}(0) \cap V(8))| = 16$,

(b) $L_0 \cap X_1 = \varnothing$, $|L_0 \cap X_2| = 4$, $|L_j(r) \cap X_l| = 2$ $(1 \leq j \leq 3, 1 \leq l \leq 2)$ for $r \in R$.

Equation (a) can be proved by straightforward calculations.

Let us prove (b). Consider $\bar{f} \in (L_1(r) \cup L_0) \cap \mathbb{P}V(8)$. If $(a:b)$ is a root of $f$ of multiplicity $m$, then so is $(a:-b)$. It follows that if $(a:b)$ is a root of $f$ of multiplicity $\geq 6$, then $(a:b) = (1:0)$ or $(a:b) = (0:1)$. Suppose $\bar{f} \in L_0$; then neither $(1:0)$ nor $(0:1)$ is a root of $f$ of multiplicity $\geq 6$. Therefore,

$$L_0 \cap X_1 = \varnothing. \tag{6.1}$$

Suppose $\bar{f} \in L_1(r) \cap X_1$. If $(1:0)$ is a root of $f$ of multiplicity $\geq 6$, then $\bar{f} = \overline{-e_1 - r_1e_4 + r_1e_7 + e_8}$. If $(0:1)$ is a root of $f$ of multiplicity $\geq 6$, then $\bar{f} = \overline{e_1 + r_1e_4 + r_1e_7 + e_8}$. It follows that

$$|L_1(r) \cap X_1| = 2. \tag{6.2}$$

Direct calculations give us

$$L_0 \cap \mathbb{P}(\delta_\lambda^{-1}(0) \cap V(8)) = \{\overline{5e_7 \pm e_8}, \overline{15e_7 \pm 5e_8 - e_9}\}. \tag{6.3}$$

Taking into account (6.1) and (6.3) we obtain

$$|L_0 \cap X_2| = 4.$$

Direct calculations give us

$$L_1(r) \cap \mathbb{P}(\delta_\lambda^{-1}(0) \cap V(8))$$
$$= \{\overline{\pm(e_1 + r_1e_4) + r_1e_7 + e_8}, \overline{\pm(ae_1 + r_1ae_4) + (90 - 5r_1^2)e_7 - 5r_1e_8 + 6e_9}\}, \tag{6.4}$$

where $a^2 = 25(r_1^2 - 36)$. Using (6.2) and (6.4), we get

$$|L_1(r) \cap X_2| = 2.$$

We have

$$\sigma \cdot L_1(r) = L_2(\sigma \cdot r), \qquad \sigma \cdot L_2(r) = L_3(\sigma \cdot r), \qquad \sigma \cdot L_3(r) = L_1(\sigma \cdot r).$$

For $2 \leq j \leq 3$, $1 \leq l \leq 2$, we obtain

$$|L_j(r) \cap X_l| = |(\sigma^{1-j} \cdot L_j(r)) \cap (\sigma^{1-j} \cdot X_l)| = |L_1(\sigma^{1-j} \cdot r) \cap X_l| = 2. \qquad \square$$

COROLLARY. *$L^0(r) \cap X_2$ is an $H$-orbit for $r \in R''$.*

*Proof.* It is clear that the stabilizer of any $\bar{x} \in L^0(r)$ in the group $H$ is trivial. Therefore, any $H$-invariant finite subset of $L^0(r)$ of 4 points is an $H$-orbit. Hence, $L^0(r) \cap X_2$ is an $H$-orbit. $\square$

*Proof of Lemma 5.2.* Set

$$(r, u''(r)) = \pi(X_2 \cap L^0(r)).$$

Statements (1) and (2) of Lemma 5.2 follow from Lemma 6.2 and its Corollary.

Let us prove statement (3) of Lemma 5.2. It can easily be checked that $x^0$ has no root of multiplicity $\geq 6$ (as an element of $V(8)$). From Lemma 6.1 it follows that $\overline{x^0} \in X_2$. We get

$$u''(0) = u''(\pi(\overline{x^0})) = \pi(\overline{x^0}) = (-5/4 : 20 : -20 : 65 : 0 : 13 : 0 : 0 : 0). \qquad \square$$

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