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On perpendicular categories of stones over quiver algebras

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Let $H = k\vec{\Delta}$ be the path algebra of a quiver $\vec{\Delta}$, where k denotes an arbitrary field. $\vec{\Delta}$ will always be assumed to be finite without oriented cycles. We denote by $\operatorname{mod} H$ the category of finite dimensional left modules over H. An indecomposable H-module X is called a stone provided $\operatorname{Ext}^1_H(X, X) = 0$. Note that for a stone X the endomorphism ring $\operatorname{End} X = k$ (see for example part 1).

For a stone X we denote by $X^\perp$ the full subcategory of $\operatorname{mod} H$ of the modules Y with $\operatorname{Hom}_H(X, Y) = 0 = \operatorname{Ext}^1_H(X, Y)$. Dually, ${}^\perp X$ consists of those H-modules Y satisfying $\operatorname{Hom}_H(Y, X) = 0 = \operatorname{Ext}^1_H(Y, X)$. The categories $X^\perp$ and ${}^\perp X$ are called the right respectively left perpendicular category of X. They are equivalent to $\operatorname{mod} A_r$ respectively $\operatorname{mod} A_l$, where A_r and A_l are again finite dimensional, hereditary k-algebras, and both have one simple module less than H (compare [GL] and [S1]). We denote by $\vec{\Delta}_r$ respectively $\vec{\Delta}_l$ the quivers with $k\vec{\Delta}_r = A_r$ and $k\vec{\Delta}_l = A_l$.

Since their introduction to Geigle, Lenzing and Schofield, perpendicular categories have become a powerful tool in the representation theory of finite dimensional algebras. They inherit essential properties from $\operatorname{mod} H$, furnish a reduction procedure and open the possibility for proofs by induction.

In general, it is difficult problem to determine the (right or left) perpendicular category of a given stone X. We briefly recall what is known. It is quite easy to see (Proposition 2.1) that the quiver $\vec{\Delta}_r$ differs from a full subquiver of $\vec{\Delta}$ by an admissible change of orientation if and only if X is preprojective or preinjective over H. Moreover, if H is tame, then there are only finitely many regular stones X over H, and the structure of its perpendicular categories is contained in [Ri1]. Hence we may restrict to the context where H is wild and X is a regular stone. Baer and Strauß proved [B2], [St] that in this situation the perpendicular categories are wild again, but combining results of Xi [X] and Kerner [K3] it follows that certain wild module categories, namely of those algebras classified in [X], do not arise as perpendicular categories of regular stones over wild algebras. On the other hand, investigations of growth numbers in [K3] prove, that for a fixed hereditary, connected, wild algebra with at least three simple modules, there are infinitely many pairwise non isomorphic algebras A_i such that $\operatorname{mod} A_i$ is equivalent to $X^\perp$ for a regular stone X over H.

The main result of the article asserts that up to τ-translation there are only finitely many stones whose perpendicular category is predescribed. This is related to the fact that the automorphism group of the bounded derived category of H is relatively small. More precisely we show.

THEOREM 1. *Let $\vec{\Delta}$ and $\vec{\Sigma}$ be quivers with n respectively $n-1$ vertices and let k be some field. Up to τ-translations there are only finitely many stones X over $k\vec{\Delta}$ with $X^{\perp} \cong \operatorname{mod} k\vec{\Sigma}$.*

We point out that this was first shown in a special case in [Ha1]. The key idea is to show that the result holds for any field if it holds for some field. Hence it is enough, to prove this result for algebras over a finite field. The independence of the field k is shown by exceptional sequences in part 1. In this way we avoid using results from [Ka].

We want to mention an equivalent formulation of the main result.

THEOREM 2. *Let $H = k\vec{\Delta}$ be a path algebra and N a nonnegative integer. Then there are only finitely many regular components containing bricks of quasi-length at least two having selfextensions of dimension N. In particular, there are only finitely many τ-orbits containing regular stones of quasi-length at least two.*

As a consequence of the main result we obtain that there are only finitely many bijections between the set of regular components of a hereditary algebra H and the set of regular components of a hereditary algebra A with one simple module less than H which are induced by tilting modules (for the definition and properties see section 6 below). Following [Ri4] we will call these bijections elementary Kerner-bijections.

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1. Exceptional sequences

In this section it will be shown that the statement of the main result only depends on the quiver $\vec{\Delta}$, and not on the ground field k. The proof is done by exceptional sequences, introduced in [CB1] and [Ri3]. We first recall the definition and central results.

Let k be a field and let H be a connected hereditary k-algebra, not necessarily a path algebra, with n pairwise nonisomorphic simple modules. A sequence $(X_1, \ldots, X_r)$ of stones X_i in $\operatorname{mod} H$ is called an exceptional sequence of length r

if $X_i \in X_j^\perp$ for all $1 \leq i < j \leq r$. An exceptional sequence of length n is called a complete sequence. If $(X_1, \ldots, X_n)$ is a complete sequence and $1 \leq i < n$ is given, then there exist unique stones $l(X_i, X_{i+1})$ and $r(X_i, X_{i+1})$ such that $(X_1, \ldots, X_{i-1}, l(X_i, X_{i+1}), X_i, X_{i+2}, \ldots, X_n)$ and $(X_1, \ldots, X_{i-1}, X_{i+1}, r(X_i, X_{i+1}), X_{i+2}, \ldots, X_n)$ are complete sequences. We denote by B_n the braid group in $n-1$ generators $\sigma_1, \ldots, \sigma_{n-1}$, the assignments $\sigma_i(X_1, \ldots, X_n) = (X_1, \ldots, X_{i-1}, l(X_i, X_{i+1}), X_i, X_{i+2}, \ldots, X_n)$ and $\sigma_i^{-1}(X_1, \ldots, X_n) = (X_1, \ldots, X_{i-1}, X_{i+1}, r(X_i, X_{i+1}), X_i, X_{i+2}, \ldots, X_n)$ define a B_n-action of the set $\mathscr{E}$ of complete sequences in mod H. The main result of [CB1] and [Ri3] says that the braid group B_n acts transitively on $\mathscr{E}$. As a consequence of the proof one gets that for a stone X in mod H there exists a simple H-module S with $\mathrm{End}_H(X) \cong \mathrm{End}_H(S)$, see [Ri3], Theorem 4. The well known corresponding results from Kac [Ka] and Schofield [S2] for path algebras follow from this theorem.

We will also need the Euler bilinear form $\langle -,- \rangle: \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$, see for example [Ri1]. It is characterized by $\langle \mathbf{dim}\, X, \mathbf{dim}\, Y \rangle = \dim_k \mathrm{Hom}_H(X, Y) - \mathbf{dim}_k \mathrm{Ext}^1_H(X, Y)$, for H-modules X, Y and we write $\langle X, Y \rangle$ for $\langle \mathbf{dim}\, X, \mathbf{dim}\, Y \rangle$.

Since a stone X in mod H is determined, up to isomorphism, by its dimension vector $\mathbf{dim}\, X$, see e.g. [K4], 8.2, an exceptional sequence $(X_1, \ldots, X_r)$ in mod H is determined by the sequence $(\mathbf{dim}\, X_1, \ldots, \mathbf{dim}\, X_r)$ in $K_0(H)$. If $(X_1, \ldots, X_n)$ is a complete sequence, then $\mathbf{dim}\, l(X_i, X_{i+1})$ and $\mathbf{dim}\, r(X_i, X_{i+1})$ only depend on $\mathbf{dim}\, X_i$, $\mathbf{dim}\, X_{i+1}$ and (X_r, X_s) for $r, s \in \{i, i+1\}$, see for example [CB1], p. 124 for explicit formulas if $H = k\vec{\Delta}$ is a path algebra. These formulas show that for path algebras $\mathbf{dim}\, l(X_i, X_{i+1})$ and $\mathbf{dim}\, r(X_i, X_{i+1})$ only depend on $\mathbf{dim}\, X_i$, $\mathbf{dim}\, X_{i+1}$ and the quiver $\vec{\Delta}$, not however on the field k. Hence we get

LEMMA 1.1. *Let H be a connected hereditary k-algebra and let $\mathscr{E}_{K_0(H)}$ denote the set of sequences $(\mathbf{dim}\, X_1, \ldots, \mathbf{dim}\, X_n)$ in $K_0(H)$, where $(X_1, \ldots, X_n)$ is a complete sequence in* mod H. *Then B_n acts transitively on $\mathscr{E}_{K_0(H)}$.*

LEMMA 1.2. *Let $(X_1, \ldots, X_n)$ be a complete sequence. Then X_i is simple for $1 \leq i \leq n$ if and only if $\langle X_i, X_j \rangle \leq 0$ for all $i \neq j$.*

Proof. If X_i is simple for $1 \leq i \leq n$ the assertion on the values of the bilinear form trivially holds. For the converse suppose that $\langle X_i, X_j \rangle \leq 0$ for $i < j$. Then by [H1], IV.1.5 we infer that $\mathrm{Hom}(X_i, X_j) = 0$. In particular, $X_1, \ldots, X_n$ are pairwise orthogonal. But then $X_1, \ldots, X_n$ are the simple modules by [Ri3].

The following result was also shown in [CB2], in a slightly different context. At the same time it gives a non-geometric proof that real Schur roots of a quiver are independent of the field k.

PROPOSITION 1.3. *Let $\vec{\Delta}$ be a connected quiver and k, K two fields. Let X be a stone in* $\operatorname{mod} k\vec{\Delta}$ *and* $X^{\perp} \cong \operatorname{mod} k\vec{\Delta}_X$. *Then there exists a unique stone Y in* $\operatorname{mod} K\vec{\Delta}$ *with* $\mathbf{dim}\, X = \mathbf{dim}\, Y$ *and* $Y^{\perp} \cong \operatorname{mod} K\vec{\Delta}_X$.

Proof. There exists a complete exceptional sequence $(X_1, \ldots, X_{n-1}, X_n = X)$ where $X_1, \ldots, X_{n-1}$ are the simple objects in $X^{\perp}$. If $(S_1, \ldots, S_n)$ is a complete sequence in $\operatorname{mod} k\vec{\Delta}$, consisting of simple $k\vec{\Delta}$-modules, then there exists a $\sigma \in B_n$ with $\sigma(S_1, \ldots, S_n) = (X_1, \ldots, X_n)$. If $(S'_1, \ldots, S'_n)$ denotes the corresponding complete sequence of simple $K\vec{\Delta}$-modules, we get by (1.1) $\sigma(S'_1, \ldots, S'_n) = (Y_1, \ldots, Y_{n-1}, Y_n = Y)$, with $\mathbf{dim}\, X_i = \mathbf{dim}\, Y_i$. In particular we have that $\mathbf{dim}\, X = \mathbf{dim}\, Y$. Since $X^{\perp}$ and $Y^{\perp}$ are full exact and extension closed subcategories and the bilinear form only depends on $\vec{\Delta}$ we get that

$$0 \geq \langle X_i, X_j \rangle_{X^{\perp}} = \langle X_i, X_j \rangle_{k\vec{\Delta}} = \langle Y_i, Y_j \rangle_{K\vec{\Delta}} = \langle Y_i, Y_j \rangle_{Y^{\perp}}$$

for $1 \leq i < j \leq n-1$. Thus by 1.2 we have that $(Y_1, \ldots, Y_{n-1})$ are the simple objects in $Y^{\perp}$. Again by [H1], IV.1.5 we have that

$$\dim_K \operatorname{Ext}^1(Y_i, Y_j) = \langle Y_i, Y_j \rangle_{K\vec{\Delta}} = \langle X_i, X_j \rangle_{k\vec{\Delta}} = \dim_k \operatorname{Ext}^1(X_i, X_j).$$

Therefore we infer that $Y^{\perp} \simeq \operatorname{mod} K\vec{\Delta}_X$.

For a connected quiver $\vec{\Delta}$ we denote by $\mathscr{S}(k)$ (*resp.* $\mathscr{R}(k)$) the set of isomorphism classes of stones (resp. regular stones) in $\operatorname{mod} k\vec{\Delta}$. Let k, K be two fields. Then (1.3) implies that there exists a bijection $\phi: \mathscr{S}(k) \to \mathscr{S}(K)$, which preserves dimension vectors. Since a stone X is preprojective (resp. preinjective) if and only if $\Phi^{-m}(\mathbf{dim}\, X) < 0$ (resp. $\Phi^{m}(\mathbf{dim}\, X) < 0$) for some $m > 0$, where Φ denotes the Coxeter transformation. Since the Coxeter transformation only depends on the quiver, ϕ also induces a bijection.

$$\phi: \mathscr{R}(k) \to \mathscr{R}(K).$$

COROLLARY 1.4. *Let $\vec{\Delta}$ be a wild connected quiver and k, K be two fields. If X is a regular stone in* $\operatorname{mod} k\vec{\Delta}$ *of quasi-length s, then $\phi(X)$ is a regular stone of quasi-length s.*

Proof. By [St] we infer that the regular stone X has quasi-length s if and only if $X^{\perp} \simeq \operatorname{mod} k\vec{\Delta}_r$, where $\vec{\Delta}_r$ is a disjoint union of a connected wild quiver and a linearly oriented quiver of Dynkin type A_{s-1}. Hence the result follows from the previous Proposition.

2. Non-regular stones

We keep the notions of the introduction and refer to [Ri1] for unexplained terminology and some representation-theoretic background.

For a hereditary algebra H we denote by $\rho(H)$ its growth number. According to [DR] this is the spectral radius of the Coxeter matrix of H.

Let $H = k\vec{\Delta}$, and let X be a stone over H. As mentioned before, $X^{\perp} = \mathrm{mod}\, A_r$ and ${}^{\perp}X = \mathrm{mod}\, A_l$, where $A_r = k\vec{\Delta}_r$ and $A_l = k\vec{\Delta}_l$, and $\vec{\Delta}_r$ respectively $\vec{\Delta}_l$ have one vertex less than $\vec{\Delta}$.

If X is a sincere stone, i.e. if $\mathrm{Hom}_H(P, X) \neq 0$ for all indecomposable projective H-modules P, then A_r and A_l coincide [U], but it is easy to construct examples where A_r and A_l differ provided X is not sincere.

A full subquiver $\vec{\Delta}'$ of a quiver $\vec{\Delta}$ is a quiver whose vertex set is contained in the vertex set of $\vec{\Delta}$ and the arrows in $\vec{\Delta}'$ and $\vec{\Delta}$ between vertices in $\vec{\Delta}'$ coincide. For a quiver $\vec{Q}$ we denote by Q its underlying graph.

PROPOSITION 2.1. *Let X be a stone over $H = k\vec{\Delta}$. The quiver $\vec{\Delta}_r$ differs from a full subquiver of $\vec{\Delta}$ by an admissible change of orientation if and only if X is preprojective or preinjective.*

Proof. Let $\vec{\Delta}_r$ be a quiver which differs from a full subquiver of $\vec{\Delta}$ by an admissible change of orientation. If H is representation finite, then X obviously is preprojective.

If H is tame, then Δ_r is a disjoint union of Dynkin diagrams, implying that A_r is representation finite. Then X is not regular, for otherwise all indecomposable modules from tubes not containing X belong to $X^{\perp}$.

Hence assume that H is wild. If A_r is tame or representation finite then X is preprojective or preinjective. Namely, if X is regular, then A_r is wild by a result of Baer [B2] and Strauß [St].

Hence we may assume that A_r is wild. It follows from [Ri1], 4.1 that the growth number is invariant under an admissible change of orientation of the quiver. According to [X] this implies that $\rho(A_r) \leq \rho(H)$. Since $\rho(H) < \rho(A_r)$ for a regular module X by a result of Kerner [K3], we obtain that X is preprojective or preinjective.

For the other implication assume the $X = \tau^{-l}P_t$ for an indecomposable projective H-module P_t with top the simple module S_t, which corresponds to the vertex t in Δ. If X is projective, then by definition $A_r = k(\vec{\Delta}\backslash\{t\})$, and $\vec{\Delta}_r$ is actually a subquiver of $\vec{\Delta}$. If X is not projective, then there is a complete slice $\mathscr{S}$ in the preprojective component of mod H with X a sink in $\mathscr{S}$. Let $\oplus_{i=1}^{n-1} U_i \oplus X$ be the direct sum of the modules in $\mathscr{S}$. It is easy to verify that $\oplus_{i=1}^{n-1} U_i$ belongs to $X^{\perp}$, and the direct summands form a complete slice in the preprojective component of mod A_r. Since complete slices are obtained by an admissible change of orientation we obtain the assertion. Similar arguments prove the assertion provided X is preinjective.

Obviously, an analogous result holds for $\vec{\Delta}_l$.

3. Finiteness conditions for tilted algebras

The next result will turn out to be essential in the later developments, but seems to be of independent interest. For unexplained terminology concerning derived categories and the computation of the Auslander-Reiten quiver of $D^b(H)$ we refer to [H1].

PROPOSITION 3.1. *Let $H = k\vec{\Delta}$ be a finite dimensional, connected, hereditary k-algebra for some quiver $\vec{\Delta}$ with n vertices. Let A be a tilted algebra of type H, and let $\mathcal{M} = \{{}_HT \mid {}_HT$ is a tilting module with $\mathrm{End}_H T \simeq A\}$. Then the number of τ_H-orbits in $\mathcal{M}$ is bounded by $3q$, where q is the order of the automorphism group of $\vec{\Delta}$.*

Proof. We first remark that it is easy to see that an automorphism of $D^b(H)$ which induces the identity on $\mathbb{Z}\vec{\Delta}[0]$ is the identity on $D^b(H)$.

Let T and T' be H-tilting modules with $A = \mathrm{End}\, T \simeq \mathrm{End}\, T'$. This yields a triangle equivalence $\sigma\colon D^b(H) \to D^b(H)$ with $\sigma(T) = T'$. In fact, let $F, F'\colon D^b(H) \to D^b(A)$ be the equivalences induced by T respectively T' and let G' be an inverse to F'. We set $\sigma = FG'$. Now σ is a triangle equivalence with $\sigma(H[0]) \cap H[0] \neq 0$.

Since σ is a triangle equivalence, there is a tilting complex $K^\bullet \in D^b(H)$ (compare [R]) such that $\mathrm{End}\, K^\bullet = H$ and $\sigma(K^\bullet) = {}_HH$. In particular, $\mathcal{S} = \mathrm{add}\, K^\bullet$ is a slice in $D^b(H)$ isomorphic to $\mathcal{S}' = \mathrm{add}_H H$. Thus σ induces a quiver isomorphism $\varepsilon_\sigma\colon \mathcal{S} \to \mathcal{S}'$. There are at most q such quiver isomorphisms. Now ${}_HH \in \mathrm{add}\, \mathbb{Z}\vec{\Delta}[0]$ and $\sigma(H[0]) \cap H[0] \neq 0$ shows that $K^\bullet \in \mathrm{add}\, \mathbb{Z}\vec{\Delta}[i] \cap H[i]$ for $i \in \{-1, 0, 1\}$. Thus up to shift in $D^b(H)$ and a quiver isomorphism we may assume that $\mathcal{S} = \mathcal{S}'$ as quivers and both are contained in $\mathbb{Z}\vec{\Delta}[0]$. But then there is an integer j such that $\mathcal{S} = \tau^j \mathcal{S}'$ where τ denotes the translation in $F'(D^b(H))$. Thus σ is of the form $\tau^j \varepsilon_\sigma[i]$ for some integer j and some $i \in \{-1, 0, 1\}$ and ε_σ is a quiver automorphism, hence the assertion.

4. Proof of the main result

We need the notion of a one-point extension algebra. Let A be a finite dimensional k-algebra and M in mod A. The one-point extension algebra $A[M]$ of

A by M is by definition the finite dimensional k-algebra

$$A[M] = \begin{pmatrix} A & M \\ 0 & k \end{pmatrix}$$

with multiplication

$$\begin{pmatrix} a & m \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} a' & m' \\ 0 & \lambda' \end{pmatrix} = \begin{pmatrix} aa' & am' + m\lambda' \\ 0 & \lambda\lambda' \end{pmatrix}$$

where $a, a' \in A$, $m, m' \in M$ and $\lambda, \lambda' \in k$.

The proof of the following lemma is analogous to the proof of Satz 5.2.4 in [L].

LEMMA 4.1. *Let X and Y be regular stones over H. Let P_X and P_Y be the projective generators of $X^{\perp}$ respectively $Y^{\perp}$. Assume that* $\operatorname{End}(X \oplus P_X) = A[M]$ *and* $\operatorname{End}(Y \oplus P_Y) = A[\tau^l M]$ *for some positive integer l. Then* $\operatorname{End}(Y \oplus P_Y) \simeq \operatorname{End}(X \oplus \tau_{X^{\perp}}^{-l} P_X)$.

We start with the proof of theorem 1. As already mentioned it is enough to consider the case that H is wild hereditary. If X is a stone and X_i, $i = 1, 2$ are in the τ-orbit of X with $X_i^{\perp} \cong \operatorname{mod} A_i$, then the hereditary algebras A_1 and A_2 differ at most by an admissible change of orientation. Especially there are infinitely many stones in the τ-orbit of X with equivalent perpendicular categories. We first consider the case where the quiver $\vec{\Sigma}$ is connected.

PROPOSITION 4.2. *Let $H = k\vec{\Delta}$ and $A = k\vec{\Sigma}$ be wild path algebras of connected quivers with n respectively $n-1$ vertices. Up to τ-translations there are only finitely many stones X over H with $X^{\perp} \simeq \operatorname{mod} A$.*

Proof. By Proposition 1.3 it is enough to show the assertion for a finite field. Let I be an infinite index set such that X_i, for $i \in I$ is a family of stones from pairwise different τ_H-orbits with $X_i^{\perp} \simeq \operatorname{mod} A$. It follows from proposition 2.1 that we may assume that all X_i are regular. Since A is connected, all modules X_i are quasi-simple [St], hence we may assume that all X_i are from pairwise different components of the Auslander-Reiten quiver of H.

Let P_{X_i} be the projective generator of $X_i^{\perp}$. Then $X_i \oplus P_{X_i}$ is a tilting module over H for all $i \in I$ [GL], and $\Lambda_i = \operatorname{End}(X_i \oplus P_{X_i})$ is a one-point extension algebra $A[M_i]$ of A by a regular module M_i [St].

If infinitely many of the algebras Λ_i are isomorphic, then by Proposition 3.1 there exist $i \neq j$ such that $X_i \oplus P_{X_i}$ and $X_j \oplus P_{X_j}$ are in the same τ-orbit. We claim that then also X_i and X_j are in the same τ-orbit. This follows easily from the following straightforward fact. Namely, X_i is the unique indecomposable direct summand of $X_i \oplus P_{X_i}$ such that there are infinitely many pairwise non-isomorphic indecomposable modules Y with $\operatorname{Ext}^1(X_i, Y) = 0$ and $\operatorname{Hom}(Y, X_i) \neq 0$.

Next we use that the modules M_i are elementary over A [KL] 5.3, [L] 5.3.1.

Denote by Φ the Coxeter matrix of A. For a regular module M we therefore have $\mathbf{dim}\, \tau_A M = \Phi(\mathbf{dim}\, M)$, see [Ri1]. It is shown in [L], 4.2.1 and [KL], 2.1 that there are only finitely many Φ-orbits of dimension vectors of elementary A-modules. Since k is finite there are only finitely many pairwise nonisomorphic modules M with the same dimension vector $\mathbf{dim}\, M$. Hence for a finite field k there are only finitely many τ-orbits of elementary modules. Therefore the elementary modules M_i are concentrated in finitely many τ_A-orbits. Thus there is an infinite index set $J \subset I$, for simplicity we assume that $J = \mathbb{N}$, such that all modules M_i, $i \in \mathbb{N}$ are pairwise non isomorphic and lie in the same τ_A-orbit $\mathcal{O}$.

Let $N \in \mathcal{O}$ be such that $A[N]$ is a tilted algebra. It is shown in [L], 5.3.2 and [KL], 5.4 that there is a positive integer t such that $A[\tau^{-l}N]$ is not a tilted algebra for all $l \geq t$, and that $A[\tau^r N]$ is a tilted algebra of type H for all $r > 0$. Hence we may assume that the modules M_i are ordered such that $\Lambda_i = A[M_i]$ and $M_{i+1} = \tau^{r_i} M_i$ for some $r_i > 0$. In particular, $\Lambda_i = A[M_i] = A[\tau^{l_i} M_1]$ where $l_i = \Sigma_{j=1}^{i-1} r_j$ for all $i > 1$. Let $m = 3q$, where q is the order of the automorphism group of $\vec{\Delta}$, and consider the algebra Λ_{m+1}. Then

$$\Lambda_{m+1} = A[M_{m+1}] = A[\tau^{r_m} M_m] = A[\tau^{r_m + r_{m-1}} M_{m-1}] = \cdots = A[\tau^{l_{m+1}} M_1].$$

Using the lemma above we obtain that

$$\begin{aligned} \Lambda_{m+1} = \operatorname{End}(X_{m+1} \oplus P_{X_{m+1}}) &= \operatorname{End}(X_m \oplus \tau_{X_m^\perp}^{-r_m} P_{X_m}) \\ &= \cdots = \operatorname{End}(X_1 \oplus \tau_{X_1^\perp}^{-l_{m+1}} P_{X_1}). \end{aligned}$$

Hence Λ_{m+1} is the endomorphism ring of $m+1$ tilting modules over H. Proposition 3.1 states that at least two of these tilting modules lie in the same τ_H-orbit, in particular, there are $1 \leq i, j \leq m+1$ with $i \neq j$, and X_i and X_j are in the same τ_H-orbit. This contradicts the choice of the modules X_i.

Next we will concentrate on the equivalence of Theorem 1 and Theorem 2 mentioned in the introduction.

LEMMA 4.3. *Let $\vec{\Delta}$ be a wild quiver and k some field. Then the following two statements are equivalent.*

(*i*) *If A is a wild path algebra then there exist up to τ-translation only finitely many quasi-simple stones over $k\vec{\Delta}$ with $X^{\perp} \simeq \operatorname{mod} A$.*
(*ii*) *Given a nonnegative integer N there are only finitely many regular components containing bricks of quasi-length 2 having self-extensions of dimension N.*

Proof. We start by observing the following. If X is a quasi-simple regular stone and $0 \to X \to Z \to \tau^{-}X \to 0$ is the Auslander-Reiten sequence starting at X, then it is shown in [H2] that $\dim H^1(A) = \dim H^1(k\vec{\Delta}) + \dim \operatorname{Ext}^1(Z, Z)$ where $X^{\perp} = \operatorname{mod} A$ and $H^1(A)$ denotes the first Hochschild cohomology group. It is easy to see that for a fixed m and a fixed number of simple modules m' there are only finitely many hereditary algebras A with m' simple modules and $\dim H^1(A) = m$, compare [H2]. The assertion now follows from the fact that X is a stone if and only if Z is a brick.

It follows from proposition 4.2 and lemma 4.3 that the following holds.

PROPOSITION 4.4. *Let $\vec{\Delta}$ be a wild quiver and k some field. Then given a nonnegative integer N there are only finitely many regular components containing bricks of quasi-length 2 having self-extensions of dimension N.*

COROLLARY 4.5. *Let $\vec{\Delta}$ be a wild quiver and k some field. Then there exist only finitely many τ-orbits of regular stones of quasi-length at least two and only finitely many τ-orbits of bricks of quasi-length at least three.*

Proof. Choosing $N = 0$ in 4.4. It follows that there are only finitely many regular components, containing stones of quasi-length two. Let $\mathscr{C}$ be a regular component and let X be quasi-simple in $\mathscr{C}$. If $X = X(1) \to X(2) \to X(3) \to \cdots$ is a sectional path of irreducible monomorphisms, then $X(r)$ is a stone if and only if $X(r+1)$ is brick, see [K2]. Moreover, if $X(r)$ is a stone for some $r > 1$, then also $X(r-1)$ is a stone. By [Ho] we infer that $r < n - 1$. Hence the general statement follows from the first case.

(4.5) finishes the proof of theorem 2. Also theorem 1 now follows from (4.2) and (4.5). If X is a stone with $X^{\perp} \cong \operatorname{mod} k\vec{\Sigma}$ and $\vec{\Sigma}$ is not connected, then X is either preprojective of preinjective or X is regular of quasi-length at least two. Hence X is in one of finitely many distinguished τ-orbits.

5. Perpendicular categories of decomposable modules

Let $H = k\vec{\Delta}$ and assume that the quiver $\vec{\Delta}$ has n vertices. Let X be a direct summand of a multiplicity free tilting module over H. As above we may define the right perpendicular category $X^{\perp}$ of X. Let $X = \bigoplus_{i=1}^{r} X_i$. Then $X^{\perp} \simeq \operatorname{mod} A$ where $A = k\vec{Q}$ is a finite dimensional hereditary k-algebra and the quiver $\vec{Q}$ has $n - r$ vertices.

In this section we show that the assertion of Theorem 1 is false if we drop the assumption that X is indecomposable.

Let $n \geq 3$ and choose a quasi-simple regular stone Y over H. Since a regular component contains only finitely many non sincere modules [B1], we may assume that $\tau^l Y$ is sincere for all $l \geq 0$. Then $Y^{\perp}$ does not contain an indecomposable preprojective H-module [S1], [U]. Let $Y^{\perp} \simeq \operatorname{mod} H'$ with $H' = k\vec{\Delta}'$. According to [K3] we have that $\rho(H) < \rho(H')$. Let P be an indecomposable direct summand of the projective generator of $Y^{\perp}$. It is easy to see that $Y \oplus \tau_{Y^{\perp}}^{-l} P$ has no selfextensions for all $l \geq 0$, hence it is a direct summand of a tilting module. It follows from proposition 2.1 that $(Y \oplus \tau_{Y^{\perp}}^{-l} P)^{\perp} = \operatorname{mod} A$ for infinitely many l, where $A = k\vec{Q}$ and Q is a subgraph of Δ'.

Since each τ_H-orbit of an indecomposable H-module contains only finitely many indecomposable modules in $Y^{\perp}$, infinitely many of the modules $\tau_{Y^{\perp}}^{-j} P$ lie in pairwise different τ_H-orbits.

6. Elementary Kerner-bijections

Let H be a wild hereditary algebra. We denote by $\Omega(H)$ the set of the regular components of the Auslander-Reiten quiver of H.

In [K2], Kerner proved that there are certain bijections $\Omega(H) \to \Omega(C)$ for any two wild hereditary algebras H and C. We call these bijections Kerner-bijections and refer for the definition and properties to [K2] and [Ha2].

Kerner-bijections are built up by finitely many maps, which are induced by a functor, and which we call elementary Kerner-bijections. They are defined as follows:

Let H be a wild hereditary algebra, and let T be a tilting module without preinjective direct summands. According to Strauß [St], we may decompose T into $T' \oplus T''$ such that

(i) $(T'')^{\perp}$ is equivalent to the module category of a connected, wild, hereditary algebra C, and
(ii) T' is preprojective in $(T'')^{\perp}$, and
(iii) the preprojective component of the algebra $C' = \operatorname{End}(T')$ is a full component of the Auslander-Reiten quiver of End T. It is the unique preprojective component of End T.

We call such a decomposition of T a Strauß decomposition. We point out that such a decomposition is unique.

Given this situation, the algebra C' is a wild concealed algebra. We say that C' is dominated by the algebra H. Let $C' = \operatorname{End} T'$ be dominated by H. An elementary

Kerner-bijection $\eta_T\colon \Omega(H) \to \Omega(C')$ is induced by a map κ_T which maps regular H-modules to regular C'-modules via $\kappa_T(M) = \tau_{C'}^{r(M)} \tau_{\mathrm{End}T}^{-2r(M)} \operatorname{Hom}_H(T, \tau_H^{r(M)} M)$ for some natural number $r(M) \gg 0$. (We refer for details to [CBK] or [Ha2].) Then $\kappa_T(M) \simeq \tau_{C'}^{r'} \tau_{\mathrm{End}T}^{-2r'} \operatorname{Hom}_H(T, \tau_H^{r'} M)$ for all $r' \geq r(M)$ [K2].

LEMMA 6.1. *Let H be wild, hereditary, and let $T = T' \oplus T''$ be the Strauß decomposition of a tilting module T over H. Let $S = \tau_{(T'')^\perp}^- T' \oplus T''$. Then $\eta_S = \eta_T$.*

Proof. Let M be the projective generator of $(T'')^\perp$, and let $(T'')^\perp \simeq \operatorname{mod} C$. According to [Ha1], we have that $\eta_T = \eta_P \eta_{T_0}$ where $T_0 = M \oplus T''$ and $P = \operatorname{Hom}_H(M, T')$. Then $\eta_S = \eta_{P'} \eta_{T_0}$ with $P' = \operatorname{Hom}_H(M, \tau_{(T'')^\perp}^- T') = \tau_C^- P$.

It follows from the definition of κ_T respectively η_T that $\eta_T = \eta_{\tau_H^- T}$ for all tilting modules T without preinjective direct summand. Hence $\eta_P = \eta_{P'}$, implying that $\eta_T = \eta_S$.

COROLLARY 6.2. *Let H be a wild hereditary algebra, and let X be a partial tilting module over H. Then there are only finitely many elementary Kerner-bijections η_T with Strauß decomposition $T = T' \oplus T''$ and $X \simeq T''$.*

Proof. Let $(T'')^\perp \simeq \operatorname{mod} C$ be as above. It follows from the fact, that a preprojective component contains only finitely many non sincere modules, that up to τ_C-shift there are only finitely many tilting modules over C which are preprojective. The assertion now follows from the lemma above.

Combining our main result with the investigations above, we obtain:

PROPOSITION 6.3. *Let $H = k\vec{\Delta}$ be a wild quiver algebra, where $\vec{\Delta}$ has n vertices. Let $C = k\vec{Q}$ be wild and assume that $\vec{Q}$ has $n-1$ vertices. Then there are only finitely many elementary Kerner-bijections $\eta_T\colon \Omega(H) \to \Omega(C)$.*

Proof. Let T be a tilting module over H without preinjective direct summand and with Strauß decomposition $T = T' \oplus T''$. Let $(T'')^\perp \simeq \operatorname{mod} C$. The module T' is a preprojective tilting module over $(T'')^\perp$. According to our main result, there are up to τ_H-shift only finitely many stones X with $X^\perp = \operatorname{mod} C$. Moreover, for every such X there are only finitely many elementary Kerner-bijections η_T with $T = T' \oplus T''$ and $X \simeq T''$. Since $\eta_T = \eta_{\tau_H^- T}$, we obtain the desired result.

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