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# Projective and Hilbert modules over group algebras, and finitely dominated spaces

BENO ECKMANN

## 0. Introduction

The objective of this note is to consider  $\ell_2$ -Betti numbers for FP-complexes and finitely dominated spaces, and to compare the corresponding Euler characteristic  $\bar{\chi}$  with the ordinary  $\chi$ . The results depend on the validity of the Bass conjectures for the group G involved. We first outline the contents.

- **0.1.** Let G be a countable group, P a finitely generated projective  $\mathbb{Z}G$ -module, and  $\ell_2G$  the Hilbert space of square-integrable real functions on G with left and right G-action by translation. Then  $P^{(2)} = \ell_2 G \otimes_G P$  is a Hilbert-G-module. We write rkP for the "naive" rank  $dim_{\mathbb{R}} \mathbb{R} \otimes_G P$  of P. We will show (Theorem 1 in Section 3) that  $P^{(2)}$  is a "free" Hilbert-G-module, i.e., a direct sum of copies of  $\ell_2G$  their number being = rkP, provided (\*) G fulfills the strong Bass conjecture, or G is residually finite. The Bass conjectures are explained in Section 2 (for more details we refer to Bass' fundamental paper [B]).
- **0.2.** A complex  $P_*$  is said of type FP, or an FP-complex, if it is of finite length n>0 ( $P_i=0$  for i>n and i<0) and if all  $P_i$  are finitely generated projective  $\mathbb{Z}G$ -modules; it is said of type FF if the  $P_i$  are free. G is always assumed to be finitely generated. In the FF-case  $\ell_2 G \otimes_G P_*$  is a complex of free Hilbert-G-modules. Under condition (\*) this is also the case for an FP-complex, and the rank of  $P_i^{(2)}$  is  $= rk P_i$ ; one has (reduced) homology groups,  $\ell_2$ -Betti numbers  $\bar{\beta}_i$ , and an  $\ell_2$ -Euler characteristic  $\bar{\chi}$  which is equal to the ordinary homological Euler characteristic  $\chi$  (Section 1, and Theorem 2 in Section 3), all depending only on the chain homotopy type of  $P_*$ .

More generally,  $w(P_*) = \Sigma_0^n (-1)^i [P_i] \in K_0(\mathbb{Z}G)$  is the Wall obstruction of  $P_*$ , where  $[P_i]$  is the class of  $P_i \in K_0(\mathbb{Z}G)$ . Then  $rk \ w(P_*)$  is the Euler characteristic, and  $\tilde{w}(P_*)$  corresponding to  $w(P_*)$  in  $\tilde{K}_0(\mathbb{Z}G)$  is the finiteness (or rather freeness) obstruction. Thus our result tells that  $\ell_2 G \otimes_G$ - annihilates  $\tilde{w}$  and leaves  $\chi$  unchanged.

- **0.3.** In topology FP-complexes occur in connection with a finitely dominated connected space X. The chain complex of the universal cover  $\tilde{X}$  is chain homotopy equivalent to an FP-complex  $P_*$  over  $\mathbb{Z}G$  where G is the fundamental group of X; it is finitely presented. One can write  $w(X) = w(P_*)$ ;  $rk \ w(X)$  is the Euler characteristic of X and  $\tilde{w}(X)$  the Wall finiteness obstruction of X. Under condition (\*) on G one has  $\bar{\chi}(X) = \chi(X)$ , and the  $\ell_2$ -finiteness obstruction of X is 0.
- **0.4.** If a complex  $P_*$  fulfills Poincaré duality of dimension n  $(PD^n)$  then it is an FP-complex, and its  $\ell_2$ -Betti numbers also fulfill  $PD^n$  (Section 4). This applies to finitely dominated spaces, but also to  $PD^n$ -groups (which are finitely generated, but need not be finitely presented).

# 1. FP-complexes and $\ell_2$ -Betti numbers

1.1. We recall that a space X is said to be finitely dominated, cf. [M], if it is a retract, in the homotopy category, of a finite CW-complex. Such a space is homotopy equivalent to a (finite dimensional, in general infinite) CW-complex. In the following "space" will always mean connected CW-complex.

Let G be the fundamental group of the space X, and  $\widetilde{X}$  the universal cover. The cellular chain complex  $C_*\widetilde{X}$  is a complex of free  $\mathbb{Z}G$ -modules. If X is finitely dominated then  $C_*\widetilde{X}$  is  $\mathbb{Z}G$ -chain homotopy equivalent to a complex  $P_*$  of type FP; i.e.,  $P_*$  has finite length n ( $P_i = 0$  for i < 0 and i > n) and all  $P_i$  are finitely generated projective  $\mathbb{Z}G$ -modules. Moreover G is finitely presentable (and for such G the converse holds: if  $C_*\widetilde{X}$  is of type FP then X is finitely dominated).

Most of our arguments deal in fact just with FP-complexes  $P_*$  over a group ring  $\mathbb{Z}G$ . Finite presentability of G is in general irrelevant.

1.2. Let  $\beta_i X = \dim_{\mathbb{R}} H_i(X; \mathbb{R})$  be the *i*-th Betti number of X. If X is finitely dominated the  $\beta_i X$  are finite and the homological Euler characteristic  $\chi X$  is defined as  $\Sigma_0^n (-1)^i \beta_i X$ . Homology can be computed from any  $P_*$  equivalent to  $C_* \tilde{X}$  through  $\mathbb{R} \otimes_G P_*$ . We write  $rk P_i = \dim_{\mathbb{R}} \mathbb{R} \otimes_G P_i$ . The usual Euler-Poincaré argument then yields

$$\chi X = \sum_{0}^{n} (-1)^{i} rk P_{i}.$$

If X itself is a finite CW-complex then we can take  $P_* = C_* \tilde{X}$  (all  $P_i$  are finitely generated free over  $\mathbb{Z}G$ , of rank =  $rk P_i$ ; such a  $P_*$  is said to be of type FF). We recall that, in that case,  $rk P_i$  is equal to the number  $\alpha_i$  of *i*-cells of X. Thus

 $\chi X = \Sigma_0^n (-1)^i \alpha_i$  becomes the ordinary "combinatorial" Euler characteristic of the finite cell-complex X.

1.3. For a finite CW-complex X (or an FF-complex  $P_*$ ) it is well-known that  $\chi X$  can be expressed by  $\ell_2$ -Betti numbers  $\bar{\beta}_i X$  as  $\chi X = \bar{\chi} X = \Sigma_0^n (-1)^i \bar{\beta}_i X$ . For  $\ell_2$ -homology and  $\ell_2$ -Betti numbers we refer, e.g., to [Lü] and references given there. Here we just recall the main facts.

Let  $\ell_2 G$  be as in 0.1. The complex  $\ell_2 G \otimes_G P_*$  consists of "free" Hilbert-G-modules  $\ell_2 G \otimes_G P_i = \ell_2 G \oplus \cdots \oplus \ell_2 G$ ,  $\alpha_i$  terms, also written  $\ell_2 G^{\alpha_i}$ , with the induced boundary operators  $\partial$ . The reduced homology group  $\overline{H}_i \widetilde{X} = \overline{H}_i P_* = \{ \text{cycles in } \ell_2 G \otimes_G P_i \} / \text{closure of } \partial (\ell_2 G \otimes_G P_{i+1}) \text{ is a Hilbert-}G\text{-module, since it can be imbedded in } \ell_2 G^{\alpha_i} \text{ (as orthogonal complement of the above closure in the cycle space). Its von Neumann dimension <math>\dim_G \overline{H}_i \widetilde{X}$  is  $\overline{\beta}_i X$ ; it is a homotopy invariant of X. Since  $\dim_G \ell_2 G \otimes_G P_i = \alpha_i$ , and since the von Neumann dimension behaves as a rank (although it is a real number  $\geq 0$ ) the Euler-Poincaré argument applied to  $\ell_2 G \otimes_G P_*$  yields

$$\bar{\chi}X = \sum_{i=0}^{n} (-1)^{i} \bar{\beta}_{i} X = \sum_{i=0}^{n} (-1)^{i} \alpha_{i}$$

whence  $\bar{\chi}X = \chi X$ .

1.4. In order to apply the same procedure to the general case of a finitely dominated space X (an FP-complex  $P_*$ ) we look closer at  $\ell_2 G \otimes_G P$  where P is a finitely generated projective  $\mathbb{Z}G$ -module. It is clearly a Hilbert-G-module since it imbeds in a free one, namely in  $\ell_2 G \otimes_G F$  where F is a finitely generated free  $\mathbb{Z}G$ -module containing P as a direct summand.

It turns out (Theorem 1 in Section 3) that if G fulfills the Bass conjecture or if G is residually finite then  $\ell_2 G \otimes_G P$  is isometrically G-isomorphic to  $\ell_2 G^k$  where  $k = rk P = \dim_{\mathbb{R}} \mathbb{R} \otimes_G P$ . (The residually finite case has already been treated by Lück [Lü]; his approach is slightly different from ours where the residually finite case is a corollary of the strong Bass case.)

Now the arguments of 1.3 apply to  $\ell_2 G \otimes_G P_*$  and to its  $\ell_2$ -Betti number  $\bar{\beta}_i X$  except that  $\alpha_i$  is replaced by  $rk P_i$ 

$$\bar{\chi}X = \sum_{0}^{n} (-1)^{i} \bar{\beta}_{i} X = \sum_{0}^{n} (-1)^{i} rk P_{i}$$

whence, according to 1.1,  $=\chi X$ . Similarly for an arbitrary FP-complex.

1.5. The advantage resulting from  $\bar{\chi}X$  instead of the classical  $\chi X$  is due to the fact that for the  $\bar{\beta}_i X$  certain vanishing theorems are known. They imply properties of  $\chi X$  or of the  $\beta_i X$ . Some immediate applications are given in Section 4: To Poincaré duality spaces, to groups of type FP, to groups fulfilling  $PD^2$  (Poincaré duality of formal dimension 2).

# 2. Rank concepts for projective modules

- **2.1.** In this section we recall the various rank concepts and their relations (Bass conjectures) since we will make essential use of them. P will always denote a finitely generated projective  $\mathbb{Z}G$ -module, G an arbitrary group.
- (a) The "homological" or "naive" rank  $rk P = \dim_{\mathbb{R}} \mathbb{R} \otimes_G P$  is being used for the definition of homology, Betti numbers, and Euler characteristic. We note that it is not clear a priori whether rk P = 0 implies P = 0.
- (b) The "Kaplansky" rank  $\kappa P$ . It is defined as follows: Let F be a finitely generated free  $\mathbb{Z}G$ -module containing P as a direct summand. In a basis of F, consisting of m elements, the endomorphism f of  $\mathbb{Z}G^m$  which projects F onto P is given by an  $m \times m$  matrix  $M_P$  with  $\mathbb{Z}G$ -entries.  $\kappa P$  is the coefficient of  $1 \in G$  of trace  $M_P$ ; it is easily seen to be independent of the choice of F and of the basis.

Another way of describing f is by a matrix  $(f_{ij})$ ,  $f_{ij}$  being the  $\mathbb{Z}G$ -endomorphism of  $\mathbb{Z}G$  determined by  $f_{ii}(1) = a_{ii} \in \mathbb{Z}G$ .

(c) The "Hattori-Stallings" rank  $r_P$ . The trace  $\in \mathbb{Z}G$  of the matrix  $M_P$  above depends on the choices. However, if one passes from  $\mathbb{Z}G$  to  $\overline{\mathbb{Z}G} = \mathbb{Z}G/\{xy - yx\}$  where  $\{xy - yx\}$  is the submodule of  $\mathbb{Z}G$  generated by all additive commutators (it suffices to take xy - yx for all  $x, y \in G$ ) then the value of trace  $M_P$  in  $\overline{\mathbb{Z}G}$ , denoted by  $r_P$ , is independent of the choices.  $\overline{\mathbb{Z}G}$  is obtained from  $\mathbb{Z}G$  by identifying conjugate elements of G; it is thus the free Abelian group generated by the conjugacy classes [x] of G. One can write  $r_P$  as

$$r_P = \sum r_P(x)[x]$$

where x is an arbitrary representative of [x].

**2.2.** One immediately notes that  $r_P(1) = \kappa P$ . Moreover,  $\Sigma_x r_P(x) = rk P$ . Indeed  $\Sigma r_P(x)$  is the sum of all coefficients in trace  $M_P \in \mathbb{Z}G$ , i.e., it is induced by the augmentation  $\varepsilon \colon \mathbb{Z}G \to \mathbb{Z}$ ; and  $\varepsilon$  turns P into the free Abelian group  $\mathbb{Z} \otimes_G P$ . By naturality  $\varepsilon_* r_P$  is the rank, with respect to  $\mathbb{Z}$ , of  $\mathbb{Z} \otimes_G P$  which is = rk P.

The Strong Bass Conjecture (SB) claims that all  $r_P(x)$ ,  $x \neq 1$ , are 0. It implies that  $\kappa P = rk P$ . That equality is the Weak Bass Conjecture (WB). It has been

proved quite generally by Linnell [Li, p. 96] that  $r_P(x)$  vanishes on all elements  $x \neq 1$  of finite order.

- (SB) has been established for various classes of groups. We mention the following:
  - 1) Finite groups (Swan's Theorem [S])
  - 2) Linear groups ([B], [E])
  - 3) Negatively curved groups
  - 4) Solvable groups [E]
  - 5) Groups of cohomology dimension  $\leq 2$  over  $\mathbb{R}$  ([E])

As for 3), this does not seem to be explicitly in the literature. It follows from the method used in [E] and the fact that in such a group the centralizer of an element of infinite order is virtually infinite cyclic.

(WB) holds for all residually finite groups [B], but it is not known whether these fulfill (SB).

**2.3.** We introduce an adhoc *notation*  $c_x(\alpha) = \text{coefficient of } x \in G \text{ in } \alpha \in \mathbb{Z}G \text{ (or } \mathbb{R}G, \text{ or } \ell_2G) \text{ to be used in this and the next sections.}$ 

As a consequence of the definitions one has

PROPOSITION 1. The von Neumann dimension  $\dim_G(\ell_2 G \otimes_G P)$  is equal to  $\kappa P$ .

*Proof.* We recall that the von Neumann dimension of a Hilbert-G-module  $H \subset \ell_2 G^m$  is defined as "trace" of the projection operator  $\varphi$  of  $\ell_2 G^m$  having H as image.  $\varphi$  is given by an  $m \times m$  matrix  $(\varphi_{ij})$  with  $\varphi_{ij} \in NG$ , the ring of bounded G-equivariant operators in  $\ell_2 G$  (the von Neumann algebra of G); "trace" is to be understood as  $c_1 \sum_{i=1}^m \varphi_{ii}(1)$ . If  $H = \ell_2 G \otimes_G P$  then  $(\varphi_{ij})$  is the same as the matrix  $M_P = (f_{ij}), f_{ij}(1) = a_{ij} \in \mathbb{Z}G$ ; indeed the endomorphism ring of  $\mathbb{Z}G$  is naturally imbedded in NG. Thus the two "traces"  $c_1(\Sigma f_{ii}(1)) = c_1(\Sigma \varphi_{ii}(1))$  are the same.

# 3. Structure of $\ell_2 G \otimes_G P$

- 3.1. If G fulfills (SB) or is residually finite it follows from Proposition 1 that  $\dim_G \ell_2 G \otimes_G P = rk P$ . In this section we will show that  $\ell_2 G \otimes_G P$  and  $\ell_2 G^{rk P}$  are even isometrically and G-equivariantly isomorphic. We need some preparations.
- 3.2. Let CG be the center of  $\mathbb{R}G$ ; it has a basis consisting of all sums  $\sigma_x = \sum_{y \in [x]} y$  for the *finite* conjugacy classes in G.

PROPOSITION 2. Let P be a finitely generated projective  $\mathbb{Z}G$ -module,  $\zeta$  an element of CG. If the Hattori-Stallings ranks  $r_P$  vanishes on finite conjugacy classes  $[x], x \neq 1$  then

$$c_1(\zeta \text{ trace } M_p) = c_1(\zeta)\kappa P.$$

*Proof.* It suffices to prove that  $c_1$  ( $\sigma_x$  trace  $M_P$ ) = 0 for  $x \neq 1$ . Now

$$c_1(\sigma_x \text{ trace } M_P) = \sum_{y \in [x]} c_1(y \text{ trace } M_P)$$

$$= \sum_{y \in [x]} c_{y^{-1}}(\text{trace } M_P)$$

$$= \sum_{y \in [x^{-1}]} c_y(\text{trace } M_P) = r_P(x^{-1}) = 0.$$

The assumption of Proposition 2 holds if G fulfills (SB), and then  $\kappa P = rk P$ . The result also holds if G is residually finite: We choose a normal subgroup N of finite index in G which does not contain the elements of  $\sigma_x$ , of trace  $M_P$ , nor of their product. Passing from  $\mathbb{Z}G$  to  $\mathbb{Z}(G/N)$  we get a projective  $\mathbb{Z}(G/N)$ -module  $P' = \mathbb{Z}(G/N) \otimes_G P$  and an element  $\sigma'_x \in C(G/N)$ . The coefficient  $c_1$  of  $\sigma_x$ , of trace  $M_P$ , and of the product remains unchanged. By Swan's theorem [S], G/N fulfills (SB); therefore

$$r_P(x^{-1}) = c_1(\sigma_x \text{ trace } M_P) = c_1(\sigma_x' \text{ trace } M_{P'}) = r_{P'}(x^{-1}) = 0$$

if  $x \neq 1$ . Moreover  $\kappa P = \kappa P' = rk P'$  and

$$rk \ P' = \dim_{\mathbb{R}}(\mathbb{R} \otimes_{G/N} P') = \dim_{\mathbb{R}}(\mathbb{R} \otimes_{G/N} \mathbb{Z}G/N \otimes_{G} P)$$
$$= \dim_{\mathbb{R}}(\mathbb{R} \otimes_{G} P) = rk \ P.$$

COROLLARY 2'. If G is residually finite then

$$c_1(\zeta \text{ trace } M_P) = c_1(\zeta)rk P.$$

3.3. We assume that  $r_P$  vanishes on finite conjugacy classes [x],  $x \neq 1$  and show that for the two Hilbert-G-modules  $\ell_2 G \otimes_G P$  and to  $\ell_2 G^{\kappa P}$  the center-valued trace (cf. [KR]),  $ctr \in$  center of NG, applied to the respective  $\Sigma \varphi_{ii}$  is the same. This implies that they are isometrically G-isomorphic.

For  $\ell_2 G \otimes_G P$  one has  $ctr(\Sigma \varphi_{ii}) = ctr(trace M_P)$  which lies in CG since  $ctr x = (1/card[x]) \Sigma_{y \in [x]} y$  if [x] is finite, and 0 otherwise. Of course for  $\ell_2 G^{\kappa P}$ , the

ctr of the matrix is  $\kappa P \cdot E$ ,  $E = identity \in NG$ . Thus

$$\xi = ctr\bigg(\sum \varphi_{ii} - \kappa P \cdot E\bigg)$$

is an element of CG, and so is its adjoint  $\xi^*$  in NG.

Since the 1-coefficient of the central-valued trace fulfills quite generally  $c_1(ctr \varphi) = c_1 \varphi(1), \varphi \in NG$ , we have

$$c_1(\xi^*\xi) = c_1(\xi^*(\text{trace } M_P - \kappa P));$$

by Proposition 2 this is equal to  $c_1(\xi^*)c_1(\text{trace } M_P - \kappa P) = 0$ . Since  $c_1(\xi^*\xi) = \langle \xi^*\xi, 1 \rangle = \langle \xi, \xi \rangle$  in  $\ell_2 G$ , it follows that  $\xi = 0$ :

PROPOSITION 3. If  $r_P$  vanishes on finite conjugacy classes [x],  $x \neq 1$  then  $\ell_2 G \otimes_G P$  and  $\ell_2 G^{\kappa P}$  are isometrically G-isomorphic.

By Corollary 2' the same arguments work in the residually finite case. We summarize the important cases as follows.

THEOREM 1. If G fulfills (SB), or if G is residually finite, then  $\ell_2 G \otimes_G P$  is isometrically G-isomorphic to  $\ell_2 G^{rk P}$ .

# 3.4. As stated in Section 1.4 this implies

THEOREM 2. If G fulfills (SB), or if G is residually finite, then for any finitely dominated space X with fundamental group G, or for any FP-complex  $P_*$  over  $\mathbb{Z}G$ , the  $\ell_2$ -Euler characteristic  $\bar{\chi}$  and the ordinary Euler characteristic  $\chi$  coincide.

3.5. An FP-complex  $P_*$  over  $\mathbb{Z}G$  is always chain homotopy equivalent to a complex where the  $P_i$  are all free except (possibly) for the top module  $P_n$ . Then for  $\ell_2 G \otimes_G P_i$  to be free of rank = rk  $P_i$ , i < n, it is not necessary to apply Theorem 1, i.e., to assume (SB) or G residually finite.

We recall that for  $P_*$  chosen as above the class of  $P_n$  in  $\tilde{K}_o(\mathbb{Z}G)$  is, up to sign, the "Wall obstruction"; it is 0 if and only if  $P_*$  is equivalent to an FF-complex. In that sense Theorem 1 tells that, under the appropriate assumption the  $\ell_2$ -Wall obstruction always vanishes.

*Remark*. We have worked throughout with homology. Everything could also be carried through in cohomology, based on  $Hom_G(P_*, \ell_2 G) = P_*^{\text{dual}} \otimes_G \ell_2 G$ ;  $P_i^{\text{dual}} = Hom_G(P_i, \mathbb{Z}G)$  is again finitely generated projective. We note that in case  $P_i$  is free,

 $P_i^{\text{dual}}$  can be identified with  $P_i$ , so that  $\bar{H}^i = \bar{H}_i$ . But this also holds in the general projective case. Indeed the projection matrix for  $P_i^{\text{dual}}$  can be taken to be the transposed of  $M_{P_i}$ ; thus the trace and the various ranks are the same.

## 4. Applications

# 4.1. Groups of type FP

These are groups G for which there exists a resolution  $P_* \to \mathbb{Z}$  over  $\mathbb{Z}G$  with  $P_*$  of type FP. We assume that the cohomology dimension cdG, the minimal length n of  $P_*$ , is  $\geq 2$  (if cdG = 1 then G is finitely generated free). It is not known whether such a group is necessarily of type FF; cf. [Br] for a discussion of that problem.

G being infinite one notes that, without further assumptions on G,  $H^o(G; \ell_2 G) = \ell_2 G^G = 0$  (an element of  $\ell_2 G$  cannot be invariant unless it is 0). Then  $\bar{H}^0 G = 0$  and  $\bar{H}_0 G = 0$ .

If G fulfills (SB) or is residually finite then  $\bar{\chi}G = \chi G$ , hence  $\Sigma_0^n (-1)^i \beta_i G = \Sigma_1^n (-1)^i \bar{\beta}_i G$ . If, moreover, G is amenable then all  $\bar{\beta}_i G$  are 0 (see [CG] or [E2]), and thus  $\chi G = 0$ .

# 4.2. Spaces with Poincaré duality

If the space X fulfills (ordinary) oriented Poincaré duality of formal dimension n

(\*) 
$$H^i(X; A) \cong H_{n-i}(X; A)$$

for all  $i \in \mathbb{Z}$  and all  $\mathbb{Z}G$ -modules A,  $G = \pi_1 X$ , then  $C_* \tilde{X}$  is equivalent to an FP-complex  $P_*$  over  $\mathbb{Z}G$  of length n. This follows from the fact that homology, and therefore cohomology, commutes with direct limits in A (the isomorphisms (\*) are assumed to be natural in A); and from the finiteness criterion of Bieri-Eckmann-Brown, see [Br]. If moreover G is finitely presented then X is finitely dominated, but this is not of importance here.

We further assume that (\*) is given by the cap-product  $e \cap -$  where e is a generator of  $H_n(G; \mathbb{Z}) = H^0(G; \mathbb{Z}) = \mathbb{Z}$ . Then  $e \cap -$  maps  $Hom_G(P_*, \mathbb{Z}G) = P_*^{\text{dual}}$  to  $P_* = \mathbb{Z}G \otimes_G P_*$ . We reverse the numbering of  $P_*^{\text{dual}}$   $(i \sim n - i)$  to make this a map of degree 0; it induces homology isomorphism since  $H_i(P_*^{\text{dual}}) = H^{n-i}(P_*; \mathbb{Z}G) = H_i(P_*)$  and is therefore a chain homotopy equivalence. Tensoring with  $\ell_2 G$  and passing to reduced homology yields

$$\bar{H}^{n-i}P_{\star}\cong \bar{H}_{i}P_{\star}$$

for all *i*, whence  $\bar{\beta}_{n-i}X = \bar{\beta}_iX$ . If *G* is infinite  $\bar{\beta}_0X = 0$  as in 4.1 and then also  $\bar{\beta}_nX = 0$ , without the special assumptions on *G*.

PROPOSITION 4. If X fulfills Poincaré duality of formal dimension n = 2k (in short  $PD^n$ ) then

$$\chi X = 2 - 2\beta_1 X + \cdots + (-1)^k \beta_k X = -2\bar{\beta}_1 X + \cdots + (-1)^k \bar{\beta}_k X,$$

provided (SB) holds for G, or G is residually finite.

4.3. Example. n = 4 of Proposition 4. In that case

$$\chi X = 2 - 2\beta_1 X + \beta_2 X = -2\bar{\beta_1} X + \bar{\beta_2} X$$

 $\bar{\beta}_1 X$  depends on G only (one can obtain a K(G, 1) by adding cells of dimension  $\geq 3$ ), it can be written  $\bar{\beta}_1 G$ .

PROPOSITION 5. If X fulfills  $PD^4$ , and if a) G fulfills (SB) or is residually finite, and b)  $\bar{\beta}_1 G = 0$  then  $\chi X \ge 0$ .

For groups with  $\bar{\beta}_1 G = 0$  see [BV].

4.4. Example. n=2 of Proposition 4. In that case we first note that if G is infinite then  $H_2\tilde{X}=\pi_2X=0$ . Indeed,  $H_2\tilde{X}=H_2(X;\mathbb{Z}G)=H^0(X;\mathbb{Z}G)=0$ . Thus X is aspherical, i.e. a K(G,1), and G is a  $PD^2$ -group. All groups of cohomology dimension 2 (over  $\mathbb{R}$ ) fulfill (SB), see [E]. We thus get

$$\chi X = \chi G = 2 - \beta_1 G = -\bar{\beta}_1 G$$

whence  $\beta_1 G \geq 2$ .

This may seem trivial since it is known that the orientable  $PD^2$ -groups are just the fundamental groups of closed orientable surfaces of genus  $g \ge 1$ , so  $\beta_1 G = 2g$ . However,  $\beta_1 G \ge 2$  was an important ingredient in the proof of that result (see [EL]).

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