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# Projective and Hilbert modules over group algebras, and finitely dominated spaces

BENO ECKMANN

## 0. Introduction

The objective of this note is to consider  $\ell_2$ -Betti numbers for  $FP$ -complexes and finitely dominated spaces, and to compare the corresponding Euler characteristic  $\bar{\chi}$  with the ordinary  $\chi$ . The results depend on the validity of the Bass conjectures for the group  $G$  involved. We first outline the contents.

**0.1.** Let  $G$  be a countable group,  $P$  a finitely generated projective  $\mathbb{Z}G$ -module, and  $\ell_2 G$  the Hilbert space of square-integrable real functions on  $G$  with left and right  $G$ -action by translation. Then  $P^{(2)} = \ell_2 G \otimes_G P$  is a Hilbert- $G$ -module. We write  $rk P$  for the “naive” rank  $\dim_{\mathbb{R}} \mathbb{R} \otimes_G P$  of  $P$ . We will show (Theorem 1 in Section 3) that  $P^{(2)}$  is a “free” Hilbert- $G$ -module, i.e., a direct sum of copies of  $\ell_2 G$  their number being  $= rk P$ , provided  $(*)$   $G$  fulfills the strong Bass conjecture, or  $G$  is residually finite. The Bass conjectures are explained in Section 2 (for more details we refer to Bass’ fundamental paper [B]).

**0.2.** A complex  $P_*$  is said of type  $FP$ , or an  $FP$ -complex, if it is of finite length  $n > 0$  ( $P_i = 0$  for  $i > n$  and  $i < 0$ ) and if all  $P_i$  are finitely generated projective  $\mathbb{Z}G$ -modules; it is said of type  $FF$  if the  $P_i$  are free.  $G$  is always assumed to be finitely generated. In the  $FF$ -case  $\ell_2 G \otimes_G P_*$  is a complex of free Hilbert- $G$ -modules. Under condition  $(*)$  this is also the case for an  $FP$ -complex, and the rank of  $P_i^{(2)}$  is  $= rk P_i$ ; one has (reduced) homology groups,  $\ell_2$ -Betti numbers  $\bar{\beta}_i$ , and an  $\ell_2$ -Euler characteristic  $\bar{\chi}$  which is equal to the ordinary homological Euler characteristic  $\chi$  (Section 1, and Theorem 2 in Section 3), all depending only on the chain homotopy type of  $P_*$ .

More generally,  $w(P_*) = \sum_0^n (-1)^i [P_i] \in K_0(\mathbb{Z}G)$  is the *Wall obstruction* of  $P_*$ , where  $[P_i]$  is the class of  $P_i \in K_0(\mathbb{Z}G)$ . Then  $rk w(P_*)$  is the Euler characteristic, and  $\tilde{w}(P_*)$  corresponding to  $w(P_*)$  in  $\tilde{K}_0(\mathbb{Z}G)$  is the finiteness (or rather freeness) obstruction. Thus our result tells that  $\ell_2 G \otimes_G$ -annihilates  $\tilde{w}$  and leaves  $\chi$  unchanged.

**0.3.** In topology *FP*-complexes occur in connection with a finitely dominated connected space  $X$ . The chain complex of the universal cover  $\tilde{X}$  is chain homotopy equivalent to an *FP*-complex  $P_*$  over  $\mathbb{Z}G$  where  $G$  is the fundamental group of  $X$ ; it is finitely presented. One can write  $w(X) = w(P_*)$ ;  $rk\ w(X)$  is the Euler characteristic of  $X$  and  $\tilde{w}(X)$  the Wall finiteness obstruction of  $X$ . Under condition (\*) on  $G$  one has  $\bar{\chi}(X) = \chi(X)$ , and the  $\ell_2$ -finiteness obstruction of  $X$  is 0.

**0.4.** If a complex  $P_*$  fulfills Poincaré duality of dimension  $n$  ( $PD^n$ ) then it is an *FP*-complex, and its  $\ell_2$ -Betti numbers also fulfill  $PD^n$  (Section 4). This applies to finitely dominated spaces, but also to  $PD^n$ -groups (which are finitely generated, but need not be finitely presented).

## 1. *FP*-complexes and $\ell_2$ -Betti numbers

**1.1.** We recall that a space  $X$  is said to be finitely dominated, cf. [M], if it is a retract, in the homotopy category, of a finite *CW*-complex. Such a space is homotopy equivalent to a (finite dimensional, in general infinite) *CW*-complex. In the following “space” will always mean connected *CW*-complex.

Let  $G$  be the fundamental group of the space  $X$ , and  $\tilde{X}$  the universal cover. The cellular chain complex  $C_*\tilde{X}$  is a complex of free  $\mathbb{Z}G$ -modules. If  $X$  is finitely dominated then  $C_*\tilde{X}$  is  $\mathbb{Z}G$ -chain homotopy equivalent to a complex  $P_*$  of type *FP*; i.e.,  $P_*$  has finite length  $n$  ( $P_i = 0$  for  $i < 0$  and  $i > n$ ) and all  $P_i$  are finitely generated projective  $\mathbb{Z}G$ -modules. Moreover  $G$  is finitely presentable (and for such  $G$  the converse holds: if  $C_*\tilde{X}$  is of type *FP* then  $X$  is finitely dominated).

Most of our arguments deal in fact just with *FP*-complexes  $P_*$  over a group ring  $\mathbb{Z}G$ . Finite presentability of  $G$  is in general irrelevant.

**1.2.** Let  $\beta_i X = \dim_{\mathbb{R}} H_i(X; \mathbb{R})$  be the  $i$ -th Betti number of  $X$ . If  $X$  is finitely dominated the  $\beta_i X$  are finite and the homological Euler characteristic  $\chi X$  is defined as  $\sum_0^n (-1)^i \beta_i X$ . Homology can be computed from any  $P_*$  equivalent to  $C_*\tilde{X}$  through  $\mathbb{R} \otimes_G P_*$ . We write  $rk\ P_i = \dim_{\mathbb{R}} \mathbb{R} \otimes_G P_i$ . The usual Euler–Poincaré argument then yields

$$\chi X = \sum_0^n (-1)^i rk\ P_i.$$

If  $X$  itself is a *finite CW*-complex then we can take  $P_* = C_*\tilde{X}$  (all  $P_i$  are finitely generated free over  $\mathbb{Z}G$ , of rank  $= rk\ P_i$ ; such a  $P_*$  is said to be of type *FF*). We recall that, in that case,  $rk\ P_i$  is equal to the number  $\alpha_i$  of  $i$ -cells of  $X$ . Thus

$\chi X = \sum_0^n (-1)^i \alpha_i$  becomes the ordinary “combinatorial” Euler characteristic of the finite cell-complex  $X$ .

**1.3.** For a *finite CW-complex*  $X$  (or an *FF-complex*  $P_*$ ) it is well-known that  $\chi X$  can be expressed by  $\ell_2$ -Betti numbers  $\bar{\beta}_i X$  as  $\chi X = \bar{\chi} X = \sum_0^n (-1)^i \bar{\beta}_i X$ . For  $\ell_2$ -homology and  $\ell_2$ -Betti numbers we refer, e.g., to [Lü] and references given there. Here we just recall the main facts.

Let  $\ell_2 G$  be as in 0.1. The complex  $\ell_2 G \otimes_G P_*$  consists of “free” Hilbert- $G$ -modules  $\ell_2 G \otimes_G P_i = \ell_2 G \oplus \cdots \oplus \ell_2 G$ ,  $\alpha_i$  terms, also written  $\ell_2 G^{\alpha_i}$ , with the induced boundary operators  $\partial$ . The *reduced* homology group  $\bar{H}_i \tilde{X} = \bar{H}_i P_* = \{\text{cycles in } \ell_2 G \otimes_G P_i\} / \text{closure of } \partial(\ell_2 G \otimes_G P_{i+1})$  is a Hilbert- $G$ -module, since it can be imbedded in  $\ell_2 G^{\alpha_i}$  (as orthogonal complement of the above closure in the cycle space). Its von Neumann dimension  $\dim_G \bar{H}_i \tilde{X}$  is  $\bar{\beta}_i X$ ; it is a homotopy invariant of  $X$ . Since  $\dim_G \ell_2 G \otimes_G P_i = \alpha_i$ , and since the von Neumann dimension behaves as a rank (although it is a real number  $\geq 0$ ) the Euler–Poincaré argument applied to  $\ell_2 G \otimes_G P_*$  yields

$$\bar{\chi} X = \sum_0^n (-1)^i \bar{\beta}_i X = \sum_0^n (-1)^i \alpha_i$$

whence  $\bar{\chi} X = \chi X$ .

**1.4.** In order to apply the same procedure to the general case of a finitely dominated space  $X$  (an *FP-complex*  $P_*$ ) we look closer at  $\ell_2 G \otimes_G P$  where  $P$  is a finitely generated projective  $\mathbb{Z}G$ -module. It is clearly a Hilbert- $G$ -module since it imbeds in a free one, namely in  $\ell_2 G \otimes_G F$  where  $F$  is a finitely generated free  $\mathbb{Z}G$ -module containing  $P$  as a direct summand.

It turns out (Theorem 1 in Section 3) that *if  $G$  fulfills the Bass conjecture or if  $G$  is residually finite then  $\ell_2 G \otimes_G P$  is isometrically  $G$ -isomorphic to  $\ell_2 G^k$  where  $k = rk P = \dim_{\mathbb{R}} \mathbb{R} \otimes_G P$ . (The residually finite case has already been treated by Lück [Lü]; his approach is slightly different from ours where the residually finite case is a corollary of the strong Bass case.)*

Now the arguments of 1.3 apply to  $\ell_2 G \otimes_G P_*$  and to its  $\ell_2$ -Betti number  $\bar{\beta}_i X$  except that  $\alpha_i$  is replaced by  $rk P_i$

$$\bar{\chi} X = \sum_0^n (-1)^i \bar{\beta}_i X = \sum_0^n (-1)^i rk P_i$$

whence, according to 1.1,  $= \chi X$ . Similarly for an arbitrary *FP-complex*.



**1.5.** The advantage resulting from  $\bar{\chi}X$  instead of the classical  $\chi X$  is due to the fact that for the  $\bar{\beta}_i X$  certain vanishing theorems are known. They imply properties of  $\chi X$  or of the  $\beta_i X$ . Some immediate applications are given in Section 4: To Poincaré duality spaces, to groups of type  $FP$ , to groups fulfilling  $PD^2$  (Poincaré duality of formal dimension 2).

## 2. Rank concepts for projective modules

**2.1.** In this section we recall the various rank concepts and their relations (Bass conjectures) since we will make essential use of them.  $P$  will always denote a finitely generated projective  $\mathbb{Z}G$ -module,  $G$  an arbitrary group.

(a) The “homological” or “naive” rank  $rk P = \dim_{\mathbb{R}} \mathbb{R} \otimes_G P$  is being used for the definition of homology, Betti numbers, and Euler characteristic. We note that it is not clear a priori whether  $rk P = 0$  implies  $P = 0$ .

(b) The “Kaplansky” rank  $\kappa P$ . It is defined as follows: Let  $F$  be a finitely generated free  $\mathbb{Z}G$ -module containing  $P$  as a direct summand. In a basis of  $F$ , consisting of  $m$  elements, the endomorphism  $f$  of  $\mathbb{Z}G^m$  which projects  $F$  onto  $P$  is given by an  $m \times m$  matrix  $M_P$  with  $\mathbb{Z}G$ -entries.  $\kappa P$  is the coefficient of  $1 \in G$  of trace  $M_P$ ; it is easily seen to be independent of the choice of  $F$  and of the basis.

Another way of describing  $f$  is by a matrix  $(f_{ij})$ ,  $f_{ij}$  being the  $\mathbb{Z}G$ -endomorphism of  $\mathbb{Z}G$  determined by  $f_{ij}(1) = a_{ij} \in \mathbb{Z}G$ .

(c) The “Hattori–Stallings” rank  $r_P$ . The trace  $\in \mathbb{Z}G$  of the matrix  $M_P$  above depends on the choices. However, if one passes from  $\mathbb{Z}G$  to  $\overline{\mathbb{Z}G} = \mathbb{Z}G / \{xy - yx\}$  where  $\{xy - yx\}$  is the submodule of  $\mathbb{Z}G$  generated by all additive commutators (it suffices to take  $xy - yx$  for all  $x, y \in G$ ) then the value of trace  $M_P$  in  $\overline{\mathbb{Z}G}$ , denoted by  $r_P$ , is independent of the choices.  $\overline{\mathbb{Z}G}$  is obtained from  $\mathbb{Z}G$  by identifying conjugate elements of  $G$ ; it is thus the free Abelian group generated by the conjugacy classes  $[x]$  of  $G$ . One can write  $r_P$  as

$$r_P = \sum r_P(x)[x]$$

where  $x$  is an arbitrary representative of  $[x]$ .

**2.2.** One immediately notes that  $r_P(1) = \kappa P$ . Moreover,  $\sum_x r_P(x) = rk P$ . Indeed  $\sum r_P(x)$  is the sum of all coefficients in trace  $M_P \in \mathbb{Z}G$ , i.e., it is induced by the augmentation  $\varepsilon: \mathbb{Z}G \rightarrow \mathbb{Z}$ ; and  $\varepsilon$  turns  $P$  into the free Abelian group  $\mathbb{Z} \otimes_G P$ . By naturality  $\varepsilon_* r_P$  is the rank, with respect to  $\mathbb{Z}$ , of  $\mathbb{Z} \otimes_G P$  which is  $= rk P$ .

The *Strong Bass Conjecture* (SB) claims that all  $r_P(x)$ ,  $x \neq 1$ , are 0. It implies that  $\kappa P = rk P$ . That equality is the *Weak Bass Conjecture* (WB). It has been

proved quite generally by Linnell [Li, p. 96] that  $r_P(x)$  vanishes on all elements  $x \neq 1$  of finite order.

(SB) has been established for various classes of groups. We mention the following:

- 1) Finite groups (Swan's Theorem [S])
- 2) Linear groups ([B], [E])
- 3) Negatively curved groups
- 4) Solvable groups [E]
- 5) Groups of cohomology dimension  $\leq 2$  over  $\mathbb{R}$  ([E])

As for 3), this does not seem to be explicitly in the literature. It follows from the method used in [E] and the fact that in such a group the centralizer of an element of infinite order is virtually infinite cyclic.

(WB) holds for all residually finite groups [B], but it is not known whether these fulfill (SB).

**2.3.** We introduce an adhoc notation  $c_x(\alpha) =$  coefficient of  $x \in G$  in  $\alpha \in \mathbb{Z}G$  (or  $\mathbb{R}G$ , or  $\ell_2 G$ ) to be used in this and the next sections.

As a consequence of the definitions one has

**PROPOSITION 1.** *The von Neumann dimension  $\dim_G(\ell_2 G \otimes_G P)$  is equal to  $\kappa P$ .*

*Proof.* We recall that the von Neumann dimension of a Hilbert- $G$ -module  $H \subset \ell_2 G^m$  is defined as “trace” of the projection operator  $\varphi$  of  $\ell_2 G^m$  having  $H$  as image.  $\varphi$  is given by an  $m \times m$  matrix  $(\varphi_{ij})$  with  $\varphi_{ij} \in NG$ , the ring of bounded  $G$ -equivariant operators in  $\ell_2 G$  (the von Neumann algebra of  $G$ ); “trace” is to be understood as  $c_1(\sum_{i=1}^m \varphi_{ii}(1))$ . If  $H = \ell_2 G \otimes_G P$  then  $(\varphi_{ij})$  is the same as the matrix  $M_P = (f_{ij}), f_{ij}(1) = a_{ij} \in \mathbb{Z}G$ ; indeed the endomorphism ring of  $\mathbb{Z}G$  is naturally imbedded in  $NG$ . Thus the two “traces”  $c_1(\sum f_{ii}(1)) = c_1(\sum \varphi_{ii}(1))$  are the same.

### 3. Structure of $\ell_2 G \otimes_G P$

**3.1.** If  $G$  fulfills (SB) or is residually finite it follows from Proposition 1 that  $\dim_G \ell_2 G \otimes_G P = rk P$ . In this section we will show that  $\ell_2 G \otimes_G P$  and  $\ell_2 G^{rk P}$  are even isometrically and  $G$ -equivariantly isomorphic. We need some preparations.

**3.2.** Let  $CG$  be the center of  $\mathbb{R}G$ ; it has a basis consisting of all sums  $\sigma_x = \sum_{y \in [x]} y$  for the finite conjugacy classes in  $G$ .

**PROPOSITION 2.** *Let  $P$  be a finitely generated projective  $\mathbb{Z}G$ -module,  $\zeta$  an element of  $CG$ . If the Hattori–Stallings ranks  $r_P$  vanishes on finite conjugacy classes  $[x]$ ,  $x \neq 1$  then*

$$c_1(\zeta \text{ trace } M_P) = c_1(\zeta) \kappa P.$$

*Proof.* It suffices to prove that  $c_1(\sigma_x \text{ trace } M_P) = 0$  for  $x \neq 1$ . Now

$$\begin{aligned} c_1(\sigma_x \text{ trace } M_P) &= \sum_{y \in [x]} c_1(y \text{ trace } M_P) \\ &= \sum_{y \in [x]} c_{y^{-1}}(\text{trace } M_P) \\ &= \sum_{y \in [x^{-1}]} c_y(\text{trace } M_P) = r_P(x^{-1}) = 0. \end{aligned}$$

The assumption of Proposition 2 holds if  $G$  fulfills (SB), and then  $\kappa P = rk P$ . The result also holds if  $G$  is *residually finite*: We choose a normal subgroup  $N$  of finite index in  $G$  which does not contain the elements of  $\sigma_x$ , of trace  $M_P$ , nor of their product. Passing from  $\mathbb{Z}G$  to  $\mathbb{Z}(G/N)$  we get a projective  $\mathbb{Z}(G/N)$ -module  $P' = \mathbb{Z}(G/N) \otimes_G P$  and an element  $\sigma'_x \in C(G/N)$ . The coefficient  $c_1$  of  $\sigma_x$ , of trace  $M_P$ , and of the product remains unchanged. By Swan's theorem [S],  $G/N$  fulfills (SB); therefore

$$r_P(x^{-1}) = c_1(\sigma_x \text{ trace } M_P) = c_1(\sigma'_x \text{ trace } M_{P'}) = r_{P'}(x^{-1}) = 0$$

if  $x \neq 1$ . Moreover  $\kappa P = \kappa P' = rk P'$  and

$$\begin{aligned} rk P' &= \dim_{\mathbb{R}}(\mathbb{R} \otimes_{G/N} P') = \dim_{\mathbb{R}}(\mathbb{R} \otimes_{G/N} \mathbb{Z}G/N \otimes_G P) \\ &= \dim_{\mathbb{R}}(\mathbb{R} \otimes_G P) = rk P. \end{aligned}$$

**COROLLARY 2'.** *If  $G$  is residually finite then*

$$c_1(\zeta \text{ trace } M_P) = c_1(\zeta) rk P.$$

**3.3.** We assume that  $r_P$  vanishes on finite conjugacy classes  $[x]$ ,  $x \neq 1$  and show that for the two Hilbert- $G$ -modules  $\ell_2 G \otimes_G P$  and to  $\ell_2 G^{\kappa P}$  the *center-valued trace* (cf. [KR]),  $ctr \in \text{center of } NG$ , applied to the respective  $\Sigma \varphi_{ii}$  is the same. This implies that they are isometrically  $G$ -isomorphic.

For  $\ell_2 G \otimes_G P$  one has  $ctr(\Sigma \varphi_{ii}) = ctr(\text{trace } M_P)$  which lies in  $CG$  since  $ctr x = (1/\text{card}[x]) \sum_{y \in [x]} y$  if  $[x]$  is finite, and 0 otherwise. Of course for  $\ell_2 G^{\kappa P}$ , the

$\text{ctr}$  of the matrix is  $\kappa P \cdot E$ ,  $E = \text{identity} \in NG$ . Thus

$$\xi = \text{ctr} \left( \sum \varphi_{ii} - \kappa P \cdot E \right)$$

is an element of  $CG$ , and so is its adjoint  $\xi^*$  in  $NG$ .

Since the 1-coefficient of the central-valued trace fulfills quite generally  $c_1(\text{ctr } \varphi) = c_1 \varphi(1)$ ,  $\varphi \in NG$ , we have

$$c_1(\xi^* \xi) = c_1(\xi^*(\text{trace } M_P - \kappa P));$$

by Proposition 2 this is equal to  $c_1(\xi^*)c_1(\text{trace } M_P - \kappa P) = 0$ . Since  $c_1(\xi^* \xi) = \langle \xi^* \xi, 1 \rangle = \langle \xi, \xi \rangle$  in  $\ell_2 G$ , it follows that  $\xi = 0$ :

**PROPOSITION 3.** *If  $r_P$  vanishes on finite conjugacy classes  $[x]$ ,  $x \neq 1$  then  $\ell_2 G \otimes_G P$  and  $\ell_2 G^{\kappa P}$  are isometrically  $G$ -isomorphic.*

By Corollary 2' the same arguments work in the residually finite case. We summarize the important cases as follows.

**THEOREM 1.** *If  $G$  fulfills (SB), or if  $G$  is residually finite, then  $\ell_2 G \otimes_G P$  is isometrically  $G$ -isomorphic to  $\ell_2 G^{\kappa P}$ .*

**3.4.** As stated in Section 1.4 this implies

**THEOREM 2.** *If  $G$  fulfills (SB), or if  $G$  is residually finite, then for any finitely dominated space  $X$  with fundamental group  $G$ , or for any FP-complex  $P_*$  over  $\mathbb{Z}G$ , the  $\ell_2$ -Euler characteristic  $\bar{\chi}$  and the ordinary Euler characteristic  $\chi$  coincide.*

**3.5.** An FP-complex  $P_*$  over  $\mathbb{Z}G$  is always chain homotopy equivalent to a complex where the  $P_i$  are all free except (possibly) for the top module  $P_n$ . Then for  $\ell_2 G \otimes_G P_i$  to be free of rank  $= rk P_i$ ,  $i < n$ , it is not necessary to apply Theorem 1, i.e., to assume (SB) or  $G$  residually finite.

We recall that for  $P_*$  chosen as above the class of  $P_n$  in  $\tilde{K}_o(\mathbb{Z}G)$  is, up to sign, the “Wall obstruction”; it is 0 if and only if  $P_*$  is equivalent to an FF-complex. In that sense Theorem 1 tells that, under the appropriate assumption the  $\ell_2$ -Wall obstruction always vanishes.

*Remark.* We have worked throughout with homology. Everything could also be carried through in cohomology, based on  $\text{Hom}_G(P_*, \ell_2 G) = P_*^{\text{dual}} \otimes_G \ell_2 G$ ;  $P_i^{\text{dual}} = \text{Hom}_G(P_i, \mathbb{Z}G)$  is again finitely generated projective. We note that in case  $P_i$  is free,

$P_i^{\text{dual}}$  can be identified with  $P_i$ , so that  $\bar{H}^i = \bar{H}_i$ . But this also holds in the general projective case. Indeed the projection matrix for  $P_i^{\text{dual}}$  can be taken to be the transposed of  $M_{P_i}$ ; thus the trace and the various ranks are the same.

## 4. Applications

### 4.1. Groups of type $FP$

These are groups  $G$  for which there exists a resolution  $P_* \rightarrow \mathbb{Z}$  over  $\mathbb{Z}G$  with  $P_*$  of type  $FP$ . We assume that the cohomology dimension  $cdG$ , the minimal length  $n$  of  $P_*$ , is  $\geq 2$  (if  $cdG = 1$  then  $G$  is finitely generated free). It is not known whether such a group is necessarily of type  $FF$ ; cf. [Br] for a discussion of that problem.

$G$  being infinite one notes that, without further assumptions on  $G$ ,  $H^0(G; \ell_2 G) = \ell_2 G^G = 0$  (an element of  $\ell_2 G$  cannot be invariant unless it is 0). Then  $\bar{H}^0 G = 0$  and  $\bar{H}_0 G = 0$ .

If  $G$  fulfills (SB) or is residually finite then  $\bar{\chi}G = \chi G$ , hence  $\Sigma_0^n (-1)^i \beta_i G = \Sigma_1^n (-1)^i \bar{\beta}_i G$ . If, moreover,  $G$  is amenable then all  $\bar{\beta}_i G$  are 0 (see [CG] or [E2]), and thus  $\chi G = 0$ .

### 4.2. Spaces with Poincaré duality

If the space  $X$  fulfills (ordinary) oriented Poincaré duality of formal dimension  $n$

$$(*) \quad H^i(X; A) \cong H_{n-i}(X; A)$$

for all  $i \in \mathbb{Z}$  and all  $\mathbb{Z}G$ -modules  $A$ ,  $G = \pi_1 X$ , then  $C_* \tilde{X}$  is equivalent to an  $FP$ -complex  $P_*$  over  $\mathbb{Z}G$  of length  $n$ . This follows from the fact that homology, and therefore cohomology, commutes with direct limits in  $A$  (the isomorphisms  $(*)$  are assumed to be natural in  $A$ ); and from the finiteness criterion of Bieri–Eckmann–Brown, see [Br]. If moreover  $G$  is finitely presented then  $X$  is finitely dominated, but this is not of importance here.

We further assume that  $(*)$  is given by the cap-product  $e \cap -$  where  $e$  is a generator of  $H_n(G; \mathbb{Z}) = H^0(G; \mathbb{Z}) = \mathbb{Z}$ . Then  $e \cap -$  maps  $\text{Hom}_G(P_*, \mathbb{Z}G) = P_*^{\text{dual}}$  to  $P_* = \mathbb{Z}G \otimes_G P_*$ . We reverse the numbering of  $P_*^{\text{dual}}$  ( $i \sim n - i$ ) to make this a map of degree 0; it induces homology isomorphism since  $H_i(P_*^{\text{dual}}) = H^{n-i}(P_*; \mathbb{Z}G) = H_i(P_*)$  and is therefore a chain homotopy equivalence. Tensoring with  $\ell_2 G$  and passing to reduced homology yields

$$\bar{H}^{n-i}P_* \cong \bar{H}_iP_*$$

for all  $i$ , whence  $\bar{\beta}_{n-i}X = \bar{\beta}_iX$ . If  $G$  is infinite  $\bar{\beta}_0X = 0$  as in 4.1 and then also  $\bar{\beta}_nX = 0$ , without the special assumptions on  $G$ .

**PROPOSITION 4.** *If  $X$  fulfills Poincaré duality of formal dimension  $n = 2k$  (in short  $PD^n$ ) then*

$$\chi X = 2 - 2\beta_1X + \cdots + (-1)^k\beta_kX = -2\bar{\beta}_1X + \cdots + (-1)^k\bar{\beta}_kX,$$

provided (SB) holds for  $G$ , or  $G$  is residually finite.

4.3. *Example.*  $n = 4$  of Proposition 4. In that case

$$\chi X = 2 - 2\beta_1X + \beta_2X = -2\bar{\beta}_1X + \bar{\beta}_2X$$

$\bar{\beta}_1X$  depends on  $G$  only (one can obtain a  $K(G, 1)$  by adding cells of dimension  $\geq 3$ ), it can be written  $\bar{\beta}_1G$ .

**PROPOSITION 5.** *If  $X$  fulfills  $PD^4$ , and if a)  $G$  fulfills (SB) or is residually finite, and b)  $\bar{\beta}_1G = 0$  then  $\chi X \geq 0$ .*

For groups with  $\bar{\beta}_1G = 0$  see [BV].

4.4. *Example.*  $n = 2$  of Proposition 4. In that case we first note that if  $G$  is infinite then  $H_2\tilde{X} = \pi_2X = 0$ . Indeed,  $H_2\tilde{X} = H_2(X; \mathbb{Z}G) = H^0(X; \mathbb{Z}G) = 0$ . Thus  $X$  is aspherical, i.e. a  $K(G, 1)$ , and  $G$  is a  $PD^2$ -group. All groups of cohomology dimension 2 (over  $\mathbb{R}$ ) fulfill (SB), see [E]. We thus get

$$\chi X = \chi G = 2 - \beta_1G = -\bar{\beta}_1G$$

whence  $\beta_1G \geq 2$ .

This may seem trivial since it is known that the orientable  $PD^2$ -groups are just the fundamental groups of closed orientable surfaces of genus  $g \geq 1$ , so  $\beta_1G = 2g$ . However,  $\beta_1G \geq 2$  was an important ingredient in the proof of that result (see [EL]).

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