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Autor: Sridharan, Raja

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Homotopy and sections of algebraic vector bundles over real affine algebras

RAJA SRIDHARAN

Introduction

Let A be a regular affine domain over $\mathbb{R}$ with $\dim A = n$. Assume that the set of real points of $\operatorname{Spec} A = X$ (which we denote by $X(\mathbb{R})$), is compact for the real topology. Let K_A be the canonical line bundle of A. We consider the set of pairs (P, i), where P is a projective A-module of rank n and $i: K_A \xrightarrow{\sim} \bigwedge^n P$ is an isomorphism.

In Section 2 of this paper, we associate to this pair an invariant in $\mathbb{Z}^S$, where S is the number of connected components of $X(\mathbb{R})$. This invariant coincides with the one defined by Barge and Ojanguren ([BO], Th. 6.2) in the case $\dim A = 2$, and $X(\mathbb{R})$ is orientable. In the higher dimensional case also, the definition of the invariant is patterned after theirs. In fact, we define more generally, an invariant (due to Nori) if A is a regular affine domain over any field k of characteristic zero and P is as above. This invariant has values in a certain quotient of the Witt group $W(k)$ of k. We refer the reader to section 2 for details.

In Section 3, we use this invariant, to give an algebraic proof due to Nori, that the tangent bundle to an even dimensional real sphere does not have a unimodular element. Different algebraic proofs have been given, by Swan [Sw3] for the real 2-sphere and Kong [Ko] for all even dimensional spheres. However, this proof also works for real closed fields.

In Section 4, we prove (4.1) that if dimension of A is 2, two stably free A-modules of rank 2 having the same invariant are isomorphic, using ideas of [BO] and [OPS]. We use this result to prove a theorem (4.5) on sections of stably free modules.

In Section 5, we use this result to answer a question of Bhatwadekar on unimodular rows.

In Section 6, we apply the results of Section 4 to prove the following theorem.

THEOREM. *Let A be a regular affine domain over $\mathbb{R}$ with* $\dim A = 2$ *and* $\operatorname{Pic} A = 0$. *Assume that $X(\mathbb{R})$ is compact. Then the following are equivalent:*

(1) $\tilde{K}_0(A)$ *is 2-torsion.*
(2) *Maximal ideals of A which correspond to complex points of* Spec A *are generated by two elements.*
(3) *If P is projective A-module of rank 2 and $P \otimes C(X(\mathbb{R}))$ is free (where $C(X(\mathbb{R}))$ is the ring of real valued continuous functions on $X(\mathbb{R})$), then P is free.*

This theorem has been proved by Barge and Ojanguren ([BO], Th. 7.3) in the case where A is the coordinate ring of the real 2; sphere.

In Section 1, we collect some known results. In the Appendix we prove a result on homotopy of sections of projective modules that is needed in Section 2.

1. Some preliminaries

In this section, we collect some well known results that will be used in the subsequent sections. We only state the results in the generality that we need them.

Let A be a finitely generated k-algebra, where k is perfect. We recall from ([Ha], Section 8, Chapter 2), the definition of $\Omega_{A/k}$. It is the quotient of the free A-module generated by symbols $d(a)$, $a \in A$ by the submodule generated by all expressions of the form

(1) $d(b+b') - d(b) - d(b')$,
(2) $d(bb') - bd(b') - b'd(b)$,
(3) $d(c)$, $c \in k$.

When there is no possibility of confusion, we write Ω_A instead of $\Omega_{A/k}$.

Let A be as above. Let $I \subset A$ be an ideal. We recall the following proposition from ([Ha] §8, page 173, Prop. 8.4).

PROPOSITION 1.1. *There exists a natural homomorphism $\delta\colon I/I^2 \to \Omega_A/I\Omega_A$ of A/I modules, given by $\delta(b) = d(b)$.*

Let A be a regular affine domain over k with $\dim A = n$. Then $\Omega_{A/k}$ is a projective A-module of rank n. We define the *canonical* line bundle of A (denoted by K_A) to be the nth exterior power of $\Omega_{A/k}$.

Let $I \subset A$ be a prime ideal such that A/I is regular with $\dim A/I = 1$. Then there exists an exact sequence ([Ha] §8, Chapter 2, Prop. 8.17)

$$0 \to I/I^2 \to \Omega_A \otimes_A A/I \to \Omega_{A/I} \to 0.$$

Since $\dim A/I = 1$, we have $\Omega_{A/I} = K_{A/I}$. Further, this sequence splits, $d(\bar{b}) \xrightarrow{s} d(b) \otimes 1$ inducing a splitting. Taking top exterior powers, we see that $K_A \otimes A/I \simeq \bigwedge^{n-1} I/I^2 \otimes \Omega_{A/I}$, the isomorphism being induced by the splitting s. (The existence of this isomorphism will be used in Section 2.)

We now recall some results on quadratic forms. We cite Lam's book [La] as a general reference. Let k be a field with char $k \neq 2$. By $W(k)$ (the *Witt ring* of k), we mean the ring of isometry classes of non-singular quadratic spaces over k, modulo hyperbolic spaces). We recall that multiplication in $W(k)$ is defined by taking tensor products and addition is defined by taking orthogonal sums of quadratic spaces. We need, in what follows, the following result of Geyer-Harder-Knebusch-Scharlau [GHKS].

Let X be a smooth projective curve over a perfect field k with char $k \neq 2$. We can define the Witt group $W(X, \Omega)$ of quadratic spaces over X with values in the line bundle Ω. There are canonical residue homomorphisms $W(k(X), \Omega_{k(X)}) \to W(k(x))$ for each closed point $x \in X$, where $k(X)$ and $k(x)$ denote the function field of X and the residue field at x respectively. Any nonsingular quadratic form q over $k(X)$ with values in $\Omega_{k(X)}$ may be written as

$$q_1\, d\pi \perp q_2\left(\frac{d\pi}{\pi}\right)$$

with q_1 and q_2 regular over $\mathcal{O}_{X,x}$ and π a parameter at x. Let $res(q)$ denote the reduction of q_2 modulo π, which is independent of the choice of π. It is proved in [GHKS] that the sequence

$$W(k(X), \Omega_{k(X)}) \xrightarrow{\text{res}} \bigoplus_{x \in X} W(k(x)) \xrightarrow{s} W(k)$$

is a complex, where the direct sum is over all closed points $x \in X$, and s is the transfer induced by the trace map $k(x) \to k$.

For all results on the Witt group $W^{-1}(A)$ of skew symmetric forms that we use, we refer to [OPS] §1. For all unexplained notation and definitions we refer to [Ba], [Ma], and [La], for projective modules, commutative algebra and quadratic forms respectively.

2. The definition of Nori's invariant

The results of this section are due to Nori.

Let A be a regular affine domain over any field k of characteristic zero with $\dim A = n$ and $X = \operatorname{Spec} A$. We consider pairs (P, i) where P is a projective

A-module of rank n and $i: K_A \xrightarrow{\sim} \bigwedge^n P$ is an isomorphism. Let $s_0: P \to I_0$ be a surjection such that I_0 is the intersection of finitely many maximal ideals m_j, $1 \leq j \leq r$ of A. We call such surjections *good* sections. Let $\Omega_{A/k}$ be the module of differentials, and $\delta: I_0/I_0^2 \to \Omega_{A/k} \otimes_A A/I_0$ be defined by $\delta(b) = d(b) \otimes 1$.

Let $\sim$ denote the operation of taking top exterior powers. We then have the following sequence of maps.

$$K_A/I_0K_A \xrightarrow{i} \overset{n}{\bigwedge} P/I_0P \xrightarrow{\tilde{s}_0} \overset{n}{\bigwedge} I_0/I_0^2 \xrightarrow{\tilde{\delta}} K_A/I_0K_A.$$

The composite of these maps gives an automorphism of K_A/I_0K_A, which is multiplication by an element $\overline{u_0} \in (A/I_0)^*$.

Let Y be a smooth projective model for $X = \operatorname{Spec} A$. Let $H = \operatorname{im}(\oplus_{x \in Y} W(k(x)) \xrightarrow{s} W(k))$.

Let $s_j: W(A/m_j) \to W(k)$ be the map induced by the trace. The following theorem defines a map (called the invariant) from pairs (P, i) to $W(k)/H$.

THEOREM. *The assignment $(P, i) \mapsto \Sigma\, s_j(\langle \overline{u_0} \rangle)$, is independent of the choice of the good section and thus is an invariant of the pair (P, i).*

Proof. We prove the theorem in several steps.

Step 1. Let $s_1: P \to I_1$ be another good section. By the appendix of this paper, there exists an ideal J in $A[t]$, such that $A[t]/J$ is a Dedekind domain and a map $s(t): P[t] \to J$ such that $s(0) = s_0$ and $s(1) = s_1$. Let $R = A[t]/J$. We have the following sequence of induced isomorphisms:

$$K_A \otimes R \xrightarrow{i} \overset{n}{\bigwedge} P[t]/JP[t] \xrightarrow{\tilde{s}(t)} \overset{n}{\bigwedge} J/J^2.$$

Tensoring with K_R, we obtain induced isomorphisms

$$K_A \otimes R \otimes K_R \xrightarrow{\sim} \overset{n}{\bigwedge} J/J^2 \otimes K_R \xrightarrow[\sim]{\text{canonical}} \overset{n+1}{\bigwedge} \Omega_{A[t]} \otimes R.$$

Now $\bigwedge^{n+1} \Omega_{A[t]} \xrightarrow[\sim]{\text{canonical}} K_A \otimes A[t]$. So we obtain an isomorphism

$$K_A \otimes K_R \xrightarrow{\sim} K_A \otimes R.$$

Tensoring with K_A^{-1} we obtain an isomorphism $\theta: K_R \xrightarrow{\sim} R$, i.e. a derivation $D: R \to R$ given by $D(b) = \theta(d(b))$. Let $D(\bar{t}) = \overline{f(t)}$. We claim that $\overline{f(0)} = \overline{u_0}$ and $\overline{f(1)} = \overline{u_1}$. This is proved in the next step.

Step 2. We check that $\overline{f(0)} = \overline{u_0}$. The proof that $\overline{f(1)} = \overline{u_1}$ is similar. After semi localising at the maximal ideals containing I_0, we may assume that $K_A \approx A$. We fix a generator λ of K_A obtained after localisation. Suppose that $i(\lambda) = p_1 \wedge \cdots \wedge p_n \in \bigwedge^n P$ with $p_i \in P$. (We assume for convenience of notation that $i(\lambda)$ is decomposable). Proceeding as in Step 1, we see that the isomorphism

$$K_A \otimes K_R \xrightarrow{\sim} K_A \otimes R$$

sends $\lambda \otimes dt$ to $(p_1 \wedge \cdots \wedge p_n) \otimes dt$, to $(d(s(t)(p_1)) \wedge \cdots \wedge d(s(t)(p_n))) \otimes dt$ to $(d(s(t)(p_1)) \wedge \cdots \wedge d(s(t)(p_n))) \wedge dt = \alpha \wedge dt$, where $\alpha \in K_A \otimes R$. Thus $\lambda \otimes dt$ goes to $\alpha \in K_a \otimes R$ under the above isomorphism. Since $\alpha \in K_A \otimes R$, $\alpha = \overline{f(t)}\lambda$, where $\overline{f(t)}$ is as in Step 1. It follows that $d(s_0(p_1)) \wedge \cdots \wedge d(s_0(p_n)) = \overline{f(0)}\bar{\lambda}$ in K_A/I_0K_A. On the other hand from the definition of the invariant it follows that

$$d(s_0(p_1)) \wedge \cdots \wedge d(s_0(p_n)) = \overline{u_0}\bar{\lambda}.$$

Therefore $\overline{f(0)} = \overline{u_0}$.

Step 3. Let K be the quotient field of $R = A[t]/J$ and C the smooth projective curve defined by the set of discrete valuations of K. In [GHKS], there is a complex

$$W(k(C), \Omega_{k(C)}) \xrightarrow{\text{res}} \oplus_{v \in C} W(\mathcal{O}_v/m_v) \xrightarrow{tr} W(k),$$

where v runs over all the discrete valuations of $k(C)$. Here

$$\mathcal{O}_v = \{x \in k(C) \mid v(x) \geq 0\},$$
$$m_v = \{x \in \mathcal{O}_v \mid v(x) > 0\}.$$

Let Y be a smooth projective model for $X = \operatorname{Spec} A$. We then have the following inclusions

$$\operatorname{Spec} A[t]/J \to (\operatorname{Spec} A) \times \mathbb{A}^1 \to (\operatorname{Spec} A) \times \mathbb{P}^1 \to Y \times \mathbb{P}^1.$$

Since C is a smooth projective curve containing Spec $A[t]/J$, there exists a unique $h: C \to Y \times \mathbb{P}^1$, extending the above inclusion. Points in C are of three kinds:

Type I $\quad \{v \in C, h(v) \in \operatorname{Spec} A \times \mathbb{A}^1\}$,

Type II $\quad \{v \in C, h(v) \in \operatorname{Spec} A \times \mathbb{P}^1 \backslash \operatorname{Spec} A \times \mathbb{A}^1\}$,

Type III $\quad \{v \in C, h(v) \in Y \times \mathbb{P}^1 \backslash \operatorname{Spec} A \times \mathbb{P}^1\}$.

Let $\theta\colon K_R \to R$ be as in Step 1. Let $\theta^{-1}(1) = w$. Then w is a nowhere vanishing differential on R, θ being an isomorphism. Since $\theta(d(t)) = f(t)$, $w = dt/f(t)$. By [GHKS], we have,

$$0 = \sum_{v \in C} \operatorname{tr} \operatorname{res} w = \sum_{v \text{ type I}} \operatorname{tr} \operatorname{res} w + \sum_{v \text{ type II}} \operatorname{tr} \operatorname{res} w + \sum_{v \text{ type III}} \operatorname{tr} \operatorname{res} w.$$

We have (by the definition of H) $\sum_{v \text{ type III}} \operatorname{tr} \operatorname{res} w = 0$ in $W(k)/H$. Since w is a nowhere vanishing differential on R, we have, $\sum_{v \text{ type I}} \operatorname{tr} \operatorname{res} w = 0$.

Therefore

$$\sum_{v \text{ type II}} \operatorname{tr} \operatorname{res} w = 0$$

in $W(k)/H$.

Similarly, we have

$$0 = \sum_{v \in C} \operatorname{tr} \operatorname{res} \frac{wt}{t-1} = \sum_{v \text{ type I}} \operatorname{tr} \operatorname{res} \frac{wt}{t-1} + \sum_{v \text{ type II}} \operatorname{tr} \operatorname{res} \frac{wt}{t-1} + \sum_{v \text{ type III}} \operatorname{tr} \operatorname{res} \frac{wt}{t-1}$$

in $W(k)$. The type III term is zero in $W(k)/H$. On the other hand, $\sum_{v \text{ type II}} \operatorname{tr} \operatorname{res} wt/(t-1) = \sum_{v \text{ type II}} \operatorname{tr} \operatorname{res} w$ which is 0 in $W(k)/H$ as proved earlier.

Thus, $\sum_{v \text{ type I}} \operatorname{tr} \operatorname{res} wt/(t-1) = 0$ in $W(k)/H$, which implies

$$\sum_{v \text{ type I}} \operatorname{tr} \operatorname{res}(tf(t)/t-1)\, dt = 0.$$

in $W(k)/H$.

Using the fact that $\overline{f(1)} = \overline{u_1}$ and $\overline{f(0)} = \overline{u_0}$, one sees easily from the previous equation that the invariant does not depend on the choice of the good section.

The definition of the invariant for $k = \mathbb{R}$ and $X(\mathbb{R})$ compact.

Let A be a regular affine domain over $\mathbb{R}$ with $\dim A = n$. Assume that the set of real points of $X = \operatorname{Spec} A$ (which we denote by $X(\mathbb{R})$) is compact for the real topology. Let K_A be the canonical line bundle of A. We consider the set of pairs (P, i), where P is a projective A-module of rank n and $i\colon K_A \xrightarrow{\sim} \bigwedge^n P$ is an isomorphism. To this pair we associate an element of $\mathbb{Z}^S$ (where S denotes the number of connected components of $X(\mathbb{R})$) in the following manner: Let $s_0\colon P \to I_0$ be a good section. Then, as before, we obtain an element $\overline{u_0} \in (A/I_0)^*$.

Let $X(\mathbb{R}) = X_1 \cup X_2 \cup \cdots \cup X_S$ be the decomposition of $X(\mathbb{R})$ into its connected components. Taking the sign of u_0 at each real point in $V(I_0)$ belonging to a fixed connected component X_k of X, we obtain one dimensional quadratic spaces over $\mathbb{R}$, the sum of whose signatures we denote by v_k. To the pair (P, i) we assign the tuple $(v_1, \ldots, v_S) \in \mathbb{Z}^S$.

This invariant coincides with the invariant defined earlier for arbitrary fields, when $X(\mathbb{R})$ is connected. This is so because in this case $H = 0$ (since the points belonging to $Y\backslash X$ are all complex points). Thus this invariant is well defined in this case.

If C be the smooth projective curve obtained in Step. 3. Let $C(\mathbb{R}) = C_1 \cup \cdots \cup C_l$ be the decomposition of $C(\mathbb{R})$ into its connected components. With the notation as above we have the following lemma due to Knebusch ([K]).

LEMMA. *The following sequence is a complex*

$$W(\mathbb{R}(C), \Omega_{\mathbb{R}(C)}) \xrightarrow{\text{res}} \bigoplus_{v \in C_i} W(\mathcal{O}_v/m_v) \xrightarrow{\text{tr}} W(\mathbb{R}).$$

Using this lemma and proceeding exactly as in Step 3 replacing C by C_i we conclude that the invariant of P is well defined even when $X(\mathbb{R})$ is not connected. (I thank Parimala for pointing this out.)

REMARK. Following a suggestion of M. Ojanguren, one can define the invariant even if $X(\mathbb{R})$ is not compact. We hope to treat it elsewhere.

3. Applications of the invariant to tangent bundles of spheres

In this section, we give an algebraic proof (due to Nori) that the tangent bundle of the real 2-sphere does not have a unimodular element. We note that in this case $H = 0$, with the notation as in the previous section. (The same proof works for any even dimensional sphere and also when the field is real closed). For other proofs see [Sw3] and [Ko].

We begin with a well known

LEMMA 3.1. *Let A be a ring. Let $(a, b, c) \in U_3[A]$ be a unimodular row of length 3. Let $x, y, z \in A$ be chosen so that $ax + by + cz = 1$. Let $P = A^3/(a, b, c)A$. Let e_i be the image in P of the ith coordinate function under the canonical map $A^3 \to A^3/(a, b, c)A$. Then $\bigwedge^2 P$ is generated by $w = ze_1 \wedge e_2 + xe_2 \wedge e_3 + ye_3 \wedge e_1$.*

Proof. It is easy to check that $aw = e_2 \wedge e_3$, $bw = e_3 \wedge e_1$ and $cw = e_1 \wedge e_2$. Thus w is a generator of $\bigwedge^2 P$.

THEOREM 3.2. *Let $A = \mathbb{R}[X, Y, Z]/(X^2 + Y^2 + Z^2 - 1)$. Let x, y, z be the images of X, Y, Z* in *A. Then $A^3/(x, y, z)$ is a non-free stably free module.*

Proof. Since $x^2 + y^2 + z^2 = 1$, we have, $xdx + ydy + zdz = 0$. Therefore $\Omega_{A/R} = Adx \oplus Ady \oplus Adz/(xdx, ydy, zdz)$. As in Lemma 3.1, it is easy to check that

$w = xdy \wedge dz + ydz \wedge dx + xdx \wedge dy$ is a generator of $\bigwedge^2 \Omega_A \simeq K_A$. In fact $zw = dx \wedge dy$, $xw = dy \wedge dz$, $yw = dz \wedge dx$. We define an isomorphism $i \colon K_A \to \bigwedge^2 P$ as follows: $xe_2 \wedge e_3 + ye_3 \wedge e_1 + ze_1 \wedge e_2 = i(w)$. We compute the invariant for (P, i) using a section s of P. Let $I = (y, z)$ and let $s\colon A^3/(x, y, z) \to I$ be defined by $s(e_1) = 0$, $s(e_2) = z$, $s(e_3) = -y$. We have the following sequence of maps as below:

$$K_A/IK_A \xrightarrow{\bar{i}} \bigwedge^2 P/IP \xrightarrow{\bar{s}} \bigwedge^2 I/I^2 \xrightarrow{\bar{\delta}} K_A/IK_A$$

given by

$$\begin{aligned} &w \to xe_2 \wedge e_3 + ye_3 \wedge e_1 + ze_1 \wedge e_2 \to xs(e_2) \wedge s(e_3) + \cdots \\ &\quad \to xd(s(e_2)) \wedge d(s(e_3)) + \cdots = xdy \wedge dz. \end{aligned}$$

We have, however, $xdy \wedge dz = x^2 w$, so that the invariant of (P, i) is the quadratic space $\langle 1, 1\rangle$. Therefore P does not have a unimodular element, since $\langle 1, 1\rangle$ is not hyperbolic. Hence the tangent bundle of the two sphere is nontrivial.

4. Stably free modules and the invariant

In this section, we prove that if the dimension of A is 2, the invariant determines the isomorphism class of any stably free module. We note that since we have defined the invariant only if the top exterior power of the projective module is isomorphic to K_A, we can define the invariant for stably free modules only if $K_A \simeq A$.

THEOREM 4.1. *Let A be a two dimensional regular affine domain over $\mathbb{R}$, with $K_A \simeq A$ and $X(\mathbb{R})$ compact. Let P_1 and P_2 be two stably free A-modules of rank 2. Suppose that the invariants of P_1 and P_2 (defined after choosing trivialisations of $\bigwedge^2 P_1$ and $\bigwedge^2 P_2$) are the same. Then P_1 and P_2 are isomorphic.*

Since $K_A \simeq A$, $X(\mathbb{R})$ is orientable. In this case Barge and Ojanguren define a map from $W^{-1}(A)$ to $H^2(X(\mathbb{R}), \mathbb{Z}) = \mathbb{Z}^S$ (where S denotes the number of connected components of $X(\mathbb{R})$ as follows. We fix an isomorphism $i\colon K_A \simeq A$. Any element of $W^{-1}(A)$ is represented by a pair (P, s), where P is a projective A-module with rank $P = 2$ and $s\colon \bigwedge^2 P \simeq A$ is an isomorphism. Then $i^{-1}s$ is an isomorphism $\bigwedge^2 P \simeq K_A$. Barge and Ojanguren ([BO], Th. 6.2) prove that the map from $W^{-1}(A) \to \mathbb{Z}^S$ sending (P, s) to the invariant of $(P, i^{-1}s)$ is an isomorphism. Therefore, in order to prove Theorem 4.1 it is enough to prove

THEOREM 4.2. *Let A be as in Theorem* 4.1. *Let* (P, s_1) *and* (P_2, s_2) *be two elements of* $W^{-1}(A)$, *where* P_1 *and* P_2 *are two stably free A-modules of rank* 2. *Suppose that* $(P_1, s_1) = (P_2, s_2)$ in $W^{-1}(A)$. Then $P_1 \simeq P_2$.

Step 1. We first prove the following

CLAIM. *Let A be as in Theorem* 4.1. *Let* $(P_0, s_0) \in W^{-1}(A)$, *where* P_0 *is a stably free A-module with rank* $P_0 = 2$. *Suppose* (P_0, s_0) *is trivial in* $W^{-1}(A)$. *Then* P_0 *is free.*

Proof. We follow the method of ([OPS], p. 497) to prove this and refer the reader to [OPS] for further details and notation. Let h denote the hyperbolic space. Since P_0 is stably free, we have, $P_0 \oplus A \simeq A^3$ which implies that $P_0 \oplus A^2 \simeq A^4$. Therefore

$$(P_0, s_0) \perp (A^2, h) \simeq (A^4, s''), \tag{1}$$

for some alternating from s'' on A^4. By [OPS], there exist rank 2 projective A-modules P and Q with trivial determinants such that

$$(A^4, s'') \simeq (P, s) \perp (Q, s'). \tag{2}$$

$$(A^4, h) \simeq (P, s) \perp (Q, \pm s'). \tag{3}$$

Case 1. Suppose $(A^4, h) \simeq (P, s) \perp (Q, s')$. Then by equations (2) and (3), $(A^4, s'') \simeq (A^4, h)$. By equation (1), $(P_0, s_0) \perp (A^2, h) \simeq (A^4, s'') \simeq (A^4, h)$ and hence by cancellation, $(P_0, s_0) \simeq (A^2, h)$ and P_0 is free.

Case 2. Suppose $(A^4, h) \simeq (P, s) \perp (Q, -s')$. We then rewrite equation (3) as

$$(A^4, h) \simeq (A^4, -h) \simeq (P, -s) \perp (Q, s'). \tag{4}$$

Adding (2) and (4), we obtain

$$(A^4, s'') \perp (A^4, h) = (Q, s') \perp (Q, s') \tag{5}$$

in $W^{-1}(A)$. Using (1) and (5) we obtain

$$(P_0, s_0) \perp (A^2, h) \perp (A^4, h) = (Q, s') \perp (Q, s') \tag{6}$$

in $W^{-1}(A)$. Since (P_0, s_0) is trivial in $W^{-1}(A)$, we see using (6) that $2(Q, s')$ is trivial in $W^{-1}(A)$. By [BO], $W^{-1}(A) \simeq \mathbb{Z}^S$ is torsion free and hence (Q, s') is trivial $W^{-1}(A)$. Therefore $(Q, s') \simeq (Q, -s')$. Now we argue as in Case 1, to conclude that P_0 is free.

Step 2. We now prove that if (P_1, s_1) and (P_2, s_2) are equal in $W^{-1}(A)$, with P_1 and P_2 stably free of rank 2, then $P_1 \simeq P_2$.

Proof. We have, $(P_1, s_1) \perp (P_2 - s_2)$ is trivial in $W^{-1}(A)$ and is isomorphic to $(P_3, s_3) \perp (A^2, h)$. Since P_1 and P_2 are stably free, so is P_3. Further, (P_3, s_3) is trivial in $W^{-1}(A)$. By Step 1, P_3 is free. Therefore

$$(P_1, s_1) \perp (P_2, -s_2) \simeq (A^2, h) \perp (A^2, h).$$

Similarly, one can show that

$$(P_2, s_2) \perp (P_2, \cdot - s_2) \simeq (A^2, h) \perp (A^2, h).$$

Comparing the above two equations, we see that $(P_1, s_1) \simeq (P_2, s_2)$, so that $P_1 \simeq P_2$. This proves the theorem.

We apply the previous result to prove a theorem on sections of stably free modules. We first recall Serre's codimension 2 correspondence ([BO], §3 or [Mu]). We only state these results in the generality that we need them.

THEOREM 4.3. *Let A be a regular noetherian ring with* $\dim A = 2$. *Let $J \subset A$ be an ideal of height* 2, *which is the intersection of finitely many maximal ideals. Then we have an isomorphism* φ: $Ext^1_A(J, A) \simeq Hom_{A/J}(\bigwedge^2 J/J^2, A/J)$.

REMARK 4.4. Let $E\colon 0 \to A \to P \xrightarrow{s} J \to 0 \in Ext^1_A(J, A)$, with J as in the theorem and P a projective A-module with rank $P = 2$. Then comparing E with the Koszul resolution $0 \to \bigwedge^2 P \to P \xrightarrow{s} J \to 0$, we obtain an isomorphism $j\colon A \simeq \bigwedge^2 P$. Tensoring with A/J, we obtain an isomorphism $\tilde{s}\tilde{j}\colon A/J \xrightarrow{\tilde{j}} \bigwedge^2 P/JP \xrightarrow{\tilde{s}} \bigwedge^2 J/J^2$, where $\tilde{j}$ and $\tilde{s}$ are induced by j and s. Then, $(\tilde{s}\tilde{j})^{-1}\colon \bigwedge^2 J/J^2 \to A/J = \varphi(E)$.

This remark will be used in the proof of the following.

THEOREM 4.5. *Let A be a regular affine domain over $\mathbb{R}$ with* $\dim A = 2$ *and $X(\mathbb{R})$ compact. Assume that $K_A \simeq A$. Let P be a stably free A-module of rank* 2. *Let $s\colon P \to I$ be a good section.* (*i.e. I is a product of distinct maximal ideals*). *Then, I is generated by* 2 *elements.*

Proof.

Step 1. We first show that $V(I)$ has an even number of real points. Let $P = A^3/(a, b, c)$. We may assume by Swan's Bertini theorem ([Sw1], Th. 1.4), that (b, c) is the interesection of finitely many maximal ideals. There exists a surjection from P to (b, c) which sends e_1 to 0, e_2 to c, e_3 to $-b$. There also exists a surjection from A^2 to (b, c). Since the invariant of a free module is trivial, $V(b, c)$ has an even number of real points. Since there exists a surjection from P onto a reduced zero dimensional ideal (b, c) whose support consists of an even number of real points, comparing invariants, we see that $V(I)$ also has an even number of real points.

Step 2. We next prove the following:

Let A be as above. Let $I \subset A$ be the intersection of finitely many maximal ideals such that $V(I)$ has an even number of real points. Then there exists a projective module P', having trivial invariant, which maps onto I.

Proof. Let $\lambda: \bigwedge^2 I/I^2 \xrightarrow{\sim} A/I$ be any isomorphism. Let $0 \to A \to P' \to I \to 0 (= \varphi^{-1}(\lambda))$, where φ is as in (4.3). By altering λ by a unit of A/I, we may assume, using (4.4), and an easy computation of invariants, that P' has trivial invariant. We note that in the computation we would have to use the fact that the number of real points in $V(I)$ is even, which is a necessary condition in order for P' to have trivial invariant.

Step 3. Let

$$0 \to A \to P \xrightarrow{s} I \to 0$$

$$0 \to A \to P' \to I \to 0$$

be the two extensions where P is as in the statement of the Theorem and P' is obtained as in Step 2. By Schanuel's Lemma $P \oplus A \xrightarrow{\sim} P' \oplus A$. Since P is stably free, so is P'. Since the invariant of P' is trivial, by the previous theorem, P' is free. Therefore I is generated by two elements.

COROLLARY 4.6. *Let A be as in* (4.5). *Suppose I and J are two co-maximal ideals of A, which are both intersections of finitely many maximal ideals. Suppose that J and $I \cap J$ are generated two elements. then, so is I.*

Proof. It follows from our hypothesis that $[A/I] = 0$ in $K_0(A)$, so that there is a stably free A-module P of rank 2 mapping on to I. By (4.5), I is generated by two elements.

REMARK 4.7. For a smooth affine two dimensional domain A over any field k, the following question, raised by M. P. Murthy (cf. [Sw4], p. 581) is of general interest.

QUESTION. *Let $\mathscr{L}$ be a rank one projective module over A such that $\mathscr{L} \oplus \mathscr{L}^{-1}$ is stably free. Is $\mathscr{L} \oplus \mathscr{L}^{-1}$ free?*

This is equivalent to the question whether $c_1(\mathscr{L})^2 = 0$, ($c_1(\mathscr{L})$ denoting the first chern class of the line bundle $\mathscr{L}$), implies that $\mathscr{L}$ is generated by two elements. We note that this is indeed the case if A is a smooth affine two dimensional domain over $\mathbb{R}$, with $X(\mathbb{R})$ compact and orientable. Since $\mathscr{L} \oplus \mathscr{L}^{-1}$ supports a hyperbolic symplectic form, its invariant in $W^{-1}(A) = \mathbb{Z}^S$, is zero. By Theorem 4.1, $\mathscr{L} \oplus \mathscr{L}^{-1}$ is free.

5. A question of Bhatwadekar

Bhatwadekar (personal communication) raised the following

QUESTION. *Let A be a noetherian ring with* $\dim A = 2$. *Let $I \subset A$ be an ideal of height* 2. *Let (b, c) and (b', c') be two sets of generators of I. Let $a \in A$ be such that a is a unit* mod I. *Is there example of a ring A and an ideal I as above with the property that*

$$A^3/(a, b, c) \not\approx A^3/(a, b', c')?$$

We give an example of a ring A with the above property. Let

$$A = \frac{\mathbb{R}[X, Y, Z]}{(X^2 + Y^2 + Z^2 - 1)}.$$

Let $a = y + z, b = x, c = yz$. Let $I = (x, yz) = (x, y) \cap (x, z) = (x, y, z - 1) \cap (x, y, z + 1) \cap (x, z, y - 1) \cap (x, z, y + 1)$. Let $P_0 = A^3/(y + z, x, yz)$. We claim that P_0 is free. In fact, $(y + z, x, yz)$ is elementarily equivalent to $(y + z, x, -y^2)$, so that by the Swan-Towber-Suslin theorem [SwT], P_0 is free. One can also prove this directly, without using the Swan-Towber-Suslin theorem.

Let $P' = A^3/(y + z, b', c')$, where b', c' will be chosen appropriately. We will show, that for a suitable choice, b', c' generate I and that P' is not free. In fact the construction is fairly explicit and one can write down a set of generators b', c' of I.

We consider the unimodular row $(y - z, x, yz)$. This can be elementarily transformed to $(y - z, x, y^2)$ which is completable by the Swan-Towber-Suslin theorem.

Let

$$C = \begin{pmatrix} y - z, x, yz \\ a_{21}, a_{22}, a_{23} \\ a_{31}, a_{32}, a_{33} \end{pmatrix}$$

be a completion of the above unimodular row in $SL_3[A]$. We claim that $b' = a_{23}x - a_{22}yz$, $c' = a_{33}x - a_{32}yz$ is another set of generators of I. In order to see this, we consider the surjection $f\colon A^3 \to I$, defined by $f(e_1) = 0, f(e_2) = -yz, f(e_3) = x$. Since $(y-z, x, yz)$, (a_{21}, a_{22}, a_{23}), (a_{31}, a_{32}, a_{33}) generate A^3,

$$f(y-z, x, yz), f(a_{21}, a_{22}, a_{23}), f(a_{31}, a_{32}, a_{33})$$

generate I. Since $f(y-z, x, yz) = 0$, we see that $f(a_{21}, a_{22}, a_{23}) = a_{23}x - a_{22}yz = b'$ and $f(a_{31}, a_{32}, a_{33}) = a_{33}x - a_{32}yz = c'$ generate I. We note that b', c' can be explicitly written down in view of the fact that Swan and Towber have written down an explicit completion. Using these generators one can check that the projective module defined by (a, b', c') does not have trivial invariant and hence is not free. However, we give a more illuminating proof of this. We need the following lemmas.

We recall $b = x$, $c = yz$, $b' = a_{23}x - a_{22}yz$, $c' = a_{33}x - a_{32}yz$, $I = (b, c) = (b', c')$.

LEMMA 5.1. *Let* $A = \mathbb{R}[X, Y, Z]/(X^2 + Y^2 + Z^2 - 1)$. *Let* $w = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$. *Then* $dx \wedge d(yz) = (y^2 - z^2)w$.

Proof. An easy computation, as in Section 3.

LEMMA 5.2. *We have,* $db' \wedge dc' = (det\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}) db \wedge dc$, *in* $\bigwedge^2 \Omega_A / I\Omega_A$.

Proof. An easy computation.

LEMMA 5.3. $det\begin{vmatrix} a_{32} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = (y-z)^{-1} \bmod I$.

Proof. Follows by looking at the matrix C and using the fact that C belongs to $SL_3(A)$.

We now compute the invariant of the projective module $A^3/(a, b', c')$. Suppose $aa_1 + b'b_1 + c'c_1 = 1$. Let $s\colon P' = A^3/(a, b', c') \to I$ be given by $s(e_1) = 0$, $s(e_2) = -c'$ and $s(e_3) = b'$. By (3.1), $c_1e_1 \wedge e_2 + a_1e_2 \wedge e_3 + b_1e_3 \wedge e_1$ generates $\bigwedge^2 P'$. We define $i'\colon \bigwedge^2 P' \to K_A$ by

$$i'(a_1e_2 \wedge e_3 + b_1e_3 \wedge e_1 + c_1e_1 \wedge e_2) = w.$$

We compute the invariant of (P', i') using the section s. Consider the sequence of maps

$$K_A/IK_A \to \bigwedge^2 P'/IP' \to \bigwedge^2 I/I^2 \to K_A/IK_A$$

given by $w \to a_1 e_2 \wedge e_3 + b_1 e_3 \wedge e_1 + c_1 e_1 \wedge e_2 \to a_1(b' \wedge c') \to a_1\, db' \wedge dc' = hw$. Then $h \bmod I$ gives rise to a quadratic space which is the invariant of P'. By 5.1, 5.2 and 5.3,

$$a_1 db' \wedge dc' = a_1(y-z)^{-1}\, dx \wedge dy = a_1(y-z)^{-1}(y^2 - z^2)w.$$

The element $a_1(y-z)^{-1}(y^2-z^2) \bmod I$ is the invariant of P'. Since multiplication by squares does not change the isometry class of a quadratic space, $a^2 a_1(y-z)^{-1}(y-z)^2(y^2-z^2) \bmod I$ is the invariant of P'. Since $aa_1 \equiv 1 \bmod I$, $a(y-z)(y^2-z^2) \bmod I$ (where $a = y+z$), is the invariant of P', i.e. $(y^2-z^2)^2 \bmod I$ is the invariant of P'. Thus the invariant of P' is $\langle 1, 1, 1, 1\rangle$, which is not hyperbolic. Hence P' is not free.

REMARK 5.4.* Setting $a_{21} = 0$, $a_{22} = 1$, $a_{23} = -x/2$, $a_{31} = -2$, $a_{32} = 0$ and $a_{33} = y - z$ in the above proof one obtains that

$$P = \frac{A^3}{(y+z, x, yz)} \not\simeq \frac{A^3}{(y+z, x^2 + 2yz, (y-z)x)}.$$

On the other hand the ideals (x, yz) and $(x^2 + 2yz, (y-z)x)$ are equal.

6. $\tilde{K}_0$ and the invariant

The aim of this section is to prove the following

THEOREM 6.1. *Let A be a regular affine domain over $\mathbb{R}$ with* $\dim A = 2$ *and* $\operatorname{Pic} A = 0$. *Let $X =$* Spec A. *Assume that $X(\mathbb{R})$ is compact. Then the following are equivalent.*

(1) *$\tilde{K}_0(A)$ is 2-torsion.*
(2) *All maximal ideals of A which correspond to complex points of* Spec A *are generated by two elements.*
(3) *Any projective A-module P of rank 2 which, after tensoring with $C(X(\mathbb{R}))$, the ring of real valued continuous functions on $X(\mathbb{R})$, becomes free is already free.*

REMARK 6.2. If A is as in the Theorem, then Barge and Ojanguren prove that $W^{-1}(A) \xrightarrow{\sim} H^2(X(\mathbb{R}), \mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}^S$ is an isomorphism (where, as remarked earlier, the isomorphism is given by sending any (P, s), where rank $P = 2$ and $s\colon \bigwedge^2 P \to A$ an

**Note added in proof.* Since writing this paper, the author has become aware of a paper of C. A. Weibel (Comm. in Alg. (12), 1994, 3011–3051) which contains a similar computation (p. 3026).

isomorphism, to the invariant of (P, s)). From their results, we can deduce the following implications. Let A be as in the Theorem. Let P be a projective A-module with rank $P = 2$. Let $s: \bigwedge^2 P \to A$ be any isomorphism. Then the following are equivalent.

The invariant of (P, s) vanishes $\Leftrightarrow$ (P, s) is trivial in $W^{-1}(A)$ $\Leftrightarrow$ the image of (P, s) in $H^2(X(\mathbb{R}), \mathbb{Z})$ is trivial $\Leftrightarrow$ P is free after tensoring with the ring of real valued continuous functions $C(X(\mathbb{R}))$ on $X(\mathbb{R})$.

Proof of the Theorem. $3 \Rightarrow 2$. Suppose that $m \subset A$ is a maximal ideal with $A/m \simeq \mathbb{C}$. Then, by Serre's codimension 2 correspondence, there exists a projective A-module P with rank $P = 2$ and $\bigwedge^2 P \xrightarrow{\sim} A$, mapping onto m. Since $A/m \xrightarrow{\sim} \mathbb{C}$, $P \otimes C(X(\mathbb{R}))$ is free. By (3), P is free, and so m is generated by 2 elements.

In order to prove $2 \Rightarrow 3$, we need the following results which are contained in [RS2].

RESULT 1. Let A be an affine algebra over $\mathbb{R}$ with $\dim A = 2$. Let J be an ideal of height 2 in A, which is generated by 2 elements. Assume that J is the intersection of finitely many maximal ideals which correspond to complex points of Spec A. Let P be a projective A-module with rank $P = 2$ and $\bigwedge^2 P \xrightarrow{\sim} A$. Assume that there exists a surjection $s: P \to J$. Then P is free.

This is proved easily, using Schanuel's Lemma and the Swan-Towber-Suslin Theorem. We refer the reader to ([RS], Th. 2).

RESULT 2. Let A be a noetherian ring with $\dim A = 2$. Let I_1 and I_2 be two comaximal ideals of height 2 in A, both of which are intersections of finitely many maximal ideals. Assume that I_1 and I_2 are both generated by 2 elements. Then so is $I_1 \cap I_2$.

This is proved in ([RS], Th. 1), using Quillen's Localisation Theorem and Serre's codimension 2 correspondence.

We now prove $2 \Rightarrow 3$.

Suppose that P is a projective A-module with rank $P = 2$, such that $P \otimes C(X(\mathbb{R}))$ is free. Then, by a theorem of Swan ([Sw2], Th. 10.1), there exists a multiplicatively closed subset S of A (consisting of elements $g \in A$ such that $g(x) > 0$ for every $x \in X(\mathbb{R})$), such that $P \otimes S^{-1}A$ is a free $S^{-1}A$-module. Let $S^{-1}P \to S^{-1}A$ be any surjective map. By multiplying by a suitable element s of S, we may assume, that there exists $t_0 = t: P \to A$, such that $t(p) = f \in S$, for some $p \in P$.

By Swan's Bertini Theorem, there exist sections $t_1, \ldots, t_n \in P^*$, such that for all $(\lambda_0, \ldots, \lambda_n)$ in some nonempty Zariski open subset U of $\mathbb{R}^{n+1}$, $\lambda_0 t_0 + \cdots + \lambda_n t_n(P)$ is a reduced zero-dimensional ideal. We prove that for a suitable choice of

$(\lambda_0, \ldots, \lambda_n)$, in this Zariski open set, $\lambda_0 t_0(p) + \cdots + \lambda_n t_n(p)$ is a function which is >0 for all points $x \in X(\mathbb{R})$. Since U is Zariski open, it contains a basic open set of the form $D(h)$ where $h \in \mathbb{R}[x_0, \ldots, x_n]$. We choose λ_0 sufficiently large so that $h(\lambda_0, \ldots, \lambda_n) \neq 0$ and $\lambda_0 t_0(p) + \cdots + \lambda_n t_n(p) = g$ is a function such that $g(x) > 0$ for every $x \in X(\mathbb{R})$. We can do this, since $X(\mathbb{R})$ is compact and $t_0(p)$ is a function f, which satisfies the property that $f(x) > 0$ for $x \in X(\mathbb{R})$. For this choice of $\lambda_0, \ldots, \lambda_n$, $\lambda_0 t_0 + \cdots + \lambda_n t_n$ is a map from $P \to A$ such that the image is a reduced zero dimensional ideal which is supported only at complex points. Now, we appeal to Results 1 and 2 to conclude that P is free.

We now prove $3 \Rightarrow 1$.

Assume 3 and let Q be an element of $\tilde{K}_0(A)$ which is not 2-torsion. Since $\operatorname{Pic} A = 0$, we may assume that rank $Q = 2$, and let $(Q, s) \perp (Q, -s) \simeq (P, s) \perp (A^2, h)$. Then, P is a projective A-module of rank 2 and one checks easily that (P, s) is trivial in $W^{-1}(A)$, Using the fact that Q is not a 2-torsion element of $\tilde{K}_0(A)$, one verifies that P is not free. By the remark preceding the proof of the Theorem, one sees that since (P, s) is trivial in $W^{-1}(A)$, $P \otimes C(X(\mathbb{R}))$ is free. This contradicts 3.

$1 \Rightarrow 3$.

By the remark preceding the proof of the theorem, it suffices to prove the following: If (P, s) is an element of $W^{-1}(A)$, where P is of rank 2 and (P, s) is trivial in $W^{-1}(A)$, then P is free. Since (P, s) is trivial in $W^{-1}(A)$ and $\operatorname{Pic} A = 0$, one sees easily that $(P, s) \perp (A^2, h) \simeq (Q, s') \perp (Q, -s')$. Since $\tilde{K}_0(A)$ is 2-torsion, one verifies easily that $Q \oplus Q$ is free, hence P is stably free. Since $(P, s) = 0$ in $W^{-1}(A)$, by Theorem 4.2, P is free.

Appendix

In this section, we prove a proposition about homotopy of sections of projective modules that was used in Section 2.

PROPOSITION. Let A be an affine domain over an infinite field with $\dim A = n$. Let P be a projective A-module with rank $P = n$. Let $s_0: P \to I_0$ and $s_1: P \to I_1$, be two sections, where I_0 and I_1 are reduced zero-dimensional ideals in A. Then there exists a section $s(t): P[t] \to J$, with $A[t]/J$ a Dedekind domain such that $s(0) = s_0$ and $s(1) = s_1$.

Proof. Let $s'(t) = ts_0 + (1-t)s_1$. Then $s'(0) = s_0$ and $s'(1) = s_1$. We modify $s'(t)$ to $s(t)$ as follows. By Swan's Bertini Theorem, there exist sections

$s_1(t), \ldots, s_n(t)\colon P[t] \to J$ and $\lambda_0, \ldots, \lambda_n \in \mathbb{R}$, such that $s(t) = \lambda_0 s'(t) + \lambda_1 t(1-t)s_1(t) + \lambda_2 t(1-t)s_2(t) + \cdots + \lambda_n t(1-t)s_n(t)$ is a map $P[t] \to A[t]$, with the property that if $J = s(t)(P)$, then $(A[t]/J)_{t(t-1)}$ is a Dedekind domain. We may, by scaling, assume that $\lambda_0 = 1$. We prove that $R = A[t]/J$ is a Dedekind domain. Since $R_{t(t-1)}$ is a Dedekind domain, it is enough to check that if m is a maximal ideal of R which contains t or $t-1$, then R_m is a discrete valuation ring. We check this if m contains t. Let p be a prime ideal of $A[t]$ which is minimal over J, which is contained in m. Since there exists a surjective map $P[t] \to J$ and $P[t]_m$ is free of rank n, J_m is generated by n elements. By Krull's principal ideal theorem, the height of p in $A[t]$ is at most n and since A is an affine domain, with $\dim A = n$, $\dim A[t]/p \geq 1$ so that $\dim R_m \geq 1$. On the other hand mR_m is a principal ideal which is generated by t. This implies that R_m is a discrete valuation ring. This completes the proof that R is a Dedekind domain.

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Raja Sridharan
School of Mathematics
Tata Institute of Fundamental Research
Bombay 400 005, India
E-mail: *sraja@tifrvax.tifr.res.in*

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