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Compactifying coverings of 3-manifolds

MICHAEL L. MIHALIK

Abstract. If a finitely presented group G is negatively curved, automatic or asynchronously automatic then G has an asynchronously bounded "almost prefix closed" combing. Results in [Br_1] and [E] imply that the fundamental group of any closed 3-manifold satisfying Thurston's geometrization conjecture has an asynchronously bounded, almost prefix closed combing.

MAIN THEOREM. If M is a compact P^2-irreducible 3-manifold, $\pi_1(M)$ has an asynchronously bounded, almost prefix closed combing, and H, a subgroup of $\pi_1(M)$, is quasiconvex with respect to this combing, then the cover of M corresponding to H is a missing boundary manifold.

§1. Introduction

A 3-manifold M is a *missing boundary manifold* if there is a compact manifold $\bar{M}$, and a subset K of the boundary of $\bar{M}$ such that $\bar{M} - K$ is homeomorphic to M. If M is a compact P^2-irreducible 3-manifold and $\hat{M}$ is the cover of M corresponding to a finitely generated subgroup of $\pi_1(M)$, then J. Simon has conjectured that $\hat{M}$ is a missing boundary manifold (see [Si]).

In §2, we define a geometric group theory notion "1-tame", for pairs of groups, motivated by the combing ideas for groups introduced in [MT].

The connection between compactifying 3-manifolds and 1-tame pairs is described by the following result, proved in §4.

If G is the fundamental group of a compact P^2-irreducible 3-manifold M, and H is a finitely generated subgroup of G then, the pair (H, G) is 1-tame if and only if the cover of M corresponding to H is a "missing boundary" manifold.

In [Si] Simon verified his conjecture for M any compact P_2-irreducible 3-manifold and H the fundamental group of a boundary component of M. In [J] W. Jaco generalized this to the case where H is a finitely generated peripheral subgroup of $\pi_1(M)$. W. Thurston showed in [T] that if M admits a geometrically finite, complete hyperbolic structure of infinite volume, then M is a missing boundary manifold. F. Bonahon [Bo] showed that any hyperbolic-3-manifold M with finitely generated fundamental group is a missing boundary manifold, provided that $\pi_1(M)$ is not a

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free product. In [HRS] J. Hass, H. Rubinstein and P. Scott show that if M is a closed P^2-irreducible 3-manifold such that $\pi_1(M)$ contains a subgroup A isomorphic to $\mathbf{Z} \times \mathbf{Z}$, then the cover of M with fundamental group A is a missing boundary manifold.

Our results on the Simon conjecture are obtained through an examination of combings of groups. If a finitely presented group G is word hyperbolic, automatic or asynchronously automatic then G has an asynchronously bounded "almost prefix closed" combing. Results in [Br_1] and [E] imply that the fundamental group of any closed 3-manifold satisfying Thurston's geometrization conjecture has an asynchronously bounded, almost prefix closed combing.

Our main result in this paper is

MAIN THEOREM. *If M is a compact P^2-irreducible 3-manifold, $\pi_1(M)$ has an asynchronously bounded, almost prefix closed combing, and H, a subgroup of $\pi_1(M)$, is quasiconvex with respect to this combing, then the cover of M corresponding to H is a missing boundary manifold.*

M. Bridson has observed that if G is the 3-dimensional Heisenberg group then any finitely generated subgroup of G is quasiconvex with respect to some asynchronously bounded almost prefix closed combing. If G is a CAT(0) group acting properly discontinuously, cocompactly and by isometries on the geodesic metric space X and H is an abelian subgroup of G then by the "Flat Torus Theorem", H acts on an n-flat in X and so H is quasiconvex with respect to geodesics in X. In particular, it G is a knot or link group (then G is CAT(0) see [BH] and [L]) and H a peripheral subgroup then H is quasiconvex in a bounded almost prefix closed combing of G. More generally, if M is a compact P_2-irreducible 3-manifold satisfying Thurston's geometrization conjecture and H the fundamental group of a boundary component of M, then H is quasiconvex in some bounded almost prefix closed combing of $\pi_1(M)$ (see [Br_2]). Combining this with our main theorem gives Simon's result [Si].

In [R], L. Reeves shows that cubed 3-manifolds have biautomatic fundamental groups. These manifolds have canonical immersed surfaces and the corresponding surface groups are quasiconvex (rational) in the fundamental group of the cubed manifolds with respect to the biautomatic structure. Hence (by the above theorem) the covers of cubed manifolds corresponding to these surface groups are missing boundary manifolds.

In [GS] S. Gersten and H. Short study the quasiconvex subgroups of biautomatic groups.

In the case that G is the fundamental group of a closed irreducible 3-manifold M, the pair (G, H) is 1-tame and H is the trivial group, results in [BT], [MT] and

[Tu] imply that the universal cover of M is homeomorphic to $\mathbf{R}^3$. It is an easy matter to see that if G is "almost convex" then the pair $(G, 1)$ is 1-tame. This gives a different proof of the main result of V. Poénaru's paper [P]. The relation between A. Casson's C_2 condition, the *QSF* condition developed by S. Brick, S. Gersten and J. Stallings and tame pairs $(G, 1)$ is discussed in [MT].

§2. Definitions, and statements of results

While our main interest in this paper is in compactifying covers of compact 3-manifolds we will work with more general spaces, namely polyhedra. If T is a triangulation of the pair of polyhedra (X, C), then the largest subcomplex of the first baracentric subdivision of T, whose realization is contained in $X - C$, is a strong deformation retract of $X - C$. This can be used to show,

LEMMA 1. *If (X, C) is a pair of polyhedra with C compact, then the following are equivalent.*

(1) *$\pi_1(X - C)$ is finitely generated (I.e. each component of $X - C$ has finitely generated fundamental group).*
(2) *$\pi_1(Cl(X - C))$ is finitely generated.*
(3) *For each component K of $X - C$, $\pi_1(Cl(K))$ is finitely generated.*
(4) *For any vertex $v \in X - C$, there is a finite subcomplex B of X containing v such that any loop in $X - C$, based at v, is homotopic rel $\{0, 1\}$ to a loop in B by a homotopy in $X - C$.*

By (4), if $C_1 \subset C_2$ are finite subcomplexes of X and $\pi_1(X - C_2)$ is finitely generated then $\pi_1(X - C_1)$ is finitely generated. So to show that $\pi_1(X - C)$ is finitely generated for all finite subcomplexes C of X, it suffices to show $\pi_1(X - C)$ is finitely generated for sufficiently large C.

DEFINITION. *Let X be a compact polyhedra, H a finitely generated subgroup of $\pi_1(X)$ and $*$ a vertex of $\tilde{X}$ ($\equiv$ the universal cover of X). The pair (X, H) is 1-tame if for each integer N there is an integer M such that for any edge path α in $Cl(\tilde{X} - St^N(H*))$ with $\alpha(0), \alpha(1) \in St^N(H*)$, α is homotopic rel$\{0, 1\}$ to an edge path β in $St^M(H*)$, by a homotopy in $Cl(\tilde{X} - St^N(H*))$.*

This definition is easily seen to be independent of base point $*$, and triangulation of $\tilde{X}$.

THEOREM 1. *If X_1 and X_2 are compact polyhedra, H_1 is a finitely generated subgroup of $\pi_1(X_1)$ and $f(\pi_1(X_1), H_1) \to (\pi_1(X_2), H_2)$ is an isomorphism of pairs then (X_1, H_1) is* 1*-tame if and only if (X_2, H_2) is* 1*-tame.*

DEFINITION. *If G is a finitely presented group and H a finitely generated subgroup of G, then the pair (G, H) is* 1*-tame if for some (any) finite polyhedra X, with $\pi_1(X) = G$, we have (X, H) is* 1*-tame.*

DEFINITION. *A* 3*-manifold M is a missing boundary manifold if there is a compact manifold $\bar{M}$, and a subset K of the boundary of $\bar{M}$ such that $\bar{M} - K$ is homeomorphic to M.*

In [Tu], T. Tucker shows that if M is a noncompact P^2-irreducible 3-manifold, and for each finite subcomplex C of (some triangulation of) M, $\pi_1(M - C)$ is finitely generated then M is a missing boundary manifold.

THEOREM 2. *If X is a finite polyhedra with $\pi_1(X) = G$ and H is a finitely generated subgroup of G, then (G, H) is* 1*-tame iff for each finite subcomplex C of $H \backslash \tilde{X}$, $\pi_1((H \backslash \tilde{X}) - C)$ is finitely generated.*

COROLLARY 3. *If M is a compact P^2-irreducible* 3*-manifold and H is a finitely generated subgroup of $\pi_1(M)$ then $H \backslash \tilde{X}$ is a missing boundary manifold iff $(\pi_1(M), H)$ is* 1*-tame.*

In the next section we define combings of groups.

If a finitely presented group G is negatively curved, automatic or asynchronously automatic then G has an asynchronously bounded "almost prefix closed" combing. Results in [Br_1] and [E] imply that the fundamental group of any closed 3-manifold satisfying Thurston's geometrization conjecture has an asynchronously bounded, almost prefix closed combing.

Our main result is an easy corollary of the following more general result.

THEOREM 4. *If a group G has an asynchronously bounded, almost prefix closed combing, and H, a subgroup of G, is quasiconvex with respect to this combing, then the pair (G, H) is* 1*-tame.*

§3. Algebraic preliminaries

Our approach to combings is a geometric one. If A is a finite set of generators for a group G, then $\Gamma(G, A)$, the Cayley graph of G with respect to A, is a

1-complex with 0-skeleton equal to G and a directed edge from vertex v to vertex w if $va = w$ for some $a \in A$. Making each edge of Γ isometric to the unit interval gives a metric d, on Γ. Let $*$ be the identity vertex of Γ. A *combing* of G (with respect to A) is a set of edge paths $\{\rho_v: [0, T_v] \to \Gamma\}$ such that $v \in G$, $\rho_v(0) = *$, $\rho_v(T_v) = v$. As a convenience ρ_v is extended to $[0, \infty)$ by setting $\rho_v(x) = v$ for all $x > T_v$. A combing is *asynchronously bounded* if there is an integer K such that for each pair of adjacent vertices u, v in Γ, there are non-decreasing surjections $\alpha, \beta: [0, \infty) \to [0, \infty)$ each of which, when restricted to an interval between adjacent integers, is either constant or an isometry and so that $d(\rho_u(\alpha(x)), \rho_v(\beta(x))) \leq K$ for all $x \in [0, \infty)$. We call (α, β) an *asynchronous adjustment for* (ρ_u, ρ_v). The constant K is called the *asynchronous fellow traveler constant.*

A combing $\{\rho_v\}$ of G (with respect to A) defines a map r of G into A^*, the free monoid. The group G is *asynchronously automatic* if $\{\rho_v\}$ is asynchronously bounded and $r(G)$ is the language accepted by some finite state automaton.

A combing $\{\rho_v\}$ is *almost prefix closed* if there is an integer K such that for each $v \in G$ and integer $n \in [0, T_v]$ there is a $w \in G$ such that $\rho_w(T_w) \in St^K(\rho_v(n))$ and the prefix $\rho_v([0, n])$ is a subset of $\mathrm{St}^K(im(\rho_w))$.

Say G is asynchronously automatic. Then $r(G) \subset A^*$ is the accept language of a finite state automaton. Let K be an integer so that if S is a state of this automaton that can be joined to an accept state, then it can be joined by a path of length $\leq K$. Then any prefix of a combing path given by r can be extended by K or fewer edges to an actual combing path. Hence any asynchronously automatic group has an asynchronously bounded, almost prefix closed combing. By Theorem 8.28 of [E] the fundamental group of a Haken 3-manifold need not be asynchronously automatic.

If G is a finitely generated group with combing $\{\rho_v\}$, then the subgroup H of G is *quasi-convex* with respect to $\{\rho_v\}$ if there is an integer L such that for each $h \in H$, $\mathrm{im}(\rho_h) \subset \mathrm{St}^L(H) \subset \Gamma(G, A)$.

§4. Proofs of Theorems 1 and 2

The following lemmas are straightforward.

LEMMA 4.1. *If X is a finite simplicial complex and $X^{(2)}$ its 2-skeleton, then (X, H) is 1-tame iff $(X^{(2)}, H)$ is 1-tame.*

Hence we need only consider the case X_1 and X_2 are 2-dimensional simplicial complexes.

LEMMA 4.2. *If* $(X_1, *_1)$ *and* $(X_2, *_2)$ *are finite complexes such that* $(\pi_1(X_1, *_1), H_1)$ *and* $(\pi_1(X_2, *_2), H_2)$ *are isomorphic pairs of groups, then there are maps* $f_1\colon (X_1, *_1) \to (X_2, *_2)$ *and* $f_2\colon (X_2, *_2) \to (X_1, *_1)$ *such that:*

(i) $(f_1 \circ f_2)_\#$ *and* $(f_2 \circ f_1)_\#$, *the induced maps on fundamental groups, are the identity.*

(ii) $(f_1)_\#(H_1) = H_2$ *(and so* $(f_2)_\#(H_2) = H_1$*).*

(iii) *For each edge* e *of* $X_1(X_2)$, $f_1(e)(f_2(e))$ *is an edge path.*

LEMMA 4.3. *Say* f_i, X_i, H_i *and* $*_i$ *are as in Lemma* 4.2. *Let* $\tilde{f}_1$ *be the lift of* f_1 *to* $(\tilde{X}_1, \tilde{*}_1)$ *that takes* $\tilde{*}_1$ *to* $\tilde{*}_2$ *and let* $\tilde{f}_2$ *be the lift of* f_2 *to* $(\tilde{X}_2, \tilde{*}_2)$ *that takes* $\tilde{*}_2$ *to* $\tilde{*}_1$. *Then there is an integer* K_1 *such that if* $x \in \tilde{X}_2$, *then* $\tilde{f}_1 \circ \tilde{f}_2(x) \in St^{K_1}(x)$ *and, if* $x \in \tilde{X}_1$ *then* $\tilde{f}_2 \circ \tilde{f}_1(x) \in St^{K_1}(x)$.

Proof of Theorem 1

Assume the notation of Lemma 4.3, and that (X_2, H_2) is 1-tame. The group $\pi_1(X_i, *_i)$ acts on the left of $\tilde{X}_i (i \in \{1, 2\})$. By elementary covering space theory, if $g_i \in \pi_1(X_i, *_i)$, then $\tilde{f}_1(g_1\tilde{*}_1) = ((f_1)_\#(g_1))\tilde{*}_2$, and $\tilde{f}_2(g_2\tilde{*}_2) = ((f_2)_\#(g_2))\tilde{*}_1$. In particular, $\tilde{f}_2 \circ \tilde{f}_1(g_1\tilde{*}_1) = g_1\tilde{*}_1$, $\tilde{f}_1 \circ \tilde{f}_2(g_2\tilde{*}_2) = g_2\tilde{*}_2$, $\tilde{f}_1(H_1\tilde{*}_1) = H_2\tilde{*}_2$ and $\tilde{f}_2(H_2\tilde{*}_2) = H_1\tilde{*}_1$. Note that if K_2 is an integer larger than the length of the edge path $f_i(e)$ for all edges e in X_i, then for any edge e in $\tilde{X}_1(\tilde{X}_2)$, $\tilde{f}_2\tilde{f}_1(e)(\tilde{f}_1\tilde{f}_2(e))$ has length $\leq K_2^2$.

Now choose an integer K_3 such that any edge loop in $\tilde{X}_1$ or $\tilde{X}_2$ of length $\leq 2K_1 + K_2^2 + 1$ is homotopically trivial in $\mathrm{St}^{K_3}(v)$ for any vertex v of the loop.

Let A be an integer and γ an edge path in $\mathrm{Cl}(\tilde{X}_1 - \mathrm{St}^A(H_1\tilde{*}_1))$ such that $\gamma(0), \gamma(1)$ are elements of $\mathrm{St}^A(H_1\tilde{*}_1)$. Then γ decomposes into edge subpaths each of which is either a path in $\mathrm{St}^{A+K_3}(H_1\tilde{*}_1)$ or an edge path α such that $\alpha(0), \alpha(1)$ are elements of $\mathrm{St}^{A+K_3}(H_1\tilde{*})$ and $\mathrm{im}(\alpha) \subset \mathrm{Cl}(\tilde{X}_1 - \mathrm{St}^{A+K_3}(H_1\tilde{*}_1))$. To see that (X_1, H_1) is 1-tame, it suffices to find an integer J so that any such α is homotopic rel$\{0, 1\}$ to an edge path in $\mathrm{St}^J(H_1\tilde{*}_1)$ by a homotopy in $\mathrm{Cl}(\tilde{X}_1 - \mathrm{St}^A(H_1\tilde{*}_1))$.

Let α be as above. Let v_0 be the initial point of α and v_1 the end point of α. Choose $\delta_0(\delta_1)$, an edge path of length $\leq K_1$ from v_0 to $\tilde{f}_2 \circ \tilde{f}_1(v_0)$ (v_1 to $\tilde{f}_2 \circ \tilde{f}_1(v_1)$). By the definition of K_3, α is homotopic rel$\{0, 1\}$ to $\langle \delta_0, \tilde{f}_2 \circ \tilde{f}_1(\alpha), \delta_1^{-1} \rangle$ by a homotopy in $\mathrm{St}^{K_3}(im(\alpha)) \subset \mathrm{Cl}(\tilde{X}_1 - \mathrm{St}^A(H_1\tilde{*}_1))$. We have reduced our problem to showing:

There is an integer J such that for any α, as above, $\tilde{f}_2 \circ \tilde{f}_1(\alpha)$ is homotopic rel$\{0, 1\}$ to an edge path in $\mathrm{St}^J(H_1\tilde{*}_1)$ by a homotopy in $\mathrm{Cl}(\tilde{X}_1 - \mathrm{St}^A(H_1\tilde{*}_1))$.

If N is a positive integer choose $L_1(N)$ such that $\tilde{f}_1(\mathrm{St}^N(\tilde{*}_1)) \subset \mathrm{St}^{L_1(N)}(\tilde{*}_2)$ and $L_2(N)$ such that $\tilde{f}_2(\mathrm{St}^N(\tilde{*}_2)) \subset \mathrm{St}^{L_2(N)}(\tilde{*}_1)$. Then $\tilde{f}_1(\mathrm{St}^N(H_1\tilde{*}_1)) \subset \mathrm{St}^{L_1(N)}(H_2\tilde{*}_2)$ and

$\tilde{f}_2(\mathrm{St}^N(H_2\tilde{*}_2)) \subset \mathrm{St}^{L_2(N)}(H_1\tilde{*}_1)$. Furthermore, we may assume that L_i is an increasing function.

Note that if $x \in \tilde{X}_2 - \mathrm{St}^{K_1+L_1(N)}(H_2\tilde{*}_2)$, then $\tilde{f}_2(x) \in \tilde{X}_1 - \mathrm{St}^N(H_1\tilde{*}_1)$. (If $\tilde{f}_2(x) \in \mathrm{St}^N(H_1\tilde{*}_1)$, then $\tilde{f}_1\tilde{f}_2(x) \in \mathrm{St}^{L_1(N)}(H_2\tilde{*}_2)$, but $x \in \mathrm{St}^{K_1}(\tilde{f}_1 \circ \tilde{f}_2(x))$ implies that $x \in \mathrm{St}^{K_1+L_1(N)}(H_2\tilde{*}_2)$.) Similarly if $x \in \tilde{X}_1 - \mathrm{St}^{K_1+L_2(N)}(H_1\tilde{*}_1)$, then $\tilde{f}_1(x) \in \tilde{X}_2 - \mathrm{St}^N(H_2\tilde{*}_2)$.

As $\tilde{f}_1(\mathrm{St}^{A+K_3}(H_1\tilde{*}_1)) \subset \mathrm{St}^{L_1(A+K_3)}(H_2\tilde{*}_2)$, we have $\tilde{f}_1(\alpha(0))$ and $\tilde{f}_1(\alpha(1))$ in $\mathrm{St}^{L_1(A+K_3)}(H_2\tilde{*}_2)$. Decompose $\tilde{f}_1(\alpha)$ as $\langle \beta_1, \alpha_1, \beta_2, \alpha_2, \ldots, \beta_n, \alpha_n, \beta_{n+1}\rangle$ where α_i and β_i are sub-edge paths of $\tilde{f}_1(\alpha)$, such that $\mathrm{im}(\beta_i) \subset \mathrm{St}^{K_1+L_1(A+K_3)}(H_2\tilde{*}_2)$ and $\mathrm{im}(\alpha_i) \subset \mathrm{Cl}(\tilde{X}_2 - \mathrm{St}^{K_1+L_1(A+K_3)}(H_2\tilde{*}_2))$ with $\alpha_i(0)$ and $\alpha_i(1)$ in $\mathrm{St}^{K_1+L_1(A+K_3)}(H_2\tilde{*}_2)$, for all i. Since (X_2, H_2) is 1-tame, there is an integer $M(K_1 + L_1(A + K_3))$ and for each i, a homotopy F_i of α_i, rel$\{0, 1\}$, to an edge path γ_i, in $\mathrm{St}^M(H_2\tilde{*}_2)$ such that $\mathrm{im}(F_i) \subset \mathrm{Cl}(\tilde{X}_2 - \mathrm{St}^{K_1+L_1(A+K_3)}(H_2\tilde{*}_2)) \subset \tilde{X}_2 - \mathrm{St}^{K+1+L_1(A)}(H_2\tilde{*}_2)$. By the above note, $\mathrm{im}(\tilde{f}_2 \circ F_i) \subset \tilde{X}_1 - St^A(H_1\tilde{*}_1)$.

Let $M_1 = \max\{M(K_1 + L_1(A + K_3)), K_1 + L_1(A + K_3)\}$. Then $\tilde{f}_2 \circ \tilde{f}_1(\alpha)$ is homotopic rel$\{0, 1\}$ to an edge path in $\tilde{f}_2(\mathrm{St}^{M_1}(H_2\tilde{*}_2)) \subset \mathrm{St}^{L_2(M_1)}(H_1\tilde{*}_1)$ by a homotopy in $\tilde{X}_1 - \mathrm{St}^A(H_1\tilde{*}_1)$, and we can select $J = L_2(M_1)$.

Similarly if (X_1, H_1) is 1-tame, we have (X_2, H_2) is 1-tame. □

Proof of Theorem 2. Say (G, H) is 1-tame. Let $*$ be a vertex of X and $\tilde{*}$ a vertex of $\tilde{X}$ over $*$. Let $q: \tilde{X} \to H\backslash\tilde{X}$ be the quotient map and let $\hat{*} = q(\tilde{*})$. To help distinguish which space a star occurs in, we use St_H^N for the N-th star in $H\backslash\tilde{X}$. If N is a positive integer and v is a vertex of $\mathrm{Bd}(\mathrm{St}_H^N(\hat{*}))$ then we show there is an integer $M(N)$ such that any loop at v in $\mathrm{Cl}(H\backslash\tilde{X} - \mathrm{St}_H^N(\hat{*}))$ is homotopic rel$\{0, 1\}$ to a loop in $\mathrm{St}_H^M(\hat{*})$, by a homotopy in $\mathrm{Cl}(H\backslash\tilde{X} - \mathrm{St}_H^N(\hat{*}))$. As $\pi_1(\mathrm{Cl}(\mathrm{St}_H^M(\hat{*}) - \mathrm{St}_H^N(\hat{*}))$ is finitely generated, we will have $\pi_1(\mathrm{Cl}(H\backslash\tilde{X} - \mathrm{St}_H^N(\hat{*}))$ is finitely generated. Let α be an edge loop at v with image in $\mathrm{Cl}(H\backslash\tilde{X} - \mathrm{St}_H^N(\hat{*}))$. Choose $\tilde{v} \in \tilde{X}$ over v and let $\tilde{\alpha}$ be the lift of α to $\tilde{v}$. Observe that for any integer K, $q^{-1}(\mathrm{St}_H^K(\hat{*})) = \mathrm{St}^K(H\tilde{*})$. Hence $\tilde{\alpha}$ has image in $\mathrm{Cl}(\tilde{X} - \mathrm{St}^N(H\tilde{*}))$ and $\tilde{\alpha}(0)$, $\tilde{\alpha}(1)$ are elements of $\mathrm{St}^N(H\tilde{*})$.

Since (G, H) is 1-tame, there is an integer M such that $\tilde{\alpha}$ is homotopic rel$\{0, 1\}$ to an edge path in $\mathrm{St}^M(H\tilde{*})$ by a homotopy in $\mathrm{Cl}(\tilde{X} - \mathrm{St}^N(H\tilde{*}))$. Composing this homotopy with q gives a homotopy of α rel$\{0, 1\}$, to an edge loop in $\mathrm{St}_H^M(\hat{*})$ by a homotopy in $\mathrm{Cl}(H\backslash\tilde{X} - \mathrm{St}_H^N(\hat{*}))$.

Conversely, assume for each integer N, each component of $\mathrm{Cl}(H\backslash\tilde{X} - \mathrm{St}_H^N(\hat{*}))$ has finitely generated fundamental group. For each component of $\mathrm{Cl}(H\backslash\tilde{X} - \mathrm{St}_H^N(\hat{*}))$ (there are only finitely many) choose a vertex base point in the boundary of this component. If C is such a component and v is the base point of C, then as $\pi_1(C, v)$ is finitely generated, we may choose $M_{C,v}$ an integer such that $\mathrm{St}_H^M(\hat{*})$ contains a finite set of edge loops which represent generators of $\pi_1(C, v)$.

Now any loop in $\mathrm{Cl}(H\backslash\tilde{X} - \mathrm{St}_H^N(\hat{*}))$ based at v is homotopic rel$\{0, 1\}$ to a loop in $\mathrm{St}_H^M(\hat{*})$ by a homotopy in $\mathrm{Cl}(H\backslash\tilde{X} - \mathrm{St}_H^N(\hat{*}))$. Choose M_1 such that if v and w are vertices of $\mathrm{Bd}(\mathrm{St}_H^N(\hat{*}))$ which are in the same component of $\mathrm{Cl}(H\backslash\tilde{X} - \mathrm{St}_H^N(\hat{*}))$ then there is an edge path between v and w, in $\mathrm{Cl}(H\backslash\tilde{X} - \mathrm{St}_H^N(\hat{*}))$ of length $\leq M_1$. Let M be an integer larger than M_1 and each of the (finitely many) $M_{C,v}$ above. Now by a change of base point argument any loop in $\mathrm{Cl}(H\backslash\tilde{X} - \mathrm{St}_H^N(\hat{*}))$ based at a vertex in $\mathrm{Bd}(\mathrm{St}_H^N(\hat{*}))$ is homotopic rel$\{0, 1\}$ to a loop in $\mathrm{St}_H^M(\hat{*})$ by a homotopy in $\mathrm{Cl}(\tilde{X} - \mathrm{St}_H^N(*))$.

Let α be an edge path in $\mathrm{Cl}(\tilde{X} - \mathrm{St}^N(H\tilde{*}))$ such that $\alpha(0)$, $\alpha(1)$ are vertices of $\mathrm{St}^N(H\tilde{*})$. Then $q \circ \alpha$ is an edge path in $\mathrm{Cl}(H\backslash\tilde{X} - \mathrm{St}_H^N(\hat{*}))$ with vertices in $\mathrm{Bd}\,\mathrm{St}_H^N(\hat{*})$. Let β be an edge path of length $\leq M_1$ from $q\alpha(1)$ to $q\alpha(0)$ in $\mathrm{Cl}(H\backslash\tilde{X} - \mathrm{St}_H^N(\hat{*}))$. Now $\langle q\alpha, \beta\rangle$ is homotopic rel$\{0, 1\}$ to an edge loop γ at $q\alpha(0)$ where γ is in $\mathrm{St}_H^M(\hat{*})$ and the image of this homotopy is in $\mathrm{Cl}(H\backslash\tilde{X} - \mathrm{St}_H^N(\hat{*}))$. Let F be such a homotopy.

We have $q \circ \alpha$ is homotopic rel$\{0, 1\}$ to $\langle\beta, \gamma^{-1}\rangle$ by a homotopy with the same image as F. Call this homotopy W. Lifting W gives a homotopy of α rel$\{0, 1\}$ to and edge path (the lift of $\langle\gamma, \beta^{-1}\rangle$ to $\alpha(0)$) in $\mathrm{St}^M(H\tilde{*})$. Furthermore this homotopy has image in $\mathrm{Cl}(\tilde{X} - \mathrm{St}^N(\tilde{*}))$, so (G, H) is 1-tame. □

§5. Proof of the main theorem

Let P be a presentation for G. Let X be the standard 2-complex corresponding to P. Let $\tilde{X}$ be the universal cover of X. The 1-skeleton of $\tilde{X}$ is the Cayley graph of P, and so the vertices of $\tilde{X}$ are the elements of G. Let $*$ represent the identity vertex. Let $q\colon \tilde{X} \to H\backslash\tilde{X}$ be the quotient map, and $* \in H\backslash\tilde{X} = q(*)$. We have an asynchronously bounded, almost prefix closed combing for G. For each vertex v of $\tilde{X}$, let $\rho_v\colon [0, \infty) \to \tilde{X}$ be the combing path for v. Let L be the quasiconvex constant for H with respect to $\{\rho_v\}$, and δ be the asynchronous fellow traveler constant. Let I be the constant that arises from $\{\rho_v\}$ being almost prefix closed. I.e. if $\alpha\colon [0, n] \to \tilde{X}$ is a prefix of a combing path, then there exists $v \in G$ such that $d(v, \alpha(n)) \leq I$ and $\mathrm{im}(\alpha) \subset \mathrm{St}^I(\mathrm{im}(\rho_v))$. Let Q be an integer such that any edge loop α in $\tilde{X}$ of length at most $2\delta + 2$ is homotopically trivial in $\mathrm{St}^Q(v)$ for all vertices $v \in \alpha$.

LEMMA 5.1. *If M is an integer, $\alpha\colon [0, n] \to \tilde{X}$ is an initial segment of a combing path and $\alpha(n) \in St^M(H)$, then $im(\alpha) \subset St^{I+L+(I+M)\delta}(H)$.*

Proof. Choose v such that $d(v, \alpha(n)) \leq I$ and $\mathrm{im}(\alpha) \subset \mathrm{St}^I(\mathrm{im}(\rho_v))$. Let $\langle e_1, \ldots, e_k\rangle$ be an edge path in $\tilde{X}$ such that $e_1(0) \in H$, $e_k(1) = v$ and $k \leq I + M$. We have $\mathrm{im}(\rho_{e_1(0)}) \subset \mathrm{St}^L(H)$ (since H is quasiconvex). Also, $\mathrm{im}(\rho_{e_1(1)}) \subseteq$

$\mathrm{St}^\delta(\mathrm{im}(\rho_{e_1(0)}))$ and $\mathrm{im}(\rho_{e_i(1)}) \subset \mathrm{St}^\delta(\mathrm{im}(\rho_{e_{i-1}(1)}))$ for $i \in \{2, 3, \ldots, k\}$ (by the definition of δ). Hence $\mathrm{im}(\rho_v) \subseteq \mathrm{St}^{L+(I+M)\delta}(H)$ and so $\mathrm{im}(\alpha) \subset \mathrm{St}^{I+L+(I+M)\delta}(H)$. $\square$

LEMMA 5.2. *If α and β are terminal segments of combing paths,* ρ_v and ρ_w *respectively, such that there is an edge e from v to w, the initial points of α and β correspond under the asynchronous matching, and γ is an edge path of length $\leq \delta$ from the initial point of α to the initial point of β then $\langle \alpha, e, \beta^{-1}, \gamma^{-1} \rangle$ is homotopically trivial in $St^Q(im(\alpha))$.*

Proof. Choose $0 = x_1 < x_2 < \cdots < x_k = 1$. Define $F: I \times I \to \tilde{X}$ as follows:

(1) $F|_{[0,1] \times \{0\}}$ and $F|_{[0,1] \times \{1\}}$ are respectively, a reparametrization of α composed with an asynchronous adjustment and a reparametization of β composed with an asynchronous adjustment, so that $F|_{[x_i, x_{i+1}] \times \{0\}}$ is an edge of α or constant, $F|_{[x_i, x_{i+1}] \times \{1\}}$ is an edge of β or constant, and $d(F(x_i, 0), F(x_i, 1)) \leq \delta$ for all i. (This is possible since α and β are asynchronous δ-fellow travelers.)

(2) $F|_{\{x_i\} \times [0,1]}$ is an edge path of length $\leq \delta$.

(3) $F|_{\{0\} \times [0,1]} = \gamma$ and $F|_{\{1\} \times [0,1]} = e$.

Now consider the edge loop $\langle F|_{[x_i, x_{i+1}] \times \{0\}}, F|_{\{x_{i+1}\} \times I}, (F|_{[x_i, x_{i+1}] \times \{1\}})^{-1}, (F|_{\{x_i\} \times I})^{-1} \rangle$. The length of this loop is $\leq 2\delta + 2$. Hence it is homotopically trivial in $\mathrm{St}^Q(F(x_i, 0)) \subset \mathrm{St}^Q(im(\alpha))$. Define F on $[x_i, x_{i+1}] \times I$ to realize this homotopy. Then $\mathrm{im}(F) \subset \mathrm{St}^Q(\mathrm{im}(\alpha))$, and the result follows. $\square$

Let α be an edge path in $\mathrm{Cl}(\tilde{X} - \mathrm{St}^N(H))$ such that $\alpha(0), \alpha(1) \subset \mathrm{St}^N(H)$. Choose $M > Q + N$. It suffices to show that α is homotopic rel$\{0, 1\}$ to a path in $\mathrm{St}^{I+(M+I)\delta+L+\delta}(H)$ by a homotopy in $\tilde{X} - \mathrm{St}^N(H)$. The path α decomposes into sub-edge paths, $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_{n-1}, \beta_{n-1}, \alpha_n$ where $\mathrm{im}(\alpha_i) \subset \mathrm{St}^M(H)$ and $\mathrm{im}(\beta_i) \subset \mathrm{Cl}(\tilde{X} - \mathrm{St}^M(H))$. It suffices to show that $\beta(\equiv \beta_i)$ is homotopic rel$\{0, 1\}$ to a path with image in $St^{I+(M+I)\delta+L+\delta}(H)$ by a homotopy in $\tilde{X} - \mathrm{St}^N(H)$.

Let $e_1, e_2, \ldots, e_n$ be the edges of β, ρ_i be the combing path corresponding to $e_i(1)$, ρ_0 the combing path corresponding to $e_1(0)$ and $T_i \equiv T_{e_i(1)}$.

If t_i is the last point of $[0, T_i]$ such that $\rho_i(t_i) \in \mathrm{St}^M(H)$, then let a_i be a point of $[0, T_i]$ such that a_i corresponds (under the asynchronous tracking) to $\rho_{i-1}(t_{i-1})$, γ_i be an edge path of length $\leq \delta$ from $\rho_{i-1}(t_{i-1})$ to $\rho_i(a_i)$ and ξ_i be the subpath of ρ_i and a_i to t_i.

It suffices to show that for each i.

(1) $\mathrm{Im}\langle \gamma_i, \xi_i \rangle \subset \mathrm{St}^{I+(M+I)\delta+L+\delta}(H)$ and

(2) $\langle \gamma_i, \xi_i \rangle$ is homotopic rel $\{0, 1\}$, by a homotopy in $\tilde{X} - \mathrm{St}^N(H)$, to $\langle \rho_{i-1}|_{[t_{i-1}, T_{i-1}]}, e_i, (\rho_i|_{[t_i, T_i]})^{-1} \rangle$.

For then, patching together these homotopies gives a homotopy rel $\{0, 1\}$ of $\beta \equiv \langle e_1, e_2, \ldots, e_n \rangle$ to $\langle \gamma_1, \xi_1, \gamma_2, \ldots, \gamma_n, \xi_n \rangle$, in $\tilde{X} - \mathrm{St}^N(H)$.

If $a_i \leq t_i$, then $\xi_i = \rho_i|_{[a_i,t_i]}$. In this case, by Lemma 5.1, $\rho_i|_{[0,t_i]}$ has image in $\mathrm{St}^{I+(M+I)\delta+L}(H)$, and $\mathrm{im}(\gamma_i) \subset \mathrm{St}^{\delta}(\rho_{i-1}(t_{i-1})) \subset \mathrm{St}^{\delta+M}(H)$, so $\mathrm{im}\langle \xi_i, \gamma_i \rangle \subset \mathrm{St}^{I+(M+I)\delta+L+\delta}(H)$. By applying Lemma 5.2 with $\rho_{i-1}|_{[t_{i-1},T_{i-1}]}$ in place of α, e_i in place of e, γ_i in place of γ and $\rho_i|_{[a_i,T_i]}$ in place of β we have $\langle \gamma_i, \xi_i \rangle$ is homotopic rel $\{0, 1\}$, by a homotopy in $\mathrm{St}^{Q}(\mathrm{im}(\rho_{i-1}|_{[t_{i-1},T_{i-1}]}))$, to $\langle \rho_{i-1}|_{[t_{i-1},T_{i-1}]}, e_i, (\rho_i|_{[t_i,T_i]})^{-1} \rangle$. As $\mathrm{im}(\rho_{i-1}|_{[t_{i-1},T_{i-1}]} \subset \mathrm{Cl}(\tilde{X} - \mathrm{St}^{M}(H))$ and $M > N + Q$, the image of this homotopy does not intersect $\mathrm{St}^{N}(H)$.

If $t_i \leq a_i$, then again $\mathrm{im}(\gamma_i) \subset \mathrm{St}^{\delta}(\rho_{i-1}(t_{i-1})) \subset \mathrm{St}^{I+(M+I)\delta+L+\delta}(H)$. Observe that $\xi_i = (\rho_i|_{[t_i,a_i]})^{-1}$ and $\mathrm{im}(\xi_i) \subseteq \mathrm{St}^{\delta}(\mathrm{im}(\rho_{i-1}|_{[0,t_{i-1}]})$. Hence by Lemma 5.1 $\mathrm{im}(\xi_i) \subseteq \mathrm{St}^{I+(M+I)\delta+L+\delta}(H)$. By Lemma 5.2, γ_i is homotopic rel $\{0, 1\}$ to $\langle \rho_{i-1}|_{[t_{i-1},T_{i-1}]}, e_i, (\rho_i|_{[a_i,T_i]})^{-1} \rangle$ by a homotopy in $\mathrm{St}^{Q}(\mathrm{im}(\rho_{i-1}|_{[t_{i-1},T_{i-1}]})) \subset \tilde{X} - \mathrm{St}^{N}(H)$. As $\mathrm{im}(\xi_i) \subset \mathrm{Cl}(\tilde{X} - \mathrm{St}^{M}(H)) \subset \tilde{X} - \mathrm{St}^{N}(H)$, $\langle \gamma_i, \xi_i \rangle$ is homotopic rel $\{0, 1\}$ to $\langle \rho_{i-1}|_{[t_{i-1},T_{i-1}]}, e_i, (\rho|_{[t_i,T_i]})^{-1} \rangle$ by a homotopy in $\tilde{X} - \mathrm{St}^{N}(H)$. $\square$

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Michael L. Mihalik
Department of Mathematics
Vanderbilt University
Nashville TN 37240
U.S.A.
E-mail address: mihalikm@ctrvax.vanderbilt.edu

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