Correction to "uniqueness for the harmonic map flow from surfaces to general targets"

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The calculation leading to estimate (3.8) in the paper is incorrect. Thus the construction of adapted p-frames described in the paper is not valid, and the existence of such frames adapted to a general time-dependent map (Theorem 3.1) remains in doubt. What is possible to obtain are *tangent* frames which are 'optimal' in a certain sense, but only for each fixed time. As described below, this turns out to be sufficient to prove the main theorem 1.1 as stated in the paper. For convenience of the reader, we recall its statement. We consider weak solutions of the heat flow for harmonic maps with initial data $u_0 \in H^1(M; N)$, where N is a k-dimensional compact embedded submanifold of $\mathbb{R}^p$, with the induced Riemannian metric. Define:

$$V^T = H^1(M \times [0, T]; N) \cap L^\infty([0, T]; H^1(M; N)) \cap L^2([0, T]; H^2(M, N)).$$

By work of M. Struwe [1], a solution $v \in V^T$ exists for sufficiently small $T > 0$ (depending on u_0), and may be continued to a global weak solution with finite singular set in $M \times (0, \infty)$. We refer to v as the 'almost regular solution'.

THEOREM 1.1. *Let $u \in H^1(M \times [0, T]; N)$ be a weak solution of the harmonic map flow with initial conditions $u_0 \in H^1(M, N)$. Assume $E_u(t) \leq E_{u_0}$ a.e. in $I = [0, T]$. Then there exists $T' \in (0, T)$ such that $u \in V^{T'}$.*

The proof proceeds in two steps. First, using a modified 'optimal frames' construction for each constant time (Lemma A) and a lemma of Hélein [16] (Lemma D), we obtain a regularity result for each $u(t)$ (Lemma B). Then we conclude the proof by the same 'perturbation argument' used in the paper (Lemma 2.2).

Theorems, lemmas, equations and references in the paper are referred to here by the same number. The notation is the same, with the following additional conventions. $\mathbb{F}^{k,p}$ denotes the space of orthonormal k-frames in $\mathbb{R}^p$. In the notation of frames e_i and connection forms ω_{ij} the indices i, j are usually omitted for brevity.

$c > 0$ denotes a generic positive constant whose value may depend on M, N, a given smooth tangent orthonormal frame $\bar{e}$ on N, and on u_0.

For simplicity the argument below is given in the case of surfaces without boundary.

The statement and proof of Theorem 3.1 (existence of adapted frames) should be changed as follows.

Let $\bar{e} = (\bar{e}_i)_{i=1}^k$, $\bar{e} \in L^\infty(I; H^1(M; \mathbb{F}^{k,p}))$, be the 'background' adapted frame obtained by composing a smooth orthonormal frame tangent to N with $u(x, t)$. Since $u(t) \to u_0$ strongly in $H^1(M; N)$ as $t \to 0$ (by the energy bound hypothesized in the theorem), we also have $\bar{e}(t) \to \bar{e}(0)$ strongly in $H^1(M; \mathbb{F}^{k,p})$. Let $\epsilon > 0$ be given. We define $T_\epsilon \in (0, T)$ as follows. Since $dim(M) = 2$, we may write:

$$\bar{e}(0) = \bar{e}^1(0) + \bar{e}^2(0),$$

where $\bar{e}^2(0) \in C^\infty(M; \mathbb{F}^{k,p})$ and $\bar{e}^1(0) \in (H^1 \cap L^\infty)(M; \mathbb{R}^p)$ satisfies $\|\bar{e}^1(0)\|_{H^1} < \epsilon/2$. Fix $T_\epsilon > 0$ such that $\|\bar{e}(t) - \bar{e}(0)\|_{H^1} < \epsilon/2$ for all $t \in [0, T_\epsilon]$ such that $\bar{e}(t) \in H^1$. Then, setting $\bar{e}^1(t) = \bar{e}(t) - \bar{e}^2(0)$, we have:

$$\|\bar{e}^1(t)\|_{H^1} < \epsilon,$$

for all $t \in [0, T_\epsilon]$ such that $\bar{e}(t) \in H^1$.

The result that replaces theorem 3.1 is:

LEMMA A. *Let $\epsilon > 0$ be given, and choose $T_\epsilon > 0$ as above. Fix $t \in [0, T_\epsilon]$ such that $u(t) \in H^1(M; N)$. We may find a tangent orthonormal frame $e(t) \in H^1(M; \mathbb{F}^{k,p})$ adapted to $u(t)$, whose connection 1-form $\omega_{ij} = \langle de_i, e_j \rangle$ admits the decomposition:*

$$\omega(t) = \omega^1(t) + \omega^2(t),$$

with $\delta\omega^1(t) = 0$, $\|\omega^1(t)\|_2 < \epsilon$ and $\omega^2(t) \in L^\infty \Omega^1(M) \otimes so(k)$, with norm bounded by a constant independent of t.

Proof. Let $\bar{\omega}_{ij}^1(t) = \{\langle d\bar{e}_i^1(t), \bar{e}_j(t) \rangle\}$, $\bar{\omega}_{ij}^2(t) = \{\langle d\bar{e}_i^2(0), \bar{e}_j(t) \rangle\}$. Consider a 'gauge transformation' $g \in H^1$; $SO(k))$ which minimizes the functional $F(g)$ given by:

$$F(g) = \int_M |dgg^t + g\bar{\omega}^1(t)g^t|^2 \, dx,$$

where the superscript 't' denotes 'transpose'. A minimizer clearly exists. The Euler-Lagrange equation for F is:

$$\delta\omega^1(t) = 0,$$

where: $\omega^1(t) = dgg^t + g\bar{\omega}^1(t)g^t$. Since g is a minimizer,

$$\|\omega^1(t)\|_2 \le \|\bar{\omega}^1(t)\|_2 < \epsilon.$$

Let $e_i(t) = \Sigma_j g_{ij}\bar{e}_j(t)$. The connection 1-forms ω of $e(t) = (e_i)(t)$ may be decomposed as:

$$\omega(t) = dgg^t + g\bar{\omega}(t)g^t = \omega^1(t) + \omega^2(t),$$

where $\omega^2(t) = g\bar{\omega}^2(t)g^t$. Since $\|\bar{\omega}^2_{ij}(t)\|_\infty \le \|d\bar{e}^2_i(0)\|_\infty \le c$, this concludes the proof.

Remark A. ω^1 is not the connection 1-form of a frame. However, defining $e^1 = g\bar{e}^1$, we have (since $\langle \bar{e}, e \rangle = g^t$):

$$\langle de^1, e \rangle = dg\langle \bar{e}^1, e \rangle + g\langle d\bar{e}^1, e \rangle$$

$$= dgg^t - dg\langle \bar{e}^2, e \rangle + g\langle d\bar{e}^1, \bar{e} \rangle\langle \bar{e}, e \rangle$$

$$= dgg^t + g\bar{\omega}^1 g^t - dg\langle \bar{e}^2, e \rangle;$$

so $\omega^1 = \{\langle de^1, e \rangle + dg\langle \bar{e}^2, e \rangle\}$. (In this calculation we have denoted, for instance, by $\langle \bar{e}^1, e \rangle$ the matrix with ij-th entry $\langle \bar{e}^1_i, e_j \rangle$.)

Lemma A gives no control on the connection 1-forms ω_{ia} (e_i tangential, e_a normal). Thus the proof of theorem 1.1 must be modified. This is accomplished by using a lemma of Hélein, which allows us to give a simpler proof than the argument attempted in the paper. The main step is the following lemma.

LEMMA B. *Consider a solution* $u: M \times I \to N$ *of the harmonic map flow to* N *satisfying the assumptions of theorem* 1.1. *There exists* $\epsilon_1 > 0$ *with the following property. Define* $T_{\epsilon_1} > 0$ *as in the paragraph preceding Lemma A. Let* $t \in (0, T_{\epsilon_1}]$ *satisfy* $u_t(t) \in L^2(M; \mathbb{R}^p)$, $u(t) \in H^1(M, N)$. *Then* $du(t) \in L^4\Omega^1_M$ *and*

$$\|du(t)\|_4 \le c(1 + \|u_t(t)\|_2),$$

for a constant c *independent of* t. *In particular,* $du \in L^2([0, T_{\epsilon_1}]; L^4)$.

The conclusion of lemma B is 'higher regularity of u in space directions'. It is then easy to conclude u is in the class $V^{T'}$ (for some $T' < T$) as claimed in theorem 1.1, by means of a 'perturbation argument' based on linear parabolic theory. This

is accomplished in the following lemma (which is essentially a restatement of Lemma 2.2 in the paper).

Consider the general non-homogeneous linear parabolic system of the form:

(L)
$$\begin{cases} \Phi_t - \Delta\Phi = -(d\Phi \cdot \omega)e + f(x, t) & \text{on } M \times I \\ \Phi(\,.\,, 0) = 0 & \text{in } M, \end{cases}$$

LEMMA C. *There exists $\epsilon_2 > 0$ (depending on M and T) with the following property. Let $I' = [0, T']$, where $T' \in (0, T]$ is arbitrary. Let $f \in L^4(I', L^{4/3})$, $\omega \in L^\infty(I', L^2\Omega^1)$ and $e \in L^\infty(M \times I')$, $\|e\|_\infty \le 1$. Assume $\|\omega\|_{L^\infty(I', L^2)} < \epsilon_2$. Let Φ be a solution of (L) in $(H^1 \cap L^\infty)(M \times I')$, such that $\|\nabla\Phi\|_{L^4(M)} \in L^2(I')$. Then $\Phi \in L^4(I', W^{2,4/3})$. (Note: ϵ_2 is independent of T'.)*

Proof. The first part of the proof of Lemma 2.2 shows that, for each $p \in (1, 4]$, there exists $\epsilon(p) > 0$ such that system (L) has unique solutions in $L_0^p(I', W^{2,4/3})$. In particular one may find $\epsilon_2 > 0$ such that (L) has a unique solution $\Phi_1 \in L_0^4(I', W^{2,4/3})$, and also a unique solution in $L_0^2(I', W^{2,4/3})$. The latter must coincide with the given Φ, since $\|\nabla\Phi\|_{L^4(M)} \in L^2(I')$ implies (by linear theory, Theorem 2.1) $\Phi \in L_0^2(I', W^{2,4/3})$. Since $L_0^4(I', W^{2,4/3}) \hookrightarrow L_0^2(I', W^{2,4/3})$, uniqueness in $L_0^2(I', W^{2,4/3})$ implies $\Phi = \Phi_1$.

Proof of Theorem 1.1. Using the background frame $\{\bar{e}_r\}_{r=1}^p$, we write the equation for u in the form (2.4a):

$$u_t - \Delta u = -\sum_{i,a} \langle du \cdot \bar{\omega}_{ia}, \bar{e}_i \rangle \bar{e}_a.$$

Let ϵ_2 be given by lemma C. Choose $T' < min\{T_{\epsilon_1}, T_{\epsilon_2}\}$. Then for $t \in [0, T']$ the following decomposition holds:

$$\bar{\omega}_{ia}(t) = \bar{\omega}_{ia}^1(t) + \bar{\omega}_{ia}^2(t),$$

where $\|\bar{\omega}_{ia}^1(t)\|_2 < \epsilon_1$ and $\bar{\omega}_{ia}^2 \in L^\infty(M \times [0, T'])$. Letting $v: M \times [0, T_0] \to \mathbb{R}^p$ be the solution of the linear heat equation with initial data u_0, we obtain for $w = u - v$:

$$\begin{cases} w_t - \Delta w = -\sum_{i,a} \langle dw \cdot \bar{\omega}_{ia}^1, \bar{e}_i \rangle \bar{e}_a + f(x, t), \\ w(\,.\,, 0) = 0, \end{cases}$$

where:

$$f(x, t) = \langle dv \cdot \bar{\omega}_{ia}^1, \bar{e}_i \rangle \bar{e}_a + \langle dw \cdot \bar{\omega}_{ia}^2, \bar{e}_i \rangle \bar{e}_a \in L^4([0, T'], L^{4/3}).$$

Indeed we have (as in subsection 2.3):

$$\left\| dv \cdot \bar{\omega}_{ia}^1 \right\|_{L^{4/3}(M_t)} \le c \left\| dv \right\|_{L^4(M_t)} \left\| \bar{\omega}_{ia}^1 \right\|_{L^2(M_t)}$$

$$\le c \left\| dv \right\|_{L^2(M_t)}^{1/2} \left\| v \right\|_{H^2(M_t)}^{1/2} \epsilon_1,$$

$$\left\| dw \cdot \bar{\omega}_{ia}^2 \right\|_{L^{4/3}(M_t)} \le c \left\| dw \right\|_{L^2(M_t)} \left\| \bar{\omega}_{ia}^2 \right\|_{L^4(M_t)},$$

which implies:

$$\left\| f \right\|_{L^4(I, L^{4/3})}^4 \le c \epsilon_1^4 \left\| v \right\|_{L^2(I, H^2)}^2 + c T' \left\| dw \right\|_{L^\infty(I, L^2)}^4 \left\| \bar{\omega}_{ia}^2 \right\|_{L^\infty(M \times I)}^4.$$

By lemma B, $dw \in L^2([0, T'], L^4)$. Thus we may apply lemma C to w and conclude $w \in L_0^4([0, T'], W^{2,4/3})$, hence $u \in L^4([0, T'], W^{2,4/3})$. From the embedding $W^{2,4/3} \hookrightarrow W^{1,4}$ and the equation of the flow this implies $u \in L^2([0, T'], H^2)$, as claimed in the theorem.

Proof of lemma B. (1) Let $\epsilon_1 = c_1 \epsilon_0$, where c_1 is the constant defined in paragraph (4) below and ϵ_0 is given by Lemma D below. Let $T_{\epsilon_1} > 0$ be chosen as in Lemma A above. Fix $t \in (0, T_{\epsilon_1}]$ such that $u_t(t) \in L^2$, and let $e(t) = (e_i)(t)$ be the adapted frame given by Lemma A, whose connection 1-forms decompose as in its statement: $\omega(t) = \omega^1(t) + \omega^2(t)$. Since $\delta \omega^1(t) = 0$, we have the Hodge decomposition:

$$\omega^1(t) = \delta B(t) + H(t),$$

where $B(t) \in H^1 \Omega_M^2 \otimes so(k)$ satisfies $\int_M B(t) = 0$ and:

$$\| B \|_{H^1} \le c \| \omega^1 \|_2 < c\epsilon,$$

and $H(t)$ is a harmonic 1-form on M.

(2) In order to use Hélein's lemma we must localize the problem. Cover M by open sets $\{ U_\alpha \}_{\alpha = 1}^N$, such that:

    (i) there exist conformal coordinate charts $\varphi_\alpha : D_2 \to M$, $\varphi_\alpha(D_2) = U_\alpha$;

    (ii) $M \subset \bigcup_{i=1}^N V_\alpha$, $V_\alpha = \varphi_\alpha(D_{1/2})$.

We denote by D_r the open disk with radius r centered at the origin in $\mathbb{C}(D = D_1)$, endowed with the metric $ds^2 = \lambda^2 |dz|^2$, pulled back from M via the local charts φ_α. Fix an index α for the remainder of the proof, and for simplicity of notation identify the maps u, e_i, etc. with their pullbacks to D_2 under φ_α.

Let

$$\alpha_i(t) = \langle u_z, e_i(t) \rangle = \frac{1}{2} \langle u_x - iu_y, e_i(t) \rangle \in L^2(D_2),$$

$$a_{ij} = \langle (e_i(t))_{\bar{z}}, e_j(t) \rangle$$

$$= \frac{1}{2} [(\omega_{ij})(\partial_x) + i\omega_{ij}(\partial_y)] \in L^2(D_2)$$

(with L^2 norm independent of t in both cases). Then:

$$(\alpha_i)_{\bar{z}} = \langle u_{z\bar{z}}, e_i \rangle + \langle u_z, (e_i)_{\bar{z}} \rangle$$

$$= \frac{1}{4} \langle \lambda^2 \Delta u, e_i \rangle + \sum_j \alpha_j a_{ij},$$

(Δ denotes the Laplacian in the Euclidean metric) so:

(1) $\qquad (\alpha_i)_{\bar{z}} + \sum_j a_{ji}\alpha_j = \frac{\lambda^2}{4} \langle u_t, e_i \rangle.$

(3) By remark A above, we have:

$$\Delta B = d\omega^1 = \{de^1 \wedge de - dg \wedge d\langle \bar{e}_2, e \rangle\}_- \in \mathcal{H}^1_{loc}(D_2).$$

($\mathcal{H}^1_{loc}$ denotes a local Hardy space, as in [16]). Therefore $B \in W^{2,1}_{loc}(D_2) \subset C^0(D_2)$ and:

$$\|B\|_{W^{2,1}(D_1)} \le c(\|de^1\|_2 \|de\|_2 + \|dg\|_2 \|e\|_{H^1}).$$

We also have the estimates (in D_1):

$$\|dg(t)\|_2 = \|\omega^1(t)g(t) - g(t)\bar{\omega}^1(t)\|_2 \le 2\|\bar{\omega}^1(t)\|_2 < 2\epsilon,$$

$$\|de^1(t)\|_2 = \|(dg)\bar{e}^1 + gd\bar{e}^1\|_2 \le c(\|dg\|_2 + \|\bar{e}^1\|_{H^1}) < c\epsilon,$$

so we conclude $\|B\|_{W^{2,1}(D_1)} \le c\epsilon.$

(4) Corresponding to the decomposition:

$$\omega = \delta B + \omega^2 + H,$$

we set:

$$\omega(\partial_{\bar{z}}) = a(t) = a^1(t) + a^2(t) + a^3(t),$$

where (and here we define c_1):

$$a^1(t) = \delta B(\partial_{\bar{z}}) \in W^{1,1}(D), \qquad \|a^1\|_{W^{1,1}} < c_1 \epsilon,$$

$$a^2(t) = \omega^2(\partial_{\bar{z}}) \in L^\infty(D),$$

$$a^3(t) = H(t)(\partial_{\bar{z}}) \in L^\infty(D),$$

with L^∞ norm bounded independently of t in the last two cases.

We may rewrite (1) in the form:

$$(2) \qquad (\alpha_i)_{\bar{z}} + \sum_j a_{ji}^1 \alpha_j = f_i$$

where:

$$f_i = \frac{\lambda^2}{4} \langle u_t(t), e_i(t) \rangle + \sum_j (a_{ij}^2 + a_{ij}^3) \alpha_j(t) \in L^2(D),$$

with:

$$\|f_i(t)\|_{L^2(D)} \leq c(\|u_t(t)\|_{L^2(D)} + \|\langle du, e_i(t) \rangle\|_{L^2(D)})$$

$$(3) \qquad\qquad\qquad \leq c(1 + \|u_t(t)\|_{L^2(D)}).$$

(5) **LEMMA D.** (*Hélein* [16]) *There exist constants $\epsilon_0 > 0$, $c_0 > 0$ such that if $a_{ij} \in W^{1,1}(D)$, $\|a_{ij}\|_{W^{1,1}} < \epsilon_0$, one may find solutions $\beta^k \in L^\infty(D, \mathbb{C}^n)$, $k = 1, \ldots, n$ of the system*:

$$(\beta_i)_{\bar{z}} = \sum_j a_{ij}\beta_j, \qquad \beta^k = (\beta_i^k)_{i=1}^n$$

and a map $M = (m_{ij}) \in L^\infty(D, GL(n, \mathbb{C}))$ such that:

$$\|M\|_{L^\infty} + \|M^{-1}\|_{L^\infty} \leq c_0 \qquad \|\beta^j - b^j\|_{L^\infty} \leq c_0$$

and $b^k = \sum_{j=1}^n m_{kj}\beta^j$, where $b^1, \ldots, b^n$ is the standard basis of $\mathbb{C}^n$.

(6) *We have (as in* [16]):

$$\alpha_j = \sum_i \delta_{ji}\alpha_i = \sum_{k,i} m_{jk}\beta_i^k\alpha_i = \sum_k m_{jk}\,\gamma^k,$$

where $\gamma^k = \Sigma_i\,\beta_i^k\alpha_i$ *and* (2) *implies*:

$$\gamma_{\bar{z}}^k = \sum_i [(\beta_i^k)_{\bar{z}}\alpha_i + \beta_i^k(\alpha_i)_{\bar{z}}]$$

(4) $= \sum_i \beta_i^k f_i \in L^2(D)$

By elliptic regularity for $\bar{\partial}$, this implies:

$$\gamma^k(t) \in H^1(D_{1/2}),$$

and hence $\gamma^k(t)$ (and therefore α_j) is in $L^p(D_{1/2})$ for each $1 < p < \infty$. In particular, this clearly implies $du(t) \in L^4(V_\alpha))$ for each α. Thus $du \in L^4\Omega_M^1$. The estimate claimed in the lemma follows from (3) and (4).

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Erratum: Antonio G. Rodicio, *Flat exterior Tor algebras and cotangent complexes,* Comment. Math. Helv. *70* (1995) 546-557.

The proof of Proposition 5, and so the one of Theorem 1 and Corollary 3, are incomplete. The problem is that it has been used as a general result in homotopical algebra that (as far as we know) is not yet proved: the fact that the Künneth spectral sequence of simplicial algebras over a simplicial ring is a spectral sequence of bigraded algebras with divided powers. Recently we have learned that our claim in Remark 6 on the existence of the bisimplicial algebra resolution P was too precipitate. Our idea to construct P only works when Y contains a field; and in this case the resolution can also be easily deduced from the bar construction. Therefore, although we are positively convinced that the results are valid in full generality, the proofs given in the paper are, at the present, only complete when the rings in consideration contain a field.

In fact, we have an alternative proof of the cohomological part of Corollary 3, for general rings, and as a consequence, in the case A is noetherian, one of the homological part. This proof will appear in our paper "On the acyclicity of the Tate complex".

Even though it is probable that the bisimplicial algebra resolution does not exist in general, it is easy to prove that the spectral sequence is of bigraded algebras. For, it is used as a bisimplicial module resolution and it is defined, using the shuffle operator, an external product in the total complex associated to the bicomplex, which is compatible with the filtrations. So it seems likely that the divided powers exist. We leave it as an interesting open problem.

On the other hand, Corollary 3 was used in the paper "Projective exterior Koszul homology and decomposition of the Tor functor" (Invent. Math. 123, 123-140 (1996)) by A. Blanco, J. Majadas and A. G. Rodicio. Since at the present it is not proved for general rings that the FLEKH property implies the vanishing of André-Quillen homology, in Proposition 7 and Theorem 8 (i), the hypothesis "I is a FLEKH-ideal" must be replaced by "$H_j(A, B, -) = 0$ for all $j \geq 3$". No additional change is necessary in the statements of the results: though in the proofs of Theorem 2 and Theorem 13, our incomplete Proposition 5 was used, alternative proofs of these results will also appear in the paper cited above.

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