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Quasi-periodic solutions for a nonlinear wave equation

JÜRGEN PÖSCHEL

1. Introduction and main result

We are going to study the nonlinear wave equation

$$u_{tt} = u_{xx} - mu - f(u) \tag{1}$$

on the finite x-interval $[0, \pi]$ with Dirichlet boundary conditions

$$u(t, 0) = 0 = u(t, \pi), \qquad -\infty < t < \infty.$$

Here, $m > 0$ is a real parameter, sometimes referred to as the "mass", and f is a real analytic, odd function of u of the form

$$f(u) = au^3 + \sum_{k \geq 5} f_k u^k, \qquad a \neq 0. \tag{2}$$

This class of equations comprises the sine-Gordon, the sinh-Gordon and the ϕ^4-equation, given by

$$mu + f(u) = \begin{cases} \sin u, \\ \sinh u, \\ u + u^3, \end{cases}$$

respectively, as well as odd perturbations of them of order five or more.

Our approach and its results are parallel to an investigation of the nonlinear Schrödinger equation $iu_t = u_{xx} - mu - f(|u|^2)u$ on the same interval undertaken by Kuksin and the author in [6]. Hence some parts of the respective expositions are quite similar. But we decided to repeat them anyway so that the reader need not refer to [6] for the essentials.

We study this equation as an infinite dimensional hamiltonian system. As the phase space one may take, for example, the product of the usual Sobolev spaces $\mathscr{P} = H_0^1([0, \pi]) \times L^2([0, \pi])$ with coordinates u and $v = u_t$. The hamiltonian is then

$$H = \frac{1}{2}\langle v, v\rangle + \frac{1}{2}\langle Au, u\rangle + \int_0^\pi g(u)\,dx,$$

where $A = d^2/dx^2 + m$ and $g = \int_0 f(s)\,\mathrm{ds}$, and $\langle\cdot\,,\cdot\rangle$ denotes the usual scalar product in L^2. The hamiltonian equations of motions are

$$u_t = \frac{\partial H}{\partial v} = v, \qquad v_t = -\frac{\partial H}{\partial u} = -Au - f(u), \tag{3}$$

hence they are equal to (1).

We will not reply on this set up, however, nor will we consider the initial value problem for this system. Rather, we will solve an embedding problem, which does not involve its flow, in order to construct large families of real analytic, global solutions. More precisely, our aim is to construct plenty of real analytical solutions that are *quasi-periodic* in time, hence *a fortiori* global. That is, they can be written in the form

$$u(t, x) = U(\omega_1 t, \ldots, \omega_n t, x),$$

where U is a real analytic function of period 2π in the first $n \geq 1$ arguments, and $\omega_1, \ldots, \omega_n$ are rationally independent real numbers, the basic frequencies of u. Thus, u admits a Fourier series expansion

$$u(t, x) = \sum_{k \in \mathbb{Z}^n} e^{ik\cdot\omega t}\, U_k(x),$$

where $k \cdot \omega = \Sigma_j k_j \omega_j$. A special case are time periodic solutions, which are quasi-periodic with exactly one basic frequency.

From a geometric point of view, such solutions arise from *embeddings* of the n-torus $\mathbb{T}^n$ into the phase space $\mathscr{P}$,

$$\mathbb{T}^n \to \mathscr{P}, \qquad \theta \mapsto (U(\theta, \cdot), DU(\theta, \cdot)),$$

where $DU = \Sigma_j \omega_j U_{\theta_j}$, such that the straight windings $\theta(t) = \omega t + \theta_0$ on the torus map into solutions of (3). Hence, in phase space they correspond to embedded invariant tori, on which in suitable coordinates the vector field is constant with linear flow. We call them *rotational tori* in the following.

The quasi-periodic solutions to be constructed are of *small amplitude*. Thus, in first approximation the higher order terms may be considered as a small perturbation of the linear equation $u_{tt} = u_{xx} - mu$. The latter is of course well understood and has plenty of quasi-periodic solutions.

To be more precise, let

$$\phi_j = \sqrt{\frac{2}{\pi}} \sin jx, \qquad \lambda_j = \sqrt{j^2 + m}$$

for $j = 1, 2, \ldots$ be the basic modes and frequencies of the linear system with Dirichlet boundary conditions. Then every solution is the superposition of their harmonic oscillations and of the form

$$u(t, x) = \sum_{j \geq 1} q_j(t)\phi_j(x), \qquad q_j(t) = I_j \cos(\lambda_j t + \varphi_j^0)$$

with amplitudes $I_j \geq 0$ and initial phases φ_j^0. Their combined motion is periodic, quasi-periodic or almost periodic, respectively, depending on whether one, finitely many or infinitely many modes are excited. In particular, for every choice

$$J = \{j_1 < j_2 < \cdots < j_n\} \subset \mathbb{N}$$

of finitely many modes there is an invariant $2n$-dimensional linear subspace E_J that is completely foliated into rotational tori with frequencies $\lambda_{j_1}, \ldots, \lambda_{j_n}$:

$$E_J = \{(u, v) = (q_1\phi_{j_1} + \cdots + q_n\phi_{j_n}, p_1\phi_{j_1} + \cdots + p_n\phi_{j_n})\} = \bigcup_{I \in \overline{\mathbb{P}^n}} \mathcal{T}_J(I),$$

where $\mathbb{P}^n = \{I \in \mathbb{R}^n : I_j > 0 \text{ for } 1 \leq j \leq n\}$ is the positive quadrant in $\mathbb{R}^n$ and

$$\mathcal{T}_J(I) = \{(u, v) : q_j^2 + \lambda_j^{-2} p_j^2 = I_j \text{ for } 1 \leq j \leq n\},$$

using the above representation of u and v. In addition, each such torus is linearly stable, and all solutions have vanishing Lyapunov exponents. This is the linear situation.

Upon restoring the nonlinearity f the invariant manifolds E_J with their quasi-periodic solutions will not persist in their entirety due to resonances among the modes and the strong perturbing effect of f for large amplitudes. In a sufficiently small neighbourhood of the origin, however, there does persist a large *Cantor subfamily* of rotational n-tori which are only slightly deformed.

That means, there exists a Cantor set $\mathscr{C} \subset \mathbb{P}^n$, a family of n-tori

$$\mathscr{T}_J[\mathscr{C}] = \bigcup_{I \in \mathscr{C}} \mathscr{T}_J(I) \subset E_J$$

over $\mathscr{C}$, and a Lipschitz continuous embedding

$$\Phi\colon \mathscr{T}_J[\mathscr{C}] \to \mathscr{E}_J \subset \mathscr{P},$$

such that the restriction of Φ to each $\mathscr{T}_J(I)$ in the family is an embedding of a rotational n-torus for the nonlinear equation. The image $\mathscr{E}_J$ of $\mathscr{T}_J[\mathscr{C}]$ we call a *Cantor manifold of rotational n-tori.*

These Cantor manifolds have a number of additional properties.

(1) The embedding Φ is a higher (fractional) order perturbation of the inclusion mapping $\Phi_0\colon E_J \hookrightarrow \mathscr{P}$ restricted to $\mathscr{T}_J[\mathscr{C}]$. Its restriction to each torus $\mathscr{T}_J(I)$ is real analytic, and it maps into the space of real analytic functions on $[0, \pi]$, with uniform domains of analyticity.

(2) The cantor set $\mathscr{C}$ has full density at the origin:

$$\lim_{r\to 0} \frac{\mu(\mathscr{C} \cap B_r)}{\mu(\mathbb{P}^n \cap B_r)} = 1,$$

where $B_r = \{I : \|I\| < r\}$ and μ denotes Lebesgue measure.

(3) By the previous properties, $\mathscr{E}_J$ has a tangent space at the origin equal to E_J:

$$T_0\mathscr{E}_J = E_J.$$

(4) The frequencies ω of the rotational tori are diophantine, whence we also call the latter *diophantine tori.* That is, there exist positive α and τ such that

$$|k \cdot \omega| \geq \frac{\alpha}{|k|^\tau}, \qquad 0 \neq k \in \mathbb{Z}^n.$$

The exponent τ can be kept fixed, while α tends to zero as the tori approach the origin.

(5) All the tori are still linearly stable, and all their orbits have zero Lyapunov exponents.

MAIN THEOREM. *Suppose the nonlinearity f is real analytic, of the form* (2) *and odd*: $f(-u) = -f(u)$. *Then for each index set* $J = \{j_1 < \cdots < j_n\}$ *with* $n \geq 2$, *satisfying*

$$\min_{1 \le i < n} j_{i+1} - j_i \le n - 1, \tag{4}$$

there exists for all $m > 0$ a Cantor manifold $\mathscr{E}_J$ of real analytic, linearly stable, diophantine n-tori for the nonlinear wave equation given by a Lipschitz continuous embedding

$$\Phi\colon \mathscr{T}_J[\mathscr{C}] \to \mathscr{E}_J,$$

which is a higher order perturbation of the inclusion map $\Phi_0\colon E_J \to \mathscr{P}$ restricted to $\mathscr{T}_J[\mathscr{C}]$. The Cantor set $\mathscr{C}$ has full density at the origin, and $\mathscr{E}_J$ has a tangent space at the origin equal to E_J. Moreover, $\mathscr{E}_J$ is contained in the space of real analytic functions on $[0, \pi]$.

For one point sets $J = \{j\}$ the same holds except for those m-values at which

$$\frac{4}{3}\left(\frac{\lambda_j}{\lambda_\nu} + \frac{\lambda_j}{\lambda_\mu}\right) = k = \frac{\lambda_\nu}{\lambda_j} + \frac{\lambda_\mu}{\lambda_j} \tag{5}$$

with an integer k and some indices $1 \le \nu < j < \mu$. There are at most finitely many such exceptions, and in particular none for $J = \{1\}$.

Remark 1. An assumption of the form (4) is made to ensure that the Cantor manifolds exist for *all* positive m. Otherwise, one might have to exclude some set of m-values, which is discrete in every compact interval in $(0, \infty)$. But the condition given here is certainly not the sharpest. Also, we did not investigate the exceptional set for one point sets J given by (5) thoroughly because for the existence of Canor discs of periodic solutions there are better results anyhow [4]. But it may well be that there are no exceptional points at all.

Remark 2. The assumption that f is odd is necessary. The solutions constructed below are real analytic sine-series, hence in a neighbourhood of $x = 0$ they are defined, odd, and satisfy the differential equation. Adding the equations for $u(t, x)$ and $u(t, -x)$ one obtains $f(u) + f(-u) = 0$.

Remark 3. One can show that the embedding Φ is not only Lipschitz across the tori, but smooth in the sense of Whitney. We did not pursue this point.

Remark 4. The frequencies of the diophantine tori are also under control. They are

$$\omega(I) = \lambda_J + A_J I + O(\|I\|^2),$$

where $\lambda_J = (\lambda_{j_1}, \ldots, \lambda_{j_n})$ and A_J is the $n \times n$-matrix with coefficients $A_{kl} = (6/\pi)(4 - \delta_{kl}/\lambda_{j_k}\lambda_{j_l})$.

Remark 5. The results remain true for odd nonlinearites f of the form

$$f(x, u) = au^3 + \sum_{k \geq 5} f_k(x)u^k, \qquad a \neq 0,$$

where the coefficients f_k are real analytic in x, or in some Sobolev space $H^s([0, \pi])$, $s > \frac{1}{2}$, with norms growing at most exponentially to ensure analyticity in u. In the latter, non-analytic case the resulting quasi-periodic solutions are of class H^{s+2} in x.

Remark 6. One may also add a general odd perturbation term

$$\epsilon g(x, u) = \epsilon \sum_{k \geq 0} g_k(x)u^k$$

to the nonlinearity f, with coefficients g_k of the same type as the f_k. Then there still exist Cantor manifolds for all sufficiently small ϵ, the smallness depending on m, n and J. However, they are not dense at the origin, but have a *hole* there, since the perturbation no longer tends to zero as we approach the origin.

Remark 7. As one of the referees points out, the proof given below and thus also the results apply to parameter values $-1 < m < 0$ as well. On the other hand, very little is known about the case $m = 0$, which is "completely resonant".

Remark 8. Exactly the same results hold for the nonlinear wave equation (1) with Neumann boundary conditions

$$u_x(t, 0) = 0 = u_x(t, \pi),$$

and nonlinearities of the form (2) which need not be odd. Indeed, the solutions constructed are real analytic cosine-series, hence even about $x = 0$, which is compatible with arbitrary nonlinearities. See also the remark following the proof of Lemma 1.

The size of the Cantor manifolds $\mathscr{E}_J$ is not uniform, but depends on m, n and J, and tends to zero as n tends to infinity. Thus, unlike the linear spaces E_J, they are *not* dense in some fixed neighbourhood of the origin. But they are *asymptotically dense* in the following sense.

COROLLARY. *The union of all Cantor manifolds $\mathscr{E}_J$ intersects every nonempty open cone in $H_0^1([0, \pi]) \times L^2([0, \pi])$ with vertex at the origin.*

We turn to the idea of proof and related results. As already indicated, we are dealing with a perturbation problem in an infinite dimensional hamiltonian system. The aim is to continue finite dimensional invariant tori with quasi-periodic motions under the influence of an infinite dimensional perturbation. This calls for an extension of the well known KAM theory for finite dimensional almost integrable hamiltonian systems, which was recently developed mainly by Kuksin – see the monograph [5] and the references therein – and also by Wayne in [12] and the author in [7, 8]. Moreover, this calls for an appropriate choice of the integrable system to apply the perturbation theory to. And here, there are essentially three possibilities.

Linear system. The Klein-Gordon equation $u_{tt} = u_{xx} - mu$ with Dirichlet boundary conditions is integrable, and all its solutions are periodic, quasi-periodic or almost periodic. However, it is also completely *degenerate*, as in a linear system there is no frequency amplitude modulation. Hence KAM theory is *not* applicable.

The situation is different, if the scalar parameter m is replaced by some potential function $Q \in L^2([0, \pi])$. This amounts to introducing infinitely many parameters into the system, which may be adjusted and thus substitute the usual nondegeneracy condition. As a result one finds a *Cantor set of potentials* Q for which there are Cantor families of small amplitude quasi-periodic solutions. This approach was taken by Wayne [12]. However, that Cantor set surely does not include any open interval of constant potentials $Q \equiv m > 0$ due to infinitely many nonresonance conditions imposed on the frequencies λ_j.

Integrable PDE. The sine-Gordon equation and the sinh-Gordon equation with periodic boundary conditions are known to be integrable, exhibiting plenty of quasi-periodic solutions. They may serve as the starting point for a perturbation theory. This approach was taken by Bobenko and Kuksin [1], and essentially the same results were obtained. However, before KAM theory may be applied a formidable amount of work needs to be done to bring the equations into suitable form. This involves the use of hyperelliptic Riemann surfaces, theta-functions, Schottky uniformization, and other tools.

Integrable ODE. Here the starting point is the equation $u_{tt} = u_{xx} - mu \mp u^3$ with Dirichlet boundary conditions. This equation is *not* integrable. But by a single symplectic coordinate transformation, its hamiltonian is brought into *Birkhoff normal form* of order four with respect to any finite number of basic modes. Then KAM theory is applicable. This approach was suggested by the author in [7], and carried out for the nonlinear Schrödinger equation by Kuksin and the author in [6]. There some aspects, such as the transformation into normal form, are even simpler than here.

We indicate a few more details of this approach. To start, we use the complete set of eigenfunctions of the operator A to write $u = \Sigma\, \lambda_j^{-1/2} q_j \phi_j$, $v = \Sigma\, \lambda_j^{1/2} p_j \phi_j$. We obtain a hamiltonian in infinitely many coordinates which is real analytic near the origin in some suitable Hilbert space of sequences and of the form

$$H = \frac{1}{2}\sum_{j\geq 1} \lambda_j(p_j^2 + q_j^2) + O(\|q\|^4).$$

Thus we have an elliptic fixed point with infinitely many distinct frequencies. In the classical hamiltonian theory, the standard tool to investigate such systems is their Birkhoff normal form. Here, in contrast to the nonlinear Schrödinger equation, no complete normal form of order four is available due to asymptotic resonances among the frequencies. Still, sufficiently many nonresonance conditions are satisfied so that for each $n \geq 1$ there is a real analytic, symplectic coordinate transformation which takes the hamiltonian into

$$H = \frac{1}{2}\sum_{j\geq 1} \lambda_j I_j + \sum_{\min(i,j)\leq n} A_{ij} I_i I_j + \cdots,$$

where $I_j = q_j^2 + p_j^2$. The dots stand for terms or order four in $q_{n+1}, \ldots, p_{n+1}, \ldots$ and of order six in all coordinates. Thus, at least the interaction of the first n modes are put into a *nonlinear integrable* normal form up to order four.

To this hamiltonian, KAM theory may be applied to continue all those tori with $I_j = 0$ for $j > n$. Since the coefficients A_{ij} are easily determined, the relevant nondegeneracy conditions are also easily verified, at least for those index sets described in the Main Theorem. This then yields its proof.

Periodic solutions. KAM theory is a very powerful tool in order to construct families of quasi-periodic solutions. For the construction of *periodic* solutions, however, other approaches are more suitable. For example, in a pioneering paper, Craig and Wayne [4] extended the Lyapunov center theorem to the infinite dimensional hamiltonian system of the nonlinear wave equation. This, too, involves small divisor problems, but to a lesser extent requiring fewer nondegeneracy conditions. As a result, they could admit periodic boundary conditions to obtain Cantor families of periodic solutions. By comparison, the KAM theoretic approach forbids (at least until now) asymptotically double frequencies and hence periodic boundary conditions. This is also the reason, why we did not bother to investigate condition (5) for periodic solutions in detail. Very recently, this approach has been extended considerably by J. Bourgain to handle also quasi-periodic solutions and higher dimensional x-domains in this way. See for example [2].

Periodic solutions may also be found by variational and topological methods, which are not restricted to a perturbative setting. The first result of this kind is due to Rabinowitz [10,11], and some overview with references is given by Brezis [3].

However, these periodic solutions are of quite a different nature. Most importantly, their time period has to be a rational multiple of their space period so that the wave operator $\partial_t^2 - \partial_x^2$, acting on the corresponding space of x- and t-periodic functions, has discrete spectrum. For this reason, they also do not come in Cantor families. Obviously, there still is a large gap between variational and perturbative results.

Plan. The rest of the paper is organized as follows. In section 2 the hamiltonian is written in infinitely many coordinates, which is then put into its partial normal form in section 3. In section 4 we recall the Cantor Manifold Theorem from [6], which allows us to complete the proof of the Main Theorem in section 5.

2. The hamiltonian

We recall that the hamiltonian of our nonlinear wave equation is

$$H = \frac{1}{2}\langle v, v\rangle + \frac{1}{2}\langle Au, u\rangle + \int_0^\pi g(u)\, dx.$$

To rewrite it as a hamiltonian in infinitely many coordinates we make the ansatz

$$u = \mathcal{S}q = \sum_{j\geq 1} \frac{q_j}{\sqrt{\lambda_j}}\phi_j, \qquad v = \mathcal{S}'p = \sum_{j\geq 1} \sqrt{\lambda_j} p_j \phi_j,$$

where $\phi_j = \sqrt{2/\pi}\sin jx$ for $j = 1, 2, \ldots$ are the normalized Dirichlet eigenfunctions of the operator A with eigenvalues $\lambda_j^2 = j^2 + m$. The coordinates are taken from some Hilbert space $\ell^{a,s}$ of all real valued sequences $w = (w_1, w_2, \ldots)$ with finite norm

$$\|w\|_{a,s}^2 = \sum_{j\geq 1} |w_j|^2 j^{2s} e^{2ja}.$$

Below we will assume that $a > 0$ and $s > 0$. We obtain the hamiltonian

$$H = \Lambda + G = \frac{1}{2}\sum_{j\geq 1} \lambda_j (p_j^2 + q_j^2) + \int_0^\pi g(\mathcal{S}q)\, dx$$

with equations of motions

$$\dot{q}_j = \frac{\partial H}{\partial p_j} = \lambda_j p_j, \qquad \dot{p}_j = -\frac{\partial H}{\partial q_j} = -\lambda_j q_j - \frac{\partial G}{\partial q_j}. \tag{6}$$

These are the hamiltonian equations of motion with respect to the standard symplectic structure $\Sigma\, dq_j \wedge dp_j$ on $\ell^{a,s} \times \ell^{a,s}$.

This transformation may be considered as formal. But instead of discussing its validity, we just take the latter hamiltonian as our new starting point and make the following simple observation.

LEMMA 1. *Let $a > 0$ and s be arbitrary. If a curve $I \to \ell^{a,s} \times \ell^{a,s}$, $t \mapsto (q(t), p(t))$ is a real analytic solution of* (6), *then*

$$u(t, x) = \sum_{j \geq 1} \frac{q_j(t)}{\sqrt{\lambda_j}} \phi_j(x)$$

is a classical solution of (1) *that is real analytic on $I \times [0, \pi]$.*

Proof. For $a > 0$ and arbitrary s, the sum in question in absolutely convergent in some complex neighbourhood of the x-interval $[0, \pi]$ and some complex disc around a given t in I, where q then takes values in the complexification of $\ell^{a,s}$. The same is true for its termwise t-derivatives of first and second order. Therefore, u is real analytic in t and x, and we may differentiate under the summation sign. With

$$\frac{\partial G}{\partial q_j} = \frac{1}{\sqrt{\lambda_j}} \langle f(u), \phi_j \rangle \tag{7}$$

we find that

$$\begin{aligned} u_{tt} &= \sum_{j \geq 1} \frac{\ddot{q}_j}{\sqrt{\lambda_j}} \phi_j \\ &= \sum_{j \geq 1} \frac{1}{\sqrt{\lambda_j}} (\lambda_j^2 q_j - \sqrt{\lambda_j} \langle f(u), \phi_j \rangle) \phi_j \\ &= -\sum_{j \geq 1} \frac{q_j}{\sqrt{\lambda_j}} A\phi_j - \sum_{j \geq 1} \langle f(u), \phi_j \rangle \phi_j \\ &= -Au - \sum_{j \geq 1} \langle f(u), \phi_j \rangle \phi_j . \end{aligned}$$

The ϕ_j, $j \geq 1$, are an orthonormal and complete family within the space of all odd L^2-functions on $[-\pi, \pi]$. Since u is odd and f is assumed to be odd, also $f(u)$ is odd, and we conclude that

$$u_{tt} = -Au - f(u),$$

as we wanted to show. □

It is worth pointing out that the ϕ_j, $j \geq 1$, do form a complete orthonormal family for all L^2-functions on $[0, \pi]$, but *not* for all analytic functions on $[0, \pi]$. This problem does not arise with Neumann boundary conditions, where the ϕ_j are the normalized Neumann eigenfunctions, hence we are dealing with cosine-series. Then u is even, hence $f(u)$ is even about $x = 0$ no matter what f is.

Next we consider the regularity of the gradient of G given by (7). To this end, let ℓ_b^2 and L^2, respectively, be the Hilbert spaces of all *bi*-infinite, square summable sequences with complex coefficients and all square-integrable complex valued functions on $[-\pi, \pi]$. Let

$$\mathscr{F}: \ell_b^2 \to L^2, \qquad q \mapsto \mathscr{F}q = \frac{1}{\sqrt{2\pi}} \sum_j q_j e^{ijx}$$

be the inverse discrete Fourier transform, which defines an isometry between the two spaces.

Let $a \geq 0$ and $s \geq 0$. The subspaces $\ell_b^{a,s} \subset \ell_b^2$ consist, by definition, of all bi-infinite sequences with finite norm

$$\|q\|_{a,s}^2 = |q_0|^2 + \sum_j |q_j|^2 |j|^{2s} e^{2|j|a}.$$

Through $\mathscr{F}$ they define subspaces $W^{a,s} \subset L^2$ that are normed by setting $\|\mathscr{F}q\|_{a,s} = \|q\|_{a,s}$. For $a > 0$, the space $W^{a,s}$ may be identified with the space of all 2π-periodic functions which are analytic and bounded in the complex strip $|\mathrm{Im}\, z| < a$ with trace functions on $|\mathrm{Im}\, z| = a$ belonging to the usual Sobolev space H^s.

LEMMA 2. *For $a \geq 0$ and $s > \frac{1}{2}$, the space $\ell_b^{a,s}$ is a Hilbert algebra with respect to convolution of sequences, and*

$$\|q * p\|_{a,s} \leq c \|q\|_{a,s} \|p\|_{a,s}$$

with a constant c depending only on s. Consequently, $W^{a,s}$ is a Hilbert algebra with respect to multiplication of functions.

The short proof is given in Appendix A.

LEMMA 3. *For $a \geq 0$ and $s > 0$, the gradient G_q is real analytic as a map from some neighbourhood of the origin in $\ell^{a,s}$ into $\ell^{a,s+1}$, with*

$$\|G_q\|_{a,s+1} = 0(\|q\|_{a,s}^3).$$

Proof. Let q be in $\ell^{a,s}$ and $\sigma = \frac{1}{2}$. Considered as a function on $[-\pi, \pi]$, $u = \mathcal{S}q$ is in $W^{a,s+\sigma}$ with $\|u\|_{a,s+\sigma} \leq \|q\|_{a,s}$ for $m \geqslant 0$. By the algebra property and the analyticity of f, the function $f(u)$ also belongs to $W^{a,s+\sigma}$ with

$$\|f(u)\|_{a,s+\sigma} \leq c\|u\|^3_{a,s+\sigma}$$

in a sufficiently small neighbourhood of the origin. By (7) the components of the gradient of G are the Fourier sine coefficients of $f(u)$ weighted with $\lambda_j^{-\sigma}$. Therefore, G_q belongs to $\ell^{a,s+1}$ with

$$\|G_q\|_{a,s+1} \leq \|f(u)\|_{a,s+\sigma} \leq c\|u\|^3_{a,s}.$$

The regularity of G_q follows from the regularity of its components and its local boundedness [9, Appendix A]. □

To summarize, we have a real analytic hamiltonian

$$H = \Lambda + G = \frac{1}{2}\sum_{j\geqslant 1} \lambda_j(p_j^2 + q_j^2) + G(q) \tag{8}$$

in some neighbourhood of the origin in the Hilbert space $\ell^{a,s} \times \ell^{a,s}$ with standard symplectic structure $\Sigma_j\, dq_j \wedge dp_j$, where

$$\lambda_j = \sqrt{j^2 + m}, \qquad G(q) = \int_0^\pi g(\mathcal{S}q)\, dx.$$

The latter depend on the parameter $m > 0$, which is not indicated in the following. The parameters $a > 0$ and $s > 0$ may be fixed arbitrarily. Since G is independent of p, the associated hamiltonian vectorfield,

$$X_G = \sum_{j\geq 1}\left(\frac{\partial G}{\partial p_j}\frac{\partial}{\partial q_j} - \frac{\partial G}{\partial q_j}\frac{\partial}{\partial p_j}\right),$$

defines a real analytic map from some neighbourhood of the origin in $\ell^{a,s} \times \ell^{a,s}$ into $\ell^{a,s+1} \times \ell^{a,s+1}$. Hence, X_G is *smoothing* of order 1. By contrast, X_Λ is *unbounded* of order 1.

For the nonlinearity u^3 we find

$$G = \frac{1}{4}\int_0^\pi |u(x)|^4\, dx = \frac{1}{4}\sum_{i,j,k,l} G_{ijkl}\, q_i q_j q_k q_l \tag{9}$$

with

$$G_{ijkl} = \frac{1}{\sqrt{\lambda_i \lambda_j \lambda_k \lambda_l}} \int_0^\pi \phi_i \phi_j \phi_k \phi_l \, dx. \tag{10}$$

It is not difficult to verify that $G_{ijkl} = 0$ *unless* $i \pm j \pm k \pm l = 0$, for *some* combination of plus and minus signs. Thus, only a codimension one set of coefficients is actually different from zero, and the sum extends only over $i \pm j \pm k \pm l = 0$. In particular, we have

$$G_{iijj} = \frac{1}{2\pi} \frac{2 + \delta_{ij}}{\lambda_i \lambda_j} \tag{11}$$

by an elementary calculation – see [6].

From now on we focus our attention on the nonlinearity u^3, since terms of order five or more will not make any difference.

3. Partial Birkhoff normal form

Next we transform the hamiltonian (8) into some partial Birkhoff form of order four so that is appears, in a sufficiently small neighbourhood of the origin, as a small perturbation of some *nonlinear* integrable system.

For the rest of this paper we introduce complex coordinates

$$z_j = \frac{1}{\sqrt{2}}(q_j + \mathrm{i} p_j), \qquad \bar{z}_j = \frac{1}{\sqrt{2}}(q_j - \mathrm{i} p_j).$$

We obtain a real analytic hamiltonian $H = \Sigma_j \lambda_j |z_j|^2 + \cdots$ on the now *complex* Hilbert space $\ell^{a,s}$ with symplectic structure $\mathrm{i} \, \Sigma_j \, dz_j \wedge d\bar{z}_j$. Real analytic means, that H is a function of z and $\bar{z}$, real analytic in the real and imaginary part of z.

In the following, $A(\ell^{a,s}, \ell^{a,s+1})$ denotes the class of all real analytic maps from some neighbourhood of the origin in $\ell^{a,s}$ into $\ell^{a,s+1}$.

MAIN PROPOSITION. *For each finite $n \geq 1$ and each $m > 0$ there exists a real analytic, symplectic change of coordinates Γ is some neighbourhood of the origin in $\ell^{a,s}$ that takes the hamiltonian $H = \Lambda + G$ with nonlinearity* (9) *into*

$$H \circ \Gamma = \Lambda + \bar{G} + \hat{G} + K,$$

where $X_{\bar{G}}, X_{\hat{G}}, X_K \in A(\ell^{a,s}, \ell^{a,s+1})$,

$$\bar{G} = \frac{1}{2} \sum_{\min(i,j) \le n} \bar{G}_{ij} |z_i|^2 |z_j|^2$$

with uniquely determined coefficient $\bar{G}_{ij} = (6/\pi) \cdot (4 - \delta_{ij}/\lambda_i \lambda_j)$, *and*

$$|\hat{G}| = O(\|\hat{z}\|_{a,s}^4), \qquad |K| = O(\|z\|_{a,s}^6),$$

$\hat{z} = (z_{n+1}, z_{n+2}, \ldots)$. *Moreover, the neighbourhood can be chosen uniformly for every compact m-interval in* $(0, \infty)$, *and the dependence of* Γ *on* m *is real analytic.*

Thus, the hamiltonian $\Lambda + \bar{G}$ is integrable with integrals $|z_j|^2, j = 1, 2, \ldots$, while the not-normalized fourth order term $\hat{G}$ is not integrable, but independent of the first n modes.

Proof. For the proof it is convenient to introduce another set of coordinates $(\ldots, w_{-2}, w_{-1}, w_1, w_2, \ldots)$ in $\ell_b^{a,s}$ by setting $z_j = w_j$, $\bar{z}_j = w_{-j}$. The hamiltonian under consideration then reads

$$\begin{aligned} H &= \Lambda + G \\ &= \sum_{j \ge 1} \lambda_j z_j \bar{z}_j + \frac{1}{4} \sum_{i,j,k,l} G_{ijkl}(z_i + \bar{z}_i) \cdots (z_l + \bar{z}_l) \\ &= \sum_{j \ge 1} \lambda_j w_j w_{-j} + \sum_{i,j,k,l}' G_{ijkl} w_i w_j w_k w_l. \end{aligned}$$

The prime indicates that the subscripted indices run through *all nonzero* integers. The coefficients are defined for arbitrary integers by setting $G_{i \ldots l} = G_{|i| \ldots |l|}$. We recall that the sum is restricted to indices i, j, k, l such that $i \pm j \pm k \pm l = 0$. This fact is crucial for the following to hold.

Formally, the transformation Γ is obtained as the time-1-map of the flow of a hamiltonian vectorfield X_F given by a hamiltonian

$$F = \sum_{i,j,k,l}' F_{ijkl} w_i w_j w_k w_l$$

with coefficients

$$\mathrm{i} F_{ijkl} = \begin{cases} \dfrac{G_{ijkl}}{\lambda_i' + \lambda_j' + \lambda_k' + \lambda_l'} & \text{for } (i,j,k,l) \in \mathcal{L}_n \setminus \mathcal{N}_n, \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

Here, $\lambda_j' = \operatorname{sgn} j \cdot \lambda_{|j|}$,

$$\mathscr{L}_n = \{(i, j, k, l) \in \mathbb{Z}^4 : 0 \neq \min(|i|, \ldots, |l|) \leq n\},$$

and $\mathscr{N}_n \subset \mathscr{L}_n$ is the subset of all $(i, j, k, l) \equiv (p, -p, q, -q)$. That is, they are of the form $(p, -p, q, -q)$ or some permutation of it. Clearly, for the latter the denominator $\lambda_i' + \lambda_j' + \lambda_k' + \lambda_l'$ vanishes identically in m.

The definition (12) is correct in view of the following lemma, which we prove at the end of this section.

LEMMA 4. *If i, j, k, l are non-zero integers, such that $i \pm j \pm k \pm l = 0$, but $(i, j, k, l) \not\equiv (p, -p, q, -q)$, then*

$$|\lambda_i' + \lambda_j' + \lambda_k' + \lambda_l'| \geq \frac{cm}{\sqrt{n^2+m}^3}, \qquad n = \min(|i|, \ldots, |l|),$$

with some absolute constant c. Hence, the denominators in (12) *are uniformly bounded away from zero on every compact m-interval in* $(0, \infty)$.

We continue with the proof of the Main Proposition. Expanding at $t = 0$ and using Taylor's formula we formally obtain

$$\begin{aligned} H \circ \Gamma &= H \circ X_F^t|_{t=1} \\ &= H + \{H, F\} + \int_0^1 (1-t)\{\{H, F\}, F\} \circ X_F^t \, dt \\ &= \Lambda + G + \{\Lambda, F\} + \{G, F\} + \int_0^1 (1-t)\{\{H, F\}, F\} \circ X_F^t \, dt, \end{aligned}$$

where $\{H, F\}$ denotes the Poisson bracket of H and F. The last line consists of terms of order six or more in w and constitutes the higher order term K. In the second to last line,

$$\{\Lambda, F\} = -\mathrm{i} \sum_{i,j,k,l}' (\lambda_i' + \lambda_j' + \lambda_k' + \lambda_l') F_{ijkl} w_i w_j w_k w_l,$$

hence

$$G + \{\Lambda, F\} = \sum_{(i,j,k,l) \in \mathscr{N}_n} + \sum_{(i,j,k,l) \notin \mathscr{L}_n} G_{ijkl} w_i w_j w_k w_l = \bar{G} + \hat{G}.$$

Re-introducing the notations $z_j, \bar{z}_j$ and counting multiplicities we find that

$$\bar{G} = \frac{1}{2} \sum_{\min(i,j) \le n} \bar{G}_{ij} |z_i|^2 |z_j|^2$$

with

$$\bar{G}_{ij} = \begin{cases} 24 G_{iijj} = \dfrac{24}{\pi} \cdot \dfrac{1}{\lambda_i \lambda_j} & \text{for } i \neq j, \\ 12 G_{iiii} = \dfrac{18}{\pi} \cdot \dfrac{1}{\lambda_i \lambda_j} & \text{for } i = j, \end{cases}$$

by (11), while $\hat{G}$ is independent of the first n coordinates. Hence, formally we have $H \circ \Gamma = \Lambda + \bar{G} + \hat{G} + K$ as claimed.

To prove analyticity and regularity of the preceding transformation we first show that

$$X_F \in A(\ell_b^{a,s}, \ell_b^{a,s+1}). \tag{13}$$

Indeed, by Lemma 4 and equation (10), and with $\tilde{w}_j = \dfrac{|w_j| + |w_{-j}|}{\sqrt{|j|}}$, we have

$$\begin{aligned} \left| \frac{\partial F}{\partial w_l} \right| &\le \sum_{\pm i \pm j \pm k = l}' |F_{ijkl}| \, |w_i w_j w_k| \\ &\le \frac{c}{\sqrt{|l|}} \sum_{\pm i \pm j \pm k = l}' \frac{|w_i w_j w_k|}{\sqrt{|ijk|}} \\ &\le \frac{c}{\sqrt{|l|}} \sum_{i+j+k=l}' \tilde{w}_i \tilde{w}_j \tilde{w}_k = \frac{c}{\sqrt{|l|}} (\tilde{w} * \tilde{w} * \tilde{w})_l. \end{aligned}$$

If $w \in \ell_b^{a,s}$, then $\tilde{w} \in \ell_b^{a,s+\sigma}$, $\sigma = \frac{1}{2}$, and for $s > 0$, the latter is a Hilbert algebra by Lemma 2. Therefore, $\tilde{w} * \tilde{w} * \tilde{w}$ also belongs to $\ell_b^{a,s+\sigma}$, and hence

$$F_w \in \ell_b^{a,s+1}$$

with

$$\|F_w\|_{a,s+1} \le c \|\tilde{w} * \tilde{w} * \tilde{w}\|_{a,s+\sigma} \le c \|w\|_{a,s}^3. \tag{14}$$

The analyticity of F_w follows from the analyticity of each component function and its local boundedness [9, Appendix A]. This proves (13).

It follows from (13) and (14) that in a sufficiently small neighbourhood of the origin in $\ell^{a,s}$ the time-1-map $X_F^t|_{t=1}$ is well defined and gives rise to a real analytic, symplectic change of coordinates Γ with the estimates

$$\|\Gamma - id\|_{a,s+1} = O(\|w\|_{a,s}^3), \qquad \|D\Gamma - I\|_{a,s+1,s}^{\mathrm{op}} = O(\|w\|_{a,s}^2),$$

where the operator norm $\|\cdot\|_{a,r,s}^{\mathrm{op}}$ is defined by

$$\|A\|_{a,r,s}^{\mathrm{op}} = \sup_{w \neq 0} \frac{\|Aw\|_{a,r}}{\|w\|_{a,s}}.$$

Obviously, $\|D\Gamma - I\|_{a,s+1,s+1}^{\mathrm{op}} \leq \|D\Gamma - I\|_{a,s+1,s}^{\mathrm{op}}$, whence in a sufficiently small neighbourhood of the origin, $D\Gamma$ defines an isomorphism of $\ell^{a,s+1}$. It follows that with $X_H \in A(\ell^{a,s}, \ell^{a,s+1})$, also

$$\Gamma^* X_H = D\Gamma^{-1} X_H \circ \Gamma = X_{H \circ \Gamma} \in A(\ell^{a,s}, l^{a,s+1}).$$

The same holds for the Lie bracket: the boundedness of $\|DX_F\|_{a,s+1,s}^{\mathrm{op}}$ implies that

$$[X_F, X_H] = X_{\{H,F\}} \in A(\ell^{a,s}, \ell^{a,s+1}).$$

These two facts show that $X_K \in A(\ell^{a,s}, \ell^{a,s+1})$. The analogue claims for $X_{\bar{G}}$ and $X_{\hat{G}}$ are obvious. □

Proof of Lemma 4. We may restrict ourselves to positive integers such that $i \leq j \leq k \leq l$. The condition $i \pm j \pm k \pm l = 0$ then reduces to two possibilities, either $i - j - k + l = 0$ or $i + j + k - l = 0$.

We have to study divisors of the form $\delta = \pm\lambda_i \pm \lambda_j \pm \lambda_k \pm \lambda_l$ for all possible combinations of plus and minus signs. To this end, we distinguish them according to their number of minus signs. To shorten notation we let for example $\delta_{++-+} = \lambda_i + \lambda_j - \lambda_k + \lambda_l$. Similarly for all other combinations of plus and minus signs.

Case 0: *No minus sign*. This is trivial.

Case 1: *One minus sign*. Here we have $\delta_{++-+}, \delta_{+-++}, \delta_{-+++} \geq \delta_{+++-}$, so it suffices to study $\delta = \delta_{+++-}$. We consider δ as a function of m and notice that

$$\delta(0) = i + j + k - l \geq 0, \qquad \delta'(m) = \frac{1}{2}\left(\frac{1}{\lambda_i} + \frac{1}{\lambda_j} + \frac{1}{\lambda_k} - \frac{1}{k_l}\right) \geq \frac{1}{\lambda_i}.$$

Since λ_i is increasing with m, it follows that

$$\delta \geq \int_0^m \frac{dm}{\lambda_i} \geq \frac{m}{\lambda_i} = \frac{m}{\sqrt{i^2+m}}.$$

Case 2: *Two minus signs.* Here we have $\delta_{-+-+}, \delta_{--++} \geq \delta_{+--+}$, and all other cases reduce to these ones by inverting the signs. So it suffices to study $\delta = \delta_{+--+}$. The function $f(t) = \sqrt{t^2+m}$ is monoton increasing and convex for $t \geq 0$. Hence we have the estimate $\lambda_l - \lambda_k \geq \lambda_{l-p} - \lambda_{k-p}$ for every $0 \leq p \leq k$. In the case $i+j+k=l$ we thus obtain $\lambda_l - \lambda_k \geq \lambda_{l-k+i} - \lambda_i = \lambda_{j+2i} - \lambda_i$, hence

$$\delta \geq \lambda_{j+2i} - \lambda_j \geq 2(\lambda_{j+1} - \lambda_j) \geq 2if'(j) \geq \frac{i}{\sqrt{1+m}},$$

using the mean value theorem and the monotonicity of f'. With the other alternative $i-j-k+l=0$ we have $j-i=l-k \neq 0$, hence $\lambda_l - \lambda_k \geq \lambda_{j+1} - \lambda_{i+1}$ and $\lambda_{j+1} - \lambda_j \geq \lambda_{i+2} - \lambda_{i+1}$. So we obtain

$$\delta \geq \lambda_{j+1} - \lambda_{i+1} - \lambda_j + \lambda_i \geq \lambda_{i+2} - 2\lambda_{i+1} + \lambda_i \geq f''(i+2)$$

by the monotonicity of f'', hence

$$\delta \geq \left.\frac{m}{\sqrt{t^2+m}^{\,3}}\right|_{i+2} \geq \frac{cm}{\sqrt{i^2+m}^{\,3}}.$$

These bounds give the claimed estimate.

Case 3 *and* 4: *Three and four minus signs.* These ones reduce to the cases 1 and 0, respectively. □

We note that the estimate of the lemma is asymptotically optimal, as it is obtained by the divisor $\lambda_{n+1} - 2\lambda_n + \lambda_{n-1}$ as $n \to \infty$.

4. The Cantor manifold theorem

In a neighbourhood of the origin in $\ell^{a,s}$ we now consider more generally hamiltonians of the form $H = \Lambda + Q + R$, where $\Lambda + Q$ is integrable and in normal form and R is a perturbation term. More precisely, letting $z = (\tilde{z}, \hat{z})$ with $\tilde{z} = (z_1, \dots, z_n)$ and $\hat{z} = (z_{n+1}, z_{n+2}, \dots)$, and

$$I = \frac{1}{2}(|z_1|^2, \ldots, |z_n|^2), \qquad Z = \frac{1}{2}(|z_{n+1}|^2, |z_{n+2}|^2, \ldots),$$

we assume that

$$\Lambda = \langle \alpha, I \rangle + \langle \beta, Z \rangle, \qquad Q = \frac{1}{2}\langle AI, I \rangle + \langle BI, Z \rangle,$$

with constant vectors α, β and constant matrices A, B. In the Birkhoff normal form lemma, $\Lambda + \bar{G}$ is of that form.

The equations of motion of the hamiltonian $\Lambda + Q$ are

$$\dot{\tilde{z}}_j = \mathrm{i}(\alpha + AI + B^T z)_j \tilde{z}_j, \qquad \dot{\hat{z}}_j = \mathrm{i}(\beta + BI)_j \hat{z}_j.$$

Thus, the complex n-dimensional manifold $E = \{\hat{z} = 0\}$ is invariant, and it is completely filled up to the origin by the invariant tori

$$\mathcal{T}(I) = \{\tilde{z} : |\tilde{z}_j|^2 = 2I_j,\ 1 \le j \le n\}, \qquad I \in \overline{\mathbb{P}^n}.$$

On $\mathcal{T}(I)$ the flow is given by the equations

$$\dot{\tilde{z}}_j = \mathrm{i}\omega_j(I)\tilde{z}_j, \qquad \omega(I) = \alpha + AI,$$

and in its normal space by

$$\dot{\hat{z}}_j = \mathrm{i}\Omega_j(I)\hat{z}_j, \qquad \Omega(I) = \beta + BI.$$

They are linear and in diagonal form. In particular, since $\Omega(I)$ is real, $\hat{z} = 0$ is an elliptic fixed point, all the tori are linearly stable, and all their orbits have zero Lyapunov exponents. We therefore call $\mathcal{T}(I)$ an *elliptic rotational torus with frequencies* $\omega(I)$.

Including the nonintegrable perturbation term R this manifold E does in general not persist in its entirety due to resonances among the oscillations. Instead, our aim is to prove the persistence of a *large portion* of E forming an invariant Cantor manifold $\mathcal{E}$ for the hamiltonian $H = \Lambda + Q + R$.

That is, there exists a family of n-tori

$$\mathcal{T}[\mathcal{C}] = \bigcup_{I \in \mathcal{C}} \mathcal{T}(I) \subset E$$

over a Cantor set $\mathscr{C} \subset \mathbb{P}^n$ and a Lipschitz continuous embedding

$$\Psi: \mathscr{T}[\mathscr{C}] \hookrightarrow \ell^{a,s},$$

such that the restriction of Ψ to each torus $\mathscr{T}(I)$ in the family is an embedding of an elliptic rotational n-torus for the hamiltonian H. We call the image $\mathscr{E}$ of $\mathscr{T}[\mathscr{C}]$ a *Cantor manifold of elliptic rotational* n-tori given by the embedding $\Psi: \mathscr{T}[\mathscr{C}] \to \mathscr{E}$.

In addition, the Cantor set $\mathscr{C}$ has full density at the origin, the embedding Ψ is close to the inclusion map $\Psi_0: E \hookrightarrow \ell^{a,s}$, and the Cantor manifold $\mathscr{E}$ is tangent to E at the origin.

For the existence of $\mathscr{E}$ the following assumptions are made.

A. *Nondegeneracy.* The normal form $\Lambda + Q$ is *nondegenerate* in the sense that

$$\begin{aligned}
&(\mathrm{A}_1) \qquad \det A \neq 0,\\
&(\mathrm{A}_2) \qquad \langle l, \beta \rangle \neq 0,\\
&(\mathrm{A}_3) \quad \langle k, \omega(I) \rangle + \langle l, \Omega(I) \rangle \not\equiv 0,
\end{aligned}$$

for all $(k, l) \in \mathbb{Z}^n \times \mathbb{Z}^\infty$ with $1 \leq |l| \leq 2$.

B. *Spectral asymptotics.* There exists $d \geq 1$ and $\delta < d - 1$ such that

$$\beta_j = j^d + \cdots + O(j^\delta),$$

where the dots stand for terms of order less than d in j. Note that the normalization of the coefficient of j^d can always be achieved by a scaling of time.

C. *Regularity.*

$$X_Q, X_R \in A(\ell^{a,s}, \ell^{a,\bar{s}}), \qquad \begin{cases} \bar{s} \geq s & \text{for } d > 1, \\ \bar{s} > s & \text{for } d = 1. \end{cases}$$

By the regularity assumption the coefficients of $B = (B_{ij})_{1 \leq j \leq n < i}$ satisfy the estimate $B_{ij} = O(i^{s-\bar{s}})$ uniformly in $1 \leq j \leq n$. Consequently, for $d = 1$ there exists a positive constant κ such that

$$\frac{\Omega_i - \Omega_j}{i - j} = 1 + O(j^{-\kappa}), \qquad i > j, \tag{15}$$

uniformly for bounded I. For $d > 1$, we set $\kappa = \infty$.

The following theorem is proven in [6] using the KAM-theorem for partial differential equations from [8]. In [6] it is applied to some nonlinear Schrödinger equation, which has $d = 2$. Here we need it for the case $d = 1$, which is more subtle.

THE CANTOR MANIFOLD THEOREM. *Suppose the hamiltonian $H = \Lambda + Q + R$ satisfies assumptions A, B and C, and*

$$|R| = O(\|\hat{z}\|_{a,s}^4) + O(\|z\|_{a,s}^g)$$

with

$$g > 4 + \frac{4-\Delta}{\kappa}, \qquad \Delta = \min(\bar{s} - s, 1).$$

Then there exists a Cantor manifold $\mathscr{E}$ of real analytic, elliptic diophantine n-tori given by a Lipschitz continuous embedding $\Psi\colon \mathscr{T}[\mathscr{C}] \to \mathscr{E}$, where $\mathscr{C}$ has full density at the origin, and Ψ is close to the inclusion map Ψ_0:

$$\|\Psi - \Psi_0\|_{a,\bar{s},B_r \cap \mathscr{T}[\mathscr{C}]} = O(r^\sigma),$$

with some $\sigma > 1$. Consequently, $\mathscr{E}$ is tangent to E at the origin.

Remark 1. The embedding Ψ can be chosen to include a parametrization of each torus in which the flow is linear (although the estimate is then worse, see [6]). Then, for each $I \in \mathscr{C}$ and $v_0 \in \mathscr{T}(I)$,

$$\psi_{I,v_0}\colon t \mapsto \Psi(e^{i\omega(I)t} v_0)$$

is a real analytic solution curve in $\ell^{a,s}$ for the hamiltonian $H = \Lambda + Q + R$. The frequencies $\omega(I)$ are diophantine for all $I \in \mathscr{C}$, so each such orbit is quasi-periodic with n basic frequencies.

Remark 2. The map Ψ is not only Lipschitz but may be shown to be smooth on $\mathscr{T}[\mathscr{C}]$ in the sense of Whitney. But we did not pursue this technical question. Moreover, Ψ may be extended to a global Lipschitz map $\bar{\Psi}\colon E \to \ell^{a,s}$ satisfying the same estimates as Ψ – see again [6]. So $\mathscr{E}$ may be viewed as part of a global Lipschitz manifold. The latter, however, has no invariant meaning for the hamiltonian system outside the Cantor set.

5. Proof of the main theorem

We can now prove the Main Theorem. By section 2 our hamiltonian to start with is

$$H = \Lambda + G = \frac{1}{2}\sum_{j\geq 1} \lambda_j (p_j^2 + q_j^2) + \frac{1}{4}\sum_{i,j,k,l} G_{ijkl} q_i q_j q_k q_l,$$

where the coefficients G_{ijkl} are given by (10) and (11), and where

$$X_G \in A(\ell^{a,s} \times \ell^{a,s}, \ell^{a,s+1} \times \ell^{a,s+1}).$$

All coefficients depend on the parameter $m > 0$, which is not indicated. The parameters $a > 0$ and $s > 0$ may be fixed arbitrarily. The domain of analyticity is then, of course, determined with respect to the norm $\|\cdot\|_{a,s}$.

Now fix $n \geq 1$ and let $J = \{1, \ldots, n\}$ first. The case of a more general set J really makes a difference only in the proof of Lemma 6 below. By section 3 there exists a real analytic, symplectic change of coordinates Γ for each $m > 0$, which takes H into $H \circ \Gamma = \Lambda + \bar{G} + \hat{G} + K$, where, with the notation of the previous section,

$$\Lambda = \langle \alpha, I \rangle + \langle \beta, Z \rangle, \qquad \bar{G} = \frac{1}{2} \langle AI, I \rangle + \langle BI, Z \rangle,$$

with $\alpha = (\lambda_1, \ldots, \lambda_n)$, $\beta = (\lambda_{n+1}, \ldots)$ and $A = (\bar{G}_{ij})_{1 \leq i,j \leq n}$, $B = (\bar{G}_{ij})_{1 \leq j \leq n < i}$, while $|\hat{G}| = O(\|\hat{z}\|_{a,s}^4)$ and $|K| = O(\|z\|_{a,s}^6)$. Moreover, the regularity of the nonlinear vector fields is preserved. Hence the transformed hamiltonian is of the form $H \circ \Gamma = \Lambda + Q + R$ with $Q = \bar{G}$, $R = \hat{G} + K$ required by the Cantor Manifold Theorem.

Suppose for a moment that the assumptions of that theorem are satisfied. We then obtain a Cantor manifold $\mathscr{E}$ of real analytic, elliptic diophantine n-tori in $\ell^{a,s}$ for the hamiltonian $H = \Lambda + G$ in complex coordinates given by an embedding

$$\Gamma \circ \Psi : \mathscr{T}[\mathscr{C}] \to \mathscr{E}.$$

These tori carry the quasi-periodic motions

$$\gamma_{I,v_0} : t \mapsto z(t) = \Gamma \circ \Psi(e^{i\omega(I)t} v_0),$$

for $I \in \mathscr{C}$ and $v_0 \in \mathscr{T}(I)$. Their real imaginary parts, $q = \operatorname{Re} z$ and $p = \operatorname{Im} z$, solve the equations of motion for the corresponding hamiltonian (8) in real coordinates. Going back to the space $W^{a,s+1/2}$ by the isomorphism

$$\mathscr{S} : \ell^{a,s} \to W^{a,s+1/2}, \qquad q \mapsto u = \sum_{j \geq 1} \frac{q_j}{\sqrt{\lambda_j}} \phi_j(x),$$

$\mathscr{E}$ is mapped into another Cantor manifold of real analytic diophantine tori in $W^{a,s+1/2}$, which by Lemma (1) carry real analytic, quasi-periodic solutions u of the given nonlinear wave equation. This will prove the Main Theorem.

We now verify the assumptions of the Cantor Manifold Theorem. We already mentioned that $X_Q, X_R \in A(\ell^{a,s}, \ell^{a,s+1})$ with $|R| = O(\|\hat{z}\|_{a,s}^4) + O(\|z\|_{a,s}^6)$. On the other hand, we have

$$\lambda_j = \sqrt{j^2+m} = j + \frac{m}{2j} + O(j^{-3}).$$

So conditions B and C are satisfied with $d = 1$, $\delta = -1$ and $\bar{s} = s+1 > s$.

Moreover, since $B_{ij} = \bar{G}_{ij} = 24/\pi \cdot \lambda_i^{-1}\lambda_j^{-1}$, we have

$$\Omega_{j-n} = (\beta + BI)_{j-n} = \lambda_j + \frac{\langle v, I\rangle}{\lambda_j}$$

with $v = 24/\pi(\lambda_1^{-1}, \ldots, \lambda_n^{-1})$. This gives the asymptotic expansion

$$\Omega_{j-n} = j + \frac{m}{2j} + \frac{\langle v, I\rangle}{j} + O(j^{-3}) = j + \frac{m_I}{j} + O(j^{-3}),$$

$m_I = \frac{1}{2}m + \langle v, I\rangle$. Thus, for $i > j$,

$$\frac{\Omega_i - \Omega_j}{i-j} = 1 - \frac{m_I}{(i+n)(j+n)} + O(j^{-3}) = 1 + O(j^{-2}),$$

uniformly for bounded I. This gives $\kappa = 2$ in (15). Consequently, also the smallness condition is satisfied, since

$$g > 4 + \frac{4-\Delta}{\kappa}$$

for $g = 6$, $\kappa = 2$ and $\Delta = 1$.

Finally, we verify the nondegeneracy condition A. Item (A_1) is always satisfied, so it suffices to consider (A_2) and (A_3).

LEMMA 5. *For all $n \geq 1$ and all $m > 0$, the matrix $A = (\bar{G}_{ij})_{1 \leq i,j, \leq n}$ is nondegenerate.*

Proof. To shorten notation we multiply A by $\pi/6$. We then have $A = C - D$, where D is the n-dimensional diagonal matrix with diagonal elements $D_j = \lambda_j^{-2}$, and C is the rank one $n \times n$-matrix with elements $C_{ij} = 4\lambda_i^{-1}\lambda_j^{-1}$. Then, by the multilinearity of the determinant function with respect to columns,

$$\begin{aligned}\det(D-C) &= \det D - \sum_{1\le i\le n} C_{ii} \det D^i \\ &= \prod_{1\le j\le n} \frac{1}{\lambda_j^2} - \sum_{1\le i\le n} \frac{4}{\lambda_i^2} \prod_{j\ne i} \frac{1}{\lambda_j^2} \\ &= (1-4n) \prod_{1\le j\le n} \frac{1}{\lambda_j^2},\end{aligned}$$

where D^i denotes the matrix D with the i-th column and row eliminated. Thus we have $\det A \neq 0$. □

So far, all these arguments do not change except for notation for the indices, if $J = \{1, \ldots, n\}$ is replaced by an arbitrary finite index set $J = \{j_1 < \cdots < j_n\}$. This is no longer true for the nondegeneracy condition (A_3).

LEMMA 6. *For an index set $J = \{j_1 < \cdots < j_n\}$ with $n \geq 2$ satisfying the assumption of the Main Theorem, one has*

$$\langle k, \omega(I)\rangle + \langle l, \Omega(I)\rangle \not\equiv 0$$

for all k, l with $1 \leq |l| \leq 2$ and all $m > 0$. So for these index sets the normal form $A + Q$ is nondegenerate for all $m > 0$.

Proof. We have to show that either $\langle \alpha, k\rangle \neq \langle \beta, l\rangle$ or $Ak \neq B^T l$, where the vectors and matrices are now defined with respect to the more general index set J.

Suppose to the contrary that $\langle \alpha, k\rangle = \langle \beta, l\rangle$ and $Ak = B^T l$. Multiplying A and B by $\pi/6$ and defining C and D as above, we then have $Dk = Ck - B^T l$, or

$$\frac{k_i}{\lambda_{j_i}} = 4\langle v, k\rangle - 4\langle w, l\rangle,$$

where $v = (\lambda_{j_1}^{-1}, \ldots, \lambda_{j_n}^{-1})$, $w = (\lambda_{j_{n+1}}^{-1}, \ldots)$, and $j_{n+1} < j_{n+2} < \cdots$ are the integers not in J. Thus, $k_i \lambda_{j_i}^{-1}$ is independent of i, whence $\langle v, k\rangle = nk_i\lambda_{j_i}^{-1}$, and thus

$$k_i = \frac{4}{4n-1} \lambda_{j_i} \langle w, l\rangle, \qquad 1 \leq i \leq n. \tag{16}$$

The assumption $\langle \alpha, k\rangle = \langle \beta, l\rangle$ then further implies

$$\frac{4}{4n-1} \sum_{1\le i\le n} \lambda_{j_i}^2 = \frac{\langle \beta, l\rangle}{\langle w, l\rangle}. \tag{17}$$

We first show that for $|l| = 1$ this is not possible for *any* J. Indeed, we then have $\langle \beta, l \rangle = \pm \lambda_\nu = \langle w, l \rangle^{-1}$ for some $\nu \notin J$, so the two equations combined give

$$k_i^2 = \frac{4}{4n-1} \cdot \frac{\lambda_{j_i}^2}{\Sigma_i \lambda_{j_i}^2}, \qquad 1 \le i \le n.$$

But this equation can not have an integer solution for any $n \ge 1$ and any $1 \le i \le n$.

So consider now the case $|l| = 2$. If we had $|\langle w, l \rangle| < (4n-1)/(4n-4)$, then (16) implies

$$0 \ne \min_{1 \le i < n} |k_{i+1} - k_i| < \min_{1 \le i < n} \frac{|\lambda_{j_{i+1}} - \lambda_{j_i}|}{n-1} \le 1$$

by assumption, which is not possible. On the other hand, $|\langle w, l \rangle| \ge (4n-1)/(4n-4) > 1$ implies $|\langle w, l \rangle| = \lambda_1^{-1} + \lambda_\nu^{-1}$ with some index $\nu \le \frac{4}{3}n$. But then one finds that

$$\frac{\langle \beta, l \rangle}{\langle w, l \rangle} = \lambda_1 \lambda_\nu \le \lambda_\nu^2 \le 2n^2 + m,$$

which leads to a contradiction to (17). This shows that $\langle \alpha, k \rangle = \langle \beta, l \rangle$ and $Ak = B^T l$ can not have integer solutions. □

LEMMA 7. *For $J = \{j\}$ one has $\langle k, \omega(I) \rangle + \langle l, \Omega(I) \rangle \not\equiv 0$ for all k, l with $1 \le |l| \le 2$ and all $m > 0$ except those at which*

$$\frac{4}{3}\left(\frac{\lambda_j}{\lambda_\nu} + \frac{\lambda_j}{\lambda_\mu}\right) = k = \frac{\lambda_\nu}{\lambda_j} + \frac{\lambda_\mu}{\lambda_j}$$

with an integer k and some indices $1 \le \nu < j < \mu$. There are at most finitely many such m-values.

Proof. We continue the preceding proof. The case $|l| = 1$ being dealt with, let $|l| = 2$. By flipping signs if necessary we have $\langle w, l \rangle = \lambda_\nu^{-1} \pm \lambda_\mu^{-1}$ with $\nu \le \mu$ not in J, and $\nu \ne \mu$ in the minus-case. Then (16) reduces to

$$k = \frac{4}{3}\left(\frac{\lambda_j}{\lambda_\nu} \pm \frac{\lambda_j}{\lambda_\mu}\right),$$

and (16) and (17) together give

$$k = \frac{\lambda_\nu}{\lambda_j} \pm \frac{\lambda_\mu}{\lambda_j}.$$

We can rule out the minus-case, since then the two equations have opposite sign. Also, $j < \nu \le \mu$ leads to $k < \frac{4}{3} \wedge k > 2$, and $\nu \le \mu < j$ leads to $k > \frac{8}{3} \wedge k < 2$, which are both impossible. So one must have $\nu < j < \mu$. This allows only finitely many choices for ν, and reduces the integer k to the finite interval $\frac{4}{3} < k < \frac{4}{3}((j/\nu) + 1)$. One then verifies that for $\mu \ge 4j^2$ there are no solutions at all, and at most one for $\mu \le 4j^2$. Hence there are at most finitely many exceptional m-values for each j. And there are clearly none for $j = 1$. □

6. The Banach algebra property

Consider the Hilbert space $\ell^{a,s}$ of all doubly infinite complex sequences $q = (\ldots, q_{-1}, q_0, q_1, \ldots)$ with

$$\|q\|_{a,s}^2 = \sum_j |q_j|^2 \lfloor j \rfloor^{2s} e^{2a|j|} < \infty, \qquad \lfloor j \rfloor = \max(|j|, 1).$$

The convolution $q * p$ of two such sequences is defined by $(q * p)_j = \Sigma_k q_{j-k} p_k$.

LEMMA. *If $a \ge 0$ and $s > \frac{1}{2}$, then $\|q * p\|_{a,s} \le c \|q\|_{a,s} \|p\|_{a,s}$ for $q, p \in \ell^{a,s}$ with a finite constant c depending only on s.*

Proof. Let $\gamma_{jk} = (\lfloor j-k \rfloor \lfloor k \rfloor / \lfloor j \rfloor)$. By the Schwarz inequality,

$$\left|\sum_k x_k\right|^2 = \left|\sum_k \frac{\gamma_{jk}^s x_k}{\gamma_{jk}^s}\right|^2 \le c_j^2 \sum_k \gamma_{jk}^{2s} |x_k|^2, \qquad c_j^2 = \sum_k \frac{1}{\gamma_{jk}^{2s}},$$

for all j. We have

$$\frac{1}{\gamma_{jk}} \le \frac{\lfloor j-k \rfloor + \lfloor k \rfloor}{\lfloor j-k \rfloor \lfloor k \rfloor} = \frac{1}{\lfloor j-k \rfloor} + \frac{1}{\lfloor k \rfloor},$$

so that

$$c_j^2 \le \sum_k \left(\frac{1}{\lfloor j-k \rfloor} + \frac{1}{\lfloor k \rfloor}\right)^{2s} \le 4^s \sum_k \frac{1}{\lfloor k \rfloor^{2s}} \stackrel{\text{def}}{=} c^2 < \infty$$

for all j. It follows that for $a = 0$,

$$\begin{aligned}\|q * p\|_{a,s}^2 &= \sum_j [j]^{2s} \Big| \sum_k q_{j-k} p_k \Big|^2 \\ &\le c^2 \sum_j [j]^{2s} \sum_k \gamma_{jk}^{2s} |q_{j-k} p_k|^2 \\ &= c^2 \sum_{j,k} [j-k]^{2s} |q_{j-k}|^2 [k]^{2s} |p_k|^2 \\ &= c^2 \|q\|_{a,s}^2 \|p\|_{a,s}^2.\end{aligned}$$

The case $a > 0$ is a simple variation of the last estimate. □

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