Discontinuity of geometric expansions.

Comment. Math. Helvetici 71 (1996) 213–228

0010–2571/96/020213–16\$1.50 + 0.20/0

Discontinuity of geometric expansions

JOACHIM LOHKAMP*

1. Introduction

The spectrum $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \nearrow +\infty$ of the Laplacian on a Riemannian manifold (M^n, g) provides great deal of insight into the geometry of (M^n, g). Certainly the most often used method to recover geometric information from the spectrum starts with the following observation:

The trace $\operatorname{tr}(H)$ of the heat kernel H of the Laplacian fulfills:

$$\operatorname{tr}(H) = \sum_{j=0}^{\infty} \exp(-t \cdot \lambda_j). \tag{1}$$

Moreover there is an asymptotic expansion for $\operatorname{tr}(H)$:

$$\operatorname{tr}(H) \sim (4\pi t)^{-n/2} \cdot (a_0 + a_1 \cdot t + a_2 \cdot t^2 + \cdots) \tag{2}$$

where each of the *uniquely determined* a_k is an expression of the form $a_k(M, g) = \int_M P_k(\mathrm{Riem}(g))\, dV_g$, that is an integral of (universal) polynomials P_k in derivatives of the curvature tensor $\mathrm{Riem}(g)$. At least a_0 and a_1 are easily interpreted: $a_0 = \mathrm{Vol}(M, g)$, $a_1 = \frac{1}{6}\int_M \mathrm{Scal}(g)\, dV_g$.

Combining (1) and (2) we see: the spectrum determines the geometric quantities a_k. Actually the a_k are not just interesting in themselves, but allow to deduce a lot of (apparently) sharper details: for instance one can (under certain circumstances) detect symmetric metrics, constant sectional or Ricci curvature metrics, gets non-trivial bounds on the curvature beside many other things described in [BGM] and [B].

Also one can recover Weyl's formula showing the particular interplay between spectrum and volume (i.e. a_0):

$$\lambda_k^{n/2} \sim c_n \cdot k/\mathrm{Vol}(M, g),$$

for some constant $c_n > 0$ depending only on n.

* Supported by the "Deutsche Forschungsgemeinschaft DFG"

All this makes these a_k becoming significant in spectral geometry. Unfortunately the "calculation" of these a_k from the spectrum turns out to be quite intricate: the asymptotic expansion is rather implicit and even more "unrealistic" it needs the whole spectrum. Particularly the latter point leads us to ask whether "sufficiently large" but *finite* parts of the spectrum could contain comparably much information (cf P. Berard's survey [B] Ch. VII where this question had also been raised.

Let us begin with a basic result (due to Y. Colin de Verdière [C1]): such a finite prescription does not lead to any restriction of the topology of the underlying manifold (in dimension $n \geq 3$).

In this paper we will "attach geometry" to those rather special metrices in [C1]. This allows us to rule out many conceivable geometric implications.

Now let us become more specific: as the complete spectrum determines a lot of geometric data one might expect, that the relaxed condition of knowing finitely many eigenvalues implies estimates of these data becoming sharper the more eigenvalues are taken into account. The point is that these estimates should *not* depend on the underlying metric as it is desired to derive a priori information from a set of eigenvalues (as is possible from the complete spectrum).

The main results of this paper will show that such a *uniform continuity* does not hold in general.

Let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ be *any* given sequence, M^n a closed manifold of dimension $n \geq 3$.

THEOREM 1. *For any* $V \in]0, +\infty[$, $S \in]-\infty, +\infty[$ *there is a sequence of metrics* g_m *on* M^n *with*:

(i) $\lambda_k(M, g_m) = \lambda_k$, *for* $k \leq m$

(ii) $\mathrm{Vol}(M, g_m) = V$, $\int_M \mathrm{Scal}(g_m)\, dV_{g_m} = S$

(iii) $a_{2k}(M, g_m) \to +\infty$, $a_{2k+1}(M, g_m) \to -\infty$, $k \geq 1$

($\lambda_k(M, g_m)$ *and* $Scal(g_m)$ *denote the* kth *eigenvalue resp. the scalar curvature of* (M, g_m).)

There are generalized versions where finite values of higher coefficients (i.e. a_i, $i \geq 2$) can be prescribed (in a one-sided unbounded interval). However, there are non-trivial relations between them (depending on dimension and topology of M) cutting down the degrees of freedom for possible choices of these coefficients. But we can always realize any arbitrary and *independent* prescription of finitely many eigenvalues, volume and (certain) curvatures. This might become even clearer from our second result which could not hold for the whole spectrum given (as there are restrictions for prescribing (bounds for) curvatures already from the knowledge of volume and total scalar curvature).

THEOREM 2. *For any* $V \in]0, +\infty[$ *there is a sequence of metrics* g_m *on* M *with*:

(i) $\lambda_k(M, g_m) = \lambda_k$, *for* $k \leq m$

(ii) $\mathrm{Vol}(M, g_m) = V$

(iii) $\mathrm{Ric}(g_m) < -m^2$

(*Ric*(*g*) *denotes the Ricci curvature of some metric g*).

Note that, on the other hand, Ric cannot be bounded below independently from volume and eigenvalues: beside bounding volumes from $\mathrm{Ric} > c > 0$, the eigenvalues can be estimated (acc. P. Li and S. T. Yau [LY]) leading to non-trivial restrictions even if volume and Ric are compatible.

Finally, we give a brief outline of our methods. The first and main step in this paper is to get a certain metric on a Riemann surface (without fixing its genus) whose spectrum starts with those given eigenvalues and whose area is arbitrarily large. The new argument for the construction of such Riemannian manifolds combines those methods already known (cf. [C3]) with the crushed ice effect (cf. [Ch] Ch. IX). From this we finally get metrics with prescribed eigenvalues and volume in higher dimensions. In the second step we get additionally the desired curvature properties. These are derived from the author's existence results for $\mathrm{Ric} < 0$-metrics in [L1] and [L2]. Finally, the higher coefficients can also be handled using those results above and further general structural insights of these terms described in [A], [BGM], [B] and [G].

Remark. Recently B. Colbois and J. Dodziuk [CD] and Y. Xu [X] have shown that there are always metrics with $\lambda_1(M, g_m) > m$ and $\mathrm{Vol}(M, g_m) = 1$ for arbitrarily large m. Their (completely different) arguments do not extend to prescribing finitely many eigenvalues with given (or at least large) volume. The latter result (which is contained in our Theorems) was posed as a conjecture by J. Dodziuk in his more recent survey [D] and the author thanks him for sending this preprint which led us to think about these problems.

2. Stable metrics

The basic technique to obtain some metric with prescribed finite part of the spectrum was developed by Y. Colin de Verdière in [C1]. In the centre there is a notion of "stability" introduced by V. Arnold. For the reader's convenience we recall some ideas of this theory and those results needed below in an appropriate form.

There are two basic techniques of how to get a metric with given eigenvalues which are actually combined in order to establish the existence of such metrics.

The first one starts with a graph. That is just a 1-dimensional simplicial complex. Here one can define formally a Laplacian adjoint to the metric measuring the length of the edges. Actually, it is an easy combinatorial consideration that allows to choose such a complex C whose spectrum starts with the given eigenvalues. Now one can "approximate" C by a sequence of smooth (hyperbolic) surfaces (of fixed type) such that each edge of C corresponds to a degenerating closed geodesic on these surfaces. A suitable choice of the lengths of these curves allows to find that the eigenvalues of the surfaces converges (after stepwise scaling of the whole metric) to those of C in a pretty uniform manner. Actually, the Laplacian on C has another marvellous "stability" property which allows us to get precisely the desired eigenvalues already on this surface.

This is what will be described in following important technical disgression.

Let $f\colon B \to \mathcal{M}(M)$ be a continuous map from a closed ball $B = \overline{B_1(0)} \subset \mathbb{R}^N$ into the space of smooth Riemannian matrics on a compact manifold M^n with smooth (or without) boundary. Then we can consider for some fixed metric g_0 the following continuous map: $\Phi(f)\colon B \to Q_n(g_0)$, defined by the Dirichlet integral

$$\Phi(f)(p)(\varphi, \varphi) := \int_M \| \nabla_{f(p)} I_p(\varphi) \|^2_{f(p)} \, dV_{f(p)}.$$

(We will call $\Phi(f)$ a *spectral map* belonging to f.) Explanation: $Q_n(g)$ is the (finite dimensional) vector space of quadratic forms on the function space $E_n(g)$ spanned by the first n eigenfunctions of the Laplace operator Δ_g belonging to the metric G. In case $\partial M \neq \phi$ we consider Neumann eigenfunctions. (Also ∇_g, $\|\cdot\|_g$, dV_g mean that they are with respect to g.)

Finally $I_p\colon E_n(g_0) \to E_n(f(p))$ is an L^2-isometry depending continuously on $p \in B$ and of course $\Phi(f)$ depends on the choice of these I_p.

This allows to work on the fixed function space $E_n(g_0)$, and one observes that the eigenvalues of $\int_M \| \nabla_{f(p)} I_p(\cdot) \|^2_{f(p)} \, dV_{f(p)}$ on $E_n(g_0)$ and $\int_M \| \nabla_{f(p)} (\cdot) \|^2_{f(p)} \, dV_{f(p)}$ on $E_n(f(p))$ are identical (with multiplicities).

It is obvious (and important) to note that there is not need to restrict to a fixed manifold since the L^2-isometry I_p is the only link. That is we can equally well start with a metric g_0 defined on a different manifold M'. Keeping this in mind we formulate:

DEFINITION 2.1. *$f\colon B \to \mathcal{M}(M')$ is called a stable family of metrics (around $g_0 = f(0)$) resp. g_0 is stable, if there is an $\varepsilon = \varepsilon(f, g_0) > 0$ such that for any map $F\colon B \to \mathcal{M}(M)$ with $\| \Phi(F) - \Phi(f) \|_{C^0(B)} < \varepsilon$ for some spectral map $\Phi(F)$, there is a point $p \in \operatorname{int} B$ with $\Phi(F)(p) = \Phi(f)(0)$. In this case we will say that F is spectrally near to f.*

(That is $F(p)$ has the same first n eigenvalues and it is obviously again a stable metric.)

Of course, this stability only refers to the behaviour with respect to the first n eigenvalues and is more a property of $\Phi(F)$ than of F. But as we will always fix such a set of eigenvalues and construct stepwise new metrics we prefer to emphasize the background metrics.

Now we can resume our discussion of the case of surfaces. The point is that the graph C can be chosen such that the metric on C is stable in the sense above. Using that the eigenvalues of our approximating surface converge one can actually deduce that we can find a surface having exactly the prescribed eigenvalues.

Now, we can use a second "technique" (valid in dimension ≥ 3) allowing us to extend this result to higher dimensions (without topological restrictions). Roughly speaking it says that (under suitable circumstances) the behaviour of eigenvalues is governed by those "parts" of the manifold carrying most of the volume. An argument of this type is described and used in §5 below.

Finally, some more technical remarks. Here and throughout this paper we consider some fixed prescribed sequence $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \dots$ and as we are looking for metrics for any given finite part we may assume $\lambda_{n_0} < \Gamma < \lambda_{n_0+1}$ for each prescribed n_0. This will be assumed in each step of construction below without further comment.

Now we will briefly discuss a simple (actually partially redundant) *criterion* to show that a sequence maps $Fm\colon B \to \mathcal{M}(M)$ becomes (eventually) spectrally near to some stable family $f\colon B \to \mathcal{M}(M)$:

Let F_m be a sequence of maps $F_m\colon B \to \mathcal{M}(M)$ with $\varphi_1^p(m), \dots, \varphi_{n_0}^p(m)$ L^2-orthonormal bases for the first n_0 eigenvalues $\lambda_i^p(m)$ of $F_m(p)$ depending continuously on p and such that $\varphi_i^p(m) \to \varphi_i^p$ in $L^2(M, f(p))$ and $\lambda_i^p(m) \to \lambda_i^p$ uniformly in $p \in B$ (where φ_i^p, λ_i^p belong correspondingly to $f(p)$).

Then (for suitably large m) F_m is spectrally near to f.

The point is that the L^2-convergence allows to define (inductively) L^2-isometries $I^p\colon E_n(f(p)) \to E_n(F_m(p))$ depending continuously on $p \in B$: For instance we can take

$$I^p(\varphi_i^p) := \text{minimizer of} \int_M |\varphi_1^p - \varphi|^2 \, dV \text{ in } E_n(F_m(p)) \text{ with } \|\varphi\|_{L^2} = 1$$

$$I^p(\varphi_k^p) := \text{minimizer of} \int_M |\varphi_k^p - \varphi|^2 \, dV \text{ in the orthogonal complement}$$

$$\text{of span } \{I(\varphi_1^p), \dots, I(\varphi_{k-1}^p)\} \text{ with } \|\varphi\|_{L^2} = 1.$$

Using these isometries and taking $I^p \circ I_p$ we get a spectral map $\Phi(F_m)$ becoming arbitrarily near to $\Phi(f)$ in C^0-topology for large m.

3. Multiple connected sums

As described above one can obtain metrics with prescribed eigenvalues forming manifolds as "hulls" of graphs whose (formally defined) Laplacian is quite easy to handle. The problem with those manifolds is precisely that their volume becomes arbitrarily small for suitable chosen sets of eigenvalues (cf. for instance [Co], as one has to scale the metric in order to prevent the lowest eigenvalues from becoming zero, which is the effect of the shrinking of lengths of geodesics "predicted" (better to say allowed) by Cheeger's inequality, namely in the borderline case of equality.

Thus we will have to add other techniques to take the following main step in getting large volumes:

PROPOSITION 3.1. *For any $A > 0$ there is a compact Riemann surface F (of some genus depending on A) equipped with some stable metric g fulfilling*:

$\lambda_k(g) = \lambda_k, k \le n_0$ *and* *area* $(F, g) > A$.

Moreover the areas of the surrounding stable family of metrics are also $> A$.

We will start from any stable metric g_0 on a surface F_0 with $\lambda_k(g_0) = \lambda_k$, $k \le n_0$ which was obtained in [C1] and [C3]. Denote a positive lower bound of the areas of the corresponding family by α. In order to enlarge the area one might think of taking connected sums. However, we have to circumvent the case where Cheeger's inequality becomes more or less an equality. Here we will involve a technique which had been isolated from this context before.

We start with defining "Besicovitch-coverings" on (F_0, g_0) and more generally for any compact manifold (M, g). Their existence is not too hard to establish (cf. Appendix of [L1] for a proof).

There is an $R_0 = R_0(M, g) <$ injectivity radius of the exponential map ($\exp_p: T_pM \to M$) such that for any $R \in]0, R_0[$ there is a number K *independent of* R and a finite set $S = S(R) < M$ with

(i) $\overline{B_R(p)}$, $p \in S$ is a (closed) covering of M.
(ii) each $z \in M$ is contained in at most K balls $B_{2R}(p)$
(iii) $\overline{B_R(p)} \cap S = \{p\}$.

Now let $r \in]0, \frac{R}{5}[$ and define $M_{r,R} := M \setminus \bigcup_{p \in S(R)} B_r(p)$. Moreover take a second copy of $M_{r,R} =: M^+$ denoted by M^- and form (completely analogously to the usual connected sum) the

multiple connected sum $M^+ \mathop{\#}_{mult} M^- := M^+ \cup M^- / \sim$

where "$\sim$" indicates the obvious boundary identifications ($M^+ \supset \partial B_r(p) \equiv \partial B_r(p) \subset M^-$). Of course, the topology of this closed manifold depends on $S(R)$.

Now we specialize to (F_0, g_0). The following result obviously implies (3.1) from iteration (up to $(\frac{3}{2})^k > \frac{A}{\alpha}$): For suitably chosen r, R we have:

MAIN LEMMA 3.2. *There is a stable family of metrics* $f\colon B \to \mathcal{M}(F_0^+ \underset{mult}{\#} F_0^-)$ *with*:

$$\lambda_k(f(0)) = \lambda_k, k \leq n_0 \quad \text{and} \quad \mathit{area}(F_0^+ \underset{mult}{\#} F_0^-, f(p)) > \tfrac{3}{2} \cdot \alpha.$$

(The analogue holds in higher dimension, but we restrict to surfaces for notational convenience. The interested reader might anticipate (from the 3-dimensional analogue) the existence of large volume metrics for arbitrary three manifolds: here one could use Dehn surgery (that is surgery in codim 2(!)) to derive the result as soon as it is proved for some manifold).

In the proof of (3.2) we will make essential use of the canonical reflection (involution) $s\colon F_0^+ \#_{mult} F_0^- \to F_0^+ \#_{mult} F_0^-$ with $s(F_0^+) = F_0^-$. This allows us to write any function on this surface as a sum of its symmetric resp. antisymmetric part $f_s := \frac{1}{2}(f + f \circ s)$, $f_a := \frac{1}{2}(f - f \circ s)$, $(f = f_s + f_a)$. Furthermore denote by C the fixed point set of s ($=$symmetry circles $= \bigcup_{p \in S(R)} \partial B_r(p)$) and note that $f_a|_C \equiv 0$ and if there is also chosen a s-invariant metric then $\partial f_s/\partial n|_C \equiv 0$ ($\partial/\partial n$ denotes the normal derivative).

Eventually, we define s-invariant metrics which will serve for (3.2): Basically we are interested in the natural metric g_0^s on $F_0^+ \#_{mult} F_0^-$ with $g_0^s \equiv g_0$ on F_0^+, $g_0^s \equiv s^*(g_0)$ on F_0^-.

Unfortunately, these metrics are not smooth along C, since C is not totally geodesic. But our problem is well-behaved under smoothing operations as described now:

For $\rho \in]0, \frac{R}{5}[$ we define a metric g_ρ on $F_0^+ \#_{mult} F_0^-$ with

$$g_\rho \equiv g_0^s \text{ on } F_0^+ \Big\backslash \bigcup_{p \in S} B_{r+\rho}(p) \cup F_0^- \Big\backslash \bigcup_{p \in S} B_{r+\rho}(p):$$

Use (cf. (6.2)) that we can assume (in our context) g_0 being just the Euclidean metric $g_{\mathbb{R}} + r^2 \cdot g_{S^2}$ on $B_{r+\rho}(p)$ and define on $B_{r+\rho}(p) \backslash B_r(p) = [r, r+\rho[\times S^1, g_\rho \equiv g_{\mathbb{R}} + f_\rho^2(r) \cdot g_{S^1}$ for a smooth f_ρ with $f_\rho(r) \equiv r$ near $r + \rho$, $f_\rho \equiv$ const. > 0 near $r, f'_\rho, f''_\rho \geq 0$. This obviously leads (for $\rho \to 0$) to smooth metrics on $F_0^+ \#_{mult} F_0^-$ arbitrarily near to g_0^s (in C^0-topology).

Finally we will collect some properties of our coverings useful to analyze these metrics g_ρ and g_0^s:

LEMMA 3.3. *For sufficiently small* $R > 0$ *we have constants* $c_i = c_i(g_0) > 0$ *with*:

(i) *area* $(\bigcup_{p \in S(R)} B_r(p)) \leq \# S(R) \cdot \max_{p \in S(R)} (\mathit{area}\ B_r(p)) \leq c_0 \cdot r^2/R^2$

(ii) $\# S(R)/|\ln r| \geq c_1 \cdot 1/R^2 \cdot |\ln r|$.

The proofs rely on the inequalities: area $F_0 < \# S(R) \cdot \max_{p \in S}$ area $B_R(p)$ and $\# S(R) \cdot \min_{p \in S}$ area $B_R(p) \leq K \cdot$ area F_0. Details are left to the reader. □

4. Crushed ice

We will analyze the spectrum of those metrics defined above exploiting effects related to the "crushed ice" phenomenon. We use the following two important auxiliary Lemmas (cf. [Ch] Ch. IX, §4 and [RT], §3) showing a remarkably different behaviour for the Dirichlet resp. Neumann eigenvalue problem on $F_0^+ = F_{r,R}$ depending on r and R.

LEMMA 4.1. *The first nontrivial eigenvalue λ_1^D for the Dirichlet problem on $(F_{r,R}, g_0)$ fulfills: $\lambda_1^D \geq c(g_0) \cdot \# S(R)/|\ln r| \cdot K^2$, for some $c > 0$.*

On the other hand we have the following result stated in a specialized and adequate form:

LEMMA 4.2. *Let $\varphi_i(m)$, $i = 1, \ldots, n_0$ be an orthonormal set of eigenfunctions for the first n_0 eigenvalues for the Neumann problem on F_{r_m,R_m} $(\Delta\varphi_i(m) = \lambda_i(m) \cdot \varphi_i(m))$ for sequences $r_m \to 0$, $R_m \to 0$ with area $\bigcup_{p \in S(R_m)} B_{r_m}(p) \to 0$ for $m \to \infty$.*

Then there is a subsequence m_k such that the trivial extensions of $\varphi_i(m_k)$ (by 0 on $B_{r_m}(p)$) converge strongly in $L^2(F_0, g_0)$ to eigenfunctions φ_i on (F_0, g_0), $\Delta\varphi_i = \lambda_i\varphi_i$, with $\lambda_i = \lim_k \lambda_i(m_k)$. Furthermore each eigenfunction φ on (F_0, g_0) is limit of such a sequence of Neumann eigenfunctions.

Both results easily extend (uniformly) to compact families of metrics $f: B \to \mathcal{M}(M)$ (instead of a single metric g_0): The constant $c(g_0)$ in (4.1) depends continuously on the metric. In (4.2) one can assume $\varphi_i(m)$ depending continuously on $(M, f(p))$ for $p \in B$ and find jointed subsequences converging uniformly in L^2, since there are uniform $H^{1,2}$-estimates and uniform convergence of area $(\bigcup_{p \in S} B_r(p)) \to 0$, which are the two main points used in the proof of (4.2) in [RT].

COROLLARY 4.3. *The map $F_m: B \to \mathcal{M}(F_{r_m,R_m})$ with $F_m(p) = (f(p))_{\rho_m}|_{F_{r_m,R_m}}$ is spectrally near to $f(p)$ for large m and suitably small $\rho_m > 0$.*

Proof. Using the L^2-convergence (4.2) (of the trivial extensions onto F_0) of the orthonormal base $\varphi_i(m)$ of Neumann eigenfunctions on (F_{r_m,R_m}, g_0) to φ_i on (F_0, g_0) we observe from our criterion in §2 that we can define a spectral map Φ making $\bar{F}_m: B \to \mathcal{M}(F_{r_m,R_m})$, $\bar{F}_m(p) = f(p)|_{F_{r_m,R_m}}$ spectrally near to $f(p)$ (*acc.* (4.2)).

Thus it is enough to show that the Neumann eigenfunctions $\varphi_i^\rho(m)$ for g_ρ (can be assumed to) converge to those for g_0 in $L^2(F_{r_m,R_m}, g_0)$ and that $\lambda_i(\varphi_i^\rho(m)) \to \lambda_i(\varphi_i(m))$ for $\rho \to 0$. But this is immediate from the definition of g_ρ:

The point is that for $\rho \to 0$ g_ρ becomes arbitrarily near to g_0 in C^0-topology (and equal to g_0 on any compact subset) which implies the convergence of eigenvalues from their Rayleigh's characterization and the fact that the $H^{1,2}$-norms for g_ρ are (uniformly) equivalent to those for g_0. Now since $\varphi_i^\rho(m)$ are eigenfunctions with L^2-norm $= 1$ and of bounded eigenvalue their $H^{1,2}$-norms for g_ρ and hence for g_0 are also bounded. That is we may assume $\varphi_i^\rho(m)$ converges weakly in $H^{1,2}$ (and (hence) strongly in L^2) on (F_{r_m,R_m}, g_0) for $\rho \to 0$. But the uniform bound of the L^2-norm of $\varphi_i^\rho(m)$ and convergence of their eigenvalues implies (via elliptic theory) the compact $C^{k,\alpha}$-convergence (of subsequences) to an eigenfunction $\varphi_i(m)$ for the Neumann problem (The latter boundary condition is seen from induction in i and the Rayleigh characterization). □

Now we are ready to give the proof of (3.2):

Choose sequences R_m, $r_m \to 0$ with $1/R_m^2 |ln r_m| \to +\infty$, but $r_m^2/R_m^2 \to 0$. From (3.3) (i) and (ii) we find that these conditions imply (for $m \to \infty$):

$$\# S(R_m)/|ln r_m| \to +\infty \quad \text{and} \quad \text{area} \bigcup_{p \in S(R_m)} B_{r_m}(p) \to 0.$$

Hence (4.1) gives $\lambda_1^D(F_{r_m,R_m}) \to +\infty$, while the Neumann eigenfunctions on F_{r_m,R_m} converge (after choosing subsequences) to the corresponding eigenfunctions on F_0, in the sense specified in (4.2) and (4.3).

Combining these results we can analyze the spectrum of $F_0^+ \#_{mult} F_0^-$ for such a sequence r_m, R_m as chosen in the beginning: Take M such that $\lambda_1^D(F_{r_m,R_m}) > \Gamma$ for $m > M$, then each eigenfunction φ on $F_0^+ \#_{mult} F_0^-$ with eigenvalue $< \Gamma$ has to be symmetric (i.e. $\varphi = \varphi_s$) as $\varphi_a|_{F_0^+}$ is an eigenfunction for the Dirichlet problem (and (therefore) has to be zero).

But $\varphi_s|_{F_0^+}$ is eigenfunction for the Neumann problem, hence we can apply our preliminary observations on F_0^+ and conclude convergence of eigenvalues $< \Gamma$ on the closed manifold $F_0^+ \#_{mult} F_0^-$ to those of F_0. Thus using (4.2) and the criterion of §2 we obviously get that for m large enough: $F\colon B \to \mathcal{M}(F_0^+ \#_{mult} F_0^-)$ with $F(p) = f(p)_\rho$ for small $\rho > 0$ is spectrally near to $f(p)$.

Finally, as area $\bigcup_{p \in S(R_m)} B_{r_m}(p) \to 0$ we see area $F_0^+ \#_{mult} F_0^- \to 2\alpha$. □

5. Large volumes

The result for surfaces above suffices to derive the following unrestricted higher dimensional result. As already mentioned in §2 this is based on a "volume effect". More precisely, one uses the simple observation (already exploited in [(3)] that the Dirichlet integral is multiplied by λ^{n-2} when the metric is scaled by some constant $\lambda > 0$.

PROPOSITION 5.1. *Every manifold M of dimension ≥ 3 admits a stable family of metrics $f\colon B \to \mathcal{M}(M)$ with $\lambda_k(g) = \lambda_k$, $k \leq n_0$ and* $\mathrm{Vol}(M, f(p)) = V, p \in B$.

Proof. The idea is to embed the surface F obtained above in M and choose a metric "concentrated" on a tube near F. To become more precise, note first that the first Neumann eigenvalue $\mu_1(r)$ of $B_r(0) \subset \mathbb{R}^{n-2}$, $n \geq 3$ becomes arbitrarily large for sufficiently small $r > 0$. Take $r > 0$ such that $\mu_1(\rho) > \Gamma + 1$ for each $\rho \in]0, r[$ (recall $\Gamma > \lambda_{n_0}$).

Then we can conclude that the first n_0 Neumann eigenfunctions $\bar{\psi}_k$ on $F \times B_\rho(0)$ (F as in (3.1)) are of the form $\bar{\psi}_k(a, b) = \psi_k(a)$, $(a, b) \in F \times B_\rho(0)$ (where ψ_k denotes the kth eigenfunction on F). Thus take a stable family of metrics $f(p)$, $p \in B$ on a surface F with area $(F, f(p)) > 2V/\mathrm{Vol}(B_r(0))$ and with $\lambda_k(f(0)) = \lambda_k$, $k \leq n_0$, then there is a $\rho \in]0, r[$ with $\mathrm{Vol}(F \times B_\rho(0)) = V$, $\rho = \rho(p)$ depending continuously on $p \in B$.

Now recall that $F \times B_r(0)$ admits an embedding i into a ball $B \subset \mathbb{R}^n$ as (trivial) neighborhood of $F \subset \mathbb{R}^3 \subset \mathbb{R}^n$. Considering B as a ball in M we can define metrics $g(f(p), \varepsilon, \delta)$ on M using a fixed base metric g_0 on M, $\delta, \varepsilon > 0$ and $h_\varepsilon \in C^\infty(M, [0, 1])$ with $h_\varepsilon \equiv 1$ on $i(F \times B_\rho(0))$, $h_\varepsilon \equiv 0$ on $M \backslash \varepsilon$-neighborhood of $i(F \times B_\rho(0))$:

$$g(f(p), \varepsilon, \delta) := h_\varepsilon(i_*(f(p) + g_{Eucl.})) + (1 - h_\varepsilon) \cdot \delta \cdot g_0.$$

Now we can use a dimension argument (as described formally in [C3]) to analyze the spectrum of this metric for $\delta \ll 1$. The eigenfunctions of the Laplacian on M are the first n orthonormal (relative) minimizers of the Dirichlet integral (cf. §2). Thus for small $\delta > 0$ and suitably sharp h_ε (i.e. $\varepsilon \ll 1$), we observe that the eigenfunctions on M are L^2-near to corresponding minimizers on $F \times B_\rho(0)$. But these are just the Neumann eigenfunctions. Also, for $\delta \to 0$ the eigenvalue converge to the Neumann eigenvalues. The reason is that the Dirichlet integral restricted to the complement of $i(F \times B_\rho(0))$ decreases of order δ^{n-2}, as the volume element is proportional to δ^n and the gradient norm increases with δ^{-1}.

That is $g(f(p), \varepsilon, \delta)$ becomes spectrally near to $f(p)$. Moreover for $\varepsilon, \delta \to 0$, $\mathrm{Vol}(M, g(f(p), \varepsilon, \delta)) \to V$ uniformly in $p \in B$.

Thus we can consider (instead of $g(f(p), \varepsilon, \delta)$):

$$f(p)_V := \left(\frac{V}{\mathrm{Vol}(M, g(f(p), \varepsilon, \delta))}\right)^{2/n} \cdot g(f(p), \varepsilon, \delta)$$

and find that they still induce a spectral map Φ uniformly approximating $\Phi(f)$. Thus we conclude, $f(p)_V$ is a stable family of metrics on M with $\lambda_k(f(p_0)_V) = \lambda_k$, $k \leq n_0$, for some $p_0 \in \mathrm{int}\, B$ and $\mathrm{Vol}(M, f(p)_V) = V$. □

6. Attaching curvature

As already mentioned our method might be understood as stepwise attaching geometry to some stable "base metric". While §3 and 5 were devoted to volumes, we will now introduce (additionally) curvature properties. In some sense the procedures are analogous: that is the metric is "changed" substantially only on balls of precisely the same type of Besicovitch coverings.

Thus let $B_R(x)$, $x \in S(R)$ be a collection of balls on (M, g) as in §3, and $f: B \to \mathcal{M}(M)$ be a stable family around $g_0 = f(0)$ with $\mathrm{Vol}(M, f(p)) \equiv V$ and $\lambda_k(g_0) = \lambda_k$, $k \leq n_0$. Then we define a family $G: B \to \mathcal{M}(M)$ as follows:

$$G(p) := h \cdot r^2 \cdot (\exp_x)_*(H_{x,r})^*(g^x_{model}) + (1 - h) \cdot f(p) \qquad \text{on } B_{2r}(x),$$

and

$$G(p) \equiv f(p) \qquad \text{on } M \Big\backslash \bigcup_{x \in S(R)} B_{2r}(x)$$

where $h \in C^\infty(B_{2r}(x), [0, 1])$ with $h(y) = \bar{h}(\mathrm{dist}(x, y))$ for some $\bar{h} = \bar{h}_r \in C^\infty(\mathbb{R}, [0, 1])$ with $\bar{h} = 1$ on $\mathbb{R}^{\leq r}$, $\bar{h} \equiv 0$ on $\mathbb{R}^{\geq 2r}$, and $H_{x,r}: T_xM \to \mathbb{R}^n$ a linear map with $\|H_{x,r}(v)\| = 1/r \|v\|$. Finally g^x_{model} denotes one of the "model metrics" g, g_t (cf. (6.1) below) or just $g_{Eucl.}$ on $\mathbb{R}^n$, $n \geq 3$. Their choice depends on $x \in S(R)$.

Thus the geometric properties are introduced only on the small subset formed by these balls. The point is that these changes, while substantially as far as the involved geometric quantities are concerned, are too well-balanced (and small) to be realized by the lower eigenvalues.

LEMMA 6.1. *There is a metric g resp. a continuous family g_t, $t \in [-1, 1]$ on $\mathbb{R}^n$, $n \geq 3$ with $g \equiv g_t \equiv g_{Eucl.}$ on $\mathbb{R}^n \backslash B_1(0)$ and:*

(i) $\mathrm{Ric}(g) < 0$ *on* $B_1(0)$

(ii) $\displaystyle\int_{B_1(0)} \mathrm{Scal}(g_t)\, dV_{g_t} = \varphi(t)$, *for some* $\varphi \in C^\infty([-1, 1], \mathbb{R})$
with $\varphi(-1) < -1$, $\varphi(1) > 1$
$(-1)^k \cdot a_k(B_1(0), g_t) > 0$, $k \leq K$, *for any given* K.

Proof. (i) is obtained from [L]. to get (ii) we start with some metric G on S^n with (S^n, G) is isometric to a ball $\subset \mathbb{R}^n$ on $B \subset S^n$ and $\mathrm{Ric}(G) > 0$ on $S^n \backslash B$.

Now we cut-off a smaller ball $B' \subset B$ from S^n and the ball $B_{1/2}(0)$ from $\mathbb{R}^n$. It is easy to define metrics near the boundary allowing to glue these two parts getting a smooth metric $\bar{G}$ on $\mathbb{R}^n$ (with $\bar{G} = g_{Eucl.}$ outside $B_1(0)$) and with pointwise semidefinite Ricci tensor. Furthermore a global scaling of S^n (noting $\lambda^{2-n} \cdot \int_M \mathrm{Scal}(g)\, dV_g = \int_M \mathrm{Scal}(\lambda^2 \cdot g)\, dV_{\lambda^2 \cdot g}$) allows to get $\bar{G}$ with:

$$\int_{B_1(0)} \mathrm{Scal}(\bar{G})\, dV_{\bar{G}} = -\frac{1}{2} \cdot \int_{B_1(0)} \mathrm{Scal}(g)\, dV_g = S > 1$$

(where g is from (i) and which can also be chosen to fulfill the latter condition.)

Now still using g of part (i) we define metrics g_λ, $\lambda \in]0,1]$, by

$g_\lambda = \bar{G}$ on $B_1(0)$, $g_\lambda = \lambda^2 \cdot f_\lambda^*(g)$ on $B_1(-3,0,0)$ and $g \equiv g_{Eucl.}$ otherwise

(where $f: \mathbb{R}^n \to \mathbb{R}^n$ denotes the map $x \to x/\lambda + (3,0,0)$).

Thus $\int_{B_1(0)} \mathrm{Scal}(g_\lambda)\, dV_{g_\lambda} = S - \lambda^{2-n} \cdot 2 \cdot S$ i.e. < -1 for $\lambda = 1$ and > 1 for small $\lambda > 0$. Moreover we find for a_2 (acc. [BGM]):

$$a_2(B_1(0), g_\lambda) \equiv \int_{B_1(0)} 5 \cdot |\mathrm{Scal}(g_\lambda)|^2 - 2 \cdot |\mathrm{Ric}(g_\lambda)|^2 + 2 \cdot |\mathrm{Riem}(g_\lambda)|^2 \, dV_{g_\lambda}$$

$$\geq \int_{B_\rho(0)} 5 \left|\sum_i^n \mathrm{Ric}(g_\lambda)(e_i, e_i)\right|^2 - 2 \sum_i^n |\mathrm{Ric}((g_\lambda)(e_i, e_i)|^2 \, dV_{g\lambda}$$

$$\overset{(*)}{\geq} \int_{B_1(0)} 3 \sum_i^n |\mathrm{Ric}(g_\lambda)(e_i, e_i)|^2 \, dV_{g_\lambda} > 0,$$

where $e_1, \ldots, e_n$ denotes an orthonormal (not necessarily continuous) frame of TM. Note that inequality $(*)$ is owing to the fact that the Ricci tensor is semidefinite in each $x \in M$.

Now we turn to a_k for $k \geq 3$, these are quite complicated and involve derivatives of curvatures and they are still far from being understood. But there is some useful partial information: The leading terms (i.e. terms involving highest order derivatives) are known see Avramidi [A] or Gilkey [G]: More precisely

$$a_k(M, g) = (-1)^k \int_M c_1(n) \cdot |\nabla^{k-2} \mathrm{Scal}(g)|^2 + c_2(n) \cdot |\nabla^{k-2} \mathrm{Ric}(g)|^2 + P(g)\, dV_g$$

where $c_1(n)$, $c_2(n) > 0$ are universal constants, while $P(g)$ is a polynomial in terms of the curvature and its derivatives up to order $k-3$. Thus in the case $(\mathbb{R}^n, g_{Eucl.})$ we just make a slight conformal change (letting volume fixed) on $B_1(0)$:

$$\mathrm{Scal}(e^{2f} \cdot g) = e^{-2f} \cdot (2(n-1) \cdot \Delta f - (n-2)(n-1)|\nabla f|^2).$$

Now a brief look at the expression for a_k convinces us that some f with $\|f\|_{C^{k-1}} \ll 1$, but with suitably bumpy k-th derivatives, can make $\int_M |\nabla^{k-2} \mathrm{Scal}(e^{2f} \cdot g)|\, dV_{e^{2f} \cdot g}$ arbitrarily large, without essential changes neither for P nor for a_l, $l < k$. (Namely, use a radially symmetric f, that is $f(x) = F(\|x\|)$ for some $F \in C^\infty(\mathbb{R}, \mathbb{R})$ and notice $|\nabla^{k-2} \Delta f| = c_n \cdot |F^{(k)}| +$ lower orders of F.)

Therefore, in our case, $(-1)^k \cdot a_k(B_1(0), g_\lambda)$ can be assumed to be >0 for finitely many k, using the same "adding" argument as in the construction of g_λ from $\bar{G}$ above.

Finally we reparametrisize g_λ onto $[-1, 1]$ getting the claim □

LEMMA 6.2. *For large R/r the family $G(p)$, $p \in B$ becomes spectrally near to $f(p)$ independent of the choice of the model metric for $x \in S(R)$.*

Proof. First of all we notice that $G(p)$ and $f(p)$ are uniformly equivalent metrics (in $p \in B$ as well as in R/r and also in the choice of a fixed collection of model metrics as these form a compact family (and scalings do not change the equivalence constants)):

$$a^2 \cdot f(p)(v, v) \leq G(p)(v, v) \leq b^2 \cdot f(p)(v, v), \qquad b > a > 0.$$

This implies considering Rayleigh's quotients:

$$\frac{a^n}{b^{n+2}} \lambda_k(f(p)) \leq \lambda_k(G(p))) \leq \frac{b^n}{a^{n+2}} \lambda_k(f(p)), \qquad b > a > 0.$$

In particular; if we take sequences r_m, $R_m \to 0$, $R_m/r_m \to +\infty$ then we can assume that orthonormal sets $\varphi_i^p(m)$ of eigenfunctions for the first n_0 eigenvalues $\lambda_i^p(m)$ on $G(p)(= G(p)(m))$ converge weakly in $H^{1,2}(f(p))$ to some ψ_i^p in $H^{1,2}(f(p))$ and strongly in $L^2(f(p))$, uniformly in $p \in B$.

We are left (from the criterion for spectrally near maps in §2) to check (that we can assume) $\psi_i^p \equiv \varphi_i^p$ and $\lim \lambda_i^p(m) = \lambda_i^p$.

This is done by induction (in "i") we give the proof for $i = 1$ ($i \geq 2$ uses the orthogonality to previous eigenfunctions): We start with: $\limsup \lambda_1^p(m) \leq \lambda_1^p$;

$$\|\varphi_1^p\|_{H^{1,2}(M,G(p))} \leq \|\varphi_1^p\|_{H^{1,2}(M \setminus \cup_{x \in S} B_{2r}(x), f(p))} + b^{2n}$$
$$\cdot \operatorname{Vol}\left(\bigcup_{x \in S} B_r(x), f(p)\right) \cdot \|\varphi_1^p\|_{C^1(M, f(p))}$$

and $\|\varphi_1^p\|_{L^2(G(p))} \to 1$, $\int_M \varphi_1^p \, dV_{G(p)} \to 0$, for $m \to +\infty$

Thus for each $\varepsilon > 0$ we have for large m:

$$\lambda_1^p(m) - \varepsilon \leq \int_M \|\nabla \varphi_1^p\|_{G(p)}^2 \, dV_{G(p)} \bigg/ \int_M |\varphi_1^p|^2 \, dV_{G(p)} \leq \lambda_1^p + \varepsilon.$$

Next the L^2-convergence of $\varphi_1^p(m) \to \psi_1^p$ implies $\|\psi_1^p\|_{L^2(f(p))} = 1$ and $\int_M \psi_1^p = 0$, hence $\int_M \|\nabla \psi_1^p\|_{f(p)}^2 \, dV_{f(p)} \geq \lambda_1^p$.

Therefore $\lambda_1^p + 1 \leq \int_M \|\nabla \psi_1^p\|_{f(p)}^2 + |\psi_1^p|^2 \, dV_{f(p)} \leq \liminf \lambda_1^p(m) + 1 (\leq \lambda_1^p + 1)$.

Hence we can assume: $\lambda_1^p = \lim \lambda_1^p(m)$, $\varphi_1^p \equiv \psi_1^p$. □

Now we are ready to prove Theorem 1:

PROPOSITION 6.3. *For each* $V \in]0, +\infty[$, $S \in]-\infty, +\infty[$ *there is a sequence* g_m *of metrics on* M *with*:

(i) $\lambda_k(g_m) = \lambda_k$, $k \leq m$

(ii) $\operatorname{Vol}(M, g_m) = V$, $\int_M \operatorname{Scal}(g_m) \, dV_{g_m} = S$

(iii) $a_{2k}(M, g_m) \to +\infty$, $a_{2k+1}(M, g_m) \to -\infty$, $k \geq 1$

Proof. From (5.1) there is a sequence G_m of stable metrics $\lambda_k(G_m) = \lambda_k$, $k \leq m$ and $\mathrm{Vol}(G_m) = V$.

The next step is to deform G_m to get additionally the correct integral scalar curvature. Thus let (for fixed n) $f\colon B \to \mathcal{M}(M)$ be a family with $f(0) = G_m$. Then $G(p)$, $p \in B$ is a spectrally near family (acc. (6.2)) for large R/r. To ensure both the spectral approximation as well as the curvature properties we have to choose r and R fulfilling

$$c_1 \cdot r^{1/2} \geq \mathrm{Vol}\left(\bigcup_{x \in S(R)} B_r(x) \right) \geq c_2 \cdot r^{3/2} \tag{$*$}$$

(for some constants c_1, $c_2 > 0$ independent of r and R and volume measured with respect to $f(p)$).

Now we can consider $G(p)$ with model metric of (ii): $\int_{B_1(0)} \mathrm{Scal}(g_\lambda)\, dV_{g_\lambda} < -1$ "inserted" and find for $R \to 0$: $\int_M \mathrm{Scal}(G(p)\, dV_{G(p)} \to -\infty$. (Of course, the analogue holds for positive integral scalar curvature > 1). This is because the metric does not change on $M \backslash \bigcup_{x \in S(R)} B_{2r}(x)$ and the metric $G(p)$ on $B_{2r}(x)$, scaled by $1/r^2$, becomes arbitrarily C^∞-near to $(B_2(0), g_{model})$. Therefore (using ($*$)) we see that (for constants $c, \bar{c} > 0$ with $c(g_0, r) \xrightarrow[r \to 0]{} 1$):

$$\int_M \mathrm{Scal}(G(p))\, dV_{G(p)} = S_0 + \int_{\cup_{x \in S(R)} B_{2r}(x)} \mathrm{Scal}(G(p))\, dV_{G(p)}$$

$$= S_0 + c(g_0, r) \cdot \# S(R) \cdot r^{n-2} \cdot \int_{\mathbb{R}^n} \mathrm{Scal}(g_{model})\, dV_{g_{model}}$$

$$< S_0 - \bar{c}(g_0) \cdot r^{(3/2)-n} \cdot r^{n-2} \xrightarrow[r \to 0]{} -\infty$$

(In the same way:

$$(-1)^k \cdot a_k(M, G(p)) > \mathrm{const.} \cdot r^{-((4k-1)/2)} \xrightarrow[r \to 0]{} +\infty).$$

Thus we insert just as many times model (ii) in $B_r(x)$, $x \in S(R)$ ($g_{Eucl.}$ elsewhere) as necessary (but at least one) to ensure $\int_M \mathrm{Scal}(G(p))\, dV_{G(p)} < S$.

We observe that $\# \{B_r(p)$ filled with this model (ii)$\} / \# S(R) \xrightarrow[R \to 0]{} 0$. Therefore we now can substitute for the "Euclidean balls $B_r(p)$" models g_λ with $\int_{B_1(0)} \mathrm{Scal}(g_\lambda)\, dV_{g_\lambda} > 1$ to increase the integral scalar curvature again, such that $\int_M \mathrm{Scal}(G(p))\, dV_{G(p)} > S$. For $R/r \to \infty$ we have $\mathrm{Vol}(G(p)) \to V$, thus we can renormalize the volume as in (5.1) to get $\mathrm{Vol}(G(p)) \equiv V$ and using (6.1) (ii) we can carry this out such that $\int_M \mathrm{Scal}(G(p))\, dV_{G(p)} = S$ for each $p \in B$. This (final) $G(p)$ is spectrally near to $f(p)$, fulfills $\mathrm{Vol}(G(p)) \equiv V$, $\int_M \mathrm{Scal}(G(p))\, dV_{G(p)} \equiv S$ and $a_{2k}(M; G(p)) \xrightarrow[r \to 0]{} +\infty$, $a_{2k+1}(M, G(p)) \xrightarrow[r \to 0]{} -\infty$.

This implies our claim from (2.1). □

REMARK 6.4. As already mentioned in the introduction we could also prescribe higher coefficients a_k, $k \geq 2$. The idea is to use this covering argument combined with the fact $a_k(U, \lambda^2 \cdot g) = \lambda^{n-2k} \cdot a_k(U, g)$. However, the possible values for $a_k(M, g)$ can no more chosen arbitrarily. There are necessary relations for (and between) these coefficients forced from dimension and topology. For instance, in dimension 4, one has $a_2(M, g) \geq \chi(M)$ (cf. [BGM] for this and other results in this direction). This is reflected in the possible values of $a_k(B_1(0), g)$ relative to each other. □

Finally we are going to prove Theorem 2. It is notable that we will have to use the Besicovitch coverings not just to get the spectral convergence but this time this same covering is used substantially to ensure the curvature condition. (Note that in the proof of Theorem 1 the curvature construction was just made to fit into the streamline prescribed from the spectral problem, while it is clear that those curvature conditions could be obtained in other ways.)

PROPOSITION 6.5. *For each* $V \in]0, +\infty[$ *there is a sequence* g_m *of metrics on* M *with*:

(i) $\lambda_k(g_m) = \lambda_k$, $k \leq m$
(ii) $\mathrm{Vol}(M, g_m) \equiv V$
(iii) $\mathrm{Ric}(g_m) < -m^2$

Proof. Here we use $G(p)$ with model (i) inserted and we can assumed $G(p)$ is spectrally near $f(p)$ with $\mathrm{Vol}(M, G(p)) \to V$, for $R/r \to +\infty$. It is result of careful constructions (and calculations) done in [L1] and [L2] that for any arbitrarily large R/r there is a conformal change $e^{2f(p)} \cdot G(p)$ such that $\mathrm{Ric}(e^{2f(p)}G(p)) < -m^2$ on M and $|e^{2f(p)} - 1| < \varepsilon_R$ with $\varepsilon_R \to 0$, for $R \to 0$.

This is enough to conclude (6.4): Rayleigh quotient characterizations of eigenvalues imply $\lambda_k(g_n) \to \lambda_k(g)$ for C^0-converging metrics $g_m \to g$ (cf. the proof of (6.1)). In the present case this also implies the L^2-convergence (of subsequences) of eigenfunctions to those of g_0. Thus we easily get (from §2) $e^{2f(p)} \cdot G(p)$ is also spectrally near to $f(p)$ and as the volume also converges to V, we finally set $\bar{G}(p) := (V/\mathrm{Vol}(e^{2f(p)} \cdot G(p)))^{2/n} \cdot e^{2f(p)} \cdot G(p)$ getting again a family with $\mathrm{Ric} < -m^2$ spectrally near to $f(p)$ but with $\mathrm{Vol}(\bar{G}(p)) = V$. (2.1) implies the claim. □

REFERENCES

[A] Avramidi, I. G.: Theoret. Math. Fiz (1989) *291* and Yad. Fiz. *49* (1989) 1185.

[B] Berard, P.: Variétés riemanniennes isospectrales non isométriques, Sem. Bourbaki, Astérisque 177/178, (1989), 127–154.

[BGM] BERGER, M., GAUDUCHON, P., MAZET, E.: *Le spectre d'une variété riemannienne*, Springer, LNM *194*, (1971).

[Ch] CHAVEL, I.: *Eigenvalues in Riemannian Geometry*, Academic Press, 1984.

[Co] COLBOIS, B.: *Petites valeurs propres du Laplacien sur une surface de Riemann compacte et graphes*, C.R. Acad. Sci. Paris, *301* (1985), 927–930.

[CD] COLBOIS, B. and DODZIUK, J.: *Riemannian metrics with large* λ_1, Proc. AMS, to appear.

[C1] COLIN DE VERDIÈRE, Y.: *Construction de Laplaciens dont une partie finie du Spectre est donnée*, Ann. Ec. Norm. Sup., *20* (1987), 599–615.

[C2] COLIN DE VERDIÈRE, Y.: *Sur une hypothèse de transversalité d'Arnold*, Comment. Math. Helv., 63 (1988), 184–193.

[C3] COLIN DE VERDIÈRE, Y.: *Sur la multiplicité de la première valeur propre non nulle du laplacien*, Comment. Math. Helv., *61* (1986), 254–270.

[D] DODZIUK, J.: *Nonexistence of universal upper bounds for the first eigenvalue of the Laplace-Beltrami Operator*, Contemp. Math., *173* (1994), 109–117.

[G] GILKEY, P. B.: *Leading terms in asymptotics of the heat equation*, Contemporary Math. *73*, AMS (1988), 79–85.

[LY] LI, P. and YAU, S. T.: *Estimates of eigenvalues of a compact Riemannian manifold*, Proc. Symp. Pure Math., *36* (1980), 205–239.

[L1] LOHKAMP, J.: *Metrics of negative Ricci curvature*, Ann. Math., *140* (1994), 655–683.

[L2] LOHKAMP, J.: *Curvature h-principles*, Ann. Math., *142* (1995), 457–498.

[RT] RAUCH, J. and TAYLOR, M.: *Potential and scattering theory on widely perturbed domains*, J. Funct. Anal. *18* (1975), 25–59.

[X] XU, Y.: *Diverging Eigenvalues and Collapsing Riemannian Metrics*, Princeton, 1992, Preprint.

Institut des Hautes Etudes Scientifiques
35 route de Chartres
91440 Bures-sur-Yvette
France

Received, September 4, 1994; December 15, 1995