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Trapping quasiminimizing submanifolds in spaces of negative curvature

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Abstract. Let M be a Hadamard manifold with all sectional curvatures bounded above by some negative constant. A well-known lemma essentially due to M. Morse states that every quasigeodesic segment in M lies within an a priori bounded distance from the geodesic arc connecting its endpoints. In this paper we establish an analogue of this fact for quasiminimizing surfaces in all dimensions and codimensions; the only additional requirement is that the sectional curvatures of M be bounded from below as well. We apply this estimate to obtain new solutions to the asymptotic Plateau problem in various settings.

0. Introduction

A rectifiable curve σ from an interval $[u, v]$ into some metric space is called a Q-quasigeodesic for some constant $Q \geq 1$ if the length of each subsegment $\sigma \mid [u', v']$ is less than or equal to Q times the distance between its endpoints $\sigma(u')$ and $\sigma(v')$. This concept has proved to be very useful in the theory of negatively curved spaces during the last decades. One of its most important features is the following well-known lemma which is essentially due to M. Morse [Ms]:

LEMMA 0.1. *Let (M, g_0) be a simply connected, complete riemannian manifold with sectional curvature $K \leq -1$. Then for every $Q \geq 1$ there exists a constant $d_0(Q)$ such that the image of every Q-quasigeodesic $\sigma: [u, v] \to M$ is contained in a d_0-neighborhood of the geodesic arc from $\sigma(u)$ to $\sigma(v)$.*

For instance, given a riemannian metric g on M which is Lipschitz equivalent to g_0, i.e. which satisfies $\alpha^2 g_0 \leq g \leq \beta^2 g_0$ for constants $0 < \alpha \leq \beta$, every minimizing geodesic in (M, g) is (β/α)-quasigeodesic with respect to g_0. In this situation, 0.1 is the a priori estimate needed to prove the existence of a complete minimizing geodesic in (M, g) asymptotic to a given pair of distinct ideal boundary points of M, cf. [Ms], [Bu], [Kl1]. A similar argument is also used in the proof of the Mostow

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rigidity theorem, see [Mw, Sect. 13] or [G-P] for a survey. Furthermore, other interesting applications of (a variant of) the lemma have been found by Gromov [Gr1] in his theory of hyperbolic groups. A recent account of this, as well as a simple proof of the lemma (in a slightly different form), is given in [Ep].

In this paper we establish an analogue of 0.1 for quasiminimizing surfaces with arbitrary dimension and codimension in (M, g_0); the only additional restriction is that we also need a lower bound for the sectional curvatures of (M, g_0). It turned out that a well-adapted notion of surface in this context is that of a rectifiable m-current modulo two $S \in \mathscr{R}_m^2 M$; these are the compact, nonoriented generalized m-dimensional "submanifolds with boundary" studied in geometric measure theory. Quasiminimality is defined as for curves: every compact 'piece' T of S has mass (i.e. area) less than or equal to Q times the mass of every $X \in \mathscr{R}_m^2 M$ with boundary $\partial X = \partial T$, cf. 1.1. We prove the following, cf. 3.3.

THEOREM 0.2. *Let (M, g_0) be a simply connected, complete riemannian n-manifold with sectional curvature $-a^2 \leq K \leq -1$, $1 \leq a < \infty$. Then for all $Q \geq 1$ and $1 \leq m < n$ there exists a constant $d_0(Q, a, m, n)$ such that every Q-quasiminimizing surface $S \in \mathscr{R}_m^2 M$ is contained in a d_0-neighborhood of the convex hull of its boundary ∂S.*

A similar result was first obtained for homotopically quasiminimizing disks in hyperbolic 3-space H^3 and then for quasiminimizing hypersurfaces in H^n by means of a symmetrization method yielding in addition the least possible value for d_0, cf. [La1] and [La2]. Likewise we will also derive an optimal version of 0.1, cf. 3.2. The proof of 0.2 is still inspired by this construction, but is less explicit and uses as a new ingredient an isoperimetric inequality due to F. Morgan [Mo1]. In case $m = n - 1$ the proof is easier and works as well for the coefficient group $\mathbf{Z}$ instead of $\mathbf{Z}_2$.

We then apply 0.2 to solve the asymptotic Plateau problem in M with respect to a riemannian metric g which is Lipschitz equivalent to some metric g_0 satisfying the assumptions of 0.2. In particular, we show that for $2 \leq m < n$, every compact $(m-1)$-dimensional topological submanifold L of the ideal sphere M_∞ of (M, g_0) is the boundary at infinity of a complete g-area minimizing "submanifold" $S \in \mathscr{R}_{m,\mathrm{loc}}^2 M$, cf. 4.3. According to the regularity results from geometric measure theory, S is a C^∞ submanifold of M up to a closed set of Hausdorff dimension at most $m - 2$. Actually this existence result is obtained as a consequence of the more general statement 4.2 which, however, is rather technical. Moreover, we prove the existence of a complete, g-area minimizing hypersurface $S \in \mathscr{R}_{n-1,\mathrm{loc}} M$ asymptotic to a given set L satisfying the following simple topological condition: $L = \mathrm{bd}\, A$ for some subset A of M_∞ with $\mathrm{cl}(\mathrm{int} A) = A$, where bd, cl and int denote boundary, closure and interior, respectively, relative to M_∞, cf. 4.4. In this case, the obtained hypersurface S is a C^∞ submanifold of M up to a singular set of Hausdorff dimension not exceeding $n-8$. Finally, using an argument due to M. Anderson [An2], we show the existence of complete, g-area minimizing hypersurfaces in M invariant under the action of a group Γ of isometries of (M, g), cf. 4.5.

The first general existence results for m-dimensional ($m \geq 2$), complete minimizing surfaces with prescribed boundary data at infinity are due to Anderson [An1], [An2] who studied the case of hyperbolic n-space H^n. Various extensions of Anderson's results were then obtained by Gromov in [Gr2], where he also posed the problem of 'trapping' quasiminimizing submanifolds in manifolds of negative curvature, cf. Sect. 1.3.C. The existence of complete minimizing hypersurfaces in manifolds which are Lipschitz equivalent to H^n was proved in [La2].

The paper is organized as follows. In Sect. 1 we fix the notation and review some definitions and basic results from the theory of flat chains modulo two in M. In Sect. 2 we discuss the isoperimetric inequalities and density bounds which will be needed in Sect. 3 to establish 3.3. The third section also contains the optimal version of 0.1. Finally, the existence results described above are presented in Sect. 4.

1. Flat chains modulo two

Throughout this section, (M, g) denotes a complete riemannian C^∞ manifold diffeomorphic to $\mathbf{R}^n$, $n \geq 2$. We review some definitions and results from geometric measure theory, in particular from the theory of flat chains modulo 2 in M. The general reference is [Fe1, 4.2.26].

For each integer $m \geq 0$ let

$$\mathscr{R}_m M \text{ and } \mathscr{F}_m M = \{R + \partial S \colon R \in \mathscr{R}_m M,\ S \in \mathscr{R}_{m+1} M\}$$

denote the additive abelian groups of compactly supported, m-dimensional (integer multiplicity) rectifiable currents and integral flat chains in M, respectively, where ∂ is the boundary operator for currents, cf. [Fe1, 4.1.24]. Note that every element T of $\mathscr{R}_m M$ has finite mass $\mathbf{M}(T)$. Then for each integer $\nu \geq 0$ one may define a subbadditive function $\mathscr{F}^\nu$ on $\mathscr{F}_m M$ by

$$\mathscr{F}^\nu(T) := \inf\{\mathbf{M}(R) + \mathbf{M}(S)\}, \tag{1}$$

where the infimum is taken over all $R \in \mathscr{R}_m M$, $S \in \mathscr{R}_{m+1} M$ and $Q \in \mathscr{F}_m M$ with $T = R + \partial S + \nu Q$. Two currents $T, T' \in \mathscr{F}_m M$ satisfying $\mathscr{F}^\nu(T - T') = 0$ are called *congruent modulo* ν. The resulting congruence classes are called m-dimensional *flat chains modulo* ν *in* M; the set of all classes is denoted

$$\mathscr{F}^{\nu}_{m}M.$$

Then $\mathscr{F}^{\nu}$, ∂ and the addition on $\mathscr{F}_{m}M$ induce corresponding operators, denoted by the same symbols, on $\mathscr{F}^{\nu}_{m}M$. In fact, $\mathscr{F}^{\nu}$ induces a metric on $\mathscr{F}^{\nu}_{m}M$ for each $m \geq 0$, and $\partial\colon \mathscr{F}^{\nu}_{m+1}M \to \mathscr{F}^{\nu}_{m}M$ is an $\mathscr{F}^{\nu}$ continuous homomorphism. The support spt S of a congruence class $S \in \mathscr{F}^{\nu}_{m}M$ is defined to be the intersection of the supports of all elements of S.

The subgroup

$$\mathscr{R}^{\nu}_{m}M \subset \mathscr{F}^{\nu}_{m}M$$

of m-dimensional *rectifiable currents modulo* ν *in* M consists of all congruence classes mod ν containing an element of $\mathscr{R}_{m}M$. This is the space of generalized m-dimensional surfaces in M we will mainly work with. For simplicity we will always take $\nu = 2$; in this case, the space $\mathscr{R}^{2}_{m}M$ can alternately be defined in terms of rectifiable sets, as will be described below (the preceding discussion is nevertheless necessary to define ∂ and $\mathscr{F}^{2}$). This latter viewpoint has the advantage to facilitate localization, moreover it yields intuitive definitions of mass and support. However, up to the cited regularity theorem 1.3, the results of this paper stated for $\nu = 2$ can easily be generalized to $\nu \geq 2$.

For each $m \geq 0$ let $\mathscr{H}^{m}$ denote the m-dimensional Hausdorff measure induced by the given riemannian metric g. Then we let

$$\mathscr{W}_{m,\mathrm{loc}}M$$

denote the class of all $\mathscr{H}^{m}$ measurable, countably $(\mathscr{H}^{m}, m)$ rectifiable sets $W \subset M$ with locally finite $\mathscr{H}^{m}$ measure. The second property expresses that $\mathscr{H}^{m}$ almost all of W is contained in the union of countably many images of Lipschitz maps from $\mathbf{R}^{m}$ into M. On $\mathscr{W}_{m,\mathrm{loc}}M$ we define the equivalence relation

$$W \sim W' \Leftrightarrow \mathscr{H}^{m}((W \setminus W') \cup (W' \setminus W)) = 0.$$

Now, given a congruence class $S \in \mathscr{R}^{2}_{m}M$ one may choose $T \in S \cap \mathscr{R}_{m}M$ and represent $T = (\mathscr{H}^{m} \llcorner W) \wedge \eta$, where $W \in \mathscr{W}_{m,\mathrm{loc}}M$ is relatively compact and η is an m-vectorfield with $|\eta| \in \mathbf{Z}$, cf. [Fel, 4.1.28]. Denoting by W^{*} the set of points in W where $|\eta|$ is odd, it turns out that W^{*} is uniquely determined by S up to sets of $\mathscr{H}^{m}$ measure zero, cf. [Fel, p. 430] or [Zi]. In fact, one obtains an injective map from $\mathscr{R}^2_m M$ into $\mathscr{W}_{m,\mathrm{loc}}M/\sim$, where the image consists exactly of the equivalence classes $[W]$ which can be represented by a relatively compact set. One is thus led to define the space of m-dimensional *locally rectifiable currents modulo* 2 in M simply by

$$\mathscr{R}^2_{m,\mathrm{loc}}M = \mathscr{W}_{m,\mathrm{loc}}M/\sim.$$

Mass ($\in [0, \infty]$) and support of $S = [W] \in \mathscr{R}^2_{m,\mathrm{loc}}M$ are then defined by

$$\mathbf{M}(S) = \mathscr{H}^m(W) \quad \text{and} \quad \operatorname{spt} S = \bigcap_{W' \in S} \operatorname{cl} W',$$

respectively, where cl denotes closure. Moreover for every Borel subset B of M one defines $S \llcorner B := [W \cap B] \in \mathscr{R}^2_{m,\mathrm{loc}}M$. Of course if B is relatively compact then $S \llcorner B \in \mathscr{R}^2_m M$. So far we have not defined ∂S for $S \in \mathscr{R}^2_{m,\mathrm{loc}}M \backslash \mathscr{R}^2_m M$; this would require a definition of the space $\mathscr{F}^2_{m,\mathrm{loc}}M$ which is irrelevant for this paper (see for instance [Mol, 2.1]). We merely note that

$$\operatorname{spt} \partial S = \bigcup_{U \subset\subset M} (U \cap \operatorname{spt} \partial(S \llcorner U)),$$

where $U \subset\subset M$ means that U is an open and relatively compact subset of M. Then $S \in \mathscr{R}^2_{m,\mathrm{loc}}M$ is said to be complete if $\operatorname{spt} \partial S = \varnothing$.

Next we define the notion of quasiminimality for elements of $\mathscr{R}^2_{m,\mathrm{loc}}M$.

DEFINITION 1.1 (quasiminimizing). Let $Q \geq 1$ and $A \subset M$. A current $S \in \mathscr{R}^2_{m,\mathrm{loc}}M$ is called *quasiminimizing with constant Q in A*, or simply *Q-minimizing in A*, if

$$\mathbf{M}(S \llcorner B) \leq Q\mathbf{M}(X)$$

whenever B is a Borel subset of M, $S \llcorner B$ and X are elements of $\mathscr{R}^2_m M$, $\partial X = \partial(S \llcorner B)$, and $\operatorname{spt}(S \llcorner B) \cup \operatorname{spt} X \subset A$. Then S is called *quasiminimizing in A* if S is Q-minimizing in A for some $Q \geq 1$. A 1-minimizing current is called (absolutely) *minimizing*.

The principal application of this definition is the following. Assume that there are constants $0 < \alpha \leq \beta$ and a riemannian metric g_0 on M such that on A, $\alpha^2 g_0 \leq g \leq \beta^2 g_0$. Then every $S \in \mathscr{R}^2_{m,\mathrm{loc}}M$ which is minimizing in A with respect to g is quasiminimizing with constant $Q = (\beta/\alpha)^m$ in A with respect to g_0. For more comments on quasiminimality see [La2] and the references there; the example [La2, 1.2] can readily be generalized to higher codimension.

The proof of the following general existence theorem for minimizing currents mod 2 is analogous to that of [Fe1, 5.1.6(1)], with the compactness theorem for integer multiplicity currents replaced by its counterpart mod 2, [Fe1, (4.2.17)$^{\nu}$ on p. 432]. A current $R \in \mathscr{R}^2_{m-1}M$ is called a *boundary in* $A \subset M$ or is said to *bound in* A if there exists $T \in \mathscr{R}^2_m M$ with $\operatorname{spt} T \subset A$ and $\partial T = R$. (Note that in $M \simeq \mathbf{R}^n$, R bounds if and only if either $m \geq 2$ and $\partial R = 0$ or $m = 1$ and $R = [W]$ for some finite set W of even cardinality.)

PROPOSITION 1.2 (existence). *Assume that $R \in \mathscr{R}^2_{m-1}M$ bounds in some compact Lipschitz neighborhood retract A in M. Then there exists $S \in \mathscr{R}^2_m M$ with* $\operatorname{spt} S \subset A$ *and $\partial S = R$ such that S is minimizing in A with respect to g.*

Regularity of minimizing currents mod 2 is discussed in [Fe2]. In particular, the following holds.

PROPOSITION 1.3 (regularity). *Let $S \in \mathscr{R}^2_{m,\mathrm{loc}}M$ be minimizing in M. In case $m \geq 2$ there exists a set $\Sigma \subset M$ of Hausdorff dimension at most $m-2$ such that* $\operatorname{spt} S \backslash (\operatorname{spt} \partial S \cup \Sigma)$ *is an m-dimensional C^∞ submanifold of M. For $m = 1$ the same conclusion holds with $\Sigma = \varnothing$.*

We conclude this section with a few more technical results about the 'slicing' and mapping of rectifiable currents mod 2 in M. The following lemma is a variant of the formula on the bottom of p. 429 in [Fe1]. A similar result for integer multiplicity currents is proved in [Si, 28.9].

LEMMA 1.4 (slicing). *Let $S \in \mathscr{R}^2_m M$ and $h: M \to \mathbf{R}$ a C^1 map. For $t \in \mathbf{R}$ set $H_t := \{x \in M: h(x) < t\}$ and $f(t) := \mathbf{M}(S \llcorner H_t)$. Then for almost all t with $H_t \cap$* $\operatorname{spt} \partial S = \varnothing$, *$f'(t)$ exists, $\partial(S \llcorner H_t) \in \mathscr{R}^2_{m-1}M$, and*

$$\mathbf{M}(\partial(S \llcorner H_t) \llcorner U) \leq |f'(t)| \sup_U |dh|$$

for all open subsets U of M. The same conclusion holds with $<$ in the definition of H_t replaced by $\leq$.

Given $S = [W] \in \mathscr{R}^2_m M$ and a Lipschitz map $\pi: M \to M$, the push-forward $\pi_\# S \in \mathscr{R}^2_m M$ of S via π is defined by $\pi_\# S = [W^*]$, where W^* consists of all $y \in \pi(W)$ with odd cardinality of $W \cap \pi^{-1}\{y\}$. Then clearly $\operatorname{spt}(\pi_\# S) \subset \pi(\operatorname{spt} S)$ and

$$\mathbf{M}(\pi_\# S) \leq \int_W J_m(\pi \mid W)\, d\mathscr{H}^m \leq \operatorname{Lip}(\pi)^m \mathbf{M}(S), \tag{2}$$

where $J_m(\pi \mid W)$ denotes the $\mathscr{H}^m$ approximate m-dimensional Jacobian of $\pi \mid W$ and $\mathrm{Lip}(\pi)$ is the Lipschitz constant of π, cf. [Fel, 3.2.20]. In case $\partial S \in \mathscr{R}^2_{m-1}M$, $\partial \circ \pi_\# = \pi_\# \circ \partial$. (This follows from the corresponding property of $\pi_\#: \mathscr{R}_m M \to \mathscr{R}_m M$, cf. [Fel, p. 371], since $\pi_\#$ commutes with the canonical projection $\mathscr{R}_m M \to \mathscr{R}^2_m M$, see [Mo2, Ex. 4.23].)

2. Isoperimetric inequalities and density bounds

Now let (M, g_0) denote an n-dimensional ($n \geq 2$), simply connected, complete riemannian C^∞ manifold with sectional curvature $K \leq -1$, and let dist denote the induced distance function on $M \times M$. We write $\mathrm{dist}(x, A) := \inf\{\mathrm{dist}(x, y): y \in A\}$ for $\emptyset \neq A \subset M$ and define $\mathrm{dist}(x, \emptyset) := \infty$. For $x \in M$ and $r \geq 0$ we set

$$B(x, r) := \{y \in M: \mathrm{dist}(x, y) \leq r\}, \qquad U(x, r) := \{y \in M: \mathrm{dist}(x, y) < r\}.$$

Further, for $m \geq 1$ and $r \geq 0$ we define

$$\beta_m(r) := m\alpha_m \int_0^r \sinh^{m-1} t \, dt,$$

where α_m denotes the volume of the unit ball in euclidean m-space. Thus for $m \geq 2$, $\beta_m(r)$ equals the volume of a metric ball with radius r in hyperbolic m-space H^m (with constant sectional curvature $K \equiv -1$). Note that

$$(m-1)\beta_m(r) \leq m\alpha_m \sinh^{m-1} r = \beta'_m(r). \tag{3}$$

We state a mass estimate for the geodesic cone $Y \in \mathscr{R}^2_m M$ from $x \in M$ over a current $R \in \mathscr{R}^2_{m-1}M$. Writing $R = [V]$ for some $V \subset M$ one defines $Y := (\exp_x \circ h)_\# X$, where $X \in \mathscr{R}^2_m(\mathbf{R} \times T_x M)$ is the current associated to the set $[0, 1] \times \exp_x^{-1}(V)$ and h is defined by

$$h: \mathbf{R} \times T_x M \to T_x M, \qquad h(t, v) := tv. \tag{4}$$

One has $\partial Y = R$ whenever R bounds.

LEMMA 2.1 (cone inequality). *Let $R \in \mathscr{R}^2_{m-1}M$ and* spt $R \subset B(x, r)$ *for some $x \in M$, $r > 0$ and $1 \leq m \leq n$. Then the geodesic cone $Y \in \mathscr{R}^2_m M$ from x over R satisfies*

$$\mathbf{M}(Y) \leq \frac{\beta_m(r)}{\beta'_m(r)} \mathbf{M}(R).$$

In particular, by (3), $(m-1)\mathbf{M}(Y) \leq \mathbf{M}(R)$.

In case $m > 1$ and R is a closed hypersurface the latter inequality is proved, for instance, in [B–Z, 34.2.6], and the argument can easily be adapted to deduce 2.1.

Proof. Let $\bar{V} := \exp_x^{-1}(V)$, where $[V] = R$. For $\mathscr{H}^{m-1}$ almost every $v \in \bar{V}$ the approximate tangent cone $\operatorname{Tan}^{m-1}(\bar{V}, v)$ of $\bar{V}$ at v is an $(m-1)$-dimensional linear subspace of T_xM. Then we choose an orthonormal basis $u_{1,v}, \ldots, u_{m-1,v}$ of $\operatorname{Tan}^{m-1}(\bar{V}, v)$ such that $u_{i,v} \perp v$ for $1 \le i \le m-2$. Let $u_{m-1,v}^{\perp}$ denote the component of $u_{m-1,v}$ perpendicular to v. Applying (2) with $W = [0, 1] \times \bar{V}$ and $\pi = \exp_x \circ h$, cf. (4), we obtain

$$\mathbf{M}(Y) \le \int_{\bar{V}} \int_0^1 |v| j(t, v) \, dt \, d\mathscr{H}^{m-1}(v), \tag{5}$$

where we have abbreviated

$$j(t, v) := |f_{*tv} t u_{1,v} \wedge \cdots \wedge f_{*tv} t u_{m-2,v} \wedge f_{*tv} t u_{m-1,v}^{\perp}|, \quad f := \exp_x,$$

meaning $j(t, v) = 1$ in case $m = 1$. Now comparison with hyperbolic space yields

$$j(t, v) \le \frac{\sinh^{m-1}(t|v|)}{\sinh^{m-1}|v|} j(1, v)$$

for $0 \le t \le 1$, cf. [H-K]. Inserting this into (5), using $m\alpha_m \sinh^{m-1} = \beta'_m$ and the fact that β_m / β'_m is nondecreasing, we get

$$\mathbf{M}(Y) \le \frac{\beta_m(r)}{\beta'_m(r)} \int_{\bar{V}} j(1, v) \, d\mathscr{H}^{m-1}(v).$$

Since $j(1, v) \le J_{m-1}(f \mid \bar{V})(v)$ and $\mathscr{H}^{m-1}(\bar{V}) = \mathbf{M}(R)$ the claimed estimate follows. □

As a consequence of 2.1 we obtain the following.

PROPOSITION 2.2 (monotonicity formula). *Assume that $S \in \mathscr{R}^2_{m,\mathrm{loc}} M$ is Q-minimizing in $U(x, r) \subset M \backslash \operatorname{spt} \partial S$, for some $x \in M$, $r > 0$ and $Q \ge 1$. Then the function mapping $0 < \varrho \le r$ to*

$$\frac{\mathbf{M}(S \llcorner U(x,\varrho))}{\beta_m(\varrho)^{1/Q}}$$

is nondecreasing.

An analogous result for stationary varifolds was proved in [An1, p. 481]. In particular, 2.2 shows that every complete Q-minimizing current $S \in \mathscr{R}^2_{m,\mathrm{loc}}M$ has exponential volume growth, cf. [Gr2, 1.3.C′]. The argument proving 2.2 is well-known.

Proof. For $0 < t \le r$, define $S_t := S \llcorner U(x,t)$ and $f(t) := \mathbf{M}(S_t)$. By 1.4 (slicing), for almost all t, $\partial S_t \in \mathscr{R}^2_{m-1}M$, $f'(t)$ exists, and $\mathbf{M}(\partial S_t) \le f'(t)$. Then the Q-minimality of S in $U(x,r)$ together with 2.1 yields

$$f(t) \le Q \frac{\beta_m(t)}{\beta'_m(t)} \mathbf{M}(\partial S_t)$$

for all almost all t, hence

$$\frac{f'(t)}{f(t)} \ge \frac{\beta'_m(t)}{Q\beta_m(t)}$$

for almost all t with $f(t) > 0$. Now for $0 < \varrho_1 < \varrho_2 < r$ with $f(\varrho_1) > 0$, integrating from ϱ_1 to ϱ_2 (and using [Fe1, 2.9.19]) we get the claim. □

For the remaining results of this section we assume that the sectional curvatures of M be bounded from below as well, such that we have $-a^2 \le K \le -1$ for some $1 \le a \le \infty$. This additional requirement will only be used through the fact that for $x \in M$ and $r > 0$, the restriction of the exponential map $\exp_x$ to the ball $B(0_x, r)$ (where 0_x denotes the origin of T_xM) is Lipschitz with constant

$$\lambda_{ar} := (ar)^{-1} \sinh(ar). \tag{6}$$

Moreover since $K \le 0$, $\exp_x^{-1}$ is Lipschitz with constant 1. Then 2.2 leads to the following uniform estimate.

COROLLARY 2.3 (lower density bound). *For every $\theta > 0$ there exists a constant $\varrho = \varrho(\theta, Q, a, m, n) > 0$ such that the following holds: Whenever $S \in \mathscr{R}^2_{m,\mathrm{loc}}M$ is Q-minimizing in $U(x, \varrho) \subset M \backslash \operatorname{spt} \partial S$, for some $x \in \operatorname{spt} S$ and $Q \ge 1$, then $\mathbf{M}(S \llcorner U(x, \varrho)) \geqslant \theta$.*

Proof. In view of 2.2 and the above remarks on $\exp_x$ it suffices to prove a euclidean analogue of 2.3 for θ close to 0. For rectifiable currents with integral coefficients this is done on p. 523 in [Fe1], and it is readily checked that the argument works for $\mathbf{Z}_2$ coefficients as well (replace the involved isoperimetric inequality by [Fe1, (4.2.10)$^\nu$ on p. 431]). □

The following lemma will be important in the proof of the trapping theorem 3.3, moreover it leads to an absolute upper density bound for quasiminimizing currents mod 2 as stated in 2.5 below.

LEMMA 2.4 (sublinear isoperimetric inequality). *Let* $R \in \mathscr{R}^2_{m-1}M$ *be a boundary in* $B(x, r)$ *for some* $x \in M$, $r > 0$ *and* $1 \leq m \leq n$. *Then there exists* $T \in \mathscr{R}^2_m M$ *with* spt $T \subset B(x, r)$ *and* $\partial T = R$ *such that*

$$\mathbf{M}(T) \leq c_0 r^{n\delta} \lambda^m_{ar}\, \mathbf{M}(R)^{1-\delta} \leq c_1 e^{amr} \mathbf{M}(R)^{1-\delta},$$

where $\delta := (n - m + 1)^{-1}$, λ_{ar} *is defined as in* (6), *and* $c_0 = c_0(m, n)$, $c_1 = c_1(a, m, n)$ *are constants.*

Proof. The current $\bar{R} := (\exp_x^{-1})_{\#} R \in \mathscr{R}^2_{m-1}(T_x M)$ bounds in $B(0_x, r)$, hence by 1.2 (existence) there exists $\bar{T} \in \mathscr{R}^2_m(T_x M)$ with spt $\bar{T} \subset B(0_x, r)$ and $\partial \bar{T} = \bar{R}$ such that $\bar{T}$ is minimizing in $B(0_x, r)$. In fact, using the convexity of $B(0_x, r)$ it is shown that $\bar{T}$ is minimizing in $T_x M$. Now, by an isoperimetric inequality due to F. Morgan, cf. [Mo1, 2.5], $\bar{T}$ satisfies

$$\mathbf{M}(\bar{T}) \leq c_0 r^{n\delta} \mathbf{M}(\bar{R})^{1-\delta},$$

where $\delta := (n - m + 1)^{-1}$, and c_0 depends only on the dimensions. Let $T := (\exp_x)_{\#} \bar{T} \in \mathscr{R}^2_m M$; then spt $T \subset B(x, r)$, $\partial T = R$, and the first inequality follows from (2) together with the properties of $\exp_x$ stated above. Since $r^{n\delta} \lambda^m_{ar} = a^{-m} r^{n\delta - m} \sinh^m(ar)$ and $n\delta - m \leq 0$, the second inequality holds with $c_1 := c_0 (2a)^{-m} e^{am}$. □

2.5 COROLLARY (upper density bound). *There exists a constant* $\theta = \theta(Q, a, m, n)$ *such that the following holds: Whenever* $S \in \mathscr{R}^2_{m,\mathrm{loc}} M$ *is* Q*-minimizing in* $U(x, r) \subset M \backslash \mathrm{spt}\, \partial S$, *for some* $x \in M$ *and* $Q \geq 1$, *then* $\mathbf{M}(S \llcorner U(x, r/2)) \leq \theta e^{a(n-1)r/2}$.

A corresponding result for minimizing currents in $\mathscr{R}^2_m \mathbf{R}^n$ was obtained in [Mo1, 2.7] as a consequence of the isoperimetric inequality already used in the proof of 2.4.

Proof. It suffices to prove the following assertion for some fixed $\varepsilon > 0$: There exists $\theta'(Q, a, m, n)$ such that $\mathbf{M}(S \llcorner U(x, r/2)) \leq \theta'$ whenever S satisfies the assumption of 2.5 with $r \leq \varepsilon$. For $r > \varepsilon$, $\mathbf{M}(S \llcorner U(x, r/2))$ is then bounded by θ' times the maximal number of disjoint open balls with radius $\varepsilon/6$ fitting into $U(x, r/2)$. This number is bounded above, up to constant factor, by $a(n-1)r/2$.

The collection of the balls with the same centers but radius $\varepsilon/3$ then covers $U(x, r/2)$, and the balls with radius $2\varepsilon/3$ are still disjoint from spt ∂S.

To prove the assertion, using the same notation as in the proof of 2.2, we note that the Q-minimality of S in $U(x, r)$ together with 1.4 (slicing) and 2.4 yields

$$f(t) \leq Qc_0 r^{n\delta} \lambda_{ar}^m f'(t)^{1-\delta}$$

for almost all $0 < t \leq r$. Rearranging this inequality and integrating from $t = r/2$ to r we obtain the desired result. □

3. Trapping

Let (M, g_0) be given as stated at the beginning of the preceding section. Since $K \leq -1$ the following comparison lemma holds.

LEMMA 3.1. *Let γ: $\mathbf{R} \to M$ be a unit speed geodesic. For $i = 1, 2$, let $x_i \in M$ and* $\operatorname{dist}(x_i, \gamma(\mathbf{R})) = \operatorname{dist}(x_i, \gamma(t_i)) = r_i$. *Then*

$$\sinh^2 \frac{\operatorname{dist}(x_1, x_2)}{2} \geq \cosh r_1 \cosh r_2 \sinh^2 \frac{|t_2 - t_1|}{2} + \sinh^2 \frac{|r_2 - r_1|}{2},$$

and equality holds (e.g.) if $(M, g_0) = H^n$ and x_1, x_2 lie in some closed totally geodesic halfplane bounded by $\gamma(\mathbf{R})$.

Proof. This is shown using an application of the Rauch comparison theorem, cf. [Kl2, 2.7.3], together with some hyperbolic trigonometry. □

Previous to the proof of the main result of this section, 3.3, we establish an optimal version of the classical lemma on quasigeodesics stated in the introduction. Additionally this illustrates the proof of 3.3 in the simplest case. Recall that a continuous curve σ: $[u, v] \to M$ is called *rectifiable* if it has finite length

$$L(\sigma) := \sup \sum_{j=1}^{k} \operatorname{dist}(\sigma(s_{j-1}), \sigma(s_j)), \tag{7}$$

where the supremum is taken over all positive integers k and all subdivisions $u = s_0 \leq s_1 \leq \cdots \leq s_k = v$ of $[u, v]$. Then σ is called *Q-minimizing* or a *Q-quasi-geodesic* for some $Q \geq 1$ if for every subinterval $[u', v']$ of $[u, v]$, $L(\sigma|[u', v']) \leq Q \operatorname{dist}(\sigma(u'), \sigma(v'))$. For x, $y \in M$ we denote by $\overline{xy} \subset M$ the closed geodesic arc from x to y.

THEOREM 3.2. *Let $\sigma: [0, u] \to M$ be a rectifiable curve which is Q-minimizing for some $Q > 1$. Then*

$$\operatorname{dist}(\sigma(s), \overline{\sigma(0)\sigma(u)}) \leq c_Q := \frac{\pi}{2}\sqrt{Q^2 - 1}$$

for all $s \in [0, u]$. The constant c_Q is optimal.

Proof. Choose x on σ such that $\bar{d} := \operatorname{dist}(x, \overline{\sigma(0)\sigma(u)})$ is maximal; we may assume $\bar{d} > 0$. Then let $\gamma: \mathbf{R} \to M$ be the unit speed geodesic with $\gamma(0) \in \overline{\sigma(0)\sigma(u)}$ and $\gamma(\bar{d}) = x$. For $t \in \mathbf{R}$, let E_t denote the geodesic hyperplane normal to γ at $\gamma(t)$, i.e., the image under the exponential map of the orthogonal complement of $\mathbf{R}\gamma'(t)$ in $T_{\gamma(t)}M$. Define $\tau: M \to \mathbf{R}$ by $\tau(y) := t$ for $y \in E_t$, and consider the closed upper halfspaces $H_t := \{y \in M: \tau(y) \geq t\}$ bounded by E_t, for $t \in \mathbf{R}$. Note that $\tau(\sigma(0)), \tau(\sigma(u)) \leq 0$.

Now for $t \in [0, \bar{d}]$, define $u_0(t) := \inf \sigma^{-1}(H_t)$ and $u_1(t) := \sup \sigma^{-1}(H_t)$, i.e., $[u_0(t), u_1(t)]$ is the minimal subinterval of $[0, u]$ containing $\sigma^{-1}(H_t)$. Then put $r_i(t) := \operatorname{dist}(\sigma(u_i(t)), \gamma(t))$, $i = 0, 1$. It is easily checked that both r_0 and r_1 are leftcontinuous functions, and, by the choice of x, $r_0(\bar{d}) = r_1(\bar{d}) = 0$. Moreover, the Q-minimality of σ yields

$$L(\sigma|[u_0(t), u_1(t)]) \leq Q(r_0(t) + r_1(t)) \tag{8}$$

for $t \in [0, \bar{d}]$. Our aim is to construct a curve $\bar{\sigma}$ in the hyperbolic plane H^2 satisfying a corresponding inequality. Thus let $\bar{\gamma}: \mathbf{R} \to H^2$ be any unit speed geodesic, and let v be a continuous unit normal vector field along $\bar{\gamma}$. Then for $i = 0, 1$, define $\sigma_i: [0, \bar{d}] \to H^2$ by

$$\sigma_i(t) := \exp(r_i(t)v(t)).$$

These curves are merely leftcontinuous. However, the definition of arclength given in (7) applies as well (but note that L also takes account of the 'jumps' of σ_i), and 3.1 yields

$$L(\sigma_0|[t, \bar{d}]) + L(\sigma_1|[t, \bar{d}]) \leq L(\sigma|[u_0(t), u_1(t)]) \tag{9}$$

for $t \in [0, \bar{d}]$. Now put $\bar{r}(t) := (r_0(t) + r_1(t))/2$ and define $\bar{\sigma}: [0, \bar{d}] \to H^2$ by

$$\bar{\sigma}(t) := \exp(\bar{r}(t)v(t)).$$

It follows from the convexity of the distance function on $H^2 \times H^2$ that

$$2L(\bar{\sigma}|[t, \bar{d}]) \leq L(\sigma_0|[t, \bar{d}]) + L(\sigma_1|[t, \bar{d}]).$$

Combining this inequality with (9) and (8) we get

$$L(\bar{\sigma}|[t, \bar{d}]) \leq Q\bar{r}(t) \tag{10}$$

for $t \in [0, \bar{d}]$. The remaining part of the proof is now analogous to the argument used in [La2, 2.4]. The idea is to compare $\bar{\sigma}$ with a (smooth) curve $\hat{\sigma}: [0, \hat{d}] \to H^2$ satisfying

$$L(\hat{\sigma}|[t, \hat{d}]) = Q\hat{r}(t) \tag{11}$$

for all t, where $\hat{\sigma}(t) = \exp(\hat{r}(t)v(t))$ for some nonnegative function $\hat{r}: [0, \hat{d}] \to \mathbf{R}$. The discussion preceding 2.3 in [La2] shows that such curves exist for all $0 < \hat{d} < c_Q$, with $\hat{r}(0) \to \infty$ as $\hat{d} \to c_Q$. Hence in case $\bar{d} \geq c_Q$, $\hat{d}$ can always be chosen such that $\hat{d} < \bar{d}$ and $\hat{r}(0) > \sup \bar{r}$. Setting $s := \sup\{t: \hat{r}(t) > \bar{r}(t)\}$ we obtain $\hat{r}(s) \geq \bar{r}(s)$. Moreover, it can be shown that $L(\hat{\sigma}|[s, \hat{d}]) < L(\bar{\sigma}|[s, \bar{d}])$, which then leads to a contradiction to (10) and (11).

For all possible choices of $\hat{d}$, the curve obtained by concatenating $\hat{\sigma}$ with the inverse of its reflection with respect to $\bar{\gamma}$ is Q-minimizing in H^2, cf. [La2, 2.3]. This shows that 3.2 is no longer true if c_Q is replaced by a smaller constant. □

Now we assume again that the sectional curvatures of M be pinched between two negative constants, w.l.o.g. $-a^2 \leq K \leq -1$.

THEOREM 3.3. *For all $Q \geq 1$ and $1 \leq m < n$ there exists a (computable) constant $d_0 = d_0(Q, a, m, n)$ such that the following holds: Whenever $S \in \mathscr{R}^2_m M$ is Q-minimizing in M then*

$$\operatorname{dist}(x, C) \leq d_0$$

for all $x \in$ spt S, *where C denotes the convex hull of* spt ∂S.

Proof. For $m = 1$, the result is a consequence of 3.2. Namely, it is easily shown that every Q-minimizing current $R \in \mathscr{R}^2_1 M$ with finite boundary mass can be written as a sum $R = R_1 + R_2 + \cdots + R_k$, where $k = \mathbf{M}(\partial R)/2$ and each R_i is Q-minimizing and indecomposable as defined in [Fe1, 4.2.25]. Then every R_i is the current associated to a Q-minimizing unit speed curve $\sigma_i: [0, \mathbf{M}(R_i)] \to M$. In order to apply this result consider $R := S \llcorner \{y \in M: \operatorname{dist}(y, C) > t\}$ for some appropriate t (such that $\mathbf{M}(\partial R) < \infty$).

Now let $m \geq 2$. Let $x \in$ spt S and set $d := \operatorname{dist}(x, C)$; we may assume $d > 0$. Then let $\gamma: \mathbf{R} \to M$ be the unit speed geodesic with $\gamma(0) \in dC$ and $\gamma(d) = x$, and let τ and H_t

be defined as in the proof of 3.2. Finally, define $S_t := S \llcorner H_t$ and $f(t) := \mathbf{M}(S_t)$ for $t \in \mathbf{R}$. Note that $C \cap H_t = \varnothing$ (and hence spt $\partial S \cap H_t = \varnothing$) for $t > 0$.

For $r \geq 0$ let $N_r := \{y \in M: \operatorname{dist}(y, \gamma(\mathbf{R})) \leq r\}$ denote the closed tubular r-neighborhood of γ. Then 3.1 shows that on $M \backslash N_r$, $|d\tau| \leq (\cosh r)^{-1}$. Using 1.4 (slicing) we infer that for almost all $t > 0$, $\partial S_t \in \mathscr{R}^2_{m-1} M$, $f'(t)$ exists, and $\mathbf{M}(\partial S_t) \leq |f'(t)|$. Moreover, for every choice of a function $r(t) \geq 0$ (to be explicitly determined below), $R_t := (\partial S_t) \llcorner (M \backslash N_{r(t)})$ satisfies

$$\mathbf{M}(R_t) \leq (\cosh r(t))^{-1} |f'(t)| \leq 2e^{-r(t)} |f'(t)|. \tag{12}$$

Now let P_t denote the push-forward of ∂S_t via the nearest point projection $M \to N_{r(t)}$, and let T_t be the isoperimetric spanning surface for P_t given by 2.4 (sublinear isoperimetric inequality). Since $\mathbf{M}(P_t) \leq \mathbf{M}(\partial S_t) \leq |f'(t)|$ we get

$$\mathbf{M}(T_t) \leq c_1 e^{amr(t)} |f'(t)|^{1-\delta}, \tag{13}$$

where $\delta = (n - m + 1)^{-1} < 1$. Further, let X_t and Y_t denote the geodesic cones from $\gamma(t)$ over $\partial S_t - P_t$ and R_t respectively. Then $X_t = Y_t \llcorner (M \backslash N_{r(t)})$, thus by 2.1 (cone inequality) and (12) it follows

$$\mathbf{M}(X_t) \leq \mathbf{M}(Y_t) \leq 2(m-1)^{-1} e^{-r(t)} |f'(t)|. \tag{14}$$

Since $\partial(T_t + X_t) = \partial S_t$, the Q-minimality of S in M yields $f(t) \leq Q\mathbf{M}(T_t + X_t)$. Using the subadditivity of mass together with (13) and (14) we get

$$f(t) \leq c_2 e^{amr(t)} |f'(t)|^{1-\delta} + c_3 e^{-r(t)} |f'(t)|$$

for almost all $t > 0$, where $c_2 = Qc_1$ and $c_3 = 2Q(m-1)^{-1}$. Substituting $r(t) = \lambda \log f(t)$ for some constant $\lambda > 0$ we obtain

$$1 \leq c_2 f(t)^{am\lambda - 1} |f'(t)|^{1-\delta} + c_3 f(t)^{-\lambda - 1} |f'(t)|$$

for almost all $t > 0$ with $f(t) \geq 1$. In order to make the two terms on the right-hand side comparable we choose λ so that $am\lambda - 1 = (-\lambda - 1)(1 - \delta)$, thus $\lambda = \delta/(am + 1 - \delta) > 0$ as required. Since f is nonincreasing it follows

$$c_4 \leq f(t)^{-\lambda - 1} |f'(t)| = \lambda^{-1} (f(t)^{-\lambda})' \tag{15}$$

for t as above and for some constant $c_4 = c_4(Q, a, m, n)$. By 2.3 (lower density bound) there exists $\varrho = \varrho(Q, a, m, n) > 0$ such that $f(t) \geq 1$ for $t \leq d - \varrho$. Thus integrating (15) from $t = 0$ to $d - \varrho$ (and using [Fe1, 2.9.19]) we conclude

$$\operatorname{dist}(x, C) = d \leq c_4^{-1}\lambda^{-1} + \varrho =: d_0,$$

as desired. □

REMARK 3.4. In case $m = n - 1$, 3.3 also holds for rectifiable currents S (with integral coefficients) in M. Namely, let x, d, γ, H_t and S_t be given as in the above proof. Denote by Z the geodesic cone from $\gamma(0)$ over S_0; then $\partial Z \in \mathcal{R}_{n-1}M$ and $(\partial Z) \llcorner H_t = S_t$ for $t > 0$. Now the same decomposition argument as in the proof of [La2, 2.4] shows that Z can be assumed to have multiplicity 1 everywhere. Then one may proceed as above, but instead of using 2.4 one takes $T_t = (S_t - \partial(Z \llcorner H_t)) \llcorner N_{r(t)}$ with the mass estimate $\mathbf{M}(T_t) \leq c_1' e^{a(n-2)r(t)}$ for some constant $c_1' = c_1'(a, n)$. Finally, one may choose $r(t)$ such that $f(t) = 2Qc_1' e^{a(n-2)r(t)}$.

4. Existence results

In this last section, (M, g_0) will always denote a simply connected, complete riemannian n-manifold with sectional curvature $-a^2 \leq K \leq -1$. Recall that the ideal boundary of (M, g_0) is defined by $M_\infty := SM/\sim$, where SM is the unit sphere bundle of M, and $v \sim w$ if and only if the geodesic rays γ_v and γ_w with initial vectors v and w, respectively, are asymptotic in the sense that $\sup_{t \geq 0} \operatorname{dist}(\gamma_v(t), \gamma_w(t)) < \infty$. There is a natural topology on

$$\bar{M} := M \cup M_\infty,$$

called *cone topology*, s. [E-O] or [An3, Sect. 0]. With this topology, $\bar{M}$ is homeomorphic to a compact ball in $\mathbf{R}^n$. The *boundary at infinity* of a subset A of M is then defined by

$$\operatorname{bd}_\infty A := M_\infty \cap \bar{A},$$

where $\bar{A}$ denotes the closure of A relative to $\bar{M}$. In [An3] it is shown that whenever $v \in M_\infty$ and V is any neighborhood of v in $\bar{M}$ then there exists a convex subset C of M with $M \setminus V \subset C$ but $v \notin \operatorname{bd}_\infty C$. In particular, this implies the useful fact that always

$$\operatorname{bd}_\infty \operatorname{conv} A = \operatorname{bd}_\infty A, \tag{16}$$

where $\operatorname{conv} A$ denotes the convex hull of A (i.e. the intersection of all convex subsets of M containing A).

Now assume that g is a riemannian metric on M which is Lipschitz equivalent to g_0, i.e. there are constants $0 < \alpha \leq \beta$ such that

$$\alpha^2 g_0(v, v) \leq g(v, v) \leq \beta^2 g_0(v, v)$$

for all $v \in TM$. For instance, the lift of an arbitrary metric on some compact quotient of (M, g_0) has this property. We apply the results of the preceding section to construct m-dimensional, complete minimizing surfaces in (M, g) with prescribed boundary data at infinity, for every $1 \leq m < n$. Throughout the section, all metric notions will generally refer to g_0 rather than to g, except that the constructed surfaces minimize area with respect to g. In terms of g_0, these surfaces are Q-minimizing with constant

$$Q = (\beta/\alpha)^m.$$

Given a subset L of M_∞, the strategy to construct a complete minimizing surface $S \in \mathscr{R}^2_{m,\mathrm{loc}} M$ asymptotic to L can roughly be described as follows. First L is approximated, in an appropriate sense, by a sequence of $(m-1)$-dimensional boundaries R_i in M (see the conditions given in 4.2 below). Then for each R_i, 1.2 yields the existence of a minimizing current $S_i \in \mathscr{R}^2_m M$ with $\partial S_i = R_i$. Now the desired surface S will be obtained as the limit of some subsequence (S_{i_j}), and the trapping inequality 3.3 will be used to infer that indeed $\mathrm{bd}_\infty \operatorname{spt} S = L$. In order to extract the convergent subsequence we need the following result which is a variant of the compactness theorem [Mo1, 2.8]. Convergence $S_{i_j} \to S \in \mathscr{R}^2_{m,\mathrm{loc}} M$ means that for every $y \in M$ there exists a neighborhood $U \subset\subset M$ of y such that $S_{i_j} \llcorner U \to S \llcorner U$ in the $\mathscr{F}^2$ topology, cf. Sect. 1.

PROPOSITION 4.1. *Let* $1 \leq m < n$ *and* $x \in M$. *Assume that for every positive integer* i, $S_i \in \mathscr{R}^2_m M$, $\partial S_i \in \mathscr{R}^2_{m-1} M$, $\operatorname{spt} \partial S_i \cap U(x, i) = \emptyset$, *and* S_i *is minimizing in* $U(x, i)$ *with respect to* g. *Then some subsequence* (S_{i_j}) *of* (S_i) *converges to a complete current* $S \in \mathscr{R}^2_{m,\mathrm{loc}} M$ *which is minimizing in* M *with respect to* g. *Moreover,*

$$\operatorname{spt} S_i \cap K \neq \emptyset \textit{ for almost all } i \Rightarrow \operatorname{spt} S \cap K \neq \emptyset \tag{17}$$

for every compact subset K *of* M.

Proof. Each S_i is Q-minimizing in $U(x, i)$ with respect to g_0, for $Q = (\beta/\alpha)^m$. Thus since $\operatorname{spt} \partial S_i \cap U(x, i) = \emptyset$, 2.5 (upper density bound) yields

$\sup_i \mathbf{M}(S_i \llcorner U(x,j)) < \infty$ for all $j > 0$. The existence of a subsequence converging to a complete element of $\mathscr{R}^2_{m,\mathrm{loc}}M$ now follows by repeated application of 1.4 (slicing) and [Fe1, (4.2.17)$^\nu$ on p. 432] in combination with a diagonal sequence argument. The remaining conclusions are obtained by adapting [Fe1, 5.4.2] (see also [Si, 34.6(2) and 31.2]). □

Now we prove the following general existence result for complete g-area minimizing currents $S \in \mathscr{R}^2_{m,\mathrm{loc}}M$.

THEOREM 4.2. *Let $1 \leq m < n$, and let L be a closed subset of M_∞ with the property that there exists a sequence of boundaries $R_i \in \mathscr{R}^2_{m-1}M$ in M satisfying the following conditions*:

(i) *For every neighborhood U of L in $\bar{M}$ there exists $k > 0$ such that* $\operatorname{spt} R_i \subset U$ *for all $i \geq k$, and*

(ii) *for every open $V \subset \bar{M}$ meeting L there exists a closed set $G \subset V \backslash L$ such that almost all R_i do not bound in $M \backslash G$.*

Then there exists a complete m-dimensional surface $S \in \mathscr{R}^2_{m,\mathrm{loc}}M$ which is minimizing in M with respect to g and asymptotic to L, i.e. $\mathrm{bd}_\infty \operatorname{spt} S = L$.

Note that the obtained minimizing current S enjoys the regularity property described in 1.3. Moreover, S satisfies the uniform density bound given in 2.5 with $Q = (\beta/\alpha)^m$. Since the condition on the limit set L given in the theorem is rather awkward we will restate the result in a weaker but more convenient form in 4.3 below. In case $m = n - 1$ the assumption on L can be shown to be equivalent to the following: $L = \mathrm{bd}\, U = \mathrm{bd}\, U'$ for some disjoint open subsets U, U' of M_∞ with $U \cup U' = M_\infty \backslash L$. This condition is in turn equivalent to that given in 4.4.

Proof. Choose $x \in M$ and $r_i > 0$ such that $\operatorname{spt} R_i \subset B(x, r_i)$ for all i. Then by 1.2 (existence) there exists $S_i \in \mathscr{R}^2_m M$ with $\operatorname{spt} S_i \subset B(x, r_i)$ and $\partial S_i = R_i$ such that S_i is minimizing in $B(x, r_i)$ with respect to g. By condition (i) we may assume that the S_i satisfy the assumptions of 4.1. Therefore some subsequence (S_{i_j}) converges to a complete current $S \in \mathscr{R}^2_{m,\mathrm{loc}}M$ which is minimizing in M with respect to g. It remains to show that $\mathrm{bd}_\infty \operatorname{spt} S = L$.

Each S_i is quasiminimizing in $B(x, r_i)$ with respect to g_0. In fact since $B(x, r_i)$ is convex it follows that S_i is quasiminimizing in M. Let $A := \bigcup_i \operatorname{spt} R_i$ and $C := \operatorname{conv} A$. Then 3.3 yields

$$\operatorname{spt} S_i \subset N := \{y \in M : \operatorname{dist}(y, C) \leq d_0\} \tag{18}$$

for all i. Thus since $S_{i_j} \to S$ and N is a closed subset of M, spt $S \subset N$. Using (16) and condition (i) of the theorem we get

$$\mathrm{bd}_\infty \operatorname{spt} S \subset \mathrm{bd}_\infty N = \mathrm{bd}_\infty C = \mathrm{bd}_\infty A \subset L.$$

Conversely, let V and G be given as in (ii); then spt $S_i \cap G \neq \varnothing$ for almost all i. In view of (18) these spt S_i then meet the compact subset $K := G \cap N$ of M. By (17) it follows spt $S \cap K \neq \varnothing$ and hence spt $S \cap V \neq \varnothing$. Since this is true for all open $V \subset \bar{M}$ meeting L we get $L \subset \mathrm{bd}_\infty$ spt S, proving the theorem. □

The following result is a consequence of 4.2. For the special case $g = g_0$ it is stated in [Gr2].

4.3 THEOREM. *Let $2 \leq m < n$, and let L be an $(m-1)$-dimensional compact topological submanifold of M_∞ (i.e. the homeomorphic image of some compact $(m-1)$-dimensional topological manifold). Then there exists a complete m-dimensional surface $S \in \mathscr{R}^2_{m,\mathrm{loc}} M$ which is minimizing in M with respect to g and asymptotic to L.*

Proof. Identify M with the open unit ball $U(0, 1) \subset \mathbf{R}^n$, where $\bar{M}$ is homeomorphic to the closed ball $B(0, 1)$. For every V as in condition (ii) of 4.2 choose a bounded open set $W \subset \mathbf{R}^n$ with $W \cap B(0, 1) = V$. We claim that for every such W there exists a compact set $K \subset W \backslash L$ such that $0 \neq i_*[L] \in H_{m-1}(\mathbf{R}^n \backslash K)$, where H_{m-1} refers to singular homology with $\mathbf{Z}_2$ coefficients, i stands for inclusion, and $[L] \in H_{m-1}(L)$ denotes the fundamental class.

Namely, using standard techniques from algebraic topology, one shows that $0 \neq i_*[L] \in H_{m-1}((\mathbf{R}^n \backslash W) \cup L)$. Next one may use the fact that L is an absolute neighborhood retract (cf. [Gb, 26.17.4]) to infer that $(\mathbf{R}^n \backslash W) \cup L$ has an open neighborhood Z such that still $0 \neq i_*[L] \in H_{m-1}(Z)$. Then $K := \mathbf{R}^n \backslash Z$ is the desired compact set.

Now approximate L by a sequence of closed singular Lipschitz chains (with $\mathbf{Z}_2$ coefficients) in $U(0, 1)$ such that the corresponding cycles $R_i \in \mathscr{R}^2_{m-1} U(0, 1) = \mathscr{R}^2_{m-1} M$ satisfy condition (i) of 4.2, and such that for all V, W and K as above, almost all members of the sequence are homologous to L in $\mathbf{R}^n \backslash K$. It remains to show that the obtained sequence (R_i) also satisfies condition (ii) of 4.2. One takes $G = K \cap B(0, 1)$. By means of the deformation theorem [Fe1, (4.2.9)$^\nu$ on p. 431] one shows that for every R_i bounding an element of $\mathscr{R}^2_m M$ in $M \backslash G$, the corresponding Lipschitz chain is homologous to zero in $\mathbf{R}^n \backslash K$. □

As mentioned above, in case $m = n - 1$ the condition on the limit set L given in 4.2 can be reformulated in a much simpler way. Moreover in this case, the fundamental estimate 3.3 has an analogue for integer multiplicity currents as noted in 3.4. Thus we get the following existence theorem for complete minimizing hypersurfaces in (M, g). Notation and definitions are the same as in [La2] where the result was obtained for the special case that (M, g) is Lipschitz equivalent to hyperbolic n-space. In the statement of the theorem, bd, cl and int refer to the sphere topology of M_∞.

THEOREM 4.4. *Let L be a subset of M_∞ satisfying $L = \operatorname{bd} A$ for some subset A of M_∞ with $A = \operatorname{cl}(\operatorname{int} A)$. Then there exists a closed set W of locally finite perimeter in M such that $\operatorname{bd}_\infty W = A$, $S := \partial[W] \in \mathscr{R}_{n-1,\mathrm{loc}} M$ is minimizing in M with respect to g, and $\operatorname{bd}_\infty \operatorname{spt} S = L$.*

According to the regularity results from geometric measure theory, spt S is a C^∞ submanifold of M up to a closed singular set of Hausdorff dimension at most $n-8$. The proof of 4.4 is analogous to that of [La2, 3.2]. In order to facilitate the discussion of 4.5 below, we sketch it in a slightly more complicated form than necessary.

Proof. We may assume $L \neq \varnothing$. Choose $x \in M$ and identify M_∞ with the unit sphere $S_x M$ in $T_x M$. Then for all subsets V of $S_x M$ let $\operatorname{cone}_x V := \{\exp_x(rv) : r \in [0, \infty),\ v \in V\}$ denote the geodesic cone from x over V in M. Moreover, define $\delta : S_x M \to [0, \pi]$ as the spherical distance of $v \in S_x M$ from $L \subset S_x M$ in $S_x M$. Now for every positive integer i let

$$T_i := \partial[B(x, i)] \llcorner \operatorname{cone}_x \{v \in A : \delta(v) \geq \delta_i\},$$

where the δ_i are positive numbers chosen such that $\partial T_i \in \mathscr{R}_{n-2} M$ and $\lim_{i \to \infty} \delta_i = 0$. Let $Q = (\beta/\alpha)^{n-1}$, and let $d_0 = d_0(Q, a, n) \geq 0$ be the constant given by 3.4. As in the proof of [La2, 3.2] it follows that there is an $\mathscr{H}^n$ measurable set W_i of finite perimeter in M such that $S_i := \partial[W_i] - T_i$ is minimizing in $B(x, i + d_0)$ with respect to g. Then 3.4 shows that S_i is actually minimizing in M (with respect to g). Some subsequence of the W_i converges weakly to a set W of locally finite perimeter in M, and $S := \partial[W]$ is a complete minimizing hypersurface in (M, g), as desired. Since spt S has $\mathscr{H}^n$ measure zero we may assume W to be closed (replace W by spt$[W]$).

The conclusion of the proof is now analogous to that of 4.2. Namely, by the assumptions on L and A, the boundaries $R_i := \partial S_i = -\partial T_i$ clearly satisfy condition (i) of 4.2. Moreover, given V as in condition (ii), one finds a geodesic $\gamma : \mathbf{R} \to M$ with $\gamma(\mathbf{R}) \subset V$ and ideal points $\gamma(-\infty) \in \operatorname{int} A$ and $\gamma(+\infty) \in M_\infty \setminus A$. Then one may take $G = \gamma(\mathbf{R})$. □

In [An2, 3.1] Anderson constructed complete minimizing hypersurfaces invariant under a discrete group of isometries acting on hyperbolic n-space. We use his argument to obtain a similar result for Γ-invariant minimizing hypersurfaces in (M, g), where $\Gamma \subset \operatorname{Iso}(M, g)$ is an arbitrary (not necessarily discrete) group of isometries of (M, g). Note that since g is Lipschitz equivalent to g_0, isometries of (M, g) extend to homeomorphisms of $\bar{M}$.

THEOREM 4.5. *Let L and A satisfy the assumptions of* 4.4, *and assume additionally that A be invariant under the action* (*extended to $\bar{M}$*) *of some subgroup $\Gamma \subset \operatorname{Iso}(M, g)$. Then there exists a closed set Ω of locally finite perimeter in M such that $\operatorname{bd}_\infty \Omega = A$, $\Sigma := \partial[\Omega] \in \mathscr{R}_{n-1,\mathrm{loc}} M$ is minimizing in M with respect to g, $\operatorname{bd}_\infty \operatorname{spt} \Sigma = L$, and Ω (and hence Σ) is Γ-invariant.*

We emphasize that for this result, to ensure the existence of a g-area minimizing hypersurface Σ asymptotic to L and invariant under Γ, it is in general not sufficient to assume merely L to be Γ-invariant (instead of A). This is shown by simple examples. Note that in case the action of Γ on M is free and properly discontinuous, and L is the limit set of Γ, the produced minimizing hypersurface Σ projects to a complete, stable minimal hypersurface in M/Γ which is smoothly embedded provided $n \leq 7$. On the other hand, 4.5 gives rise, for instance, to catenoid- or helicoid-like hypersurfaces whenever (M, g) possesses a corresponding (continuous) group of isometries. In hyperbolic 3-space, minimal surfaces exhibiting such symmetries were constructed by Mori [Mr] and Polthier [Po].

Proof. In the following we construct a (possibly constant) sequence $\Omega_1 \supset \Omega_2 \supset \cdots$ of closed subsets of M such that for every $j \geq 1$, Ω_j has locally finite perimeter, $\partial[\Omega_j] \in \mathscr{R}_{n-1,\mathrm{loc}} M$ is asymptotic to L and minimizing with respect to g, $\operatorname{bd}_\infty \Omega_j = A$, and for every $\gamma \in \Gamma$, $\Omega_{j+1} \subset \gamma\Omega_j$. The desired set Ω will then be obtained as the limit of some subsequence of the Ω_j.

First we apply 4.4 and let Ω_1 be equal to the obtained set W. Now for every integer $j \geq 1$, assuming that Ω_j is already defined (and has the properties stated above), we construct Ω_{j+1} as follows. We repeat the proof of 4.4 with the additional requirement that for each i, $\operatorname{spt} T_i$ is contained in the closed set

$$M_j := \bigcap_{\gamma \in \Gamma} \gamma\Omega_j.$$

Since $\operatorname{bd}_\infty M_j = A$ (which follows from the Γ-invariance of A together with 3.4), this can be achieved by choosing the δ_i appropriately (possibly $T_i = 0$ for small i). Then for every $\gamma \in \Gamma$ and every i, since both $\partial[\gamma\Omega_j]$ and $S_i = \partial[W_i] - T_i$ are minimizing in (M, g), a simple area comparison argument shows that we may assume $W_i \subset \gamma\Omega_j$ and hence $W_i \subset M_j$. We get $W \subset M_j$ and put $\Omega_{j+1} := W$.

Now we extract some subsequence (Ω_{j_k}) of (Ω_j) converging weakly to a set Ω of locally finite perimeter in M. Since each $\partial[\Omega_j]$ is minimizing in (M, g) so is $\Sigma := \partial[\Omega]$. By the same argument as in the proof of 4.4 we may assume Ω to be closed. In order to show that $\mathrm{bd}_\infty \Omega = A$ and $\mathrm{bd}_\infty \operatorname{spt} \Sigma = L$ we use again 3.4. Finally, since

$$\Omega_{j_{k+1}} \subset \gamma \Omega_{j_k}$$

for all k and $\gamma \in \Gamma$, it follows $\Omega \subset \gamma\Omega$ for all $\gamma \in \Gamma$. Hence Ω is Γ-invariant as desired. □

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