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# Almost periodic Sturm-Liouville operators with Cantor homogeneous spectrum

MIKHAIL SODIN AND PETER YUDITSKII

To the memory of B. Ya. Levin (1906–1993) who was a teacher of our teachers and who gave us so much

Being based on the infinite dimensional Jacobi inversion found earlier, we establish the direct generalization of the well-known properties of finite-band Sturm-Liouville operators in the case of operators with a homogeneous and, generally speaking, Cantor-type spectrum, and with pseudocontinuable Weyl functions.

In our investigations the group of unimodular characters of the fundamental group of the resolvent set plays a role of the isospectral manifold of the operator. The generalized Abel map conjugates the nonlinear evolution of spectral data with a linear motion on this torus. In particular, the operators we consider turn out to be uniformly almost periodic.

### §1. Statement of main results

1.1. Consider the Sturm-Liouville equation

$$L[q]y = -y'' + q(x)y = \lambda y, \quad -\infty < x < \infty, \tag{1.1.1}$$

with a real bounded continuous potential q(x). We denote by  $C(x, \lambda)$  and  $S(x, \lambda)$  the fundamental solutions of Equations (1.1.1) satisfying initial conditions

$$C(0, \lambda) = S'(0, \lambda) = 1,$$
  $C'(0, \lambda) = S(0, \lambda) = 0.$ 

By virtue of the classical Weyl theorem (see, for example, Titchmarsh [25, Ch. 2]), for each nonreal  $\lambda$  Equation (1.1.1) has solutions

$$\psi_{+}(x, \lambda) = C(x, \lambda) + m_{+}(\lambda)S(x, \lambda)$$
, such that  $\psi_{\pm} \in L^{2}(\mathbb{R}_{\pm})$ .

The functions  $m_{\pm}$  are holomorphic outside the real axis,  $m_{\pm}(\bar{\lambda}) = \overline{m_{\pm}(\lambda)}$  and

$$\mathfrak{F}m_{+}(\lambda)/\mathfrak{F}\lambda > 0, \qquad \mathfrak{F}m_{-}(\lambda)/\mathfrak{F}\lambda < 0.$$

The functions  $m_{\pm}(\lambda)$  are called the Weyl functions; they are defined uniquely by virtue of the boundedness from below of the potential q(x).

We denote by  $g(x, y; \lambda)$  the Green function of L[q] which is defined as the kernel of the resolvent  $R_{\lambda} = (L[q] - \lambda)^{-1}$ . Then (see Titchmarsh [25, Ch. 2])

$$\frac{1}{g(x,x;\lambda)} = \frac{m_{-}(\lambda) - m_{+}(\lambda)}{\psi_{+}(x,\lambda)\psi_{-}(x,\lambda)}.$$
(1.1.2)

Without loss of generality, we assume that the origin is the lower bound of the spectrum of L[q].

1.2. DEFINITION. Let E be a closed set

$$E = [0, \infty) \setminus \bigcup_{j \ge 1} (a_j, b_j)$$
 (1.2.1)

satisfying the conditions:

(i) E is homogeneous (Carleson [5], Jones and Marshall [10]), i.e., there is an  $\varepsilon > 0$  such that for all  $\lambda \in E$  and all  $\delta > 0$ 

$$|(\lambda - \delta, \lambda + \delta) \cap E| \ge \varepsilon \delta;$$
 (1.2.2)

(ii) the sum of lengths of gaps in E is finite:

$$\sum_{j\geq 1} (b_j - a_j) < \infty. \tag{1.2.3}$$

A potential q belongs to the class Q(E) if the spectrum of L[q] coincides with E and the Weyl functions are pseudocontinuable:

$$m_{+}(\lambda + i0) = m_{-}(\lambda - i0)$$
 for a.e.  $\lambda \in E$ . (1.2.4.)

1.3. Let us stop for a moment at this definition and make some comments.

First, we note that the homogeneity condition (1.2.2) can be written in the equivalent form which looks slightly more invariant: there is an  $\varepsilon > 0$  such that for all  $\lambda \in E$  and all  $\delta > 0$ 

$$\int_{(\lambda-\delta,\lambda+\delta)\cap E} \frac{dt}{1+t^2} \ge \varepsilon \int_{(\lambda-\delta,\lambda+\delta)} \frac{dt}{1+t^2}.$$

It will allow us to apply later function theory results obtained in [24].

Equations (1.1.2) and (1.2.4) imply that the potential  $q \in Q(E)$  is reflectionless in the sense of Craig [6]: for every  $x \in \mathbb{R}$ 

$$\Re g(x, x; \lambda \pm i0) = 0$$
 for a.e.  $\lambda \in E$ . (1.3.1)

It may be proved (see Appendix) that, vice versa, the Craig condition (1.3.1) implies condition (1.2.4).

If there is a finite number of gaps in E, then (1.2.4) implies that there is a rational function  $m(\lambda)$  on the hyperelliptic Riemann surface  $\mathcal{R}_E$  of the function

$$\sqrt{\frac{1}{\lambda} \prod_{j \ge 1} \frac{\lambda - a_j}{\lambda - b_j}}$$

such that  $m_+(\lambda) = m(\lambda)$  and  $m_-(\lambda) = m(\lambda^*)$  where \* means the involution of the sheets of the surface  $\mathcal{R}_E$ . Hence in this case Q(E) coincides with the well-known class of finite-band Sturm-Liouville operators (see, e.g., McKean and van Moerbeke [19], Dubrovin, Matveev and Novikov [7], Moser [22], and the recent book Belokolos, Bobenko, Enol'skii, Its, and Matveev [4b]). With the special choice of E the class Q(E) also contains infinite-band periodic potentials investigated by Marchenko and Ostrovskii [17, 18] (see also Marchenko [16]), by McKean and Trubowitz [20, 21], and by Garnett and Trubowitz [9]. Such potentials are connected with hyperelliptic Riemann surfaces of infinite genus.

From the other hand, condition (1.2.4) naturally arises in the spectral theory of ergodic (or random) Sturm-Liouville operators due to well-known Kotani's theorem (see Pastur and Figotin [23a] or Carmona and Lacroix [5a], see also Belokolos, Bobenko, Enol'skii, Its, and Matveev [4b, Sect. 8.1]). Putting together with the Pastur-Ishii result, it asserts that if L is an ergodic Sturm-Liouville operator with the density of an absolutely continuous spectrum positive a.e. on a Borelian set  $A \in \mathbb{R}$ , then condition (1.2.4) holds a.e. on A. That is, all ergodic (particularly, almost periodic) Sturm-Liouville operators with a homogeneous spectrum E and with the density of absolutely continuous spectrum positive a.e. on E belong to the class Q(E).

### 1.4. Set

$$E^{(N)} = [0, \infty) \setminus \bigcup_{j=1}^{N} (a_j, b_j).$$

APPROXIMATION THEOREM. For a fixed homogeneous set E and for each sequence of finite-band potentials  $q_N \in Q(E^{(N)})$ , N = 1, 2, ..., there is a subsequence which converges uniformly on the whole axis  $\mathbb{R}$  and the set of all limit potentials coincides with Q(E).

In particular, Q(E) is compact in the topology of uniform convergence on the whole axis, and since finite-band potentials are uniformly almost periodic (see, for example, Levitan [14] or Moser [22]), we obtain that every potential of the class Q(E) is uniformly almost periodic.

Under more restrictive conditions imposed on the set E, the almost periodicity of potentials of the class Q(E) has been proved in Levitan [14, 15], Pastur and Tkachenko [23], Egorova [8, 8a].

1.5. Let  $\Omega = \mathbb{C} \setminus E$  be the resolvent set of the operator L[q], and let  $\pi(\Omega) = \pi(\Omega, -1)$  be the fundamental group of  $\Omega$  with the marked point z = -1 (in fact, its choice is inessential). By  $\pi^*(\Omega)$  we denote the group of unimodular characters of  $\pi(\Omega)$  endowed with the topology dual to the discrete one on  $\pi(\Omega)$ . Further, we will use the additive form of notations for the compact abelian group  $\pi^*(\Omega)$ :

$$\pi^*(\Omega) = \{ \alpha(\gamma) \in \mathbb{R} \mod \mathbb{Z} : \gamma \in \pi(\Omega), \, \alpha(\gamma_1 \circ \gamma_2) = \alpha(\gamma_1) + \alpha(\gamma_2) \}.$$

The group  $\pi^*(\Omega)$  is a finite-dimensional torus if  $\Omega$  is finitely connected, and is an infinite-dimensional torus if  $\Omega$  is infinitely connected. We will use this torus for a parameterization of operators L[q],  $q \in Q(E)$  with a given E.

Let us consider the conformal map of the upper half-plane onto the slitted quarter-plane (see Figure 1):

$$w: \mathbb{C}_+ \to \{\Re w < 0, \, \Im w > 0\} \setminus \bigcup_{j \ge 1} \{\Im w = \pi i \delta_j, \, -h_j \le \Re w \le 0\}, \tag{1.5.1}$$

normalized by the conditions w(0) = 0 and

$$w(-\lambda) \sim -\lambda^{1/2}, \qquad \lambda \to \infty.$$
 (1.5.2)

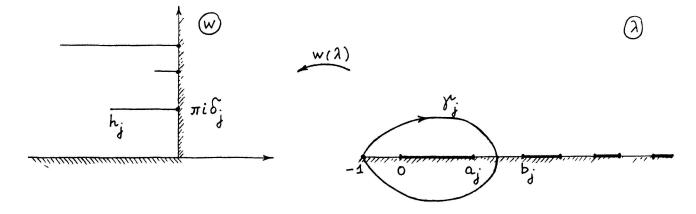


Figure 1

It maps the spectrum E onto the imaginary semi-axis and the gaps in E onto the slits. The normalization (1.5.2) is possible by virtue of condition (1.2.3) and well-known results by Akhiezer and Levin [4, p. 127] (see also Levin [13]). Such maps were introduced into the spectral theory of Sturm-Liouville operators by Marchenko and Ostrovskii [17, 18] (see also Marchenko [16]). Later, they were used by Garnett and Trubowitz [9], and by Pastur and Tkachenko [23].

Let us continue the function  $w(\lambda)$  analytically across all intervals  $(a_j, b_j)$  and  $(-\infty, 0)$  into the lower half-plane. We obtain a multivalued function on  $\Omega$  whose real part is single-valued. The ramification of  $\mathfrak{F}w$  generates a character  $\delta = \delta(E) \in \pi^*(\Omega)$ . Namely,  $\delta(\gamma_j) = \delta_j$ , where numbers  $\delta_j$  are defined in (1.5.1) and  $\{\gamma_j\} \subset \pi(\Omega)$  being a system of generators of the group  $\pi(\Omega)$ , consisting of loops  $\gamma_j$ , which begin and end at  $\lambda = -1$ , and contain  $E_j = E \cap [b_j, \infty)$  inside and  $E \setminus E_j$  outside (see Figure 1).

Now we are able to formulate our main result.

1.6. MAIN THEOREM. There exists a homeomorphism between the compacts Q(E) and  $\pi^*(\Omega)$  conjugating the shift of the potential  $q(x) \mapsto q(x+t)$  and the linear motion  $\alpha \to \alpha + \delta t$  on  $\pi^*(\Omega)$ , where  $\delta = \delta(E)$ .

COROLLARY. Every potential of the class Q(E) is a uniform almost periodic function whose frequency module is spanned by  $\{\delta_i\}$ .

In the finite-band case  $w(\lambda)$  coincides with the normed abelian integral of the second kind with a pole at infinity, and  $\pi^*(\Omega)$  is a finite-dimensional torus isomorphic to the real part of the Jacobian of the corresponding hyperelliptic Riemann surface  $\mathcal{R}_E$ . In this case our Main Theorem is a restatement of the well-known results due to Dubrovin, Matveev and Novikov [7], and McKean and van Moerbeke [19] (see also Moser [22]). Such results are going back to works by Akhiezer originally published in the early sixties in a series of papers in Soviet Math. Doklady and in Proceedings of the Kharkov Math. Society. Later, they were summed up in Akhiezer [1-3] (see also Akhiezer and Rybalko [4a]). In fact, in his papers Akhiezer considered only operators acting on the semi-axis ( $\mathbb{R}_+$  or  $\mathbb{Z}_+$ ).

1.7. Let us introduce a class of divisors

$$\mathscr{D}(E) = \left\{ D = \bigcup_{j \geq 1} (\lambda_j, \varepsilon_j) : \lambda_j \in [a_j, b_j], \, \varepsilon_j = \pm 1 \right\}.$$

If  $\lambda_j$  coincides with one of the points  $a_j$ ,  $b_j$  we arrange  $(\lambda_j, +1) \equiv (\lambda_j, -1)$ . We endow  $\mathcal{D}(E)$  with the compact topology of the product of circles  $I_j^2$ , where  $I_j^2$  is a two-sheeted covering of  $I_j = [a_j, b_j]$  with ends identified.

Following Craig [6], we associate with every potential  $q \in Q(E)$  the collection of spectral data  $D \in \mathcal{D}(E)$  of the operator L[q]. The function  $g(0, 0; \lambda)$  is a Nevanlinna function (it preserves the upper half-plane) and by virtue of (1.3.1) its multiplicative representation may be rewritten in the form

$$g(0,0;\lambda) = \frac{1}{2\sqrt{-\lambda}} \prod_{j\geq 1} \frac{\lambda - \lambda_j}{\sqrt{(a_i - \lambda)(b_i - \lambda)}}, \quad \lambda_j \in [a_j, b_j].$$
 (1.7.2)

(see, for example, Appendix in Krein and Nudelman [11] or Craig [6]). By (1.1.2) with x = 0 we obtain

$$g(0,0;\lambda) = (m_{-}(\lambda) - m_{+}(\lambda))^{-1}, \tag{1.7.2}$$

and if  $\lambda_j \in (a_j, b_j)$  then  $\lambda_j$  is a pole of one of the functions  $m_{\pm}(\lambda)$  (i.e.,  $\lambda_j$  is an eigenvalue of L[q] acting on one of the semi-axes  $\mathbb{R}_{\pm}$ ). If  $\lambda_j$  was a pole of both of the functions  $m_{\pm}(\lambda)$  then  $\lambda_j$  would belong to the spectrum of L[q] what is impossible. Thus, we may define  $\varepsilon_j = \pm 1$  depending on which of the functions  $m_{\pm}(\lambda)$  has a pole at  $\lambda_j \in (a_j, b_j)$ , and the map  $Q(E) \to \mathcal{D}(E)$  is well-defined.

The shift of the potential  $q(x) \mapsto q(x+t)$  defines a continuous curve  $\{D(t)\}_{t \in \mathbb{R}} \subset \mathcal{D}(E)$ , D(0) = D, and the potential q(t) can be recovered by this curve using the trace formula proved for this class by Craig [6]

$$q(t) = \sum_{j>1} (a_j + b_j - 2\lambda_j(t)), \tag{1.7.3}$$

where  $\lambda_i(t) \in [a_i, b_i]$  correspond to the divisor D(t).

1.8. UNIQUENESS THEOREM. The map  $Q(E) \to \mathcal{D}(E)$  is a homeomorphism of the compacts Q(E) and  $\mathcal{D}(E)$ .

This theorem establishes a continuous parameterization of operators of the class Q(E) by divisors from  $\mathcal{D}(E)$ . In the periodic case this parameterization was found in Marchenko and Ostrovskii [17]. We should also mention that a bijection between sets Q(E) and  $\mathcal{D}(E)$  was established in Craig [6] under certain conditions imposed on the spectrum E which seem to be more restrictive than the homogeneity; on the other hand in that paper the set of potentials Q(E) was endowed with a weaker topology of the uniform convergence on each compact subset of  $\mathbb{R}$ .

1.9. The key to the proofs of our theorems is the fact that the generalized infinite dimensional Abel map  $A: \mathcal{D}(E) \to \pi^*(\Omega)$ , being a homeomorphism of these

compacts, linearize, as in the finite-band case, the curve D(t) mapping it onto the line  $\alpha + \delta(E)t$  on  $\pi^*(\Omega)$ .

Here is a plan of the rest of the paper. In Sect. 2 we remind the definition of the Abel map from Sodin and Yuditskii [24]. In Sect. 3 we prove a "half" of the Uniqueness Theorem, namely we prove that the collection of spectral data D of the operator L[q] defines the potential q uniquely, i.e. the map  $Q(E) \to \mathcal{D}(E)$  is injective. In Sect. 4 we bring some auxiliary facts concerning a "finite-band approximation" of  $\mathcal{D}(E)$  by  $\mathcal{D}(E^{(N)})$  and of  $\pi^*(\Omega)$  by  $\pi^*(\Omega^{(N)})$ , where  $\Omega^{(N)} = \overline{\mathbb{C}} \setminus E^{(N)}$ ; and in Sect. 5 we prove Approximation Theorem. Simultaneously, our Main Theorem and Uniqueness Theorem will be also proved.

## §2. The Abel map $A: \mathcal{D}(E) \to \pi^*(\Omega)$

2.1. Let  $\omega(\lambda, F)$  be the harmonic measure of a set  $F \subset E$  at  $\lambda \in \Omega$  with respect to the domain  $\Omega$ . The Abel map was defined in [24] as

$$A(D)[\gamma_k] = \frac{1}{2} \sum_j \varepsilon_j \int_{\lambda_j}^{b_j} \omega(d\lambda, E_k) \mod \mathbb{Z}, \qquad k = 1, 2, \dots$$
 (2.1)

where  $D = \bigcup_j (\lambda_j, \varepsilon_j) \in \mathcal{D}(E)$ ,  $E_k = E \cap [b_k, \infty)$ , and  $\{\gamma_k\}$  being the system of loops generating the group  $\pi(\Omega)$  (see Sect. 1.5). As it was checked in this paper, the homogeneity of E yields the convergence of the series in the right-hand side of (2.1) (see also Sect. 4.2 below). This definition of the Abel map agrees with the classical one in the finite-band case.

2.2. In the just mentioned paper we have proved

THEOREM A. If a set E is homogeneous, then the Abel map gives a homeomorphism between the compacts  $\mathcal{D}(E)$  and  $\pi^*(\Omega)$ .

2.3. In the sequel, we denote by  $G(z, z_0)$  the Green function (for the usual Laplacian) of the domain  $\Omega$  with the pole at  $z = z_0$ , and we denote the complex Green function of  $\Omega$  with a zero at  $z = z_0$  by  $\Phi(z, z_0) = \exp[-G(z, z_0) - i * G(z, z_0)]$ . The function  $\Phi(z, z_0)$  is character-automorphic (Widom [26]): it has a single-valued modulus and after analytic continuation along the loop  $\gamma_k$  the variation of its argument  $-*G(z, z_0)$  equals

$$-2\pi[\omega(z_0,E_k)-\operatorname{Ind}_{\tau_k}(z_0)].$$

Hence a character  $\alpha = \alpha[\Phi(\cdot, z_0)]$  associated with  $\Phi(\cdot, z_0)$  equals

$$\alpha[\Phi(\cdot, z_0)](\gamma_k) = -\omega(z_0, E_k) \mod \mathbb{Z}, \qquad k = 1, 2, \dots$$
 (2.3.1)

# §3. An operator L[q], $q \in Q(E)$ , is uniquely defined by the collection of its spectral data D

- 3.1. Now, let us consider the Weyl functions  $m_{\pm}$ . Since  $m_{+}$  and  $-m_{-}$  preserve the upper half-plane, these functions have Nevanlinna representations as Cauchy integrals of nonnegative measured  $d\sigma_{\pm}$ . These measures are spectral measures of the restrictions of the operator L[q] on the semi-axis  $\mathbb{R}_{\pm}$  correspondingly. By the Marchenko uniqueness theorem (see, for example, Levitan [14]) these measures define uniquely the potential q(x). So we have to prove that these measures in turn are defined by the divisor D.
- 3.2. LEMMA. Let  $q \in Q(E)$ . Then the Nevanlinna measures  $d\sigma_{\pm}$  of the functions  $m_{+}$  and  $-m_{-}$  are defined uniquely by the divisor  $D = \bigcup_{j} (\lambda_{j}, \varepsilon_{j})$ .

Proof. In the proof we will use the relations

$$-\frac{1}{g(0,0)} = m_+ - m_-, \tag{3.2.1}$$

$$m_{+}(\lambda + i0) = \overline{m_{-}(\lambda + i0)}$$
 for a.e.  $\lambda \in E$ . (3.2.2)

Set

$$\tilde{m}_{+}(\lambda) = m_{+}(\lambda) - m_{+}(-1),$$

$$\tilde{m}_{-}(\lambda) = m_{-}(\lambda) - m_{+}(-1),$$

and consider the product  $F(\lambda) = \tilde{m}_{+}(\lambda)\tilde{m}_{-}(\lambda)$ . The argument arg  $F(\lambda)$  varies in the upper half-plane from  $-\pi$  to  $\pi$ . It follows from (3.2.2) that

$$arg F(\lambda + i0) = 0$$
 for a.e.  $\lambda \in E$ .

Since both functions  $\tilde{m}_{\pm}(\lambda)$  are real in the gaps, arg  $F(\lambda)$  takes there values 0 and  $\pm \pi$ .

Now, let us look at the behaviour of  $\tilde{m}_{\pm}(\lambda)$  in the gap  $(-\infty, 0)$ . The function  $\tilde{m}_{+}(\lambda)$  increases there (because its Nevanlinna measure does not support this gap), consequently

$$\tilde{m}_{+}(\lambda) \begin{cases} >0, & \lambda \in (-1,0), \\ <0, & \lambda \in (-\infty,-1). \end{cases}$$

Since  $g(0, 0, \lambda) > 0$  as  $\lambda \in (-\infty, 0)$  (it follows, for example, from representation (1.7.1)), we have

$$\tilde{m}_{-}(\lambda) = \tilde{m}_{+}(\lambda) + \frac{1}{g(0,0,\lambda)} > \tilde{m}_{+}(\lambda), \qquad \lambda \in (-\infty,0).$$

In addition,  $\tilde{m}_{-}(\lambda)$  is decreasing there and hence  $\tilde{m}_{-}(\lambda) > 0$ ,  $\lambda \in (-\infty, 0)$ . Thus,

$$\arg F(\lambda) = \begin{cases} \pi, & \lambda \in (-\infty, -1), \\ 0, & \lambda \in (-1, 0). \end{cases}$$

Similarly, we establish that only one of the functions  $\tilde{m}_{\pm}(\lambda)$  may have zero  $\lambda_j^{(1)}$  on  $(a_j, b_j)$ , and we set  $\varepsilon_j^{(1)} = +1$  if  $\lambda_j^{(1)}$  is zero of  $\tilde{m}_{\pm}(\lambda)$  and  $\varepsilon_j^{(1)} = -1$  if  $\lambda_j^{(1)}$  is zero of  $\tilde{m}_{\pm}(\lambda)$  (see Figure 2).

Taking into account the fact that  $\log F(\lambda)$  is represented on  $\Re \lambda > 0$  by the Cauchy integral of the boundary values of arg  $F(\lambda + i0)$ , we obtain a multiplicative representation

$$F(\lambda) = C^2(\lambda + 1) \prod_j \frac{\lambda - \lambda_j^{(1)}}{\lambda - \lambda_j}, \qquad \lambda_j^{(1)} \in [a_j, b_j].$$
 (3.2.3)

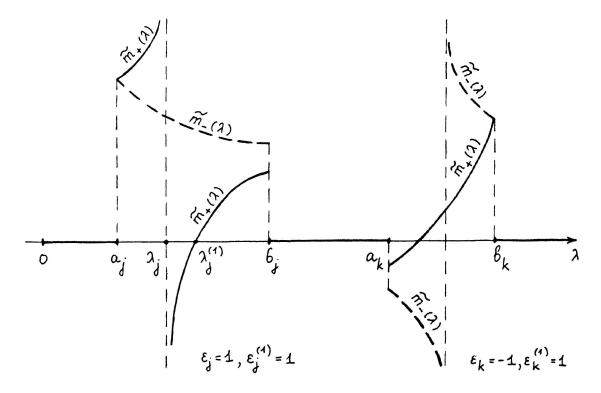


Figure 2

Simultaneously, we have defined the divisor

$$D^{(1)} = \bigcup_j \; (\lambda_j^{(1)}, \, \varepsilon_j^{(1)}), \qquad D^{(1)} \in \mathcal{D}(E).$$

Now we will use some arguments from Sodin and Yuditskii [24]. By Theorem D from this paper, the functions  $F(\lambda)$  and  $\tilde{m}_{\pm}(\lambda)$  are functions of bounded type on  $\Omega = \mathbb{C} \setminus E$  (it means that they are represented as a quotient of bounded functions), and, moreover, these functions have no singular inner factors (the latter means that the logarithm of modulus is represented in E as a sum of the Green potential and the Poisson integral of its boundary values). The factorization  $F(\lambda) = \tilde{m}_{+}(\lambda)\tilde{m}_{-}(\lambda)$  should be considered as a representation of the positive function on E, given by (3.2.3), as a square of modulus of boundary values of the function  $\tilde{m}_{+}(\lambda)$  which has no singular inner component and whose zeros and poles are known. It allows us to write a multiplicative representation

$$\tilde{m}_{+}(\lambda) = C\sqrt{(\lambda+1)\Phi(\lambda,-1)} \sqrt{\prod_{j} \frac{\lambda - \lambda_{j}^{(1)}}{\lambda - \lambda_{j}} \frac{\Phi(\lambda,\lambda_{j}^{(1)})^{\varepsilon_{j}^{(1)}}}{\Phi(\lambda,\lambda_{j})^{\varepsilon_{j}}}}$$
(3.2.4)

(see details in the cited paper by the authors).

The left-hand side of (3.2.4) is single-valued on  $\Omega$ , hence, the right-hand side also should be single-valued. Evaluating the character of the right-hand side of (3.2.4) and making use of (2.3.1), we obtain

$$0 = -\frac{1}{2}\omega(-1, E_k) - \frac{1}{2}\sum_{j} \left[\varepsilon_j^{(1)}\omega(\lambda_j^{(1)}, E_k) - \varepsilon_j\omega(\lambda_j, E_k)\right],$$

or

$$A(D^{(1)}) = A(D) + \tau, \tag{3.2.5}$$

where  $\tau(\gamma_k) = -\frac{1}{2}\omega(-1, E_k) \mod \mathbb{Z}, k = 1, 2, \ldots$  defines a fixed character from  $\pi^*(\Omega)$ .

By Theorem A (Sect. 2.2) Equation (3.2.5) implies that the divisor  $D^{(1)} \in \mathcal{D}(E)$  is defined uniquely by the divisor D.

A constant C is evaluated from the condition

$$-\frac{1}{g(0,0;-1)} = -\tilde{m}_{-}(-1).$$

Thus, the functions  $\tilde{m}_{\pm}$  are determined uniquely by the divisor D and the Lemma is proved.

## §4. "Finite-band" approximation of D(E) and $\pi^*(\Omega)$

4.1. As before, we set

$$E^{(N)} = [0, \infty) \setminus \bigcup_{j=1}^{N} (a_j, b_j), \qquad \Omega^{(N)} = \mathbb{C} \setminus E^{(N)}.$$

Let  $\{\gamma_j\}$  be a system of generators of the fundamental group  $\pi(\Omega)$  introduced in Sect. 1.5. Note that  $\{\gamma_j\}_{j\leq N}$  is a system of generators of  $\pi(\Omega^{(N)})$  and hence an arbitrary character  $\alpha^{(N)} \in \pi^*(\Omega^{(N)})$  may be extended to a character  $\alpha \in \pi^*(\Omega^{(N)})$  by setting

$$\alpha(\gamma_j) = \begin{cases} \alpha^{(N)}(\gamma_j), & j \le N, \\ 0, & j > N. \end{cases}$$

It defines a continuous embedding  $\pi^*(\Omega^{(N)}) \subseteq \pi^*(\Omega)$ . And, vice versa, by  $\alpha^{(N)}$  we denote a "projection" of the character  $\alpha \in \pi^*(\Omega)$  on  $\pi^*(\Omega^{(N)})$  which is defined as

$$\alpha^{(N)}(\gamma_i) = \alpha(\gamma_i), \qquad 1 \le j \le N.$$

Similarly, every divisor  $D^{(N)} \in \mathcal{D}(E^{(N)})$  may be complemented to a divisor  $D = D^{(N)} \cup \bigcup_{j \geq N} (b_j), D \in \mathcal{D}(E)$ , and, vice versa, for a given divisor  $D \in \mathcal{D}(E)$  we denote its "projection" onto  $\mathcal{D}(E^{(N)})$  by  $D^{(N)}$ . Then

$$\alpha^{(N)} \to \alpha, \qquad D^{(N)} \to D \qquad \text{as } N \to \infty,$$
 (4.1.1)

as it follows directly from the definition of convergence in  $\pi^*(\Omega)$  and  $\mathcal{D}(E)$ .

We denote by  $A^{(N)}$ :  $\mathcal{D}(E^{(N)}) \to \pi^*(\Omega^{(N)})$  the classical Abel map which due to above may be considered as a map  $A^{(N)}$ :  $\mathcal{D}(E) \to \pi^*(\Omega)$ . Our next goal is to prove that

$$A^{(N)}(D) \to A(D)$$
 for every  $D \in \mathcal{D}(E)$ , (4.1.2)

and

$$\delta(E^{(N)}) \to \delta(E),\tag{4.1.3}$$

as  $N \to \infty$ .

4.2. Proof of (4.1.2). We should prove that for every k it holds

$$\sum_{j \le N} \varepsilon_j \int_{\lambda_j}^{b_j} \omega(d\lambda, E_k^{(N)}, \Omega^{(N)}) \to \sum_j \varepsilon_j \int_{\lambda_j}^{b_j} \omega(d\lambda, E_k, \Omega)$$
(4.2.1)

In the proof of this relation we will use two consequences of the homogeneity of E which are pertaining to potential theory. At first, we will use the regularity of E with respect to the Dirichlet problem on  $\Omega = \mathbb{C} \setminus E$ . Landkof [12]. It follows, for example, from the Wiener criterion. Secondly, we will use that if E is homogeneous then  $\mathbb{C} \setminus E = \Omega$  satisfies the Parreau-Widom condition

$$\sum_{\{c_j: \ VG(c_j, \lambda_0) = 0\}} G(c_j, \lambda_0) < \infty \tag{4.2.2}$$

(see Jones and Marshall [10]).

First of all, we will show that for every  $\lambda \in \Omega$  and every k

$$\omega(\lambda, E_k^{(N)}, \Omega^{(N)}) \to \omega(\lambda, E_k, \Omega), \qquad N \to \infty.$$
 (4.2.3)

To this end, we consider a harmonic function on  $\Omega^{(N)}$ 

$$\omega(\lambda, E_k^{(N)}, \Omega^{(N)}) - \omega(\lambda, E_k, \Omega), \qquad N > k,$$

and remark that the homogeneity of E implies the regularity of E with respect to the Dirichlet problem on  $\mathbb{C}\backslash E$ . Hence, by the regularity of E,

$$\max_{\lambda \in E^{(N)}} \left| \omega(\lambda, E_k^{(N)}, \Omega^{(N)}) - \omega(\lambda, E_k, \Omega) \right| \to 0, \qquad N \to \infty$$

(see, for example, Landkof [12]).

Now, in order to prove (4.2.1), we will show that the series in the left-hand side of (4.2.1) is majorized by a converging series consisting of positive terms which do not depend on N.

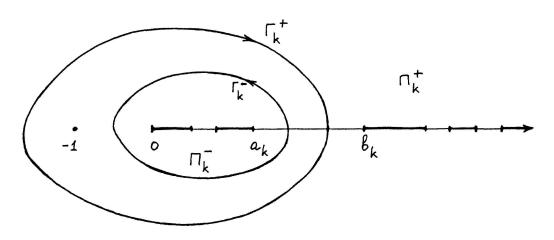


Figure 3

Let  $\Pi_k^+$  be a connected vicinity of  $[b_k, \infty)$  on the Riemann sphere and let  $\Pi_k^-$  be a connected vicinity of  $[0, a_k]$  (see Figure 3). We assume that  $\Pi_k^+ \cap \Pi_k^- = \emptyset$  and that  $-1 \in \mathbb{C} \setminus (\overline{\Pi_k^+ \cup \Pi_k^-})$ . Denote  $\Gamma_k^{\pm} = \partial \Pi_k^{\pm}$  and set

$$\mu_k^{\pm} = \min_{\lambda \in \Gamma_k^{\pm}} G(\lambda, -1),$$

where  $G(\lambda, -1)$  is the Green function of  $\Omega$  with a pole at  $\lambda = -1$ . By the Maximum Principle applied in  $\Pi_k^{\pm} \cap \Omega^{(N)}$ , we obtain

$$\begin{split} & \mu_k^- \omega(\lambda, E_k^{(N)}, \Omega^{(N)}) \leq G(\lambda, -1), \qquad \lambda \in \Pi_k^-, \\ & \mu_k^+ (1 - \omega(\lambda, E_k^{(N)}, \Omega^{(N)})) \leq G(\lambda, -1), \qquad \lambda \in \Pi_k^+, \end{split}$$

Therefore, the convergent series

$$\frac{1}{\min(\mu_k^+, \mu_k^-)} \sum_{\{c_j: VG(c_j, -1) = 0\}} G(c_j, -1)$$

is a majorant we were looking for.

4.3. Proof of (4.1.3). For this purpose, we will use the connection between conformal maps onto comb-like domains (as in (1.5.1)) and subharmonic majorants (Levin [13]). Denote by  $w_N$  the conformal map (1.5.1) corresponding to the set  $E^{(N)}$  and put  $v = \Re w$ ,  $v_N = \Re w_N$ . We will prove that

$$v_N(\lambda) \to v(\lambda), \qquad N \to \infty,$$
 (4.3.1)

uniformly on each compact in  $\Omega$ . This relation implies that for every k the variation of  $\mathfrak{F}w_N$  along the loop  $\gamma_k$  converges to the variation of  $\mathfrak{F}w$  along  $\gamma_k$  as  $N \to \infty$ , what is equivalent to (4.1.3).

Define a class  $K_E$  of subharmonic functions  $u(\lambda)$ ,  $\lambda \in \mathbb{C}$ , nonnegative on E and such that

$$\limsup_{\lambda \to \infty} \frac{u(\lambda)}{|\lambda|^{\frac{1}{2}}} \le 1.$$

Similarly, we define the class  $K_{E^{(N)}}$ . As it follows from Levin's results (see Levin [13, Theorem 2.5]), the asymptotic (1.5.2) yields

$$v(\lambda) = \sup\{u(\lambda) : u \in K_E\}$$
$$v_N(\lambda) = \sup\{u(\lambda) : u \in K_{E(N)}\},$$

Since  $K_{E^{(N)}} \subset K_E$ , then  $v_N(\lambda) \leq v(\lambda)$ ,  $\lambda \in \mathbb{C}$ . By the theorem of uniqueness (Levin [13, Theorem 3.2]) every limit function for the normal family  $\{v_N(\lambda)\}$  coincides with  $v(\lambda)$ , i.e., (4.3.1) holds.

## §5. The proof of Approximation Theorem

Now everything is ready for the proofs of our results. In Sect. 5.1 we will show that the set of potentials  $Q(E^{(N)})$  is precompact in the topology of the uniform convergence on the real axis and that every limit (as  $N \to \infty$ ) potential belongs to Q(E). In Sect. 5.2 we will show that every potential from Q(E) is a uniform limit of potentials from  $Q(E^{(N)})$ ,  $N \to \infty$ . It will prove our Approximation Theorem. Simultaneously, our Main Theorem and Uniqueness Theorem will also be proved.

5.1. Let  $q_N \in Q(E^{(N)})$  be a sequence of finite-band potentials and let  $D^{(N)} \in \mathcal{D}(E^{(N)}) \hookrightarrow \mathcal{D}(E)$  be a corresponding sequence of spectral data. Since  $\mathcal{D}(E)$  is compact, we may assume that

$$D^{(N)} \to D$$
,  $N \to \infty$ .

We should prove that

$$q_N(t) \to q(t)$$
 uniformly on  $\mathbb{R}$ , (5.1.1)

and that

$$q \in Q(E). \tag{5.1.2}$$

Define a curve  $\{D^{(N)}(t)\}_{t\in\mathbb{R}}$  which solves the classical Jacobi inversion problem

$$A^{(N)}(D^{(N)}(t)) = A^{(N)}(D^{(N)}) + \delta(E^{(N)})t, \qquad t \in \mathbb{R}.$$
(5.1.3)

Taking into account that  $\mathcal{D}(E)$  is compact and that the functions  $A^{(N)}$  and A are continuous on  $\mathcal{D}(E)$ , we obtain by (4.1.2) that  $A^{(N)}$  converges to A uniformly on  $\mathcal{D}(E)$ . Hence, making use of (4.1.1) and (4.1.3), we may pass to the limit in the right-hand side of (5.1.3) for every  $t \in \mathbb{R}$ :

$$A^{(N)}(D^{(N)}(t)) \to A(D) + \delta(E)t, \qquad N \to \infty,$$

This relation together with compactness of  $\mathcal{D}(E)$  and  $\pi^*(E)$ , with the relation (4.1.2), and with Theorem A yield

$$D^{(N)}(t) \to D(t)$$
 uniformly on  $\mathbb{R}$ ,

where

$$A(D(t)) = A(D(0)) + \delta(E)t, \qquad D(0) = D,$$

$$D(t) = \bigcup_{j} (\lambda_{j}(t), \varepsilon_{j}(t)).$$

Now passing to the limit in the trace formula

$$q_N(t) = \sum_{j=1}^{N} (a_j + b_j - 2\lambda_j^{(N)}(t)),$$

we obtain (5.1.1), where the limit potential q(t) may be recovered by the trace formula (1.7.3).

The relation (5.1.1) implies (see for example Craig [6]) that the spectrum of the limit operator L[q] coincides with the set  $E = \bigcap_{N \ge 1} E^{(N)}$  and that

$$g_N(x, x; \lambda) \to g(x, x; \lambda), \qquad N \to \infty, \qquad x \in \mathbb{R}.$$
 (5.1.4)

uniformly with respect to  $\lambda$  lying on each compact in  $\Omega$ . Then (5.1.4) together with Lemma 5.2 from Craig [6] imply the reflectionless of q (1.3.1), what is equivalent to (1.2.4) by Appendix. So (5.1.2) is verified.

5.2. Now we show that an arbitrary potential  $q \in Q(E)$  may be approximated by potentials from  $Q(E^{(N)})$  uniformly on the real axis.

Let  $D = D(0) \in \mathcal{D}(E)$  be a divisor corresponding to the potential q. We denote, as before, by  $D^{(N)}$  a "projection" of D on  $\mathcal{D}(E^{(N)})$ . There is a finite-band potential  $q_N \in Q(E^{(N)})$  which corresponds to  $D^{(N)}$ . As we have proved in Sect. 5.1, the set of potentials  $\{q_N\}$  is precompact in the topology of the uniform convergence on the real axis and each limit potential belongs to Q(E). By (4.1.1) all limit potentials for  $\{q_N\}$  have the same divisor D of their spectral data and by the result proven in Sect. 3 every limit potential should coincide with q(t). So Approximation Theorem is proved.

Since Main Theorem and Uniqueness Theorem are true for the finite-band situation, our arguments together with Theorem A prove both of these two theorems as well.

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## **Appendix**

Reflectionless in the Craig sense implies pseudocontinuability of the Weyl functions

Let q(x) be a bounded continuous potential. We will show that the Craig condition

$$\Re g(x, x; \lambda + i0) = 0$$
 for a.e.  $\lambda \in E = \sigma(L[q])$  (A1)

implies that

$$m_{+}(\lambda + i0) = \overline{m_{-}(\lambda + i0)}$$
 for a.e.  $\lambda \in E$ . (A2)

Since the Weyl solutions  $\psi_{\pm}(x,\lambda)$  of the Sturm-Liouville equation can be represented in the form

$$\psi_{\pm}(x,\lambda) = \exp\left\{\int_0^x m_{\pm}(s,\lambda) \, ds\right\},\tag{A3}$$

where  $m_{\pm}(s, \lambda)$  are the Weyl functions of the potential q(x + s) (Titchmarsh [25]), the diagonal of the resolvent kernel  $g(x, x; \lambda)$  equals

$$g(x, x; \lambda) = \frac{\psi_+(x, \lambda)\psi_-(x, \lambda)}{m_-(\lambda) - m_+(\lambda)}$$

$$= \frac{1}{m_-(\lambda) - m_+(\lambda)} \exp\left\{ \int_0^x \left[ m_-(s, \lambda) + m_+(s, \lambda) \right] ds \right\}.$$

Consequently,

$$\frac{d}{dx}\log g(x, x, \lambda) = m_{-}(x, \lambda) + m_{+}(x, \lambda), \qquad \mathfrak{F}\lambda \neq 0,$$

whence

$$\frac{d}{dx}\arg g(x, x, \lambda) = \mathfrak{F}[m_{-}(x, \lambda) + m_{+}(x, \lambda)]. \tag{A4}$$

Denote the right-hand side of (A4) by  $u(x, \lambda)$  and set  $\lambda = t + i\varepsilon$ ,  $\varepsilon > 0$ .

Let  $\psi(t)$  be an arbitrary continuous function with a compact support and let  $0 \le x_1 < x_2 \le 1$  be arbitrary values. Integrating twice (A4) and changing the order of integration, we obtain

$$\int_{x_1}^{x_2} dx \int_{\varepsilon} \frac{\psi(t)}{1+t^2} u(x, t+i\varepsilon) dt$$

$$= \int_{\varepsilon} dt \int_{x_1}^{x_2} \frac{\psi(t)}{1+t^2} \frac{d}{dx} \arg g(x; x; t+i\varepsilon) dx$$

$$= \int_{\varepsilon} \frac{\psi(t)}{1+t^2} \left\{ \arg g(x_2, x_2; t+i\varepsilon) - \arg g(x_1, x_1; t+i\varepsilon) \right\} dt. \tag{A5}$$

Since  $0 < \arg g(x, x; t + i\varepsilon) \le \pi$ , we may pass to the limit in the right-hand side of (A5). Using condition (A1), we obtain

$$\lim_{\varepsilon \to 0} \int_{x_1}^{x_2} dx \int_{E} \frac{\psi(t)}{1+t^2} u(x, t+i\varepsilon) dt = 0.$$
 (A6)

Observe, that the internal integral in (A6) is bounded uniformly with respect to  $x \in [0, 1]$  and  $\varepsilon \in [0, 1/2]$ :

$$\left| \int_{E} \frac{\psi(t)}{1+t^{2}} u(x, t+i\varepsilon) dt \right|$$

$$\leq M_{\psi} \int_{E} \frac{dt}{1+t^{2}} \left\{ \mathfrak{F}m_{+}(x, t+i\varepsilon) - \mathfrak{F}m_{-}(x, t+i\varepsilon) \right\}$$

$$\leq M_{\psi} \int_{E} \frac{dt}{1+t^{2}} \mathfrak{F} \frac{1}{g(x, x; t+i\varepsilon)}$$

$$\leq M_{\psi} C \sup_{x \in [0,1]} \frac{1}{|g(x, x; i)|}.$$

The latter supremum is finite since the function  $x \mapsto g(x, x; i)$  is continuous. It allows us to rewrite (A6) in the form

$$\int_{x_1}^{x_2} dx \left\{ \lim_{\varepsilon \to 0} \int_E \frac{\psi(t)}{1 + t^2} u(x, t + i\varepsilon) dt \right\} = 0.$$
 (A7)

Now we consider a family of charges

$$\rho_{\varepsilon}(x, dt) = u(x, t + i\varepsilon) \frac{\chi_{E}(t)}{1 + t^{2}} dt, \qquad x \in [0, 1], \qquad \varepsilon \in [0, 1/2],$$

where  $\chi_E$  is the indicator-function of the set E. The family  $\rho_{\varepsilon}(x, dt)$  converges weakly to a certain charge  $\rho_0(x, dt)$ , as  $\varepsilon \to 0$ . Together with (A7) it implies that

$$\int_{x_1}^{x_2} dx \int_{E} \psi(t) \rho_0(x, dt)$$

$$= \int_{x_1}^{x_2} dx \left\{ \lim_{\epsilon \to 0} \int_{E} \psi(t) \rho_{\epsilon}(x, dt) \right\} = 0.$$

Since  $x_1$  and  $x_2$  are arbitrary values, we conclude that

$$\int_{E} \psi(t) \rho_{0}(x, dt) = 0 \quad \text{for a.e. } x \in [0, 1].$$

The absolutely continuous part of the charge  $\rho_0$  equals

$$\mathfrak{F}[m_{-}(x, t+i0) + m_{+}(x, t+i0)] \frac{dt}{1+t^2},$$

whence for a.e.  $x \in [0, 1]$ 

$$\mathfrak{F}[m_{-}(x, t+i0) + m_{+}(x, t+i0)] = 0 \qquad \text{for a.e. } t \in E.$$
 (A8)

Further, (A1) and the equation

$$\frac{1}{g(x, x; \lambda)} = m_{-}(x, \lambda) - m_{+}(x, \lambda)$$

imply that for every  $x \in [0, 1]$ 

$$\Re[m_{-}(x, t+i0) - m_{+}(x, t+i0)] = 0$$
 for a.e.  $t \in E$ . (A9)

Comparing (A8) and (A9), we obtain that for some  $x \in (0, 1)$ 

$$m_{+}(x, t + i0) = \overline{m_{-}(x, t + i0)}$$
 for a.e.  $t \in E$ .

Note, that by virtue of (A3)

$$\begin{split} m_{\pm}(x, \lambda) &= \frac{d}{dx} \log \psi_{\pm}(x, \lambda) \\ &= \frac{C'(x, \lambda) + m_{\pm}(\lambda)S'(x, \lambda)}{C(x, \lambda) + m_{\pm}(\lambda)S(x, \lambda)}, \end{split}$$

where all four functions C, S, C' and S' are real as  $\lambda \in E$ . Thus, we have obtained the condition (A2).

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