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Almost periodic Sturm-Liouville operators with Cantor homogeneous spectrum

MIKHAIL SODIN AND PETER YUDITSKII

To the memory of B. Ya. Levin (1906–1993) who was a teacher of our teachers and who gave us so much

Being based on the infinite dimensional Jacobi inversion found earlier, we establish the direct generalization of the well-known properties of finite-band Sturm-Liouville operators in the case of operators with a homogeneous and, generally speaking, Cantor-type spectrum, and with pseudocontinuable Weyl functions.

In our investigations the group of unimodular characters of the fundamental group of the resolvent set plays a role of the isospectral manifold of the operator. The generalized Abel map conjugates the nonlinear evolution of spectral data with a linear motion on this torus. In particular, the operators we consider turn out to be uniformly almost periodic.

§1. Statement of main results

1.1. Consider the Sturm-Liouville equation

$$L[q]y = -y'' + q(x)y = \lambda y, \quad -\infty < x < \infty, \tag{1.1.1}$$

with a real bounded continuous potential $q(x)$. We denote by $C(x, \lambda)$ and $S(x, \lambda)$ the fundamental solutions of Equations (1.1.1) satisfying initial conditions

$$C(0, \lambda) = S'(0, \lambda) = 1, \qquad C'(0, \lambda) = S(0, \lambda) = 0.$$

By virtue of the classical Weyl theorem (see, for example, Titchmarsh [25, Ch. 2]), for each nonreal λ Equation (1.1.1) has solutions

$$\psi_\pm(x, \lambda) = C(x, \lambda) + m_\pm(\lambda)S(x, \lambda), \quad \text{such that } \psi_\pm \in L^2(\mathbb{R}_\pm).$$

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The functions $m_\pm$ are holomorphic outside the real axis, $m_\pm(\bar{\lambda}) = \overline{m_\pm(\lambda)}$ and

$$\mathfrak{I}m_+(\lambda)/\mathfrak{I}\lambda > 0, \qquad \mathfrak{I}m_-(\lambda)/\mathfrak{I}\lambda < 0.$$

The functions $m_\pm(\lambda)$ are called *the Weyl functions*; they are defined uniquely by virtue of the boundedness from below of the potential $q(x)$.

We denote by $g(x, y; \lambda)$ the Green function of $L[q]$ which is defined as the kernel of the resolvent $R_\lambda = (L[q] - \lambda)^{-1}$. Then (see Titchmarsh [25, Ch. 2])

$$\frac{1}{g(x, x; \lambda)} = \frac{m_-(\lambda) - m_+(\lambda)}{\psi_+(x, \lambda)\psi_-(x, \lambda)}. \tag{1.1.2}$$

Without loss of generality, we assume that the origin is the lower bound of the spectrum of $L[q]$.

1.2. DEFINITION. Let E be a closed set

$$E = [0, \infty) \setminus \bigcup_{j \geq 1} (a_j, b_j) \tag{1.2.1}$$

satisfying the conditions:

(i) E is homogeneous (Carleson [5], Jones and Marshall [10]), i.e., there is an $\varepsilon > 0$ such that for all $\lambda \in E$ and all $\delta > 0$

$$|(\lambda - \delta, \lambda + \delta) \cap E| \geq \varepsilon\delta; \tag{1.2.2}$$

(ii) the sum of lengths of gaps in E is finite:

$$\sum_{j \geq 1} (b_j - a_j) < \infty. \tag{1.2.3}$$

A potential q belongs to the class $Q(E)$ if the spectrum of $L[q]$ coincides with E and the Weyl functions are pseudocontinuable:

$$m_+(\lambda + i0) = m_-(\lambda - i0) \quad \text{for a.e. } \lambda \in E. \tag{1.2.4.}$$

1.3. Let us stop for a moment at this definition and make some comments.

First, we note that the homogeneity condition (1.2.2) can be written in the equivalent form which looks slightly more invariant: there is an $\varepsilon > 0$ such that for all $\lambda \in E$ and all $\delta > 0$

$$\int_{(\lambda - \delta, \lambda + \delta) \cap E} \frac{dt}{1 + t^2} \geq \varepsilon \int_{(\lambda - \delta, \lambda + \delta)} \frac{dt}{1 + t^2}.$$

It will allow us to apply later function theory results obtained in [24].

Equations (1.1.2) and (1.2.4) imply that the potential $q \in Q(E)$ is reflectionless in the sense of Craig [6]: for every $x \in \mathbb{R}$

$$\Re g(x, x; \lambda \pm i0) = 0 \quad \text{for a.e. } \lambda \in E. \tag{1.3.1}$$

It may be proved (see Appendix) that, vice versa, the Craig condition (1.3.1) implies condition (1.2.4).

If there is a finite number of gaps in E, then (1.2.4) implies that there is a rational function $m(\lambda)$ on the hyperelliptic Riemann surface $\mathscr{R}_E$ of the function

$$\sqrt{\frac{1}{\lambda}\prod_{j\geq 1}\frac{\lambda - a_j}{\lambda - b_j}}$$

such that $m_+(\lambda) = m(\lambda)$ and $m_-(\lambda) = m(\lambda^*)$ where * means the involution of the sheets of the surface $\mathscr{R}_E$. Hence in this case $Q(E)$ coincides with the well-known class of finite-band Sturm-Liouville operators (see, e.g., McKean and van Moerbeke [19], Dubrovin, Matveev and Novikov [7], Moser [22], and the recent book Belokolos, Bobenko, Enol'skii, Its, and Matveev [4b]). With the special choice of E the class $Q(E)$ also contains infinite-band periodic potentials investigated by Marchenko and Ostrovskii [17, 18] (see also Marchenko [16]), by McKean and Trubowitz [20, 21], and by Garnett and Trubowitz [9]. Such potentials are connected with hyperelliptic Riemann surfaces of infinite genus.

From the other hand, condition (1.2.4) naturally arises in the spectral theory of ergodic (or random) Sturm-Liouville operators due to well-known Kotani's theorem (see Pastur and Figotin [23a] or Carmona and Lacroix [5a], see also Belokolos, Bobenko, Enol'skii, Its, and Matveev [4b, Sect. 8.1]). Putting together with the Pastur-Ishii result, it asserts that if L is an ergodic Sturm-Liouville operator with the density of an absolutely continuous spectrum positive a.e. on a Borelian set $A \in \mathbb{R}$, then condition (1.2.4) holds a.e. on A. That is, *all ergodic (particularly, almost periodic) Sturm-Liouville operators with a homogeneous spectrum E and with the density of absolutely continuous spectrum positive a.e. on E belong to the class* $Q(E)$.

1.4. Set

$$E^{(N)} = [0, \infty)\setminus\bigcup_{j=1}^{N}(a_j, b_j).$$

APPROXIMATION THEOREM. *For a fixed homogeneous set E and for each sequence of finite-band potentials* $q_N \in Q(E^{(N)})$, $N = 1, 2, \ldots$, *there is a subsequence which converges uniformly on the whole axis* $\mathbb{R}$ *and the set of all limit potentials coincides with* $Q(E)$.

In particular, $Q(E)$ is compact in the topology of uniform convergence on the whole axis, and since finite-band potentials are uniformly almost periodic (see, for example, Levitan [14] or Moser [22]), we obtain that every potential of the class $Q(E)$ is uniformly almost periodic.

Under more restrictive conditions imposed on the set E, the almost periodicity of potentials of the class $Q(E)$ has been proved in Levitan [14, 15], Pastur and Tkachenko [23], Egorova [8, 8a].

1.5. Let $\Omega = \mathbb{C} \backslash E$ be the resolvent set of the operator $L[q]$, and let $\pi(\Omega) = \pi(\Omega, -1)$ be the fundamental group of Ω with the marked point $z = -1$ (in fact, its choice is inessential). By $\pi^*(\Omega)$ we denote the group of unimodular characters of $\pi(\Omega)$ endowed with the topology dual to the discrete one on $\pi(\Omega)$. Further, we will use the additive form of notations for the compact abelian group $\pi^*(\Omega)$:

$$\pi^*(\Omega) = \{\alpha(\gamma) \in \mathbb{R} \bmod \mathbb{Z} \colon \gamma \in \pi(\Omega), \alpha(\gamma_1 \circ \gamma_2) = \alpha(\gamma_1) + \alpha(\gamma_2)\}.$$

The group $\pi^*(\Omega)$ is a finite-dimensional torus if Ω is finitely connected, and is an infinite-dimensional torus if Ω is infinitely connected. We will use this torus for a parameterization of operators $L[q]$, $q \in Q(E)$ with a given E.

Let us consider the conformal map of the upper half-plane onto the slitted quarter-plane (see Figure 1):

$$w \colon \mathbb{C}_+ \to \{\Re w < 0, \Im w > 0\} \backslash \bigcup_{j \geq 1} \{\Im w = \pi i \delta_j, -h_j \leq \Re w \leq 0\}, \tag{1.5.1}$$

normalized by the conditions $w(0) = 0$ and

$$w(-\lambda) \sim -\lambda^{1/2}, \qquad \lambda \to \infty. \tag{1.5.2}$$

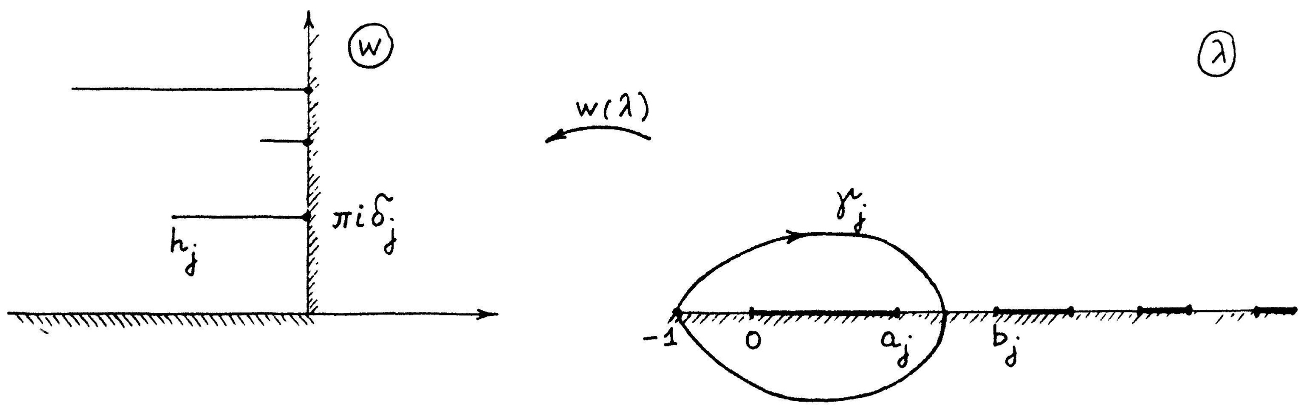

Figure 1

It maps the spectrum E onto the imaginary semi-axis and the gaps in E onto the slits. The normalization (1.5.2) is possible by virtue of condition (1.2.3) and well-known results by Akhiezer and Levin [4, p. 127] (see also Levin [13]). Such maps were introduced into the spectral theory of Sturm-Liouville operators by Marchenko and Ostrovskii [17, 18] (see also Marchenko [16]). Later, they were used by Garnett and Trubowitz [9], and by Pastur and Tkachenko [23].

Let us continue the function $w(\lambda)$ analytically across all intervals (a_j, b_j) and $(-\infty, 0)$ into the lower half-plane. We obtain a multivalued function on Ω whose real part is single-valued. The ramification of $\mathfrak{J}w$ generates a character $\delta = \delta(E) \in \pi^*(\Omega)$. Namely, $\delta(\gamma_j) = \delta_j$, where numbers δ_j are defined in (1.5.1) and $\{\gamma_j\} \subset \pi(\Omega)$ being a system of generators of the group $\pi(\Omega)$, consisting of loops γ_j, which begin and end at $\lambda = -1$, and contain $E_j = E \cap [b_j, \infty)$ inside and $E \backslash E_j$ outside (see Figure 1).

Now we are able to formulate our main result.

1.6. MAIN THEOREM. *There exists a homeomorphism between the compacts $Q(E)$ and $\pi^*(\Omega)$ conjugating the shift of the potential $q(x) \mapsto q(x+t)$ and the linear motion $\alpha \to \alpha + \delta t$ on $\pi^*(\Omega)$, where $\delta = \delta(E)$.*

COROLLARY. *Every potential of the class $Q(E)$ is a uniform almost periodic function whose frequency module is spanned by $\{\delta_j\}$.*

In the finite-band case $w(\lambda)$ coincides with the normed abelian integral of the second kind with a pole at infinity, and $\pi^*(\Omega)$ is a finite-dimensional torus isomorphic to the real part of the Jacobian of the corresponding hyperelliptic Riemann surface $\mathcal{R}_E$. In this case our Main Theorem is a restatement of the well-known results due to Dubrovin, Matveev and Novikov [7], and McKean and van Moerbeke [19] (see also Moser [22]). Such results are going back to works by Akhiezer originally published in the early sixties in a series of papers in Soviet Math. Doklady and in Proceedings of the Kharkov Math. Society. Later, they were summed up in Akhiezer [1–3] (see also Akhiezer and Rybalko [4a]). In fact, in his papers Akhiezer considered only operators acting on the semi-axis ($\mathbb{R}_+$ or $\mathbb{Z}_+$).

1.7. Let us introduce a class of divisors

$$\mathscr{D}(E) = \left\{ D = \bigcup_{j \geq 1} (\lambda_j, \varepsilon_j) : \lambda_j \in [a_j, b_j], \varepsilon_j = \pm 1 \right\}.$$

If λ_j coincides with one of the points a_j, b_j we arrange $(\lambda_j, +1) \equiv (\lambda_j, -1)$. We endow $\mathscr{D}(E)$ with the compact topology of the product of circles I_j^2, where I_j^2 is a two-sheeted covering of $I_j = [a_j, b_j]$ with ends identified.

Following Craig [6], we associate with every potential $q \in Q(E)$ the collection of spectral data $D \in \mathcal{D}(E)$ of the operator $L[q]$. The function $g(0, 0; \lambda)$ is a Nevanlinna function (it preserves the upper half-plane) and by virtue of (1.3.1) its multiplicative representation may be rewritten in the form

$$g(0, 0; \lambda) = \frac{1}{2\sqrt{-\lambda}} \prod_{j \geq 1} \frac{\lambda - \lambda_j}{\sqrt{(a_j - \lambda)(b_j - \lambda)}}, \qquad \lambda_j \in [a_j, b_j]. \tag{1.7.2}$$

(see, for example, Appendix in Krein and Nudelman [11] or Craig [6]). By (1.1.2) with $x = 0$ we obtain

$$g(0, 0; \lambda) = (m_-(\lambda) - m_+(\lambda))^{-1}, \tag{1.7.2}$$

and if $\lambda_j \in (a_j, b_j)$ then λ_j is a pole of one of the functions $m_\pm(\lambda)$ (i.e., λ_j is an eigenvalue of $L[q]$ acting on one of the semi-axes $\mathbb{R}_\pm$). If λ_j was a pole of both of the functions $m_\pm(\lambda)$ then λ_j would belong to the spectrum of $L[q]$ what is impossible. Thus, we may define $\varepsilon_j = \pm 1$ depending on which of the functions $m_\pm(\lambda)$ has a pole at $\lambda_j \in (a_j, b_j)$, and the map $Q(E) \to \mathcal{D}(E)$ is well-defined.

The shift of the potential $q(x) \mapsto q(x + t)$ defines a continuous curve $\{D(t)\}_{t \in \mathbb{R}} \subset \mathcal{D}(E)$, $D(0) = D$, and the potential $q(t)$ can be recovered by this curve using the trace formula proved for this class by Craig [6]

$$q(t) = \sum_{j \geq 1} (a_j + b_j - 2\lambda_j(t)), \tag{1.7.3}$$

where $\lambda_j(t) \in [a_j, b_j]$ correspond to the divisor $D(t)$.

1.8. UNIQUENESS THEOREM. *The map $Q(E) \to \mathcal{D}(E)$ is a homeomorphism of the compacts $Q(E)$ and $\mathcal{D}(E)$.*

This theorem establishes a continuous parameterization of operators of the class $Q(E)$ by divisors from $\mathcal{D}(E)$. In the periodic case this parameterization was found in Marchenko and Ostrovskii [17]. We should also mention that a bijection between sets $Q(E)$ and $\mathcal{D}(E)$ was established in Craig [6] under certain conditions imposed on the spectrum E which seem to be more restrictive than the homogeneity; on the other hand in that paper the set of potentials $Q(E)$ was endowed with a weaker topology of the uniform convergence on each compact subset of $\mathbb{R}$.

1.9. The key to the proofs of our theorems is the fact that the generalized infinite dimensional Abel map $A: \mathcal{D}(E) \to \pi^*(\Omega)$, being a homeomorphism of these compacts, linearize, as in the finite-band case, the curve $D(t)$ mapping it onto the line $\alpha + \delta(E)t$ on $\pi^*(\Omega)$.

Here is a plan of the rest of the paper. In Sect. 2 we remind the definition of the Abel map from Sodin and Yuditskii [24]. In Sect. 3 we prove a "half" of the Uniqueness Theorem, namely we prove that the collection of spectral data D of the operator $L[q]$ defines the potential q uniquely, i.e. the map $Q(E) \to \mathscr{D}(E)$ is injective. In Sect. 4 we bring some auxiliary facts concerning a "finite-band approximation" of $\mathscr{D}(E)$ by $\mathscr{D}(E^{(N)})$ and of $\pi^*(\Omega)$ by $\pi^*(\Omega^{(N)})$, where $\Omega^{(N)} = \bar{\mathbb{C}}\backslash E^{(N)}$; and in Sect. 5 we prove Approximation Theorem. Simultaneously, our Main Theorem and Uniqueness Theorem will be also proved.

§2. The Abel map $A\colon \mathscr{D}(E) \to \pi^*(\Omega)$

2.1. Let $\omega(\lambda, F)$ be the harmonic measure of a set $F \subset E$ at $\lambda \in \Omega$ with respect to the domain Ω. The Abel map was defined in [24] as

$$A(D)[\gamma_k] = \frac{1}{2}\sum_j \varepsilon_j \int_{\lambda_j}^{b_j} \omega(d\lambda, E_k) \bmod \mathbb{Z}, \qquad k = 1, 2, \ldots . \tag{2.1}$$

where $D = \bigcup_j (\lambda_j, \varepsilon_j) \in \mathscr{D}(E)$, $E_k = E \cap [b_k, \infty)$, and $\{\gamma_k\}$ being the system of loops generating the group $\pi(\Omega)$ (see Sect. 1.5). As it was checked in this paper, the homogeneity of E yields the convergence of the series in the right-hand side of (2.1) (see also Sect. 4.2 below). This definition of the Abel map agrees with the classical one in the finite-band case.

2.2. In the just mentioned paper we have proved

THEOREM A. *If a set E is homogeneous, then the Abel map gives a homeomorphism between the compacts $\mathscr{D}(E)$ and $\pi^*(\Omega)$.*

2.3. In the sequel, we denote by $G(z, z_0)$ the Green function (for the usual Laplacian) of the domain Ω with the pole at $z = z_0$, and we denote the complex Green function of Ω with a zero at $z = z_0$ by $\Phi(z, z_0) = \exp[-G(z, z_0) - i * G(z, z_0)]$. The function $\Phi(z, z_0)$ is character-automorphic (Widom [26]): it has a single-valued modulus and after analytic continuation along the loop γ_k the variation of its argument $- *G(z, z_0)$ equals

$$-2\pi[\omega(z_0, E_k) - \mathrm{Ind}_{\gamma_k}(z_0)].$$

Hence a character $\alpha = \alpha[\Phi(\cdot, z_0)]$ associated with $\Phi(\cdot, z_0)$ equals

$$\alpha[\Phi(\cdot, z_0)](\gamma_k) = -\omega(z_0, E_k) \bmod \mathbb{Z}, \qquad k = 1, 2, \dots . \tag{2.3.1}$$

§3. An operator $L[q]$, $q \in Q(E)$, is uniquely defined by the collection of its spectral data D

3.1. Now, let us consider the Weyl functions $m_\pm$. Since m_+ and $-m_-$ preserve the upper half-plane, these functions have Nevanlinna representations as Cauchy integrals of nonnegative measured $d\sigma_\pm$. These measures are spectral measures of the restrictions of the operator $L[q]$ on the semi-axis $\mathbb{R}_\pm$ correspondingly. By the Marchenko uniqueness theorem (see, for example, Levitan [14]) these measures define uniquely the potential $q(x)$. So we have to prove that these measures in turn are defined by the divisor D.

3.2. LEMMA. *Let $q \in Q(E)$. Then the Nevanlinna measures $d\sigma_\pm$ of the functions m_+ and $-m_-$ are defined uniquely by the divisor $D = \bigcup_j (\lambda_j, \varepsilon_j)$.*

Proof. In the proof we will use the relations

$$-\frac{1}{g(0, 0)} = m_+ - m_-, \tag{3.2.1}$$

$$m_+(\lambda + i0) = \overline{m_-(\lambda + i0)} \qquad \text{for a.e. } \lambda \in E. \tag{3.2.2}$$

Set

$$\tilde{m}_+(\lambda) = m_+(\lambda) - m_+(-1),$$
$$\tilde{m}_-(\lambda) = m_-(\lambda) - m_+(-1),$$

and consider the product $F(\lambda) = \tilde{m}_+(\lambda)\tilde{m}_-(\lambda)$. The argument $\arg F(\lambda)$ varies in the upper half-plane from $-\pi$ to π. It follows from (3.2.2) that

$$\arg F(\lambda + i0) = 0 \qquad \text{for a.e. } \lambda \in E.$$

Since both functions $\tilde{m}_\pm(\lambda)$ are real in the gaps, $\arg F(\lambda)$ takes there values 0 and $\pm\pi$.

Now, let us look at the behaviour of $\tilde{m}_\pm(\lambda)$ in the gap $(-\infty, 0)$. The function $\tilde{m}_+(\lambda)$ increases there (because its Nevanlinna measure does not support this gap), consequently

$$\tilde{m}_+(\lambda) \begin{cases} >0, & \lambda \in (-1, 0), \\ <0, & \lambda \in (-\infty, -1). \end{cases}$$

Since $g(0, 0, \lambda) > 0$ as $\lambda \in (-\infty, 0)$ (it follows, for example, from representation (1.7.1)), we have

$$\tilde{m}_-(\lambda) = \tilde{m}_+(\lambda) + \frac{1}{g(0, 0, \lambda)} > \tilde{m}_+(\lambda), \qquad \lambda \in (-\infty, 0).$$

In addition, $\tilde{m}_-(\lambda)$ is decreasing there and hence $\tilde{m}_-(\lambda) > 0$, $\lambda \in (-\infty, 0)$. Thus,

$$\arg F(\lambda) = \begin{cases} \pi, & \lambda \in (-\infty, -1), \\ 0, & \lambda \in (-1, 0). \end{cases}$$

Similarly, we establish that only one of the functions $\tilde{m}_\pm(\lambda)$ may have zero $\lambda_j^{(1)}$ on (a_j, b_j), and we set $\varepsilon_j^{(1)} = +1$ if $\lambda_j^{(1)}$ is zero of $\tilde{m}_+(\lambda)$ and $\varepsilon_j^{(1)} = -1$ if $\lambda_j^{(1)}$ is zero of $\tilde{m}_-(\lambda)$ (see Figure 2).

Taking into account the fact that $\log F(\lambda)$ is represented on $\mathfrak{J}\lambda > 0$ by the Cauchy integral of the boundary values of $\arg F(\lambda + i0)$, we obtain a multiplicative representation

$$F(\lambda) = C^2(\lambda + 1) \prod_j \frac{\lambda - \lambda_j^{(1)}}{\lambda - \lambda_j}, \qquad \lambda_j^{(1)} \in [a_j, b_j]. \tag{3.2.3}$$

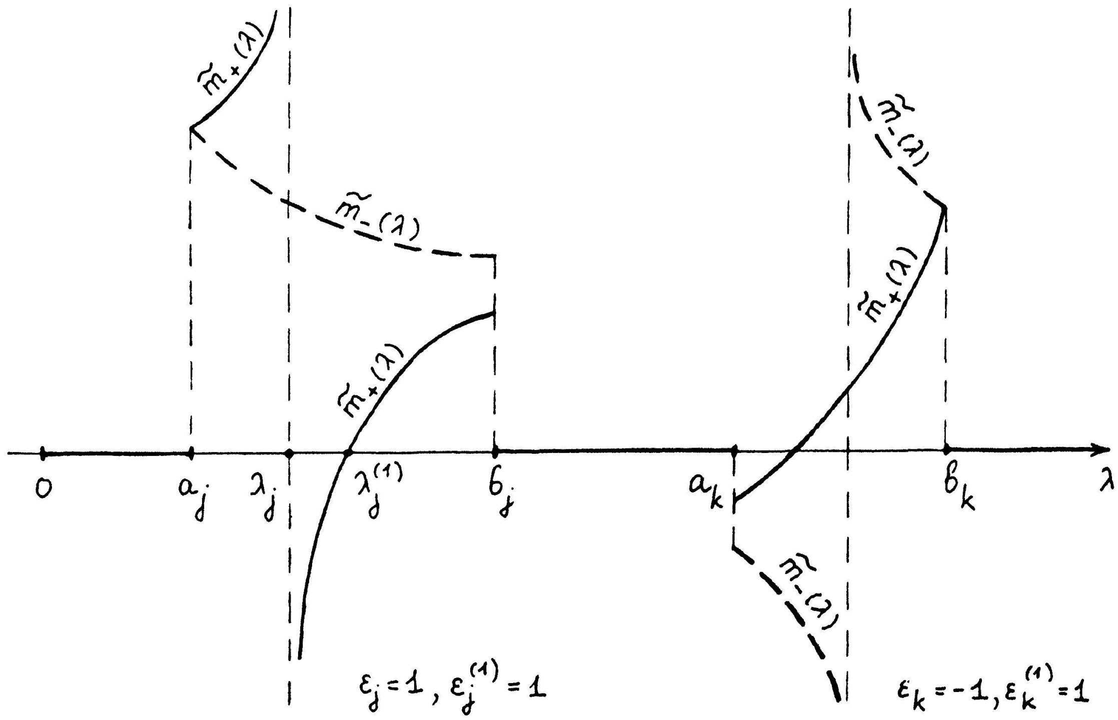

Figure 2

Simultaneously, we have defined the divisor

$$D^{(1)} = \bigcup_j (\lambda_j^{(1)}, \varepsilon_j^{(1)}), \qquad D^{(1)} \in \mathscr{D}(E).$$

Now we will use some arguments from Sodin and Yuditskii [24]. By Theorem D from this paper, the functions $F(\lambda)$ and $\tilde{m}_{\pm}(\lambda)$ are functions of bounded type on $\Omega = \mathbb{C}\backslash E$ (it means that they are represented as a quotient of bounded functions), and, moreover, these functions have no singular inner factors (the latter means that the logarithm of modulus is represented in E as a sum of the Green potential and the Poisson integral of its boundary values). The factorization $F(\lambda) = \tilde{m}_+(\lambda)\tilde{m}_-(\lambda)$ should be considered as a representation of the positive function on E, given by (3.2.3), as a square of modulus of boundary values of the function $\tilde{m}_+(\lambda)$ which has no singular inner component and whose zeros and poles are known. It allows us to write a multiplicative representation

$$\tilde{m}_+(\lambda) = C\sqrt{(\lambda+1)\Phi(\lambda, -1)}\sqrt{\prod_j \frac{\lambda - \lambda_j^{(1)}}{\lambda - \lambda_j} \frac{\Phi(\lambda, \lambda_j^{(1)})^{\varepsilon_j^{(1)}}}{\Phi(\lambda, \lambda_j)^{\varepsilon_j}}} \tag{3.2.4}$$

(see details in the cited paper by the authors).

The left-hand side of (3.2.4) is single-valued on Ω, hence, the right-hand side also should be single-valued. Evaluating the character of the right-hand side of (3.2.4) and making use of (2.3.1), we obtain

$$0 = -\frac{1}{2}\omega(-1, E_k) - \frac{1}{2}\sum_j [\varepsilon_j^{(1)}\omega(\lambda_j^{(1)}, E_k) - \varepsilon_j\omega(\lambda_j, E_k)],$$

or

$$A(D^{(1)}) = A(D) + \tau, \tag{3.2.5}$$

where $\tau(\gamma_k) = -\frac{1}{2}\omega(-1, E_k) \bmod \mathbb{Z}$, $k = 1, 2, \ldots$ defines a fixed character from $\pi^*(\Omega)$.

By Theorem A (Sect. 2.2) Equation (3.2.5) implies that the divisor $D^{(1)} \in \mathscr{D}(E)$ is defined uniquely by the divisor D.

A constant C is evaluated from the condition

$$-\frac{1}{g(0, 0; -1)} = -\tilde{m}_-(-1).$$

Thus, the functions $\tilde{m}_{\pm}$ are determined uniquely by the divisor D and the Lemma is proved.

§4. "Finite-band" approximation of D(E) and $\pi^*(\Omega)$

4.1. As before, we set

$$E^{(N)} = [0, \infty)\setminus\bigcup_{j=1}^{N} (a_j, b_j), \qquad \Omega^{(N)} = \mathbb{C}\setminus E^{(N)}.$$

Let $\{\gamma_j\}$ be a system of generators of the fundamental group $\pi(\Omega)$ introduced in Sect. 1.5. Note that $\{\gamma_j\}_{j\le N}$ is a system of generators of $\pi(\Omega^{(N)})$ and hence an arbitrary character $\alpha^{(N)} \in \pi^*(\Omega^{(N)})$ may be extended to a character $\alpha \in \pi^*(\Omega^{(N)})$ by setting

$$\alpha(\gamma_j) = \begin{cases} \alpha^{(N)}(\gamma_j), & j \le N, \\ 0, & j > N. \end{cases}$$

It defines a continuous embedding $\pi^*(\Omega^{(N)}) \hookrightarrow \pi^*(\Omega)$. And, vice versa, by $\alpha^{(N)}$ we denote a "projection" of the character $\alpha \in \pi^*(\Omega)$ on $\pi^*(\Omega^{(N)})$ which is defined as

$$\alpha^{(N)}(\gamma_j) = \alpha(\gamma_j), \qquad 1 \le j \le N.$$

Similarly, every divisor $D^{(N)} \in \mathscr{D}(E^{(N)})$ may be complemented to a divisor $D = D^{(N)} \cup \bigcup_{j\ge N}(b_j)$, $D \in \mathscr{D}(E)$, and, vice versa, for a given divisor $D \in \mathscr{D}(E)$ we denote its "projection" onto $\mathscr{D}(E^{(N)})$ by $D^{(N)}$. Then

$$\alpha^{(N)} \to \alpha, \qquad D^{(N)} \to D \qquad \text{as } N \to \infty, \tag{4.1.1}$$

as it follows directly from the definition of convergence in $\pi^*(\Omega)$ and $\mathscr{D}(E)$.

We denote by $A^{(N)}: \mathscr{D}(E^{(N)}) \to \pi^*(\Omega^{(N)})$ the classical Abel map which due to above may be considered as a map $A^{(N)}: \mathscr{D}(E) \to \pi^*(\Omega)$. Our next goal is to prove that

$$A^{(N)}(D) \to A(D) \qquad \text{for every } D \in \mathscr{D}(E), \tag{4.1.2}$$

and

$$\delta(E^{(N)}) \to \delta(E), \tag{4.1.3}$$

as $N \to \infty$.

4.2. *Proof of* (4.1.2). We should prove that for every k it holds

$$\sum_{j\le N} \varepsilon_j \int_{\lambda_j}^{b_j} \omega(d\lambda, E_k^{(N)}, \Omega^{(N)}) \to \sum_j \varepsilon_j \int_{\lambda_j}^{b_j} \omega(d\lambda, E_k, \Omega) \tag{4.2.1}$$

In the proof of this relation we will use two consequences of the homogeneity of E which are pertaining to potential theory. At first, we will use the regularity of E with respect to the Dirichlet problem on $\Omega = \mathbb{C}\backslash E$. Landkof [12]. It follows, for example, from the Wiener criterion. Secondly, we will use that if E is homogeneous then $\mathbb{C}\backslash E = \Omega$ satisfies the Parreau-Widom condition

$$\sum_{\{c_j:\, \nabla G(c_j, \lambda_0) = 0\}} G(c_j, \lambda_0) < \infty \tag{4.2.2}$$

(see Jones and Marshall [10]).

First of all, we will show that for every $\lambda \in \Omega$ and every k

$$\omega(\lambda, E_k^{(N)}, \Omega^{(N)}) \to \omega(\lambda, E_k, \Omega), \qquad N \to \infty. \tag{4.2.3}$$

To this end, we consider a harmonic function on $\Omega^{(N)}$

$$\omega(\lambda, E_k^{(N)}, \Omega^{(N)}) - \omega(\lambda, E_k, \Omega), \qquad N > k,$$

and remark that the homogeneity of E implies the regularity of E with respect to the Dirichlet problem on $\mathbb{C}\backslash E$. Hence, by the regularity of E,

$$\max_{\lambda \in E^{(N)}} \left|\omega(\lambda, E_k^{(N)}, \Omega^{(N)}) - \omega(\lambda, E_k, \Omega)\right| \to 0, \qquad N \to \infty$$

(see, for example, Landkof [12]).

Now, in order to prove (4.2.1), we will show that the series in the left-hand side of (4.2.1) is majorized by a converging series consisting of positive terms which do not depend on N.

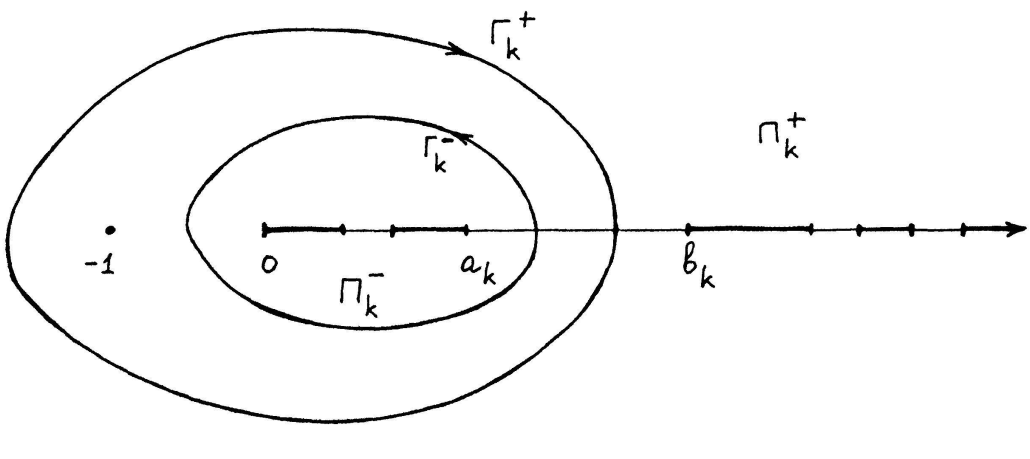

Figure 3

Let Π_k^+ be a connected vicinity of $[b_k, \infty)$ on the Riemann sphere and let Π_k^- be a connected vicinity of $[0, a_k]$ (see Figure 3). We assume that $\Pi_k^+ \cap \Pi_k^- = \varnothing$ and that $-1 \in \mathbb{C}\backslash(\Pi_k^+ \cup \Pi_k^-)$. Denote $\Gamma_k^\pm = \partial \Pi_k^\pm$ and set

$$\mu_k^\pm = \min_{\lambda \in \Gamma_k^\pm} G(\lambda, -1),$$

where $G(\lambda, -1)$ is the Green function of Ω with a pole at $\lambda = -1$. By the Maximum Principle applied in $\Pi_k^\pm \cap \Omega^{(N)}$, we obtain

$$\mu_k^- \omega(\lambda, E_k^{(N)}, \Omega^{(N)}) \le G(\lambda, -1), \qquad \lambda \in \Pi_k^-,$$

$$\mu_k^+ (1 - \omega(\lambda, E_k^{(N)}, \Omega^{(N)})) \le G(\lambda, -1), \qquad \lambda \in \Pi_k^+,$$

Therefore, the convergent series

$$\frac{1}{\min(\mu_k^+, \mu_k^-)} \sum_{\{c_j:\, \nabla G(c_j, -1) = 0\}} G(c_j, -1)$$

is a majorant we were looking for.

4.3. *Proof of* (4.1.3). For this purpose, we will use the connection between conformal maps onto comb-like domains (as in (1.5.1)) and subharmonic majorants (Levin [13]). Denote by w_N the conformal map (1.5.1) corresponding to the set $E^{(N)}$ and put $v = \Re w$, $v_N = \Re w_N$. We will prove that

$$v_N(\lambda) \to v(\lambda), \qquad N \to \infty, \tag{4.3.1}$$

uniformly on each compact in Ω. This relation implies that for every k the variation of $\Im w_N$ along the loop γ_k converges to the variation of $\Im w$ along γ_k as $N \to \infty$, what is equivalent to (4.1.3).

Define a class K_E of subharmonic functions $u(\lambda)$, $\lambda \in \mathbb{C}$, nonnegative on E and such that

$$\limsup_{\lambda \to \infty} \frac{u(\lambda)}{|\lambda|^{\frac{1}{2}}} \le 1.$$

Similarly, we define the class $K_{E^{(N)}}$. As it follows from Levin's results (see Levin [13, Theorem 2.5]), the asymptotic (1.5.2) yields

$$v(\lambda) = \sup\{u(\lambda) : u \in K_E\}$$

$$v_N(\lambda) = \sup\{u(\lambda) : u \in K_{E^{(N)}}\},$$

Since $K_{E^{(N)}} \subset K_E$, then $v_N(\lambda) \le v(\lambda)$, $\lambda \in \mathbb{C}$. By the theorem of uniqueness (Levin [13, Theorem 3.2]) every limit function for the normal family $\{v_N(\lambda)\}$ coincides with $v(\lambda)$, i.e., (4.3.1) holds.

§5. The proof of Approximation Theorem

Now everything is ready for the proofs of our results. In Sect. 5.1 we will show that the set of potentials $Q(E^{(N)})$ is precompact in the topology of the uniform convergence on the real axis and that every limit (as $N \to \infty$) potential belongs to $Q(E)$. In Sect. 5.2 we will show that every potential from $Q(E)$ is a uniform limit of potentials from $Q(E^{(N)}), N \to \infty$. It will prove our Approximation Theorem. Simultaneously, our Main Theorem and Uniqueness Theorem will also be proved.

5.1. Let $q_N \in Q(E^{(N)})$ be a sequence of finite-band potentials and let $D^{(N)} \in \mathscr{D}(E^{(N)}) \hookrightarrow \mathscr{D}(E)$ be a corresponding sequence of spectral data. Since $\mathscr{D}(E)$ is compact, we may assume that

$$D^{(N)} \to D, \qquad N \to \infty.$$

We should prove that

$$q_N(t) \to q(t) \qquad \text{uniformly on } \mathbb{R}, \tag{5.1.1}$$

and that

$$q \in Q(E). \tag{5.1.2}$$

Define a curve $\{D^{(N)}(t)\}_{t \in \mathbb{R}}$ which solves the classical Jacobi inversion problem

$$A^{(N)}(D^{(N)}(t)) = A^{(N)}(D^{(N)}) + \delta(E^{(N)})t, \qquad t \in \mathbb{R}. \tag{5.1.3}$$

Taking into account that $\mathscr{D}(E)$ is compact and that the functions $A^{(N)}$ and A are continuous on $\mathscr{D}(E)$, we obtain by (4.1.2) that $A^{(N)}$ converges to A uniformly on $\mathscr{D}(E)$. Hence, making use of (4.1.1) and (4.1.3), we may pass to the limit in the right-hand side of (5.1.3) for every $t \in \mathbb{R}$:

$$A^{(N)}(D^{(N)}(t)) \to A(D) + \delta(E)t, \qquad N \to \infty,$$

This relation together with compactness of $\mathscr{D}(E)$ and $\pi^*(E)$, with the relation (4.1.2), and with Theorem A yield

$$D^{(N)}(t) \to D(t) \qquad \text{uniformly on } \mathbb{R},$$

where

$$A(D(t)) = A(D(0)) + \delta(E)t, \qquad D(0) = D,$$

$$D(t) = \bigcup_j (\lambda_j(t), \varepsilon_j(t)).$$

Now passing to the limit in the trace formula

$$q_N(t) = \sum_{j=1}^{N} (a_j + b_j - 2\lambda_j^{(N)}(t)),$$

we obtain (5.1.1), where the limit potential $q(t)$ may be recovered by the trace formula (1.7.3).

The relation (5.1.1) implies (see for example Craig [6]) that the spectrum of the limit operator $L[q]$ coincides with the set $E = \bigcap_{N \geq 1} E^{(N)}$ and that

$$g_N(x, x; \lambda) \to g(x, x; \lambda), \qquad N \to \infty, \qquad x \in \mathbb{R}. \tag{5.1.4}$$

uniformly with respect to λ lying on each compact in Ω. Then (5.1.4) together with Lemma 5.2 from Craig [6] imply the reflectionless of q (1.3.1), what is equivalent to (1.2.4) by Appendix. So (5.1.2) is verified.

5.2. Now we show that an arbitrary potential $q \in Q(E)$ may be approximated by potentials from $Q(E^{(N)})$ uniformly on the real axis.

Let $D = D(0) \in \mathscr{D}(E)$ be a divisor corresponding to the potential q. We denote, as before, by $D^{(N)}$ a "projection" of D on $\mathscr{D}(E^{(N)})$. There is a finite-band potential $q_N \in Q(E^{(N)})$ which corresponds to $D^{(N)}$. As we have proved in Sect. 5.1, the set of potentials $\{q_N\}$ is precompact in the topology of the uniform convergence on the real axis and each limit potential belongs to $Q(E)$. By (4.1.1) all limit potentials for $\{q_N\}$ have the same divisor D of their spectral data and by the result proven in Sect. 3 every limit potential should coincide with $q(t)$. So Approximation Theorem is proved.

Since Main Theorem and Uniqueness Theorem are true for the finite-band situation, our arguments together with Theorem A prove both of these two theorems as well.

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Appendix

Reflectionless in the Craig sense implies pseudocontinuability of the Weyl functions

Let $q(x)$ be a bounded continuous potential. We will show that the Craig condition

$$\Re g(x, x; \lambda + i0) = 0 \qquad \text{for a.e. } \lambda \in E = \sigma(L[q]) \tag{A1}$$

implies that

$$m_+(\lambda + i0) = \overline{m_-(\lambda + i0)} \qquad \text{for a.e. } \lambda \in E. \tag{A2}$$

Since the Weyl solutions $\psi_\pm(x, \lambda)$ of the Sturm-Liouville equation can be represented in the form

$$\psi_\pm(x, \lambda) = \exp\left\{\int_0^x m_\pm(s, \lambda)\, ds\right\}, \tag{A3}$$

where $m_\pm(s, \lambda)$ are the Weyl functions of the potential $q(x + s)$ (Titchmarsh [25]), the diagonal of the resolvent kernel $g(x, x; \lambda)$ equals

$$\begin{aligned} g(x, x; \lambda) &= \frac{\psi_+(x, \lambda)\psi_-(x, \lambda)}{m_-(\lambda) - m_+(\lambda)} \\ &= \frac{1}{m_-(\lambda) - m_+(\lambda)} \exp\left\{\int_0^x [m_-(s, \lambda) + m_+(s, \lambda)]\, ds\right\}. \end{aligned}$$

Consequently,

$$\frac{d}{dx} \log g(x, x, \lambda) = m_-(x, \lambda) + m_+(x, \lambda), \qquad \Im\lambda \neq 0,$$

whence

$$\frac{d}{dx}\arg g(x, x, \lambda) = \Im[m_-(x, \lambda) + m_+(x, \lambda)]. \tag{A4}$$

Denote the right-hand side of (A4) by $u(x, \lambda)$ and set $\lambda = t + i\varepsilon$, $\varepsilon > 0$.

Let $\psi(t)$ be an arbitrary continuous function with a compact support and let $0 \le x_1 < x_2 \le 1$ be arbitrary values. Integrating twice (A4) and changing the order of integration, we obtain

$$\begin{aligned}
&\int_{x_1}^{x_2} dx \int_E \frac{\psi(t)}{1+t^2} u(x, t+i\varepsilon)\,dt \\
&\quad = \int_E dt \int_{x_1}^{x_2} \frac{\psi(t)}{1+t^2}\frac{d}{dx} \arg g(x; x; t+i\varepsilon)\,dx \\
&\quad = \int_E \frac{\psi(t)}{1+t^2}\{\arg g(x_2, x_2; t+i\varepsilon) - \arg g(x_1, x_1; t+i\varepsilon)\}\,dt.
\end{aligned} \tag{A5}$$

Since $0 < \arg g(x, x; t+i\varepsilon) \le \pi$, we may pass to the limit in the right-hand side of (A5). Using condition (A1), we obtain

$$\lim_{\varepsilon\to 0}\int_{x_1}^{x_2} dx \int_E \frac{\psi(t)}{1+t^2} u(x, t+i\varepsilon)\,dt = 0. \tag{A6}$$

Observe, that the internal integral in (A6) is bounded uniformly with respect to $x \in [0, 1]$ and $\varepsilon \in [0, 1/2]$:

$$\begin{aligned}
&\left|\int_E \frac{\psi(t)}{1+t^2} u(x, t+i\varepsilon)\,dt\right| \\
&\quad \le M_\psi \int_E \frac{dt}{1+t^2}\{\Im m_+(x, t+i\varepsilon) - \Im m_-(x, t+i\varepsilon)\} \\
&\quad \le M_\psi \int_E \frac{dt}{1+t^2}\Im\frac{1}{g(x, x; t+i\varepsilon)} \\
&\quad \le M_\psi C \sup_{x\in[0,1]}\frac{1}{|g(x, x; i)|}.
\end{aligned}$$

The latter supremum is finite since the function $x \mapsto g(x, x; i)$ is continuous. It allows us to rewrite (A6) in the form

$$\int_{x_1}^{x_2} dx\left\{\lim_{\varepsilon\to 0}\int_E \frac{\psi(t)}{1+t^2} u(x, t+i\varepsilon)\,dt\right\} = 0. \tag{A7}$$

Now we consider a family of charges

$$\rho_\varepsilon(x, dt) = u(x, t + i\varepsilon)\frac{\chi_E(t)}{1+t^2}dt, \qquad x \in [0, 1], \qquad \varepsilon \in [0, 1/2],$$

where χ_E is the indicator-function of the set E. The family $\rho_\varepsilon(x, dt)$ converges weakly to a certain charge $\rho_0(x, dt)$, as $\varepsilon \to 0$. Together with (A7) it implies that

$$\int_{x_1}^{x_2} dx \int_E \psi(t)\rho_0(x, dt)$$

$$= \int_{x_1}^{x_2} dx \left\{\lim_{\varepsilon\to 0} \int_E \psi(t)\rho_\varepsilon(x, dt)\right\} = 0.$$

Since x_1 and x_2 are arbitrary values, we conclude that

$$\int_E \psi(t)\rho_0(x, dt) = 0 \qquad \text{for a.e. } x \in [0, 1].$$

The absolutely continuous part of the charge ρ_0 equals

$$\mathfrak{I}[m_-(x, t + i0) + m_+(x, t + i0)]\frac{dt}{1+t^2},$$

whence for a.e. $x \in [0, 1]$

$$\mathfrak{I}[m_-(x, t + i0) + m_+(x, t + i0)] = 0 \qquad \text{for a.e. } t \in E. \tag{A8}$$

Further, (A1) and the equation

$$\frac{1}{g(x, x; \lambda)} = m_-(x, \lambda) - m_+(x, \lambda)$$

imply that for every $x \in [0, 1]$

$$\mathfrak{R}[m_-(x, t + i0) - m_+(x, t + i0)] = 0 \qquad \text{for a.e. } t \in E. \tag{A9}$$

Comparing (A8) and (A9), we obtain that for some $x \in (0, 1)$

$$m_+(x, t + i0) = \overline{m_-(x, t + i0)} \qquad \text{for a.e. } t \in E.$$

Note, that by virtue of (A3)

$$m_\pm(x, \lambda) = \frac{d}{dx} \log \psi_\pm(x, \lambda)$$

$$= \frac{C'(x, \lambda) + m_\pm(\lambda) S'(x, \lambda)}{C(x, \lambda) + m_\pm(\lambda) S(x, \lambda)},$$

where all four functions C, S, C' and S' are real as $\lambda \in E$. Thus, we have obtained the condition (A2).

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