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Flat exterior Tor algebras and cotangent complexes¹

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Introduction

Let A be a ring, B and C A -algebras (commutative with unit) and $D = B \otimes_A C$. It is well known [M, Theorem 2.2, p. 225] that $\text{Tor}^A(B, C)$ is a strictly anticommutative graded D -algebra. So we have a homomorphism of graded D -algebras

$$\gamma : \wedge_D \text{Tor}_1^A(B, C) \rightarrow \text{Tor}^A(B, C).$$

In [A₂], M. André has introduced for $n \geq 0$ and W a D -module, homology modules $H_n(A, B, C, W)$ generalizing in some way the classical homology functors of André–Quillen $H_n(R, S, -)$ (see [A₁], [Q₂], [Q₃]).

The purpose of this paper is to relate properties of γ and the vanishing of the functors $H_n(A, B, C, -)$, $n \geq 3$. More precisely, our main result is the following

THEOREM 1. *Let A be a ring, B and C A -algebras and $D = B \otimes_A C$. The following conditions are equivalent:*

(1) *The D -module $\text{Tor}_1^A(B, C)$ is flat and the canonical homomorphism*

$$\gamma : \wedge_D \text{Tor}_1^A(B, C) \rightarrow \text{Tor}^A(B, C)$$

is an isomorphism.

(2) $H_j(A, B, C, -) = 0$ for $j \geq 3$.

This theorem has as a consequence two important results, the first one is already known but the second isn't.

COROLLARY 2. *Let A be a ring, I an ideal of A and $B = A/I$. The following conditions are equivalent:*

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(1) *The B -module I/I^2 is flat and the canonical homomorphism*

$$\wedge_B I/I^2 \longrightarrow \mathrm{Tor}^A(B, B)$$

is an isomorphism.

(2) $H_j(A, B, -) = 0$ for $j \geq 2$.

This result is due to Quillen [Q₂, Theorem 10.3], [Q₃, Theorem 6.13].

COROLLARY 3. *Let A be a ring, I an ideal of A , $B = A/I$, and E the Koszul complex associated to an arbitrary set of generators of I . The following conditions are equivalent:*

(1) *The B -module $H_1(E)$ is flat and the canonical homomorphism of graded algebras*

$$\wedge_B H_1(E) \longrightarrow H(E)$$

is an isomorphism.

(2) $H_j(A, B, -) = 0$ for $j \geq 3$.

Moreover, the following conditions are equivalent:

(1') *The B -module $H_1(E)$ is projective and the canonical homomorphism of graded algebras*

$$\wedge_B H_1(E) \longrightarrow H(E)$$

is an isomorphism.

(2') $H^j(A, B, -) = 0$ for $j \geq 3$.

The proof of Theorem 1 is divided in two parts. In the first one we use an analogue to the fundamental spectral sequence of Quillen [Q₂, Theorem 6.8] to relate the vanishing of $H_n(A, B, C, -)$ with the structure of the homology algebra of a certain derived tensor product $D \overset{L}{\otimes}_Y D$. In the second part we use a spectral sequence

$$E_{p,q}^2 = \mathrm{Tor}_p^{H(Y)}(D, D)_q \Rightarrow H(D \overset{L}{\otimes}_Y D)$$

to compare $\mathrm{Tor}^A(B, C)$ with $H(D \overset{L}{\otimes}_Y D)$.

Proofs. First recall the definition of $H_n(A, B, C, W)$. Let A be a ring, B and C two A -algebras, $D = B \otimes_A C$ and W a D -module. Let X be a cofibrant simplicial

A -algebra resolution of B , let $Y = X \otimes_A C$, and let Z be a cofibrant simplicial Y -algebra resolution of D . Then

$$\mathbf{L}_{B-C|A} := \Omega_{Z|Y} \otimes_Z D$$

is a cofibrant simplicial D -module, whose normalization is a chain complex of projective D -modules independent up to homotopy equivalence of the choice of X and Z , and which therefore represents an object unique up to isomorphism of the derived category of the category of D -modules. For a D -module W

$$H_n(A, B, C, W) = H_n(\mathbf{L}_{B-C|A} \otimes_D W).$$

Notice that if J is the simplicial ideal kernel of the surjective canonical homomorphism $Z \otimes_Y D \rightarrow D$, then $\mathbf{L}_{B-C|A} = J/J^2$.

The resolution Z can be obtained by the ‘‘step by step’’ construction [A₁, Chap. IX]. So we can assume $Y_n = Z_n$ for $n = 0, 1$ and so

$$H_n(A, B, C, W) = 0 \quad \text{for } n = 0, 1.$$

For each p , J_p is the ideal generated by the variables of the polynomial D -algebra $(Z \otimes_Y D)_p$. In particular $J_0 = 0$ and J_p is a regular ideal. Quillen’s convergence theorem [Q₃, Theorem 6.12] implies $H_p(J^n) = 0$ for $p < n$.

Therefore the spectral sequence resulting from filtering $Z \otimes_Y D$ by the powers of J , is a convergent spectral sequence located in the first quadrant

$$E_{p,q}^2 = H_{p+q}(S_D^n \mathbf{L}_{B-C|A}) \Rightarrow H(D \overset{L}{\otimes}_Y D). \quad (\text{I})$$

since $H(Z \otimes_Y D) = H(D \overset{L}{\otimes}_Y D)$ as follows from [Q₁, Theorem 6-(a), p.II.6.8], because the Y_n -algebra Z_n is free for all n .

This spectral sequence is an analogue to Quillen’s fundamental spectral sequence.

With the shuffle product \otimes [Q₁, p.II.6.6] $Z \otimes_Y D$ with the differential induced by the face operators, is a strictly anticommutative differential graded D -algebra with a system of divided powers. Moreover $Z \otimes_Y D \supset J \supset J^2 \supset \dots$ is a filtration of $Z \otimes_Y D$ by differential graded ideals. So the spectral sequence is a spectral sequence of bigraded algebras with divided powers.

Since $H_0(\mathbf{L}_{B-C|A}) = H_1(\mathbf{L}_{B-C|A}) = 0$ we have [Q₂, Corollary 7.30] $H_j(S_D^n \mathbf{L}_{B-C|A}) = 0$ if $j < 2n$, i.e., $E_{p,q}^2 = 0$ for $p < q$, and there exists a canonical map $\delta: \Gamma_D H_2(\mathbf{L}_{B-C|A}) \rightarrow H(D \overset{L}{\otimes}_Y D)$ which is the unique homomorphism of graded

D -algebras with divided powers extending the edge isomorphism $H_2(\mathbf{L}_{B-C|A}) = E_{1,1}^2 = H_2(D \overset{L}{\otimes}_Y D)$.

Recall now some notation from [Q₂]. For a D -module T , $K(T, n)$ will be the simplicial D -module whose normalization is the complex with T in dimension n and zero in the remaining ones. We have a canonical morphism

$$\mathbf{L}_{B-C|A} \longrightarrow K(H_2(\mathbf{L}_{B-C|A}), 2)$$

which is a 2-equivalence (i.e., it induces isomorphisms in homology in dimensions ≤ 2).

For an object X , cX will be the constant simplicial object with $(cX)_q = X$, and whose faces and degeneracies are the identity map of X .

Finally, Σ is the suspension functor, so that

$$H_{q+1}(\Sigma X) = H_q(X).$$

The proof of the following proposition is analogous to that of Theorem 10.3 of [Q₂]. We give the details for convenience of the reader.

PROPOSITION 4. *The following conditions are equivalent*

- (i) $H_j(A, B, C, -) = 0$ for $j \geq 3$
- (ii) The D -module $H_2(D \overset{L}{\otimes}_Y D)$ is flat and the canonical homomorphism

$$\phi : \Gamma_D H_2(D \overset{L}{\otimes}_Y D) \rightarrow H(D \overset{L}{\otimes}_Y D)$$

is an isomorphism

Proof. From the universal coefficient spectral sequence

$$E_{p,q}^2 = \text{Tor}_p^D(H_q(\mathbf{L}_{B-C|A}), -) \Rightarrow H(A, B, C, -)$$

we deduce that condition (i) is equivalent to: $H_2(D \overset{L}{\otimes}_Y D)$ is D -flat and

$$\mathbf{L}_{B-C|A} \longrightarrow K(H_2(\mathbf{L}_{B-C|A}), 2)$$

is an n -equivalence for all n .

Note that $K(H_2(\mathbf{L}_{B-C|A}), 2)$ is homotopically equivalent to $\Sigma\Sigma(c(H_2(\mathbf{L}_{B-C|A})))$. Therefore, using [Q₂, 7.21] we obtain

$$\begin{aligned}
H_{p+q}(S^q_B K(H_2(\mathbf{L}_{B-C|A}), 2)) &= H_{p+q}(S^q_B \Sigma \Sigma(c(H_2(\mathbf{L}_{B-C|A})))) \\
&= H_p(\wedge^q_B \Sigma(c(H_2(\mathbf{L}_{B-C|A})))) \\
&= H_{p-q}(\Gamma^q_B c(H_2(\mathbf{L}_{B-C|A}))) \\
&= \begin{cases} 0 & \text{if } p - q \neq 0 \\ \Gamma^q_B H_2(\mathbf{L}_{B-C|A}) & \text{if } p - q = 0. \end{cases}
\end{aligned}$$

So, assuming that $\mathbf{L}_{B-C|A} \rightarrow K(H_2(\mathbf{L}_{B-C|A}), 2)$ is an n -equivalence, $n \geq 2$, then by [Q₂, 7.3] so are the induced maps of symmetric powers, hence we have

$$\begin{aligned}
E_{p,q}^2 &= H_{p+q}(S^q_B \mathbf{L}_{B-C|A}) = H_{p+q}(S^q_B K(H_2(\mathbf{L}_{B-C|A}), 2)) \\
&= \begin{cases} 0 & \text{if } p + q \leq n, p \neq q \\ \Gamma^q_B H_2(\mathbf{L}_{B-C|A}) & \text{if } p + q \leq n, p = q. \end{cases}
\end{aligned}$$

If (i) holds we can take $n = \infty$ and so

$$E_{p,q}^2 = \begin{cases} 0 & \text{if } p \neq q \\ \Gamma^q_B H_2(\mathbf{L}_{B-C|A}) & \text{if } p = q. \end{cases}$$

So the spectral sequence (I) degenerates showing that the edge homomorphism $\delta : \Gamma_D H_2(\mathbf{L}_{B-C|A}) \rightarrow H(D \overset{L}{\otimes}_Y D)$ is an isomorphism. Therefore $\phi : \Gamma_D H_2(D \overset{L}{\otimes}_Y D) \rightarrow H(D \overset{L}{\otimes}_Y D)$ is an isomorphism.

Now assume that (ii) holds. We will prove by induction on n that $\mathbf{L}_{B-C|A} \rightarrow K(H_2(\mathbf{L}_{B-C|A}), 2)$ is an n -equivalence for all n . Assuming $n \geq 2$ and that it is an n -equivalence, to see that it is an $(n+1)$ -equivalence we have to prove that $E_{n,1}^2 = H_{n+1}(\mathbf{L}_{B-C|A}) = 0$.

Since

$$E_{p,q}^2 = \begin{cases} 0 & \text{if } p + q \leq n, p \neq q \\ \Gamma^q_B H_2(\mathbf{L}_{B-C|A}) & \text{if } p + q \leq n, p = q \end{cases}$$

the only possible non zero differential coming from $E_{n,1}^2$ is

$$E_{n,1}^2 = E_{n,1}^p \xrightarrow{d^p} E_{p,p}^p = E_{p,p}^2 \quad \text{with } n = 2p.$$

As the edge homomorphism is an isomorphism we have $d^p = 0$. So $E_{n,1}^2 = E_{n,1}^\infty = 0$.

Since $H(Y) = \text{Tor}^A(B, C)$, Theorem 1 is a consequence of Proposition 4 and the following general result.

PROPOSITION 5. Let Y be a simplicial ring and $D = H_0(Y)$. Then the following conditions are equivalent:

(ii) The D -module $H_2(D \overset{L}{\otimes}_Y D)$ is flat and the canonical homomorphism

$$\phi : \Gamma_D H_2(D \overset{L}{\otimes}_Y D) \longrightarrow H(D \overset{L}{\otimes}_Y D)$$

is an isomorphism.

(iii) The D -module $H_1(Y)$ is flat and the canonical homomorphism

$$\gamma : \wedge_D H_1(Y) \longrightarrow H(Y)$$

is an isomorphism.

Before proceeding to the proof of Proposition 5 we will need some remarks.

Remark 6. By [Q₁, Theorem 6-b), p.II.6.8] there exists a spectral sequence

$$E_{p,q}^2 = \text{Tor}_p^{H(Y)}(D, D)_q \Rightarrow H(D \overset{L}{\otimes}_Y D). \tag{II}$$

Since Y is a simplicial ring and D is a simplicial Y -algebra, this spectral sequence is of bigraded algebras with divided powers.

In fact we have the following. Let Y be a simplicial ring and D a simplicial Y -algebra. Then, on the lines of the construction of [Q₁, pp.II.6.13–6.14], it is possible to generalize the “step by step” method to obtain a bisimplicial Y -algebra P and a morphism $P \rightarrow D$ such that:

- (1) For each j , $P_{*,j}$ is a free simplicial Y_j -algebra resolution of D_j .
- (2) For each i , the graded $H(Y)$ -algebra $H(P_{i,*})$ is free as an $H(Y)$ -module and the induced sequence

$$\cdots \longrightarrow H(P_{2,*}) \longrightarrow H(P_{1,*}) \longrightarrow H(P_{0,*})$$

is a resolution of $H(D)$.

The details of the construction of P are in [B].

Now, if E is another simplicial Y -algebra and $Y \xrightarrow{i} Q \xrightarrow{p} E$ is a factorization of the canonical morphism $Y \rightarrow E$ with i cofibration and p trivial fibration, then we have a bisimplicial Y -algebra

$$M_{i,j} = P_{i,j} \otimes_{Y_j} Q_j.$$

From the two spectral sequences of a double complex, we obtain

$$E_{p,q}^2 = \text{Tor}_p^{H(Y)}(H(D), H(E))_q \Rightarrow H(D \overset{L}{\otimes}_Y E).$$

This spectral sequence is of bigraded algebras with divided powers since it comes from a bisimplicial algebra.

Remark 7. Let L be a flat D -module and consider on the bigraded algebra $\wedge_D L \otimes_D \Gamma_D L$ the unique D -derivation of bidegree $(1, -1)$ such that $d(y \otimes 1) = 0, d(1 \otimes \gamma_p x) = x \otimes \gamma_{p-1} x, x, y \in L$. Then $(\wedge_D L \otimes_D \Gamma_D^* L, d_*)$ is a flat resolution of the $\wedge_D L$ -module D : by Lazard's Theorem, we can assume that L is a free D -module of finite type and then by Künneth formula we can take $L = D$. In this case it is clear.

This flat resolution $M_* = \wedge_D L \otimes_D \Gamma_D^* L$ is graded in the following way: $M_{*,i} = \wedge_D^i L \otimes_D \Gamma_D^* L$. Using this resolution we deduce

$$\text{Tor}_p^{\wedge_D L}(D, D)_q = \begin{cases} 0 & \text{if } p \neq q \\ \Gamma_D^p L & \text{if } p = q. \end{cases}$$

Now we come to the proof of Proposition 5. Consider the spectral sequence of Remark 6

$$E_{p,q}^2 = \text{Tor}_p^{H(Y)}(D, D)_q \Rightarrow H_{p+q}(D \overset{L}{\otimes}_Y D).$$

For it, the following hold:

$$E_{p,q}^2 = \text{Tor}_p^{H_0(Y)}(D, D) = 0 \quad \text{if } p > 0$$

$$E_{0,q}^2 = (D \otimes_{H(Y)} D)_q = \begin{cases} 0 & \text{if } q > 0 \\ D & \text{if } q = 0 \end{cases} \tag{III}$$

$$E_{1,q}^2 = (H_+(Y) \otimes_{H(Y)} D)_q = (H_+(Y)/H_+(Y)^2)_q.$$

In particular we get an edge isomorphism $\alpha_2: H_2(D \overset{L}{\otimes}_Y D) \rightarrow E_{1,1}^2$ and an isomorphism $E_{1,1}^2 = H_1(Y)$, which show that the flatness assumptions in (ii) and (iii) are equivalent.

Let $A = \wedge_D H_1(Y)$ and consider the homomorphism of bigraded D -algebras with divided powers

$$\Gamma_D E_{1,1}^2 = \text{Tor}^A(D, D) \xrightarrow{\gamma_{**}} \text{Tor}^{H(Y)}(D, D)$$

where the equality is due to Remark 7 and $\gamma_{*,*}$ is induced by γ . Since γ is bijective in degrees $\leq n$, then $\gamma_{p,q}$ is bijective for $q \leq n$, hence in the spectral sequence (II) we have

$$E_{p,q}^2 = \begin{cases} 0 & \text{if } p \neq q, q \leq n \\ \Gamma_D^p E_{1,1}^2 & \text{if } p = q, q \leq n \end{cases} \tag{IV}$$

When (iii) holds we can take $n = \infty$ and get an isomorphism of graded D -algebras with divided powers $\beta : \Gamma_D E_{1,1}^2 \xrightarrow{\sim} H(D \overset{L}{\otimes}_Y D)$, hence $\phi = \beta \circ \Gamma_D \alpha_2$ is bijective.

Conversely, let (ii) hold. Since γ_n is an isomorphism for $n = 0, 1$, assume by induction that γ_j is bijective for $j \leq n$ and some $n \geq 1$. We have for $p \leq n$ a diagram

$$\begin{array}{ccc} H_{2p}(D \overset{L}{\otimes}_Y D) & \xrightarrow{\alpha_{2p}} & E_{p,p}^2 \\ \phi_{2p} \uparrow & & \uparrow \psi_{2p} \\ \Gamma_D^p H_2(D \overset{L}{\otimes}_Y D) & \xrightarrow{\Gamma_D^p \alpha_2} & \Gamma_D^p E_{1,1}^2 \end{array}$$

which is commutative because α_{2p} is an edge homomorphism in a spectral sequence (II) of D -algebras with divided powers. Note that ϕ_{2p} is an isomorphism by condition (ii), and ψ_{2p} is an isomorphism by (IV). So α_{2p} is an isomorphism and therefore all differentials of the spectral sequence are zero on $E_{p,p}^r$ when $p \leq n$ and $r \geq 2$. In particular, no differential lands in $E_{1,2}^r, E_{1,n+1}^r$, or $E_{2,n+1}^r$ for $n \geq 2$ and $r \geq 2$. Any differential leaving one of these modules lands into some $E_{p,*}^r$ with $p \leq 0$, which is trivial. Thus $E_{1,2}^2 = E_{1,2}^\infty, E_{1,n+1}^2 = E_{1,n+1}^\infty$ and $E_{2,n+1}^2 = E_{2,n+1}^\infty$ for $n \geq 2$. We have $E_{p,q}^\infty = 0$ if $p + q$ is odd, and the diagram implies $E_{p,q}^\infty = 0$ when $p \neq q$ and $p + q$ is even $\leq 2n$. Therefore $E_{1,2}^2 = E_{1,n+1}^2 = E_{2,n+1}^2 = 0$ for $n \geq 2$.

By (III) we have $\text{Coker}(\gamma_{n+1}) = E_{1,n+1}^2$, hence γ_{n+1} is surjective. In order to determine $\text{Ker}(\gamma_{n+1})$ we consider an exact sequence of D -modules

$$F''_{n+1} \xrightarrow{\eta} F'_{n+1} \rightarrow \Lambda_{n+1} \xrightarrow{\gamma_{n+1}} H_{n+1}(Y) \rightarrow 0$$

in which F''_{n+1} and F'_{n+1} are free. It produces a complex of graded Λ -modules

$$\Lambda \otimes_D F''_{n+1} \xrightarrow{\Lambda \otimes_D \eta} \Lambda \otimes_D F'_{n+1} \rightarrow \Lambda \xrightarrow{\gamma} H(Y) \rightarrow 0$$

which is exact in degrees $\leq n + 1$. Thus, by using appropriate graded free Λ -modules G, G', G'' with $G_j = G'_j = G''_j = 0$ for $j \leq n + 1$ we can modify it to obtain the beginning of a graded free resolution of the graded Λ -module $H(Y)$ in the form

$$(\Lambda \otimes_D F''_{n+1}) \oplus G'' \longrightarrow (\Lambda \otimes_D F'_{n+1}) \oplus G' \longrightarrow \Lambda \oplus G \longrightarrow H(Y) \longrightarrow 0.$$

With its help we see that

$$\text{Tor}_1^A(D, H(Y))_j = \begin{cases} 0 & \text{if } j \leq n \\ \text{Coker}(\eta) = \text{Ker}(\gamma_{n+1}) & \text{if } j = n + 1. \end{cases}$$

Since γ is surjective in degrees $\leq n + 1$, the projection $Q = H(Y)/H_1(Y)H(Y) \rightarrow H(Y)/H_+(Y) = D$ is bijective in these degrees and thus induces isomorphisms

$$\text{Tor}_2^{H(Y)}(D, Q)_j = \text{Tor}_2^{H(Y)}(D, D)_j = E_{2,j}^2 \quad \text{for } 2 \leq j \leq n + 1.$$

These observations and Remark 7 show that the standard change of rings exact sequence

$$\begin{aligned} \text{Tor}_2^A(D, D) &\longrightarrow \text{Tor}_2^{H(Y)}(D, Q) \longrightarrow (D \otimes_{H(Y)} \text{Tor}_1^A(D, H(Y))) \\ &\longrightarrow \text{Tor}_1^A(D, D) \longrightarrow \text{Tor}_1^{H(Y)}(D, Q) \longrightarrow 0 \end{aligned}$$

reduces in degree $n + 1$ to an exact sequence

$$\text{Tor}_2^A(D, D)_{n+1} \xrightarrow{\gamma_{2,n+1}} E_{2,n+1}^2 \longrightarrow \text{Tor}_1^A(D, H(Y))_{n+1} \longrightarrow 0.$$

For $n = 1$ the map $\gamma_{2,2}$ is bijective, and for $n > 1$ the module $E_{2,n+1}^2$ is trivial, hence

$$\text{Ker}(\gamma_{n+1}) = \text{Tor}_1^A(D, H(Y))_{n+1} = 0 \quad \text{for } n \geq 1.$$

Thus γ_{n+1} is injective, so the induction step is complete and the Proposition is proved.

Corollary 2 follows from Theorem 1 taking $C = B$ and so $D = B$ and $H_n(A, B, C, -) = H_{n-1}(A, B, -)$ [A₂, Example 5].

For Corollary 3, let τ be the set of generators τ_m to which the Koszul complex E is associated. Let R be the free A -algebra with variables t_m and consider the A -algebra homomorphisms $\beta : R \rightarrow A$, $\omega : R \rightarrow A$, such that $\beta(t_m) = 0$, $\omega(t_m) = \tau_m$. Denote by A_β and A_ω the corresponding R -algebra structures on A . Then there exists an isomorphism of graded B -algebras $H_*(E) = \text{Tor}_*^R(A_\beta, A_\omega)$. Moreover [A₂, Example 6], $H_n(R, A_\beta, A_\omega, -) = H_n(A, B, -)$ for $n \geq 3$, and the first part of Corollary 3 follows from Theorem 1.

In order to prove the second part of Corollary 3, we will need some facts about the cotangent complex $L_{B|A}$. Let X be a simplicial resolution of the A -algebra B obtained by the “step by step” construction [A₁, Chap. IX], [A₂, p. 327]. In particular consider the first three steps. We begin by choosing a system of generators t_α of the ideal I . The first step is a simplicial A -algebra K with $K_0 = A$, K_1 the polynomial A -algebra on the variables T_α and K_{1+h} , $h > 0$, is the polynomial A -algebra on the variables

$$\sigma_h^{i_1} \sigma_{h-1}^{i_2} \cdots \sigma_1^{i_h}(T_\alpha), \quad 0 \leq i_h < \cdots < i_2 < i_1 \leq h,$$

where σ denotes the degeneration operators. The face operators are determined by

$$\varepsilon_1^0(T_\alpha) = 0, \quad \varepsilon_1^1(T_\alpha) = t_\alpha.$$

In order to construct the second step, we choose representants $s_v \in K_1$ of a set of generators of the B -module

$$\pi_1(K) = \frac{M \cap N}{MN}$$

where M is the ideal of K_1 generated by the elements T_α and N the ideal of K_1 generated by the elements $T_\alpha - t_\alpha$. The second step is a simplicial K -algebra F with $F_0 = K_0$, $F_1 = K_1$, F_2 is the polynomial K_2 -algebra on the variables S_v , and F_{2+h} , $h > 0$, is the polynomial K_{2+h} -algebra on the variables

$$\sigma_{1+h}^{i_1} \sigma_h^{i_2} \cdots \sigma_2^{i_h}(S_v), \quad 0 \leq i_h < \cdots < i_2 < i_1 \leq 1 + h.$$

The face operators are determined by

$$\varepsilon_2^0(S_v) = 0, \quad \varepsilon_2^1(S_v) = 0, \quad \varepsilon_2^2(S_v) = s_v.$$

Similarly the third step is constructed by choosing representants $z_w \in F_2$ of a set of generators of the B -module $\pi_2(F)$ to obtain a simplicial F -algebra G with $G_0 = F_0$, $G_1 = F_1$, $G_2 = F_2$, G_3 is the polynomial F_3 -algebra on the variables Z_w and G_{3+h} , $h > 0$, is the polynomial F_{3+h} -algebra on the variables

$$\sigma_{2+h}^{i_1} \sigma_{1+h}^{i_2} \sigma_h^{i_3} \cdots \sigma_3^{i_h}(Z_w), \quad 0 \leq i_h < \cdots < i_2 < i_1 \leq 2 + h.$$

The face operators are determined by

$$\varepsilon_3^0(Z_w) = 0, \quad \varepsilon_3^1(Z_w) = 0, \quad \varepsilon_3^2(Z_w) = 0, \quad \varepsilon_3^3(Z_w) = z_w.$$

We have $L_{B|A} = J/J^2$ where J is the augmentation ideal of the simplicial B -algebra $X \otimes_A B$. Denote by N the normalization functor from simplicial B -modules. We have

$$(N(J/J^2))_3 = \bigoplus_w BZ_w, \quad (N(J/J^2))_2 = \bigoplus_v BS_v$$

and the image of the differential d_3 of $N(J/J^2)$ coincides with the image of the canonical homomorphism [A₂, Remarque 23]

$$\pi_2(F) \rightarrow \bigoplus_v BS_v.$$

Therefore [A₂, Proposition 24]

$$\text{Coker } d_3 = \pi_1(K).$$

Moreover

$$(N(J/J^2))_1 = \bigoplus_\alpha BT_\alpha$$

and the differential d_3 places in a commutative diagram

$$\begin{array}{ccc} \bigoplus_v BS_v & \xrightarrow{d_3} & \bigoplus_\alpha BT_\alpha \\ \pi \searrow & & \nearrow \phi \\ & \pi_1(K) & \end{array}$$

where π is the homomorphism sending S_v on the generator represented by s_v and ϕ the canonical homomorphism [A₂, Remarque 23].

On the other hand

$$\pi_1(K) = \frac{M \cap N}{MN} = \text{Tor}_1^{K_1}(A, A)$$

where in the first variable in Tor is the structure given by ε_1^0 and in the second the one given by ε_1^1 . If E denotes the Koszul complex associated to the elements t_α , then this Tor is isomorphic to $H_1(E)$. Moreover, through this isomorphism $\pi_1(K) = H_1(E)$, the homomorphism $\phi : \pi_1(K) \rightarrow \bigoplus_\alpha BT_\alpha$ corresponds to the canonical homomorphism

$$H_1(E) \longrightarrow E_1 \otimes_A B = \bigoplus_\alpha BT_\alpha$$

induced by the inclusions of cycles and boundaries $Z_1(E) \subset E_1$, $B_1(E) \subset IE_1$. Thus we have the following proposition:

PROPOSITION 8. *Let A be a ring, I an ideal of A , $B = A/I$ and E the Koszul complex associated to an arbitrary set of generators of I . Then we can choose $\mathbf{L}_{B|A}$ satisfying:*

- (i) *The cokernel of the differential d_3 of $\mathbf{L}_{B|A}$ is a B -module isomorphic to $H_1(E)$.*
- (ii) *There exists a morphism of complexes*

$$\mathbf{L}_{B|A} \longrightarrow (H_1(E) \xrightarrow{\phi} E_1 \otimes_A B)$$

where the second complex is concentrated in degrees 2 and 1. This morphism induces isomorphisms in homology in dimensions ≤ 2 .

Now the cohomological part of Corollary 3 follows from the homological part and Proposition 8.

Remark 9. From Theorem 1 it follows that $H_j(A, B, C, -) = 0$ for all $j \geq 2$ if and only if $\text{Tor}_p^A(B, C) = 0$ for all $p \geq 1$. This result is due to André [A₂, Remarque 39].

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