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Flat exterior Tor algebras and cotangent complexes¹

ANTONIO G. RODICIO

Introduction

Let A be a ring, B and C A-algebras (commutative with unit) and $D = B \otimes_A C$. It is well known [M, Theorem 2.2, p. 225] that $Tor^A(B, C)$ is a strictly anticommutative graded D-algebra. So we have a homomorphism of graded D-algebras

$$\gamma: \wedge_D \operatorname{Tor}_1^A(B, C) \to \operatorname{Tor}_1^A(B, C).$$

In $[A_2]$, M. André has introduced for $n \ge 0$ and W a D-module, homology modules $H_n(A, B, C, W)$ generalizing in some way the classical homology functors of André-Quillen $H_n(R, S, -)$ (see $[A_1]$, $[Q_2]$, $[Q_3]$).

The purpose of this paper is to relate properties of γ and the vanishing of the functors $H_n(A, B, C, -)$, $n \ge 3$. More precisely, our main result is the following

THEOREM 1. Let A be a ring, B and C A-algebras and $D = B \otimes_A C$. The following conditions are equivalent:

(1) The D-module $Tor_1^A(B, C)$ is flat and the canonical homomorphism

$$\gamma: \wedge_D \operatorname{Tor}_1^A(B, C) \to \operatorname{Tor}_A^A(B, C)$$

is an isomorphism.

(2)
$$H_j(A, B, C, -) = 0$$
 for $j \ge 3$.

This theorem has as a consequence two important results, the first one is already known but the second isn't.

COROLLARY 2. Let A be a ring, I an ideal of A and B = A/I. The following conditions are equivalent:

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(1) The B-module I/I^2 is flat and the canonical homomorphism

$$\wedge_B I/I^2 \longrightarrow \operatorname{Tor}^A(B, B)$$

is an isomorphism.

(2)
$$H_j(A, B, -) = 0$$
 for $j \ge 2$.

This result is due to Quillen [Q₂, Theorem 10.3], [Q₃, Theorem 6.13].

COROLLARY 3. Let A be a ring, I an ideal of A, B = A/I, and E the Koszul complex associated to an arbitrary set of generators of I. The following conditions are equivalent:

(1) The B-module $H_1(E)$ is flat and the canonical homomorphism of graded algebras

$$\wedge_B H_1(E) \longrightarrow H(E)$$

is an isomorphism.

(2)
$$H_i(A, B, -) = 0$$
 for $j \ge 3$.

Moreover, the following conditions are equivalent:

(1') The B-module $H_1(E)$ is projective and the canonical homomorphism of graded algebras

$$\wedge_B H_1(E) \longrightarrow H(E)$$

is an isomorphism.

(2')
$$H^{j}(A, B, -) = 0$$
 for $j \ge 3$.

The proof of Theorem 1 is divided in two parts. In the first one we use an analogue to the fundamental spectral sequence of Quillen $[Q_2, Theorem 6.8]$ to relate the vanishing of $H_n(A, B, C, -)$ with the structure of the homology algebra of a certain derived tensor product $D \otimes_Y D$. In the second part we use a spectral sequence

$$E_{p,q}^2 = \operatorname{Tor}_p^{H(Y)}(D, D)_q \Rightarrow H(D \overset{L}{\otimes}_Y D)$$

to compare $\operatorname{Tor}^{A}(B, C)$ with $H(D \overset{L}{\otimes}_{Y} D)$.

Proofs. First recall the definition of $H_n(A, B, C, W)$. Let A be a ring, B and C two A-algebras, $D = B \otimes_A C$ and W a D-module. Let X be a cofibrant simplicial

A-algebra resolution of B, let $Y = X \otimes_A C$, and let Z be a cofibrant simplicial Y-algebra resolution of D. Then

$$\mathbf{L}_{B-C|A} \coloneqq \Omega_{Z|Y} \otimes_{Z} D$$

is a cofibrant simplicial D-module, whose normalization is a chain complex of projective D-modules independent up to homotopy equivalence of the choice of X and Z, and which therefore represents an object unique up to isomorphism of the derived category of the category of D-modules. For a D-module W

$$H_n(A, B, C, W) = H_n(\mathbf{L}_{B-C|A} \otimes_D W).$$

Notice that if J is the simplicial ideal kernel of the surjective canonical homomorphism $Z \otimes_Y D \to D$, then $L_{B-C|A} = J/J^2$.

The resolution Z can be obtained by the "step by step" construction $[A_1, Chap. IX]$. So we can assume $Y_n = Z_n$ for n = 0, 1 and so

$$H_n(A, B, C, W) = 0$$
 for $n = 0, 1$.

For each p, J_p is the ideal generated by the variables of the polynomial D-algebra $(Z \otimes_Y D)_p$. In particular $J_0 = 0$ and J_p is a regular ideal. Quillen's convergence theorem $[Q_3, Theorem 6.12]$ implies $H_p(J^n) = 0$ for p < n.

Therefore the spectral sequence resulting from filtering $Z \otimes_Y D$ by the powers of J, is a convergent spectral sequence located in the first quadrant

$$E_{p,q}^2 = H_{p+q}(S_D^q \mathbf{L}_{B-C|A}) \Rightarrow H(D \overset{L}{\otimes}_Y D). \tag{I}$$

since $H(Z \otimes_Y D) = H(D \otimes_Y D)$ as follows from $[Q_1, Theorem 6-(a), p.II.6.8]$, because the Y_n -algebra Z_n is free for all n.

This spectral sequence is an analogue to Quillen's fundamental spectral sequence.

With the shuffle product \bigotimes [Q₁, p.II.6.6] $Z \otimes_Y D$ with the differential induced by the face operators, is a strictly anticommutative differential graded D-algebra with a system of divided powers. Moreover $Z \otimes_Y D \supset J \supset J^2 \supset \cdots$ is a filtration of $Z \otimes_Y D$ by differential graded ideals. So the spectral sequence is a spectral sequence of bigraded algebras with divided powers.

Since $H_0(\mathbf{L}_{B-C|A}) = H_1(\mathbf{L}_{B-C|A}) = 0$ we have $[Q_2, Corollary 7.30]$ H_j $(S_D^n \mathbf{L}_{B-C|A}) = 0$ if j < 2n, i.e., $E_{p,q}^2 = 0$ for p < q, and there exists a canonical map $\delta: \Gamma_D H_2(\mathbf{L}_{B-C|A}) \to H(D \bigotimes_Y D)$ which is the unique homomorphism of graded

D-algebras with divided powers extending the edge isomorphism $H_2(\mathbf{L}_{B-C|A}) = E_{1,1}^2 = H_2(D \bigotimes_Y D)$.

Recall now some notation from $[Q_2]$. For a *D*-module *T*, K(T, n) will be the simplicial *D*-module whose normalization is the complex with *T* in dimension *n* and zero in the remaining ones. We have a canonical morphism

$$\mathbf{L}_{B-C|A} \longrightarrow K(H_2(\mathbf{L}_{B-C|A}), 2)$$

which is a 2-equivalence (i.e., it induces isomorphisms in homology in dimensions ≤ 2).

For an object X, cX will be the constant simplicial object with $(cX)_q = X$, and whose faces and degeneracies are the identity map of X.

Finally, Σ is the suspension functor, so that

$$H_{q+1}(\Sigma X) = H_q(X).$$

The proof of the following proposition is analogous to that of Theorem 10.3 of $[Q_2]$. We give the details for convenience of the reader.

PROPOSITION 4. The following conditions are equivalent

- (i) $H_j(A, B, C, -) = 0$ for $j \ge 3$
- (ii) The D-module $H_2(D \overset{L}{\otimes}_Y D)$ is flat and the canonical homomorphism

$$\phi: \Gamma_D H_2(D \overset{L}{\otimes}_Y D) \to H(D \overset{L}{\otimes}_Y D)$$

is an isomorphism

Proof. From the universal coefficient spectral sequence

$$E_{p,q}^2 = \operatorname{Tor}_p^D(H_q(\mathbf{L}_{B-C|A}), -) \Rightarrow H(A, B, C, -)$$

we deduce that condition (i) is equivalent to: $H_2(D \overset{L}{\otimes}_Y D)$ is D-flat and

$$\mathbf{L}_{B-C|A} \longrightarrow K(H_2(\mathbf{L}_{B-C|A}), 2)$$

is an n-equivalence for all n.

Note that $K(H_2(\mathbf{L}_{B-C|A}), 2)$ is homotopically equivalent to $\Sigma\Sigma(c(H_2(\mathbf{L}_{B-C|A})))$. Therefore, using $[Q_2, 7.21]$ we obtain

$$\begin{split} H_{p+q}(S_{D}^{q}K(H_{2}(\mathbf{L}_{B-C|A}),2)) &= H_{p+q}(S_{D}^{q}\Sigma\Sigma(c(H_{2}(\mathbf{L}_{B-C|A})))) \\ &= H_{p}(\wedge_{D}^{q}\Sigma(c(H_{2}(\mathbf{L}_{B-C|A})))) \\ &= H_{p-q}(\Gamma_{D}^{q}c(H_{2}(\mathbf{L}_{B-C|A}))) \\ &= \begin{cases} 0 & \text{if } p-q \neq 0 \\ \Gamma_{D}^{q}H_{2}(\mathbf{L}_{B-C|A}) & \text{if } p-q = 0. \end{cases} \end{split}$$

So, assuming that $L_{B-C|A} \to K(H_2(L_{B-C|A}), 2)$ is an *n*-equivalence, $n \ge 2$, then by $[Q_2, 7.3]$ so are the induced maps of symmetric powers, hence we have

$$E_{p,q}^{2} = H_{p+q}(S_{D}^{q} \mathbf{L}_{B-C|A}) = H_{p+q}(S_{D}^{q} K(H_{2}(\mathbf{L}_{B-C|A}), 2))$$

$$= \begin{cases} 0 & \text{if } p+q \leq n, p \neq q \\ \Gamma_{D}^{q} H_{2}(\mathbf{L}_{B-C|A}) & \text{if } p+q \leq n, p = q. \end{cases}$$

If (i) holds we can take $n = \infty$ and so

$$E_{p,q}^2 = \begin{cases} 0 & \text{if } p \neq q \\ \Gamma_D^q H_2(\mathbf{L}_{B-C|A}) & \text{if } p = q. \end{cases}$$

So the spectral sequence (I) degenerates showing that the edge homomorphism $\delta: \Gamma_D H_2(\mathbf{L}_{B-C|A}) \to H(D \overset{L}{\otimes}_Y D)$ is an isomorphism. Therefore $\phi: \Gamma_D H_2(D \overset{L}{\otimes}_Y D) \to H(D \overset{L}{\otimes}_Y D)$ is an isomorphism.

Now assume that (ii) holds. We will prove by induction on n that $L_{B-C|A} \to K(H_2(L_{B-C|A}), 2)$ is an n-equivalence for all n. Assuming $n \ge 2$ and that it is an n-equivalence, to see that it is an (n+1)-equivalence we have to prove that $E_{n,1}^2 = H_{n+1}(L_{B-C|A}) = 0$.

Since

$$E_{p,q}^{2} = \begin{cases} 0 & \text{if } p+q \leq n, p \neq q \\ \Gamma_{D}^{q} H_{2}(\mathbf{L}_{B-C|A}) & \text{if } p+q \leq n, p = q \end{cases}$$

the only possible non zero differential coming from $E_{n,1}^2$ is

$$E_{n,1}^2 = E_{n,1}^p \xrightarrow{d^p} E_{p,p}^p = E_{p,p}^2$$
 with $n = 2p$.

As the edge homomorphism is an isomorphism we have $d^p = 0$. So $E_{n,1}^2 = E_{n,1}^{\infty} = 0$.

Since $H(Y) = \text{Tor}^A(B, C)$, Theorem 1 is a consequence of Proposition 4 and the following general result.

PROPOSITION 5. Let Y be a simplicial ring and $D = H_0(Y)$. Then the following conditions are equivalent:

(ii) The D-module $H_2(D \otimes_Y D)$ is flat and the canonical homomorphism

$$\phi: \Gamma_D H_2(D \overset{L}{\otimes}_Y D) \longrightarrow H(D \overset{L}{\otimes}_Y D)$$

is an isomorphism.

(iii) The D-module $H_1(Y)$ is flat and the canonical homomorphism

$$\gamma: \wedge_D H_1(Y) \longrightarrow H(Y)$$

is an isomorphism.

Before proceeding to the proof of Proposition 5 we will need some remarks.

Remark 6. By [Q₁, Theorem 6-b), p.II.6.8] there exists a spectral sequence

$$E_{p,q}^2 = \operatorname{Tor}_p^{H(Y)}(D, D)_q \Rightarrow H(D \overset{L}{\otimes}_Y D). \tag{II}$$

Since Y is a simplicial ring and D is a simplicial Y-algebra, this spectral sequence is of bigraded algebras with divided powers.

In fact we have the following. Let Y be a simplicial ring and D a simplicial Y-algebra. Then, on the lines of the construction of $[Q_1, pp.II.6.13-6.14]$, it is possible to generalize the "step by step" method to obtain a bisimplicial Y-algebra P and a morphism $P \rightarrow D$ such that:

- (1) For each j, $P_{*,j}$ is a free simplicial Y_j -algebra resolution of D_j .
- (2) For each i, the graded H(Y)-algebra $H(P_{i,*})$ is free as an H(Y)-module and the induced sequence

$$\cdots \longrightarrow H(P_{2,*}) \longrightarrow H(P_{1,*}) \longrightarrow H(P_{0,*})$$

is a resolution of H(D).

The details of the construction of P are in [B].

Now, if E is another simplicial Y-algebra and $Y \xrightarrow{i} Q \xrightarrow{p} E$ is a factorization of the canonical morphism $Y \to E$ with i cofibration and p trivial fibration, then we have a bisimplicial Y-algebra

$$M_{i,j} = P_{i,j} \otimes_{Y_j} Q_j.$$

From the two spectral sequences of a double complex, we obtain

$$E_{p,q}^2 = \operatorname{Tor}_p^{H(Y)}(H(D), H(E))_q \Rightarrow H(D \overset{L}{\otimes}_Y E).$$

This spectral sequence is of bigraded algebras with divided powers since it comes from a bisimplicial algebra.

Remark 7. Let L be a flat D-module and consider on the bigraded algebra $\wedge_D L \otimes_D \Gamma_D L$ the unique D-derivation of bidegree (1, -1) such that $d(y \otimes 1) = 0$, $d(1 \otimes \gamma_p x) = x \otimes \gamma_{p-1} x$, $x, y \in L$. Then $(\wedge_D L \otimes_D \Gamma_D^* L, d_*)$ is a flat resolution of the $\wedge_D L$ -module D: by Lazard's Theorem, we can assume that L is a free D-module of finite type and then by Künneth formula we can take L = D. In this case it is clear.

This flat resolution $M_* = \wedge_D L \otimes_D \Gamma_D^* L$ is graded in the following way: $M_{*,i} = \wedge_D^{i-*} L \otimes_D \Gamma_D^* L$. Using this resolution we deduce

$$\operatorname{Tor}_{p}^{\wedge_{D}L}(D,D)_{q} = \begin{cases} 0 & \text{if } p \neq q \\ \Gamma_{D}^{p}L & \text{if } p = q. \end{cases}$$

Now we come to the proof of Proposition 5. Consider the spectral sequence of Remark 6

$$E_{p,q}^2 = \operatorname{Tor}_p^{H(Y)}(D, D)_q \Rightarrow H_{p+q}(D \overset{L}{\otimes}_Y D).$$

For it, the following hold:

$$E_{p,q}^2 = \text{Tor}_p^{H_0(Y)}(D, D) = 0$$
 if $p > 0$

$$E_{0,q}^2 = (D \otimes_{H(Y)} D)_q = \begin{cases} 0 & \text{if } q > 0 \\ D & \text{if } q = 0 \end{cases}$$
 (III)

$$E_{1,q}^2 = (H_+(Y) \otimes_{H(Y)} D)_q = (H_+(Y)/H_+(Y)^2)_q$$

In particular we get an edge isomorphism $\alpha_2: H_2(D \otimes_Y D) \to E_{1,1}^2$ and an isomorphism $E_{1,1}^2 = H_1(Y)$, which show that the flatness assumptions in (ii) and (iii) are equivalent.

Let $\Lambda = \bigwedge_D H_1(Y)$ and consider the homomorphism of bigraded *D*-algebras with divided powers

$$\Gamma_D E_{1,1}^2 = \operatorname{Tor}^A(D, D) \xrightarrow{\gamma_{*,*}} \operatorname{Tor}^{H(Y)}(D, D)$$

where the equality is due to Remark 7 and $\gamma_{*,*}$ is induced by γ . Since γ is bijective in degrees $\leq n$, then $\gamma_{p,q}$ is bijective for $q \leq n$, hence in the spectral sequence (II) we have

$$E_{p,q}^{2} = \begin{cases} 0 & \text{if } p \neq q, \, q \leq n \\ \Gamma_{D}^{p} E_{1,1}^{2} & \text{if } p = q, \, q \leq n \end{cases}$$
 (IV)

When (iii) holds we can take $n = \infty$ and get an isomorphism of graded D-algebras with divided powers $\beta: \Gamma_D E_{1,1}^2 \xrightarrow{\sim} H(D \otimes_Y D)$, hence $\phi = \beta \circ \Gamma_D \alpha_2$ is bijective.

Conversely, let (ii) hold. Since γ_n is an isomorphism for n = 0, 1, assume by induction that γ_j is bijective for $j \le n$ and some $n \ge 1$. We have for $p \le n$ a diagram

$$H_{2p}(D \overset{L}{\otimes}_{Y} D) \xrightarrow{\alpha_{2p}} E_{p,p}^{2}$$

$$\downarrow^{\phi_{2p}} \qquad \uparrow^{\phi_{2p}} \qquad \uparrow^{\phi_{2p}}$$

$$\Gamma_{D}^{p} H_{2}(D \overset{L}{\otimes}_{Y} D) \xrightarrow{\Gamma_{D}^{p} \alpha_{2}} \Gamma_{D}^{p} E_{1,1}^{2}$$

which is commutative because α_{2p} is an edge homomorphism in a spectral sequence (II) of D-algebras with divided powers. Note that ϕ_{2p} is an isomorphism by condition (ii), and ψ_{2p} is an isomorphism by (IV). So α_{2p} is an isomorphism and therefore all differentials of the spectral sequence are zero on $E_{p,p}^r$ when $p \le n$ and $r \ge 2$. In particular, no differential lands in $E_{1,2}^r$, $E_{1,n+1}^r$, or $E_{2,n+1}^r$ for $n \ge 2$ and $r \ge 2$. Any differential leaving one of these modules lands into some $E_{p,*}^r$ with $p \le 0$, which is trivial. Thus $E_{1,2}^2 = E_{1,2}^\infty$, $E_{1,n+1}^2 = E_{1,n+1}^\infty$ and $E_{2,n+1}^2 = E_{2,n+1}^\infty$ for $n \ge 2$. We have $E_{p,q}^\infty = 0$ if p + q is odd, and the diagram implies $E_{p,q}^\infty = 0$ when $p \ne q$ and p + q is even $\le 2n$. Therefore $E_{1,2}^2 = E_{1,n+1}^2 = E_{2,n+1}^2 = 0$ for $n \ge 2$.

By (III) we have $\operatorname{Coker}(\gamma_{n+1}) = E_{1,n+1}^2$, hence γ_{n+1} is surjective. In order to determine $\operatorname{Ker}(\gamma_{n+1})$ we consider an exact sequence of *D*-modules

$$F''_{n+1} \xrightarrow{\eta} F'_{n+1} \longrightarrow \Lambda_{n+1} \xrightarrow{\gamma_{n+1}} H_{n+1}(Y) \longrightarrow 0$$

in which F''_{n+1} and F'_{n+1} are free. It produces a complex of graded Λ -modules

$$\Lambda \otimes_D F''_{n+1} \xrightarrow{\Lambda \otimes_D \eta} \Lambda \otimes_D F'_{n+1} \longrightarrow \Lambda \xrightarrow{\gamma} H(Y) \longrightarrow 0$$

which is exact in degrees $\leq n+1$. Thus, by using appropriate graded free Λ -modules G, G', G'' with $G_j = G_j'' = G_j'' = 0$ for $j \leq n+1$ we can modify it to obtain the beginning of a graded free resolution of the graded Λ -module H(Y) in the form

$$(\Lambda \otimes_D F''_{n+1}) \oplus G'' \longrightarrow (\Lambda \otimes_D F'_{n+1}) \oplus G' \longrightarrow \Lambda \oplus G \longrightarrow H(Y) \longrightarrow 0.$$

With its help we see that

$$\operatorname{Tor}_{1}^{\Lambda}(D, H(Y))_{j} = \begin{cases} 0 & \text{if } j \leq n \\ \operatorname{Coker}(\eta) = \operatorname{Ker}(\gamma_{n+1}) & \text{if } j = n+1. \end{cases}$$

Since γ is surjective in degrees $\leq n+1$, the projection $Q=H(Y)/H_1(Y)H(Y)\to H(Y)/H_+(Y)=D$ is bijective in these degrees and thus induces isomorphisms

$$\operatorname{Tor}_{2}^{H(Y)}(D, Q)_{j} = \operatorname{Tor}_{2}^{H(Y)}(D, D)_{j} = E_{2,j}^{2}$$
 for $2 \le j \le n+1$.

These observations and Remark 7 show that the standard change of rings exact sequence

$$\operatorname{Tor}_{2}^{\Lambda}(D,D) \longrightarrow \operatorname{Tor}_{2}^{H(Y)}(D,Q) \longrightarrow (D \otimes_{H(Y)} \operatorname{Tor}_{1}^{\Lambda}(D,H(Y)))$$

$$\longrightarrow \operatorname{Tor}_{1}^{\Lambda}(D,D) \longrightarrow \operatorname{Tor}_{1}^{H(Y)}(D,Q) \longrightarrow 0$$

reduces in degree n + 1 to an exact sequence

$$\operatorname{Tor}_{2}^{\Lambda}(D,D)_{n+1} \xrightarrow{\gamma_{2,n+1}} E_{2,n+1}^{2} \longrightarrow \operatorname{Tor}_{1}^{\Lambda}(D,H(Y))_{n+1} \longrightarrow 0.$$

For n = 1 the map $\gamma_{2,2}$ is bijective, and for n > 1 the module $E_{2,n+1}^2$ is trivial, hence

Ker
$$(\gamma_{n+1}) = \text{Tor}_1^{\Lambda}(D, H(Y))_{n+1} = 0$$
 for $n \ge 1$.

Thus γ_{n+1} is injective, so the induction step is complete and the Proposition is proved.

Corollary 2 follows from Theorem 1 taking C = B and so D = B and $H_n(A, B, C, -) = H_{n-1}(A, B, -)$ [A₂, Example 5].

For Corollary 3, let τ be the set of generators τ_m to which the Koszul complex E is associated. Let R be the free A-algebra with variables t_m and consider the A-algebra homomorphisms $\beta: R \to A$, $\omega: R \to A$, such that $\beta(t_m) = 0$, $\omega(t_m) = \tau_m$. Denote by A_{β} and A_{ω} the corresponding R-algebra structures on A. Then there exists an isomorphism of graded B-algebras $H_*(E) = \operatorname{Tor}_*^R(A_{\beta}, A_{\omega})$. Moreover $[A_2, Example 6]$, $H_n(R, A_{\beta}, A_{\omega}, -) = H_n(A, B, -)$ for $n \ge 3$, and the first part of Corollary 3 follows from Theorem 1.

In order to prove the second part of Corollary 3, we will need some facts about the cotangent complex $L_{B|A}$. Let X be a simplicial resolution of the A-algebra B obtained by the "step by step" construction $[A_1, Chap. IX]$, $[A_2, p. 327]$. In particular consider the first three steps. We begin by choosing a system of generators t_{α} of the ideal I. The first step is a simplicial A-algebra K with $K_0 = A$, K_1 the polynomial A-algebra on the variables T_{α} and K_{1+h} , h > 0, is the polynomial A-algebra on the variables

$$\sigma_h^{i_1} \sigma_{h-1}^{i_2} \cdots \sigma_1^{i_h}(T_{\alpha}), \qquad 0 \leq i_h < \cdots < i_2 < i_1 \leq h,$$

where σ denotes the degeneration operators. The face operators are determined by

$$\varepsilon_1^0(T_\alpha) = 0, \qquad \varepsilon_1^1(T_\alpha) = t_\alpha.$$

In order to construct the second step, we choose representants $s_v \in K_1$ of a set of generators of the *B*-module

$$\pi_1(K) = \frac{M \cap N}{MN}$$

where M is the ideal of K_1 generated by the elements T_{α} and N the ideal of K_1 generated by the elements $T_{\alpha} - t_{\alpha}$. The second step is a simplicial K-algebra F with $F_0 = K_0$, $F_1 = K_1$, F_2 is the polynomial K_2 -algebra on the variables S_v , and F_{2+h} , h > 0, is the polynomial K_{2+h} -algebra on the variables

$$\sigma_{1+h}^{i_1} \sigma_h^{i_2} \cdots \sigma_2^{i_h}(S_n), \qquad 0 \leq i_h < \cdots < i_2 < i_1 \leq 1+h.$$

The face operators are determined by

$$\varepsilon_2^0(S_v) = 0, \qquad \varepsilon_2^1(S_v) = 0, \qquad \varepsilon_2^2(S_v) = s_v.$$

Similarly the third step is constructed by choosing representants $z_w \in F_2$ of a set of generators of the *B*-module $\pi_2(F)$ to obtain a simplicial *F*-algebra *G* with $G_0 = F_0$, $G_1 = F_1$, $G_2 = F_2$, G_3 is the polynomial F_3 -algebra on the variables Z_w and G_{3+h} , h > 0, is the polynomial F_{3+h} -algebra on the variables

$$\sigma_{2+h}^{i_1} \sigma_{1+h}^{i_2} \sigma_h^{i_3} \cdots \sigma_3^{i_h} (Z_w), \qquad 0 \le i_h < \cdots < i_2 < i_1 \le 2+h.$$

The face operators are determined by

$$\varepsilon_3^0(Z_w) = 0, \qquad \varepsilon_3^1(Z_w) = 0, \qquad \varepsilon_3^2(Z_w) = 0, \qquad \varepsilon_3^3(Z_w) = z_w.$$

We have $L_{B|A} = J/J^2$ where J is the augmentation ideal of the simplicial B-algebra $X \otimes_A B$. Denote by N the normalization functor from simplicial B-modules. We have

$$(N(J/J^2))_3 = \bigoplus_{w} BZ_w, \qquad (N(J/J^2))_2 = \bigoplus_{v} BS_v$$

and the image of the differential d_3 of $N(J/J^2)$ coincides with the image of the canonical homomorphism $[A_2, Remarque 23]$

$$\pi_2(F) \to \bigoplus_v BS_v.$$

Therefore $[A_2, Proposition 24]$

Coker
$$d_3 = \pi_1(K)$$
.

Moreover

$$(N(J/J^2))_1 = \bigoplus_{\alpha} BT_{\alpha}$$

and the differential d_3 places in a commutative diagram

$$\bigoplus_{v} BS_{v} \xrightarrow{d_{3}} \bigoplus_{\alpha} BT_{\alpha}$$

$$\pi_{1}(K)$$

where π is the homomorphism sending S_v on the generator represented by s_v and ϕ the canonical homomorphism [A₂, Remarque 23].

On the other hand

$$\pi_1(K) = \frac{M \cap N}{MN} = \operatorname{Tor}_1^{K_1}(A, A)$$

where in the first variable in Tor is the structure given by ε_1^0 and in the second the one given by ε_1^1 . If E denotes the Koszul complex associated to the elements t_{α} , then this Tor is isomorphic to $H_1(E)$. Moreover, through this isomorphism $\pi_1(K) = H_1(E)$, the homomorphism $\phi: \pi_1(K) \to \bigoplus_{\alpha} BT_{\alpha}$ corresponds to the canonical homomorphism

$$H_1(E) \longrightarrow E_1 \otimes_{\mathcal{A}} B = \bigoplus_{\alpha} BT_{\alpha}$$

induced by the inclusions of cycles and boundaries $Z_1(E) \subset E_1$, $B_1(E) \subset IE_1$. Thus we have the following proposition:

PROPOSITION 8. Let A be a ring, I an ideal of A, B = A/I and E the Koszul complex associated to an arbitrary set of generators of I. Then we can choose $\mathbf{L}_{B|A}$ satisfying:

- (i) The cokernel of the differential d_3 of $L_{B|A}$ is a B-module isomorphic to $H_1(E)$.
- (ii) There exists a morphism of complexes

$$\mathbf{L}_{B|A} \longrightarrow (H_1(E) \stackrel{\phi}{\longrightarrow} E_1 \otimes_A B)$$

where the second complex is concentrated in degrees 2 and 1. This morphism induces isomorphisms in homology in dimensions ≤ 2 .

Now the cohomological part of Corollary 3 follows from the homological part and Proposition 8.

Remark 9. From Theorem 1 it follows that $H_j(A, B, C, -) = 0$ for all $j \ge 2$ if and only if $\operatorname{Tor}_p^A(B, C) = 0$ for all $p \ge 1$. This result is due to André [A₂, Remarque 39].

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