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# Constructing taut foliations 

Rachel Roberts

## 0. Introduction

In ([5]-[13]), David Gabai uses taut foliations to prove an impressive collection of results. In this paper, we use a refinement of Gabai's techniques to construct taut foliations which permit a pleasingly simple description. For certain knots $k$, the foliations constructed allow one to conclude that $k$ has Property P: no nontrivial Dehn surgery on $k$ yields a simply-connected manifold. In particular, it follows that a large class of alternating knots satisfies Property P. More precisely, we obtain the following.

THEOREM 0.1. Let $k$ be a knot in $S^{3}$ with alternating projection in which there is nontrivial nesting among the Seifert cycles. Then any manifold obtained from $S^{3}$ by nontrivial Dehn surgery along $k$ contains a taut foliation. Hence, nontrivial Dehn surgery along $k$ yields a manifold with infinite fundamental group.

THEOREM 0.2. Let $k$ be a nontrivial knot in $S^{3}$ with alternating projection in which there is no nesting among the Seifert cycles. Then either all half-twists are positive and any manifold obtained from $S^{3}$ by Dehn surgery along $k$ with nonnegative surgery coefficient contains a taut foliation or else all half-twists are negative and any manifold obtained by Dehn surgery with nonpositive surgery coefficient contains a taut foliation.

Motivated by the fact that any closed manifold can be realized by Dehn surgery along some knot or link in $S^{3}$, the construction proceeds by first yielding taut foliations in certain knot exteriors. Each foliation possesses only noncompact leaves and meets the boundary torus in parallel curves of some slope $r$; it therefore extends to a foliation without compact leaves in the closed manifold obtained by Dehn filling with coefficient $r$. In Section 1 we describe the construction for alternating knots. In Section 2 we give a technical description of those knots for which the construction may proceed.

Wanting to promote the theory of essential laminations (see [15]), we hasten to add that the objects first recognized were the underlying laminations and it is on the
level of laminations that most of the interesting behaviour occurs (see Theorem 2.3 and Example 2.4). The foliations were in fact discovered as we sought to generalize the essential lamination constructions of Allen Hatcher [19] - an approach which has proved useful on at least two other occasions (see [4, 22]).

This paper is a restatement of results found in [25]. The author would like to extend warmest thanks to Allen Hatcher for his encouragement and guidance as her thesis advisor.

## Definitions and basic constructions

Let $M$ be a compact orientable 3-manifold. If $\mathscr{F}$ is a transversely oriented codimension-1 foliation of $M$ meeting $\partial M$ transversely, the $\mathscr{F}$ is said to be taut if each leaf of $\mathscr{F}$ intersects a closed transversal to $\mathscr{F}$. The foliations constructed in this paper contain no compact leaves and hence are necessarily taut. (See [16], for example.)

The taut foliations in this paper will be constructed using branched surfaces. A branched surface $B$ is a space modelled locally on the object of Figure 1. B intersects $\partial M$ transversely in a train track $\tau$, a space modelled locally on the object of Figure 2. When $B$ lies in a 3 -manifold $M$ (respectively, $\tau$ lies in a surface $F$ ), it possesses a regular neighbourhood which is fibred as shown in Figure 3 (respectively, Figure 4). Given such a fibred neighbourhood $N(B)$, we shall denote by $\partial_{h} N(B)$ that portion of $\partial N(B)$ which lies transverse to the fibres and by $\partial_{v} N(B)$ that portion of


Figure 1. Branched surface model.


Figure 2. Train track model.


Figure 3. Fibred regular neighbourhood of the branched surface of Figure 1.


Figure 4. Fibred regular neighbourhood of the train track of Figure 2.
$\partial N(B)$ which contains subarcs of the fibres (see Figure 3). We say that a lamination $\lambda$ is carried by the branched surface $B$ (respectively, the train track $\tau$ ) if it can be isotoped to lie in a fibred neighbourhood of $\mathbf{B}$ (respectively, of $\tau$ ) transverse to all the fibres.

A measure on a train track is an assignment of weights $\alpha_{i} \geq 0$ to the branches so that the $\alpha_{i}$ satisfy the branching equations; i.e. if we have weights assigned as in Figure 5, then $a=b+c$. Similarly, a measure on a branched surface is an assignment of weights $\alpha_{i} \geq 0$ to the branches so that the $\alpha_{i}$ satisfy all branching equations; i.e.: if we have weights assigned as in Figure 6, then $d=b+c$, $e=c+f, f=a+b$. A branched surface or a train track with an assigned measure is said to be measured.


C
Figure 5. Measured train track.


Figure 6. Measured branched surface.

We shall assume the well-known result that a measured branched surface in a 3-manifold M (respectively, a measured train track $\tau$ in some surface F ) defines a codimension-1 lamination carried by $B$ in $\mathbf{M}$ (respectively, by $\tau$ in F). (See for example, $[1,23,21,18])$.

In this paper, we shall construct laminations using a slightly more general object, the $\beta$-measured branched surface. A branched surface $B$ is said to be $\beta$-measured if for some finite set $\beta=\left\{\beta_{1}, \ldots, \beta_{n}\right\}$ of simple arcs, pairwise disjoint and disjoint from the branch loci, the branched surface $B^{\prime}$ obtained by cutting $B$ open along the arcs of $\beta$ is measured.

Similarly, a train track $\tau$ is said to be $\beta$-measured if the train track $\tau^{\prime}$ obtained by cutting $\tau$ open along the set along the set $\beta$ of points $\left\{p_{1}, \ldots, p_{n}\right\}$ is measured.

Notation. Denote $A \backslash \stackrel{N}{N}(B)$ by $A \mid B$. In particular, if $F$ is a compact surface properly embedded in a compact 3 -manifold $M$, then $M \mid F$ will denote the compact
manifold $M \backslash \mathcal{N}(F)$. And if $\beta$ is a compact 1-manifold properly embedded in $F$, then $F \mid \beta$ will denote the compact surface $F \backslash \backslash ْ(\beta)$.

CONSTRUCTION 0.3. $\beta$-measured branched surfaces (and $\beta$-measured train tracks) generate laminations.

Let $\tau$ be a $\beta$-measured train track with $\beta=\left\{p_{1}, p_{2}, \ldots, p_{n}\right\}$ and $\tau^{\prime}=\pi \mid\left\{p_{1}, \ldots, p_{n}\right\}$. Identify $\stackrel{N}{N}\left(p_{i}\right)=\left(p_{i}-1, p_{i}+1\right)$ for each $i, 1 \leq i \leq n$. By assigning a measure $\mu^{\prime}$ to $\tau^{\prime}$ we define a lamination $\lambda^{\prime}$ lying in a fibred neighbourhood of $\tau^{\prime}$ everywhere transverse to the fibres. In particular, $\lambda^{\prime}$ intersects transversely the fibres $I^{-}\left(p_{i}\right)$ and $I^{+}\left(p_{i}\right)$ containing $p_{i}-1$ and $p_{i}+1$ respectively, $1 \leq i \leq n$. By attaching the endpoints of leaves $\lambda^{\prime} \cap I^{-}\left(p_{i}\right)$ to endpoints $\lambda^{\prime} \cap I^{+}\left(p_{i}\right)$ according to some monotonic bijection $\left\{f_{i}: I^{-}\left(p_{i}\right) \rightarrow I^{+}\left(p_{i}\right)\right\}$ we define laminations carried by $\tau$. (See Figure 7.)

Similarly, if $B$ is a $\beta$-measured branched surface with $B^{\prime}=B \mid\left(\beta_{1} \cup \beta_{2} \cup \cdots\right.$ $\cup \beta_{n}$ ) assigned a measure $\mu^{\prime}$, then $\mu^{\prime}$ determines a lamination fully carried by $B^{\prime}$ and by gluing products $I^{2} \times I \rightarrow I^{2} \times I$ across each cutting arc $\beta_{i}$, we obtain a lamination carried by $B$ (see Figure 7).

## Sutured manifolds and disc decompositions

We now present those elements of Gabai's theory of sutured manifolds necessary for the construction given in this paper. For a more detailed exposition, see Gabai's trilogy [5, 11, 12].


Figure 7. Gluing.

A sutured manifold is a pair $(M, \gamma)$ consisting of a compact, oriented 3-manifold $\mathbf{M}$ together with a set $\gamma \subset \partial M$ of pairwise disjoint annuli $\mathbf{A}(\gamma)$ and tori $T(\gamma)$. The interior of each component of $A(\gamma)$ contains a homologically nontrivial oriented simple closed curve. Call such a simple closed curve a suture and denote the set of sutures by $s(\gamma)$. The components of $R(\gamma)=\partial M \backslash \dot{\gamma}$ can be oriented so that components lying on opposite sides of a given suture $s^{\prime}$ are oppositely oriented with corresponding boundary components sharing the orientation of $s^{\prime} .(M, \gamma)$ is said to be taut if $M$ is irreducible and $R(\gamma)$ is Thurston-norm minimizing in $H_{2}(M, \gamma)$.

Let $(M, \gamma)$ be a sutured manifold and let $S$ be an incompressible, $\partial$-incompressible properly embedded surface in $M$. Suppose $\partial S$ intersects both $\gamma$ and $s(\gamma)$ transversely with each arc of $\partial S \cap A(\gamma)$ intersecting the corresponding suture exactly once. Suppose further that no component of $\partial S$ bounds a disc $D$ in $M \backslash \dot{\gamma}$ and that no component of $S$ is a disc $D$ with $\partial D$ contained in $M \backslash \dot{\gamma}$. Then $S$ defines a sutured manifold decomposition

$$
(M, \gamma) \stackrel{S}{\rightarrow}\left(M^{\prime}, \gamma^{\prime}\right)
$$

where

$$
\begin{aligned}
& M^{\prime}=M \mid S, \\
& \gamma^{\prime}=\left(\gamma \cap M^{\prime}\right) \cup N\left(S_{+}^{\prime} \cap R_{-}(\gamma)\right) \cup N\left(S_{-}^{\prime} \cap R_{+}(\gamma)\right) \backslash \dot{\gamma}^{\prime}, \\
& R_{+}\left(\gamma^{\prime}\right)=\left(\left(R_{+}(\gamma) \cap M^{\prime}\right) \cup S_{+}^{\prime}\right) \backslash \dot{\gamma}^{\prime} \\
& R_{-}\left(\gamma^{\prime}\right)=\left(\left(R_{-}(\gamma) \cap M^{\prime}\right) \cup S_{-}^{\prime}\right) \backslash \dot{\gamma}^{\prime} .
\end{aligned}
$$

The sutured manifold decomposition is said to be well-groomed if both ( $M, \gamma$ ) and ( $M^{\prime}, \gamma^{\prime}$ ) are taut, no subset of toral components of $S \cap R(\gamma)$ is homologically trivial in $H_{2}(M)$, and for every component $V$ of $R(\gamma)$, either $S \cap V$ is a union of parallel, coherently oriented, nonseparating simple closed curves or $S \cap V$ is a union of parallel arcs such that for each component $\delta$ of $\partial V,|\delta \cap \partial S|=|\langle\delta, \partial S\rangle|$ (Definition $0.2,[11]$ ).

A sutured manifold decomposition is called a disc decomposition if the cutting surface $S$ is a disc. A sutured manifold ( $M, \gamma$ ) is disc-decomposable if there exists a sequence of disc decompositions

$$
(M, \gamma) \stackrel{D_{1}}{\sim}\left(M_{1}, \gamma_{1}\right) \xrightarrow{D_{2}} \cdots \stackrel{D_{n}}{\sim}\left(M_{n}, \gamma_{n}\right)
$$

where $\left(M_{n}, \gamma_{n}\right)$ is a disjoint union of copies of the sutured manifold $\left(D^{2} \times I, \partial D^{2} \times I\right)$.


Figure 8. Constructing branched surfaces.

CONSTRUCTION 0.4. Constructing Branched Surfaces from Sutured Manifold Decompositions.

Note that if $B$ is a transversely oriented branched surface properly embedded in a compact orientable 3-manifold $M$, then ( $M \backslash \stackrel{N}{ }(B)$, $\partial_{v} N(B)$ ) is a sutured manifold. Furthermore, given a sutured manifold decomposition

$$
\left(M \backslash \stackrel{N}{ }(B), \partial_{v} N(B)\right) \stackrel{s}{\rightarrow}\left(M^{\prime}, \gamma^{\prime}\right)
$$

where $B$ is a transversely oriented branched surface in an oriented 3-manifold $M$, we can view $S$ as intersecting $B$ as modelled in Figure 8.

By choosing an orientation on $S$, we induce branching as shown in Figure 8. The resulting object is again a transversely oriented branched surface $B^{\prime}$ (denote it also by $\langle B, S\rangle$ ) with

$$
\left(M^{\prime}, \gamma^{\prime}\right)=\left(M \backslash \stackrel{N}{N}\left(B^{\prime}\right), \partial_{v} N\left(B^{\prime}\right)\right) .
$$

We shall say that $B$ is a branched surface of depth $n$ if it can be constructed by applying Construction 0.2 to a sequence of sutured manifold decompositions

$$
(M, \partial M) \stackrel{s_{0}}{\sim}\left(M_{0}, \gamma_{0}\right) \stackrel{s_{1}}{\sim}\left(M_{1}, \gamma_{1}\right) \stackrel{s_{2}}{\sim} \cdots \stackrel{s_{n}}{\sim}\left(M_{n}, \gamma_{n}\right)
$$

for some manifold $M$ with $\partial M$ a (possibly empty) union of tori, where $n$ is minimal over all such sequences.

## 1. The construction for alternating knots

In [12], Gabai proves that any knot exterior contains taut foliations which intersect the boundary torus in parallel curves of slope 0 and remain taut after longitudinal Dehn filling. In this section we see that when $k$ is an alternating knot, this result generalizes to yield taut foliations in $S^{3} \backslash N(k)$ realizing an infinite interval of boundary slopes and remaining taut after the corresponding Dehn fillings.

Let $M=S^{3} \backslash N(k)$ where $k$ is an alternating knot. For these knot exteriors, there is a particularly simple construction of $\beta$-measured branched surfaces $\mathbf{B}$ with nice boundary behaviour. One constructs $B$ as a depth one branched surface $B=B_{1} \supset B_{0}$ where $B_{0}$ is the surface obtained by applying Seifert's algorithm to a regular alternating projection of $k$ and $B_{1}$ is obtained from $B_{0}$ by adding one of the discs properly embedded in $M=S^{3} \backslash N\left(B_{0}\right)$.

More precisely, we proceed as follows. Recall that given a regular projection of a link $k$ in the plane, Seifert's algorithm yields a Seifert surface R for $k$ as follows. After choosing an orientation for $k$, a system of Seifert cycles is obtained by removing crossing points according to orientation as shown in Figure 9. The Seifert cycles bound disjoint discs which can be connected using half-twisted bands as dictated by the original crossings. (See [26], p. 120).

CONVENTION. We call a right-handed half-twist positive and a left-handed half-twist negative. (See Figure 10.)

As in [7, p. 393), it proves useful to divide the set of alternating links into two classes. We shall say a link $k$ is nonplanar if it possesses an alternating projection in


Figure 9. Removing the crossings.


Figure 10. a. Positive half-twist b. Negative half-twist.
which there is nontrivial nesting among the Seifert cycles, i.e., for some Seifert cycle $\gamma$, both complementary regions of $\gamma$ contain Seifert cycles. (By trivial nesting we mean any nesting which may be removed by untwisting half-twisted bands which separate R.) Otherwise, we say that $k$ is planar.

THEOREM 1.1. Let $k$ be a nonplanar alternating knot. Then $M=S^{3} \backslash \stackrel{N}{ }(k)$ contains a set $\mathscr{F}$ of foliations such that for any finite number $r$, there is an element of $\mathscr{F}$ which intersects the boundary torus of $M$ in parallel curves of slope $r$. The foliations realizing the nonzero boundary slopes have no compact leaves.

COROLLARY 1.2. Let $k$ be a nonplanar alternating knot. Then any manifold obtained from $S^{3}$ by nontrivial Dehn surgery along $k$ contains a taut foliation.

Proof. Let $\hat{M}$ be the manifold obtained from $S^{3}$ by Dehn surgery along $k$ with rational coefficient $r \neq 0$. Theorem 1.1 guarantees the existence of a foliation in $S^{3} \backslash \stackrel{N}{N}(k)$ with only noncompact leaves and meeting $\partial N(k)$ in parallel simple closed curves of slope $r . F$ can therefore be completed to a foliation $\hat{F}$ in $\hat{M}$ by capping off the boundary components of the leaves of $F$ with discs in $\hat{M} \backslash \hat{M}$. Since $\hat{F}$ can have no compact leaves, it is necessarily taut (see [2]). The corresponding result when $r=0$ follows similarly and may be found in [12].

Recall that a knot $k$ is said to satisfy Property $P$ if no nontrivial surgery along $k$ yields a simply-connected manifold.

COROLLARY 1.3. Nonplanar alternating knots satisfy Property $P$.
Proof. Let $\hat{M}$ be a manifold obtained from $S^{3}$ by nontrivial Dehn surgery along $k$. By Theorem 1.1, $\hat{M}$ contains a taut foliation; hence, by [24] or [15], $M$ has infinite fundamental group.

The corresponding results for $k$ planar are less satisfying. We first note that if $k$ is planar, then it has an alternating projection in which no Seifert cycle separates and hence all half-twists have the same sign. For ease of presentation, we shall say that the alternating knot is positive planar if all half-twists are positive and negative planar if all half-twists are negative.

THEOREM 1.4. Let $k$ be a negative (respectively, positive) planar knot. Then $M=S^{3} \backslash \stackrel{N}{ }(k)$ contains a set $\mathscr{F}$ of foliations such that for any nonpositive (respectively, nonnegative) finite number $r$, there is a foliation of $\mathscr{F}$ which intersects the boundary torus of $M$ in parallel curves of slope $r$. The foliations realizing the nonzero boundary slopes have no compact leaves.

COROLLARY 1.5. Let $k$ be a negative (respectively, positive) planar alternating knot. Then any manifold obtained from $S^{3}$ by Dehn surgery along $k$ with nonpositive (respectively, nonnegative) surgery coefficient contains a taut foliation.

Proof (Theorems 1.1 and 1.4).
Case 1. Consider first the case that $k$ is nonplanar. Choose a regular alternating projection in which there is non-trivial nesting and let R be the surface generated by Seifert's construction. Since we can untwist trivial twists as necessary we may assume that no half-twisted band separates R.


Figure 11. Disc D and annular subsurface.

Let $\gamma$ be a Seifert cycle separating the plane into two regions $A$ and $A^{\prime}$, both of which contain Seifert cycles.

Let $G$ be the planar graph obtained by letting vertices correspond to the cycle $\gamma$ and those Seifert cycles in A which are maximal (not nested within another Seifert cycle of A) and letting edges correspond to the half-twisted bands connecting these maximal cycles.

Since no half-twisted band separates R , no edge of $G$ separates and hence $G$ contains a cycle $\sigma$; choose $\sigma$ innermost. Note that since $R$ is orientable, $\sigma$ contains an even number of edges. Furthermore, since $k$ is alternating, the edges of $G$ correspond to consistently oriented half-twists. In particular, $\sigma$ corresponds to an annular subsurface X of R and a properly embedded disc $D \subset M=S^{3} \backslash \stackrel{\circ}{N}(k)$ as shown in Figure 11.

Similarly, corresponding to $A^{\prime}$, define $G^{\prime}, \sigma^{\prime}, X^{\prime}$ and $D^{\prime}$.
Since $k$ is alternating, the edges of $G$ and those of $G^{\prime}$ have opposite parity. Without loss of generality, we may assume that the edges of $G$ are negatively twisted while those of $G^{\prime}$ are positively twisted.

Case 2. If the knot $k$ is negative planar, let $G$ denote the planar graph obtained by letting vertices correspond to the Seifert cycles and letting edges correspond to the half-twisted bands. Then define $\sigma, X$ and $D$ as above. If $k$ is positive planar, then proceed similarly to construct $G^{\prime}, \sigma^{\prime}, X^{\prime}$ and $D^{\prime}$.

By choosing orientations on $R, D$ and $D^{\prime}$, we obtain depth 1 branched surfaces $B=\langle R, D\rangle \supset B_{0}=R$ and $B^{\prime}=\left\langle R, D^{\prime}\right\rangle \supset B_{0}=R$. (See Figure 12.)

LEMMA 1.6. $B$ and $B^{\prime}$ are $\beta$-measured and there are sets of laminations $\mathscr{L}$ and $\mathscr{L}^{\prime}$, fully carried by $B$ and $B^{\prime}$ respectively, which intersect $\partial N(k)$ in parallel curves. The laminations in $\mathscr{L}$ realize boundary slopes $(-\infty, 0)$ and those in $\mathscr{L}^{\prime}$ realize ( $0, \infty$ ).


Figure 12. $B$.


Figure 13. $B$ is $\beta$-measured.


Figure 14. Train track near knot crossings.

Proof. By symmetry it suffices to prove the result for $B$. Let $b$ denote the (even) length of the cycle $\sigma$. As shown in Figure 13, one can choose $b / 2$ fusing arcs $\beta_{1}$ and assign weight $x>0$ to the branch corresponding to $D$ and weights $1,1+x$ to the branches corresponding to $R$ in the resulting cut-open branched surface.

So $B$ is $\beta$-measured and by choosing gluing functions along the $\beta_{i}$ we define laminations $\lambda_{x}$ fully carried by $B$.

Figure 14 depicts the behaviour of $B$ at the crossing points and the corresponding piece of the train track $\tau=B \cap \partial N(k)$.
$\tau=B \cap \partial N(k)$ is obtained by piecing together $b / 2$ copies of each of the tracks of Figure 14, in the order dictated by $k$. Note that by choosing linear gluing functions along the $\beta_{i}$, we induce linear scalings in $\tau$ over each subtrack of $\tau$ bounded by


Figure 15. After scaling.
endpoints of $\beta_{i}$. After rescaling, the tracks of Figure 14 have measures as shown in Figure 15.

So $\tau$ is measured and for each $p_{i}, i$ odd ( $i$ even), there is a slope contribution of $x /(x+1)\left(-x\right.$, respectively). Hence, the lamination $\lambda_{x}$ intersects $\partial N(k)$ in parallel curves of slope

$$
m_{\lambda}=\frac{b}{2}\left(-x+\frac{x}{x+1}\right)=\frac{b}{2}\left(\frac{-x^{2}}{x+1}\right) .
$$

As $x$ ranges over $(0, \infty)$, the boundary slopes $(-\infty, 0)$ are realized.
LEMMA 1.7. The laminations $\lambda_{x}$ contain no compact leaves.
Proof. By symmetry it suffices to consider a lamination $\lambda$ fully carried by the branched surface B. Suppose, by way of contradiction, that the lamination $\lambda$ contained a compact leaf $L$. Since $L$ is carried by $B$ it induces a non-negative measure on $B$. We show that no such measure exists.

Let the measure induced on the branch of $B$ corresponding to $D$ be $n$. Since the slope of $\partial \lambda$ is nonzero, necessarily $n>0$. The $b$ arcs of $\partial D$ cut the annular subsurface $X$ into $b$ regions with the branch conditions ensuring that for some nonnegative number $m$ these regions have weights $m$ and $m+n$ as shown in Figure 16.

Let $\partial_{-} X$ and $\partial_{+} X$ denote the two boundary components of $X$. Since $k$ is connected, there is an arc of $k$ lying outside $X$ and joining $\partial_{-} X$ to $\partial_{+} X$. But this means that there is a region weighted both $m$ and $m+n$. Hence $m=m+n$ and necessarily $n=0$, contrary to assumption.
-LEMMA 1.8. Let $\left(M_{1}, \gamma_{1}\right)$ be the sutured manifold generated by the decomposition


Figure 16. $B$.
$\left(S^{3} \backslash{ }^{\prime}(R), A(k)\right) \stackrel{D_{1}}{\sim}\left(M_{1}, \gamma_{1}\right)$
where $D_{1}$ denotes either $D$ or $D^{\prime}$. Then $\left(M_{1}, \gamma_{1}\right)$ is taut.
Proof. The following is the argument of [7, p. 393], modified slightly to suit our needs.

Case 1. Consider first the case that $k$ is a planar alternating knot. Recall that if we let $G$ denote the planar graph obtained by letting vertices correspond to the Seifert cycles and letting edges correspond to the half-twisted bands, then $D_{1}$ arises in a natural way from an innermost cycle $\sigma$ of $G$. (See Figure 11.) Figure 17 depicts the disc decomposition arising from one of the two possible choices of orientation of $D_{1}$.

We see that

$$
\left(S^{3} \backslash 尺(R), A(k)\right) \stackrel{D_{1}}{\sim}\left(M_{1}, \gamma_{1}\right)
$$

where

$$
\left(M_{1}, \gamma_{1}\right)=\left(S^{3} \backslash \stackrel{N}{N}\left(R_{1}\right), A\left(l_{1}\right)\right)
$$



Figure 17. A disc decomposition.
for some planar alternating link $l_{1}$ and corresponding Seifert surface $R_{1}$. Since $\chi\left(\partial N\left(R_{1}\right)\right)=\chi(\partial N(R))+2, \chi\left(R_{1}\right)=\chi(R)+1$. So an induction on the Euler characteristic reveals that $\left(M_{1}, \gamma_{1}\right)$ is disc decomposable and hence taut.

Case 2. Proceed now to the case that $k$ is nonplanar. Let $C_{1}, \ldots, C_{m}$ denote the separating Seifert cycles. Focus attention on one such $C_{i} . C_{i}$ separates the plane into two regions $A_{1}$ and $A_{2}$, each of which contains Seifert cycles. Let $R_{1}$ denote the


Figure 18. $N(R)$ in a neighbourhood of $C_{i}$.
surface generated by $C_{i}$ and any Seifert cycles contained in $A_{1}$, together with the corresponding half-twisted bands. Similarly, define $R_{2}$ corresponding to $A_{2}$.

Corresponding to $C_{i}$ there is a natural choice of disc decomposition, which we describe as follows. Isotope $R$ so that the disc $B_{i}^{\prime}$ bounded by $C_{i}$ is planar, lying in some sphere $S$, with $R_{1} \backslash B_{i}^{\prime}$ lying above $S$ and $R_{2} \backslash B_{i}^{\prime}$ lying below $S$; i.e., R is a Murasugi sum of $R_{1}$ and $R_{2}$. Then $N(R)$ in a neighbourhood of $C_{i}$ is a tube with the sutures describing 'flaps' hanging up or down (see Figure 18).

Choose $B_{i}=S^{3} \backslash B_{i}^{\prime}$, oriented so that $\partial B_{i}$ is oriented consistently with $C_{i}$. Performing the disc decomposition reveals that

$$
\left(S^{3} \backslash \stackrel{N}{(R)}, A(k)\right) \stackrel{B_{1}}{\rightarrow}\left(S^{3} \backslash \circ\left(R_{1}\right), A\left(\partial R_{1}\right) \sqcup\left(S^{3} \backslash \circ\left(R_{2}\right), A\left(\partial R_{2}\right)\right)\right.
$$

Repeating the process for each $i, 1 \leq i \leq m$, we obtain the sequence

$$
\left(S^{3} \backslash \stackrel{N}{ }(R), A(k)\right) \stackrel{B_{1}}{\rightarrow}\left(M_{1}^{\prime}, \gamma_{1}^{\prime}\right) \xrightarrow{B_{2}} \cdots \xrightarrow{B_{m}}\left(M_{m}^{\prime}, \gamma_{m}^{\prime}\right)
$$

where $\left(M_{m}^{\prime}, \gamma_{m}^{\prime}\right)$ is a disjoint union of sutured manifolds of the form $\left(S^{3} \backslash N(F), A(l)\right)$ for some planar alternating link $l$ and corresponding planar surface
$F$. In particular, included among the components are the complements of the surfaces corresponding to the planar graphs $G$ and $G^{\prime}$ respectively. So we may extend the above sequence by the decomposition

$$
\left(M_{m}^{\prime}, \gamma_{m}^{\prime}\right) \stackrel{D_{1}}{\sim}\left(M_{m+1}^{\prime}, \gamma_{m+1}^{\prime}\right)
$$

and by Case $1,\left(M_{m+1}^{\prime}, \gamma_{m+1}^{\prime}\right)$ is taut.
Finally, we notice that we may choose the discs $D_{1}, B_{1}, \ldots, B_{m}$ to be pairwise disjoint and hence we may rearrange the above sequence to get

$$
\left(S^{3} \backslash \stackrel{\circ}{N}(R), A(k)\right) \stackrel{D_{1}}{\sim}\left(M_{1}, \gamma_{1}\right) \stackrel{B_{1}}{\sim} \cdots \stackrel{B_{m}}{\rightarrow}\left(M_{m+1}^{\prime}, \gamma_{m+1}^{\prime}\right)
$$

Since $\left(M_{m+1}^{\prime}, \gamma_{m+1}^{\prime}\right)$ is taut, so is $\left(M_{1}, \gamma_{1}\right)$.
We next think more carefully about the way in which $\lambda_{x}$ lies in $N(B)$. In particular, we are interested in the effect on $B$ of splitting open along compact subsurfaces of leaves of $\lambda_{z}$. We call $P$ a compact surface of contact for the pair $\left(\lambda_{x}, B\right)$ if $B$ splits open along $P$, a compact subsurface of some leaf of $\lambda_{x}$, so that the components $(\partial P \times I) \backslash \partial N(k)$ bounding the $P \times I$ region created correspond exactly to some subset $\delta$ of $\partial_{c} N(B)$. In the proof of Corollary 1.10 we shall see that planar compact surfaces of contact provide a potential obstruction to extending the laminations $\lambda_{x}$ to foliations with nice boundary behaviour. Happily we have the following:

PROPOSITION 1.9. There are no compact surfaces of contact for $\left(\lambda_{x}, B\right)$.
Proof. Suppose that there exists a compact surface $P$ of contact for $\left(\lambda_{x}, B\right)$. We shall use $P$ to decompose the knot $k$ into multiple components, thereby arriving at a contradiction.

We begin by partitioning the $b$ pieces of $X \backslash N(\partial D)$ into two sets. (See Figures 11 and 12 to recall notation. In particular, recall that $\sigma$ is the core of $X$.) We do this as follows.

Let $\tau_{\sigma} \subset B$ be the train track pictured in Figure 19. The cross section of $\lambda_{x}$ over $\tau_{\sigma}$ is a lamination $\Delta_{x}$ carried by $\tau_{\sigma}$ consisting of parallel curves of 'slope' $(b / 2) x$.

CLAIM 0. $x$ is rational.
Proof. Since $B$ has a compact surface of contact $P$, necessarily $\tau$ splits open along a closed interval $J \subset P \cap N(\tau)$ such that the components $\partial J \times I$ bounding the $J \times I$ region created correspond exactly to some pair of components of $\partial_{v} N(\tau)$. But if some pair of components of $\partial_{v} N(\tau)$ bounds an $I \times I$ region in this way, then by


Figure 19. $\tau_{\sigma}$.
symmetry each component of $\partial_{v} N(\tau)$ pairs up to bound such an $I \times I$ region. Therefore $\Delta_{x}$ is obtained from $\tau_{\sigma}$ by a finite amount of splitting; hence $\Delta_{x}$ is a collection of parallel simple closed curves and $x$ is necessarily rational.

Let $N\left(\tau_{\sigma}\right)$ denote a 2-dimensional fibred neighbourhood of $\tau_{\sigma}$ carrying $\Delta_{x}$. Choose an orientation for $\sigma$ and denote by $\pi_{i}, 1 \leq i \leq m,\left(\varpi_{j}, 1 \leq j \leq n\right)$ those arcs of $\partial_{v} N\left(\tau_{0}\right) \cap \delta$ for which splitting from the corresponding cusp leads one about $\sigma$ in the positive (respectively, negative) direction. Denote by $a_{i}\left(b_{j}\right)$ the branch point of $\tau_{\sigma}$ corresponding to $\pi_{i}$ (respectively $w_{j}$ ) We shall sometimes refer to these branch points as distinguished branch points. Notice that, after relabelling as necessary, each $\pi_{i}$ pairs with $m_{i}$ to bound a rectangular component of $(P \times I) \cap N\left(\tau_{\sigma}\right)$. In particular, $n=m$.

We now assign a somewhat artificial but useful pairing to the branch points $a_{i}, b_{j}$. Start at one of the branch points $a_{i}$ and move into and along $\sigma$ in the positive
direction. Define a counter, initially set to 1 and updated each time a distinguished branch point is passed: add $+1(-1)$ if the branch point corresponds to an $a_{i}$ (respectively, $b_{j}$ ). Let $b_{i}$ be the point corresponding to the first branch point for which the counter is again zero and let $\left[a_{i}, b_{i}\right]$ denote the interval of $\sigma$ which has been traversed.

We use this pairing to partition the components $I$ of $\sigma \backslash\left\{\right.$ branch points of $\left.\tau_{\sigma}\right\}$ into two sets. We shall say that $I$ is marked if it is contained within some $\left[a_{i}, b_{i}\right]$ interval. Otherwise, we shall say that $I$ is unmarked. Note that when $I$ is marked, the set of branch points in $\left[a_{i}, b_{i}\right]$ lying to the left of $I$ and corresponding to points in $\left\{a_{m}\right\}$ outnumber those corresponding to points in $\left\{b_{n}\right\}$. So $N(I)$ must have nonempty intersection with $P \times I$. We shall use this fact later, in the proof of Claim 2. (Unmarked intervals may or may not have nonempty intersection with $P \times I$.)

This partition of the components of $\sigma \backslash\left\{\right.$ branch points of $\left.\tau_{\sigma}\right\}$ induces a partition of the $b$ regions of $X \backslash \partial D$. We shall say that a region $Y$ of $X \backslash \partial D$ is marked if the corresponding arc $Y \cap \sigma$ is marked. Otherwise, we shall say that $Y$ is unmarked.

Recalling that our goal is in fact to partition $k$ into multiple components, we establish the following two facts.

CLAIM 1. Suppose that an arc of $k \backslash \partial X$ passes between two regions $Y_{1}$ and $Y_{2}$ of equal weight (either 1 or $1+x$ ). Then $Y_{1}$ and $Y_{2}$ are of the same type: either both are marked or both are unmarked.

Proof. Imitating the construction of $\tau_{\sigma}$, we construct a train track $\tau_{\mu} \subset B$ as follows. Let $\eta$ be an arc of $R \backslash X$ connecting two regions $Y_{1}$ and $Y_{2}$ of equal weight. Choose a subarc $v$ of $\sigma$ intersecting $\partial D \cap X$ transversely in a minimal number of points and connecting the endpoints of $\eta$ and let $\mu$ be the simple closed curve obtained by concatenating $\eta$ and $\nu$. Denote by $\tau_{\mu} \subset B$ the train track obtained from $\mu$ by connecting branch points of $\partial D \cap \mu$ by disjoint arcs in $D$. $\tau_{\mu}$ contains some subset $\tau \cup \varpi$ of the distinguished branch points $a_{i}, b_{i}$ of $\tau_{\sigma}$. Each element of $\pi$ pairs up with a unique element of $m$ to bound a rectangular component of $(P \times I) \cap N\left(\tau_{\mu}\right)$; so in particular $|\pi|=|\varpi|$. But this is true only if $Y_{1}$ and $Y_{2}$ are of the same type. For suppose $Y_{1}$ is marked. Then $Y_{1} \cap\left[a_{i}, b_{i}\right] \neq \varnothing$ for some $a_{i}$. So either $Y_{2} \cap\left[a_{i}, b_{i^{\prime}}\right] \neq \varnothing$ or else one of $a_{i}, b_{i^{\prime}}$ lies in $v$. But if $a_{i}\left(b_{i^{\prime}}\right)$ lies in $v,|\pi|=|\varpi|$ implies that there is a $b_{j}$ ( $a_{j}$, respectively) in $v$ such that $Y_{2} \cap\left[a_{j}, b_{j}\right] \neq \varnothing$. In either case, $Y_{2}$ is also marked. By symmetry, it follows that $Y_{1}$ is marked if and only if $Y_{2}$ is marked. Hence, $Y_{1}$ and $Y_{2}$ are necessarily of the same type.

CLAIM 2. Suppose that an arc of $k \backslash \partial X$ passes between two regions $Y_{1}$ and $Y_{2}$ weighted respectively $1+x$ and 1 . Then $Y_{1}$ and $Y_{2}$ are unmarked.


Figure 20. $\tau_{0}$ and $\tau_{\mu}$.

Proof. Let $\eta$ be an arc in $R \backslash X$ connecting the regions $Y_{1}$ and $Y_{2}$. Choose an arc $v$ in $X$ intersecting $\partial D \cap X$ transversely in a minimal number of points and connecting the endpoints of $\eta$ and let $\mu$ be the simple closed curve obtained by concatenating $\eta$ and $\nu$. Denote by $\tau_{\mu} \subset B$ the train track with end obtained from $\mu$ by letting the 'end' correspond to the branch point $\partial D \cap v$ having nonempty intersection with $\bar{Y}_{1}$ and by connecting the remaining branch points in pairs by disjoint arcs in $D$ (see $\left(^{*}\right)$ in Figure 20). $\tau_{\mu}$ and $\tau_{\sigma}$ are related as shown in Figure 20.

As noted earlier, the compactness of $P$ implies that $x$ is necessarily rational. So we may write $x=p / q$ where $p$ and $q$ are relatively prime positive integers. We shall show that if at least one of $Y_{1}$ and $Y_{2}$ is marked then $P$ contains a properly embedded noncompact arc $\omega$.

Case 1. Suppose first that $Y_{1}$ is marked. Then, as previously noted, there is a rectangular component of $(P \times I) \cap N\left(\tau_{\sigma}\right)$, corresponding to the pair ( $\pi_{i}, w_{i}$ ) say, passing through $N\left(Y_{1}\right)$. So we may let $\omega$ initially denote a path in $P \times 0$ starting at $\pi_{i} \cap(P \times 0)$ and following $P \times 0$ into $N\left(Y_{1}\right)$. We next show that it is possible to extend $\omega$ so that it passes infinitely about $N\left(\tau_{\mu}\right)$ without encountering either an earlier point of $\omega$ or a point $\varpi_{j} \cap(P \times 0)$.

To do so we first introduce as bookkeeping device a notion of height of the leaves of $\Delta_{x}$ in the branches of $\sigma$. Recall that associated with the measured track $\tau_{\sigma}$ there is a singular foliation $\varphi_{x}$ obtained by starting with fibred rectangles $I \times$ [ 0 , branch weight] corresponding to branches of $\tau_{\sigma}$ and identifying these rectangles at branch points using the branch equations ([18], p. 74). Define the height of a leaf of $\varphi_{x}$ in the fibred rectangle $I \times[0$, branch weight $]$ to be the image of projec-
tion onto the second factor. We define the height of a leaf in $\Delta_{x}$ carried by a branch of $\tau_{\sigma}$ to be the image of the composition

$$
\Delta_{x} \rightarrow \varphi_{x} \rightarrow[0, \text { branch weight }]
$$

where the first map is the function given by pinching $\Delta_{x}$ down along the vertical fibres to obtain $\varphi_{x}$.

Returning to the construction of $\omega$, note that $\tau_{\sigma}$ can be split to simple closed curve(s) of width $1 / q$. So the endpoint of $\omega$ in $N\left(Y_{1}\right)$ can lie only at one of the heights $1 / q, 2 / q, \ldots,(p+q) / q$ in $N\left(Y_{1}\right)$. Hence $\omega$ enters $N\left(\tau_{\mu}\right)$ at height $n_{1} / q$ for some integer $n_{1}, 1 \leq n_{1} \leq p+q$. If $n_{1}=p+q$, we may follow $\omega$ along branch (*) of $\tau_{\sigma}$ (see Figure 20) and reenter $\left(N\left(\tau_{\mu}\right)\right.$ at level $q / q$. So we may suppose $1 \leq n_{1}<p+q$.

We next show that it is possible to extend $\omega$ so that it passes infinitely often about $\tau_{\mu}$ without encountering a self-intersection or an endpoint $\varpi_{j} \cap(P \times 0)$. From $N\left(Y_{1}\right)$, extend $\omega$ to follow $P \times 0$ about $\eta$. The path thus described exits $N(\eta)$ at level

$$
\left(\frac{1}{1+\frac{p}{q}}\right)\left(\frac{n_{1}}{q}\right)=\frac{n_{1}}{p+q}
$$

Continuing $\omega$ about $\tau_{\mu}$ until it again enters $N\left(Y_{1}\right)$, we see that $\omega$ meets an endpoint $\varpi_{j} \cap(P \times 0)$ or returns to the initial level only if

$$
\frac{n_{1}}{p+q}=\frac{m}{q}
$$

for some integer $m$. But this would imply

$$
(p+q) \mid n_{1}
$$

which is impossible since $1 \leq n_{1}<p+q$. We may therefore extend $\omega$ so that it reenters $\eta$ at height $n_{1} /(p+q)+n_{2}^{\prime} / q$ for some integer $n_{2}^{\prime}$.

More generally, after $r$ such passes through $\eta$, the path $\omega$ is at height

$$
\frac{n_{1} q^{r-1}}{(p+q)^{r}}+\frac{n_{2} q^{r-2}}{(p+q)^{r-1}}+\cdots+\frac{n_{r}}{p+q}
$$

So $\omega$ further extended through $\tau_{\mu}$ to $N\left(Y_{1}\right)$ can meet an endpoint $w_{j} \cap(P \times 0)$ or return to an earlier height only if

$$
\frac{n_{1} q^{r-1}}{(p+q)^{r}}+\frac{n_{2} 1 q^{r-2}}{(p+q)^{r-1}}+\cdots+\frac{n_{r}}{p+q}=\frac{m_{1} q^{r-2}}{(p+q)^{r-1}}+\cdots+\frac{m_{r-1}}{p+q}+\frac{m_{r}}{q}
$$

Again, elementary number theory reveals that this is possible only if $(p+q) \mid n_{1}$. Since $1 \leq n_{1}<p+q$, equality never holds.

Hence, $\omega$ may be extended infinitely often to yield a properly embedded noncompact arc in $P \times 0$. Since $P$ is compact, this is impossible; so $Y_{1}$ must be unmarked.

Case 2. Case 2 proceeds similarly. We suppose, by way of contradiction, that $Y_{2}$ is marked. This implies that there is an arc $\omega$ in $P \times 0$ which enters $Y_{2}$ at some height $n_{1} / q, 1 \leq n_{1} \leq q$. Start at $\pi_{i} \cap(P \times 0)$ and follow $P \times 0$ into $N\left(Y_{2}\right)$. If $n_{1}=q$ then $\omega$ may be extended to follow $P \times 0$ about $\eta$ and along branch (*) of $\tau_{\sigma}$ (see Figure 14), reentering $Y_{1}$ at height $q / q$. The argument for Case 1 therefore applies. So suppose $n_{1}<q$. The argument proceeds as in Case 1 except that we extend $\omega$ to follow $P \times 0$ about $\tau_{\mu}$ in the opposite direction; so the scaling factor is inverted and $P \times 0$ might lead infinitely often through branch $\left(^{*}\right)$ of $\tau_{\sigma}$.

In the proof of Lemma 1.7, we saw that there is a subarc of $k \backslash \partial X$ joining two regions of $X \backslash \partial D$ of unequal weight. Hence, Claim 2 guarantees the existence of a $(1+x)$-weighted region $Y$ which is unmarked.

In Figure 21 we see a subset of $X$ containing the region $Y$ and also the local behaviour of $s\left(\gamma_{1}\right)$. (Notice that we abuse notation and identify $k$ with $A(k)=\partial N(k) \backslash \partial R$.)


Figure 21. A subset of $X$.

Recall that $\delta$ denotes the subset of $\partial_{n} N(B)=A\left(\gamma_{1}\right)$ bounding the compact region $P \times I$. Since $Y$ is unmarked, the components of $s\left(\gamma_{1}\right)$ labelled $s^{\prime}$ in Figure 21 do not belong to $\delta$. So the subarc $\rho \subset k$ shown in Figure 21 is disjoint from both $\delta$ and marked regions. Consider starting at a point of $\rho$ and travelling along $k . \rho$ leaves $X$ from an unmarked region. So, by Claim 1, the subarc of $k \backslash \partial X$ passing from $\rho$ back into $X$ leads to an unmarked region. Examining Figure 21, we see that within $X$, an arc of $k \cap X$ may pass from an unmarked region to a marked region only if it begins as an arc coincident with $\delta$. Furthermore, an arc of $k \cap X$ disjoint from $\delta$ may reach an arc coincident with $\delta$ only from a marked region.

Hence, the component of $k$ containing $\rho$ is disjoint from $\delta$. Since $k$ is connected, necessarily $\delta=\varnothing$; so $P=\varnothing$.

COROLLARY 1.10. The laminations $\lambda_{x}$ extend to foliations which meet $\partial N(k)$ in parallel curves and which possess only noncompact leaves.

Proof. We apply Gabai's inductive construction (Description 2, [12]). Return to the taut sutured manifold $\left(M_{1}, \gamma_{1}\right)$. Since taut, it possesses a well-groomed sutured manifold hierarchy (Theorem 4.2, [5]):

$$
\left(M_{1}, \gamma_{1}\right) \xrightarrow{s_{2}}\left(M_{2}, \gamma_{2}\right) \xrightarrow{s_{3}} \cdots \xrightarrow{s_{n}}\left(M_{n}, \gamma_{n}\right)
$$

Notice that we may assume that the $S_{i}, i \geq 2$, intersect the sutures $s\left(\gamma_{i-1}\right)$ away from $k$. For each $i, 1 \leq i \leq n$, let $B_{i}$ denote the branched surface obtained by applying Construction 0.4 to this sequence. We construct the desired foliation by inductively adding leaves corresponding to $S_{2}, \ldots, S_{n}$.

Consider $\partial S_{2} \cap M_{1}$. We first extend the surface $S_{2}$ to a surface $S_{2}^{\prime}$ properly embedded in the complement of $\lambda_{1}$ and intersecting a finite collection $L_{1}, \ldots, L_{s}$ of leaves of $\lambda_{1}$ in properly embedded curves $C$ which are pairwise disjoint and disjoint from $\partial \lambda_{1}$.

By definition of well-groomed, we know that for each component $V$ of $R\left(\gamma_{1}\right)$, either $S_{2} \cap V$ is a union of parallel, coherently oriented nonseparating simple closed curves or it is a union of arcs such that for each component $\delta$ of $\partial V,\left|\delta \cap \partial S_{2}\right|=\left|\left\langle\delta, \partial S_{2}\right\rangle\right|$.

Consider first an arc $\sigma$ of $\partial S_{2} \cap A\left(\gamma_{1}\right)$. Proposition 1.9 guarantees the existence of an infinite strip extending $S_{2}$ at $\sigma$; i.e., there exists $\nabla=[0,1] \times I$ properly embedded in a piece of the complement of $\lambda_{1}$ cut off by $A\left(\lambda_{1}\right)$ and such that $\sigma$ is the image of $0 \times I$.

Consider next a component $\gamma$ of $\partial S_{2}$ contained in $A\left(\gamma_{1}\right)$. By definition of sutured manifold decomposition, $\gamma$ and the corresponding suture are consistently oriented. So we extend $S_{2}$ by an annulus $\gamma \times I$ where $\gamma \times 0=\gamma$ and $\gamma_{1}$ lies in a leaf parallel to $\partial_{+} \pi_{j}$ for some $j, 1 \leq j \leq m$.

Let $S_{2}^{\prime}$ be the leaf obtained by the above extensions. let $\mathscr{C}$ be the collection of curves $\partial S_{2}^{\prime}$. To eliminate any intersections $\left(C_{i} \times 0\right) \cap\left(C_{j} \times 1\right) \subset L$, double the leaf $L$ (Operation 2.1.2, [14]). Thicken the resulting leaves $L$ (Operation 2.1.1, [14]), cut each leaf of $L \times I$ open along the corresponding curves of $C$, and reglue. We thus obtain a lamination $\lambda_{2}$, fully carried by $B_{2}$ and intersecting $\partial N(k)$ in coherently oriented curves of slope $m_{\lambda}$. It remains only to show that $\lambda_{2}$ contains no compact leaves. To see this, we note that simple closed curves $\partial S_{2} \cap \lambda_{1}$ are nonseparating and hence give rise to infinite spiralling in the above construction. Similarly, the remaining simple closed curves of $\partial S_{2}^{\prime} \cap \lambda_{1}$ do not bound compact leaves by Proposition 1.9. Arcs $\partial S_{2} \cap R\left(\gamma_{1}\right)$ result in leaves containing properly embedded copies of $\mathbf{R}$. Hence, in all cases, there can be no compact leaves.

We now repeat the above construction $n-2$ times to obtain $\lambda_{n}$, fully carried by the branched surface $B_{n}$, intersecting $\partial N(k)$ in coherently oriented curves of slope $m_{\lambda}$, and containing no compact leaves. This is possible since at any step of the process, the sutures $A\left(\gamma_{j}\right) \backslash A\left(\gamma_{1}\right)$ cut off an infinite product ( $N \times I, \partial N \times I$ ) in the complement of $\lambda_{j}$; so infinite strip extension always exist.

Since the complementary regions of $B_{n}$ are products $\lambda_{n}$ extends to a foliation by taking copies of its boundary leaves in the canonical way.

Hence we have constructed a set $\mathscr{F}$ of foliations without compact leaves such that for any nonzero number $r$ in the interval $\mathbf{R}$ (for nonplanar $k$ ), ( $-\infty, 0$ ] (for negative planar $k$ ), or $[0, \infty$ ) (for positive planar $k$ ), there is an element of $\mathscr{F}$ which intersects the boundary torus of $M$ in parallel curves of slope $r$. Since there are no compact leaves, these foliations remain taut after Dehn filling with coefficient $r$. As mentioned in the introduction to this section, Gabai ([12]) has constructed taut foliations which intersect the boundary torus in parallel curves of slope 0 and remain taut after longitudinal Dehn filling.

## 2. Generalizations

In this section we give a technical description of those knots for which the construction of Section 1 can proceed. We also describe a generalization of this construction which is more widely applicable (see Example 2.4) but which yields essential laminations rather than taut foliations.

The proofs of Section 1 generalize in a straightforward way to give us the following:

THEOREM 2.1. Let $k$ be a knot for which there exists a sequence

$$
\left(S^{3} \backslash \stackrel{N}{N}(k), \partial N(k)\right) \stackrel{R}{\rightarrow}\left(M^{\prime}, \gamma\right) \stackrel{S}{\rightarrow}\left(M^{\prime \prime}, \gamma^{\prime \prime}\right)
$$



Figure 22.
of sutured manifold decompositions satisfying:
(i) $\left(M^{\prime}, \gamma^{\prime}\right)$ and $\left(M^{\prime \prime}, \gamma^{\prime \prime}\right)$ are taut,
(ii) $B=\langle R, S\rangle$ contains no planar surfaces of contact,
(iii) $R$ is a Seifert surface for $k$, and
(iv) $\partial S \cap R_{+}\left(\gamma^{\prime}\right)$ and $\partial S \cap R_{-}\left(\gamma^{\prime}\right)$ form a set of pairwise disjoint arcs.

The arcs of intersection $\partial S \cap A\left(\gamma^{\prime}\right)$ may be modelled on one of the four crossings of Figure 22. Let $p_{1}, p_{2}, n_{1}, n_{2}$ denote the number of crossings of the type of Figure $22 a, b, c, d$, respectively. Then $M=S^{3} \backslash N(k)$ contains a set $\mathscr{F}$ of foliations such that for any number $r$ in

$$
I=\left\{\left(p_{1}-n_{1}\right) \frac{x}{x+1}+\left(p_{2}-n_{2}\right) x|x \geq 0\rangle\right.
$$

there is a foliation of $\mathscr{F}$ which meets the boundary torus of $M$ transversely in parallel curves of slope $r$. The foliations realizing the nonzero boundary slopes have no compact leaves.

COROLLARY 2.2. Let $k, I$ be as described in Theorem 2.1. Then any manifold obtained from $S^{3}$ by performing Dehn surgery along $k$ with coefficient in I contains a taut foliation.

We note that the disc-decomposable knots of Gabai ([7,10]) provide an obvious source of examples and many yield to this construction. However, an explicit listing of those knots amenable to this method of attack seems difficult to realize.

By relaxing the requirement that $\left(M^{\prime}, \gamma^{\prime}\right)$ and $\left(M^{\prime \prime}, \gamma^{\prime \prime}\right)$ be taut, we obtain a more general construction. We shall abuse the terminology of Delman [(3)] as follows: Let $\lambda$ be an essential lamination in a knot exterior $M$ such that $\lambda$ meets $\partial M$ transversely in simple closed curves of slope $r$. Call $\lambda$ persistent if it caps off to yield an essential lamination $\hat{\lambda}$ in the closed manifold $\hat{M}$ obtained from $M$ by performing Dehn filling along $k$ with coefficient $r$.

THEOREM 2.3. Let $k$ be a knot for which there exists a sequence

$$
\left(S^{3} \backslash \stackrel{\circ}{N}(k), \partial N(k)\right) \stackrel{R}{\hookrightarrow}\left(M^{\prime}, \gamma^{\prime}\right) \stackrel{S}{\rightarrow}\left(M^{\prime \prime}, \gamma^{\prime \prime}\right)
$$

of sutured manifold decompositions such that:
(i) the branched surface $B=\langle R, S\rangle$ obtained by applying Construction 0.4 is essential and contains no planar surfaces of contact,
(ii) $R$ is a spanning surface for $k$, and
(iii) $\partial S \cap R_{+}\left(\gamma^{\prime}\right)$ and $\partial S \cap R_{-}\left(\gamma^{\prime}\right)$ form a set of pairwise disjoint arcs. The arcs of intersection $\partial S \cap A\left(\gamma^{\prime}\right)$ may be modelled on one of the four crossings of Figure 22. Let $p_{1}, p_{2}, n_{1}, n_{2}$ denote the number of crossings of the type of Figure $22 a, b, c, d$, respectively. Let $r$ denote the slope of $\partial R$. Then $M=S^{3} \backslash N(k)$ contains a set $\mathscr{L}$ of persistent laminations such that for any number $m$ in

$$
I=\left\{\left.r+\left(p_{1}-n_{1}\right) \frac{x}{x+1}+\left(p_{2}-n_{2}\right) x \right\rvert\, x \geq 0\right\}
$$

there is a lamination in $\mathscr{L}$ which meets the boundary torus of $M$ transversely in parallel curves of slope $m$.

Proof. The proofs of Section 1 generalize in an obvious way to yield a set $\mathscr{L}$ of laminations which meet $\partial N(k)$ in parallel curves. Since the branched surface $\langle R, S\rangle$
is essential, so are the laminations of $\mathscr{L}$. Therefore we need merely show that if $\lambda$ is an element of $\mathscr{L}$ meeting $\partial N(k)$ in simple closed curves of slope $r$, then the lamination $\hat{\lambda}$ obtained from $\lambda$ by capping off with discs is still essential.

We do so by describing a branched surface $\hat{B}$ (see [18]) which fully carries $\hat{\lambda}$. Let $T$ denote the solid torus in $\hat{M}$ bounded by $\partial N(k)$. Choose a tubular neighbourhood $N=F \times(-1,1)$ for $\partial T$ with $F \times 0=\partial T$. Note that the complementary regions of $\partial T \cap B$ are digons. Therefore, by choosing $N$ small enough, we may assume that $(F \times(-1,0)) \backslash B$ consists of product regions of the form digon $\times(-1,0)$. For each such region digon $\times(-1,0)$, add a branch to $B$ by inserting a leaf corresponding to digon $\times\{-1 / 2\}$ and pinching according to some choice of local transverse orientation. Call the resulting branched surface $B^{\prime}$. Next extend $B^{\prime}$ into $F \times(0,1)$ by first adjoining $(B \cap \partial T) \times(0,1)$ and then splitting $B^{\prime}$ near $B^{\prime} \cap(F \times 1)$ along the train track $(B \cap \partial T) \times 1$ so that one gets a branched surface $B^{\prime \prime}$ such that $B^{\prime \prime} \cap(F \times 1)$ consists of simple closed curves of slope $r$ and $B^{\prime \prime}$ coincides with $B^{\prime}$ outside $T$. finally, extend $B^{\prime \prime}$ to $\hat{B}$ by capping off the simple closed curves $B^{\prime \prime} \cap(F \times 1)$ with disjoing discs in $T \backslash(F \times(0,1])$.

We note that the only compact leaf carried by $B$ is the spanning surface $R$. So if $m \neq r$, then $\hat{B}$ can carry no compact leaves. And if $m=r$ but genus $R \geq 2$ then $\hat{B}$ carries no torus or sphere leaves. Finally, if $m=r$ and $R$ caps off to give a torus leaf carried by $\hat{B}$ then necessarily $R$ is a minimal genus Seifert surface and Gabai's construction ([12]) applies. Furthermore, each complementary region of $\hat{B}$ contained in $T$ is a $D^{2} \times I$ and the remaining complementary regions of $\hat{B}$ are isotopic to complementary regions of $B$. Hence, $\hat{B}$ satisfies all conditions of the definition of essential branched surface (see Definition 2.1, [15]) except possibly the condition that it contain no disc of contact.

We may therefore conclude that $\hat{\lambda}$ is essential if we can show that there are no discs of contact for $(\hat{\lambda}, \hat{B})$. Suppose, by way of contradiction, that $D$ were such a disc of contact. If $\partial D$ is contained in the vertical boundary of one of the $D^{2} \times I$ complementary components contained in $T$ then it caps off to reveal a sphere carried by $\hat{B}$. We have already noted that no such compact leaf exists. And if $\partial D$ is contained in the vertical boundary of one of the remainder complementary regions then it corresponds to a planar surface of contact for $(\lambda, B)$. But, by hypothesis, no such surface of contact exists. Hence there are no discs of contact for $(\hat{\lambda}, \hat{B})$. So $\hat{\lambda}$ is essential.

Again, many examples of knots for which there exists an appealing collection of such persistent laminations may be generated but an explicit listing has thus far proved unrealizable.

We close with an example which demonstrates that the construction of Theorem 2.3 may succeed in situations for which the original construction of Theorem 2.1 fails.


Figure 23. $K$.

EXAMPLE 2.4. Let $k$ be the 2-bridge knot shown in Figure 23.
By Proposition 1 of [20] we know that there is a unique Seifert surface for $k$. Since this surface ( see Figure 23a) contains only positive half-twists, the construction of Theorem 2.1 yields only taut foliations realizing the nonnegative boundary slopes.

However, the construction of Theorem 2.3 gives persistent laminations realizing all finite boundary slopes as follows. The isotopy of $k$ indicated in Figure 23b yields


Figure 24. $K$.


Figure 25. $B_{1}$.
the planar projection of Figure 24 . Let $R$ denote the spanning surface shown and let $r$ denote the slope of $i R$.

Corresponding to this projection are branched surfaces $B_{1}$ and $B_{2}$ generating laminations realizing boundary slopes $[r, x)$ and $(-x, r]$ respectively (see Figures


Figure 26. $B_{2}$.

25a and 26a). Regarded as a sutured manifold, $N(B)_{1}$ ) is homeomorphic to a regular neighbourhood of the incompressible and $\partial$-incompressible surface of Figure 25 b; so $B_{1}$ is essential. Furthermore, the proof of Proposition 1.9 guarantees that if $\lambda_{1}$ is one of the laminations generated by $B_{1}$ then $\left(\lambda_{1}, B_{1}\right)$ contains no planar surface of contact. Hence, the laminations generated by $B_{1}$ are persistent. Similarly, $B_{2}$ is essential and the laminations generated by $B_{2}$ are persistent.

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