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# Local constant of $\text{Ind}_K^L 1$

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*Abstract.* Henniart has computed the local constant  $\varepsilon(\text{Ind}_K^L 1)$  for an extension  $L$  over  $K$  of local fields of odd degree in [H]. In this paper, we show that his formula is a consequence of results of Serre [S4] and of Deligne [D2]. Further we compute the local constant for an extension of even degree, assuming the residual characteristic is not equal to 2.

## 1. Local constant of $\text{Ind}_K^L 1$

Let  $K$  be a local field. Namely  $K$  is a complete discrete valuation field with finite residue field  $F$  of order  $q$ . We assume that the characteristic of  $F$  is not equal to 2. For a separable extension  $L$  of degree  $n$  over  $K$ , let  $V = V_{L/K} = \text{Ind}_K^L 1$  be the induced representation of the absolute Galois group  $G_K = \text{Gal}(K^{\text{sep}}/K)$  from the unit representation 1 of  $G_L$ . Let  $d = d_{L/K} \in K^\times/K^{\times 2}$  be the discriminant of  $L$  over  $K$  and  $\delta = \delta_{L/K} = \det V_{L/K}$  be the character  $G_K \rightarrow \mathbb{Z}/2$  corresponding to  $d_{L/K} \in K^\times/K^{\times 2} = H^1(K, \mathbb{Z}/2)$ . We consider the local constant ([D1])

$$\varepsilon(V^0) = \begin{cases} \varepsilon(V, \psi, \mu) \cdot \varepsilon(\delta, \psi, \mu)^{-n} & n \text{ odd} \\ \varepsilon(V, \psi, \mu) \cdot \varepsilon(1, \psi, \mu)^{-(n-1)} \cdot \varepsilon(\delta, \psi, \mu)^{-1} & n \text{ even} \end{cases}$$

for the virtual representation

$$V^0 = \begin{cases} V - n\delta & n \text{ odd} \\ V - ((n-1)1 + \delta) & n \text{ even.} \end{cases}$$

Since  $\dim V^0 = 0$  and  $\det V^0 = 1$ , the local constant  $\varepsilon(V^0)$  is independent of an additive character  $\psi$  or a Haar measure  $\mu$  and we drop them in the notation. Since  $V$  is an orthogonal representation, the Artin conductor

$$\alpha(V^0) = \begin{cases} a(V) - na(\delta) & n \text{ odd} \\ a(V) - ((n-1)a(1) + a(\delta)) & n \text{ even} \end{cases}$$

is an even integer by [S3] Théorème 1. We put

$$\bar{\varepsilon}(V^0) = \varepsilon(V^0)q^{-(a(V^0)/2)}.$$

It is known to be  $\pm 1$ . As usual  $(\cdot)_F : F^\times/F^{\times 2} \rightarrow \{\pm 1\}$  denotes the Legendre symbol and  $(\cdot, \cdot)_K : K^\times/K^{\times 2} \times K^\times/K^{\times 2} \rightarrow \{\pm 1\}$  denotes the Hilbert symbol.

**THEOREM.** *Let  $L$  be a separable extension of degree  $n$  of a local field  $K$  with residual characteristic  $\neq 2$ .*

I. ([H] Proposition 2) *If  $n$  is odd, for the virtual representation  $V_{L/K}^0 = V_{L/K} - n \delta_{L/K}$ , we have*

$$\bar{\varepsilon}(V_{L/K}^0) = (d_{L/K}, 2)_K.$$

II. *Assume  $n$  is even. Let  $V_{L/K}^0$  be the virtual representation  $V_{L/K} - ((n-1)1 + \delta_{L/K})$ . Let  $D_{L/K}$  be the different of  $L$  over  $K$  and put  $D = \text{ord}_L D_{L/K}$ .*

1. *If  $D$  is even, we have*

$$\bar{\varepsilon}(V_{L/K}^0) = 1.$$

2. *Assume  $D$  is odd. Let  $e$  be the ramification index,  $f = n/e$  be the residual degree and  $\pi_K$  be an arbitrary prime element of  $K$ .*

2A. *If  $e$  is odd, then  $f$  is even and we have*

$$\bar{\varepsilon}(V_{L/K}^0) = \left(\frac{-1}{F}\right)^{\binom{n}{2}} \times (d_{L/K}, \pi_K)_K.$$

2B. *Assume  $e$  is even. Let  $K_1$  be the maximum unramified extension of  $K$  in  $L$  and  $E$  be the residue field of  $L$ . Let  $\pi_L$  be a prime element of  $L$  and  $g$  be the minimal polynomial of  $\pi_L$  over  $K_1$ . The class  $\alpha = g'(\pi_L)/\pi_L^D \in E^\times/E^{\times 2}$  is independent of  $\pi_L$  and we put  $d' = N_{E/F}(\alpha) \cdot d_{E/F} \in F^\times/F^{\times 2}$ . Then*

$$\bar{\varepsilon}(V_{L/K}^0) = \left(\frac{(-1)^{\binom{f}{2}} d'}{F}\right) \times \begin{cases} \left(\frac{-1}{F}\right)^{e/2-1} \cdot (d_{L/K}, 2)_K & f \text{ odd} \\ (d_{L/K}, \pi_K)_K & f \text{ even.} \end{cases}$$

Before starting the proof we briefly recall [D2] and [S4]. For a continuous orthogonal representation  $V$  of the absolute Galois group  $G_K$  of a field  $K$ , its Stiefel–Whitney class  $w_i(V) \in H^i(K, \mathbb{Z}/2)$  is defined [D2] (1.3). The first one

$w_1(V) \in H^1(K, \mathbb{Z}/2)$  is the determinant character regarded as an element in  $\text{Hom}(G_K, \mathbb{Z}/2) = H^1(K, \mathbb{Z}/2)$ . The total Stiefel–Whitney class  $w(V) = 1 + w_1(V) + w_2(V) + \cdots \in 1 + H^1(K, \mathbb{Z}/2) + H^2(K, \mathbb{Z}/2) + \cdots$  satisfies the multiplicativity  $w(V) = w(V_1) \cdot w(V_2)$  for the orthogonal direct sum  $V = V_1 \oplus V_2$ . Hence the Stiefel–Whitney class is defined for a virtual orthogonal representation  $V = V_1 - V_2$  by  $w(V) = w(V_1)w(V_2)^{-1}$ . When  $K$  is a local field of characteristic  $\neq 2$ , the second Stiefel–Whitney class is related to the local constant as follows. We identify  $H^2(K, \mathbb{Z}/2)$  with  $\{\pm 1\}$  by the isomorphism  $\text{inv}_K$ .

**THEOREM D.** ([D2] Théorème (1.5), [S3] Théorème 1) *Let  $K$  be a local field of characteristic  $\neq 2$  with residue field of order  $q$  and let  $V^0$  be a virtual orthogonal representation of the absolute Galois group  $G_K$ . Assume that  $\dim V^0 = 0$  and  $\det V^0 = 1$ . Then the Artin conductor  $a(V^0)$  is an even integer and the local constant  $\bar{\varepsilon}(V^0) = \varepsilon(V^0) \cdot q^{-a(V^0)/2}$  is  $\pm 1$  and*

$$\bar{\varepsilon}(V^0) = w_2(V^0).$$

For a field  $K$  of characteristic  $\neq 2$ , we call a quadratic  $K$ -module a  $K$ -vector space of finite dimension with a non-degenerate quadratic form. For a quadratic  $K$ -module  $W$ , its Stiefel–Whitney class  $w_i(W) \in H^i(K, \mathbb{Z}/2)$  is defined ([S3] 1.2). The first one  $w_1(W) \in H^1(K, \mathbb{Z}/2)$  is the discriminant regarded as an element in  $K^\times/K^{\times 2} = H^1(K, \mathbb{Z}/2)$ . The total Stiefel–Whitney class  $w(W) = 1 + w_1(W) + w_2(W) + \cdots \in 1 + H^1(K, \mathbb{Z}/2) + H^2(K, \mathbb{Z}/2) + \cdots$  satisfies the multiplicativity  $w(W) = w(W_1) \cdot w(W_2)$  for the orthogonal direct sum  $W = W_1 \oplus W_2$ . Hence the Stiefel–Whitney class is defined for a virtual orthogonal representation  $W = W_1 - W_2$  by  $w(W) = w(W_1)w(W_2)^{-1}$ . For elements  $a, b \in K^\times$ , let  $\{a\}$  denote the class of  $a$  in  $H^1(K, \mathbb{Z}/2) = K^\times/K^{\times 2}$  and let  $\{a, b\} = \{a\} \cup \{b\} \in H^2(K, \mathbb{Z}/2)$  denote the cup-product. For a quadratic  $K$ -module  $(W, Q)$  with an orthogonal basis  $(e_i)_i$ , its total Stiefel–Whitney class is  $w(W) = \prod_i (1 + \{Q(e_i)\})$  by definition. The Stiefel–Whitney class of the representation  $V_{L/K}$  is related to the Stiefel–Whitney class of a quadratic form as follows.

**THEOREM S.** ([S4] Théorème 1) *Let  $L$  be a finite separable extension of a field  $K$  of characteristic  $\neq 2$ . Let  $V = \text{Ind}_{G_K}^{G_L} 1$  be the induced orthogonal representation and  $W$  be the quadratic  $K$ -module  $(L, \text{Tr}_{L/K}(x^2))$ . Then by putting  $d = d_{L/K} = w_1(W) \in H^1(K, \mathbb{Z}/2)$ , we have*

$$w_2(V) = w_2(W) + \{d, 2\}$$

*in  $H^2(K, \mathbb{Z}/2)$ .*



In the rest of this section, we deduce Theorem for odd  $n$  from Theorems D and S. The proof for even  $n$  is more complicated and will be given at the end of the next section.

We prepare some terminology for quadratic  $K$ -modules. For  $c \in K^\times$ , let  $(c)$  denote the quadratic  $K$ -module  $(K, cx^2)$ . The dimension of a maximal totally isotropic subspace of a quadratic  $K$ -module  $W$  is called the index of  $W$ . If  $\dim W = 2 \text{ index } W$ , the quadratic module  $W$  is called hyperbolic.

*Proof of Theorem for odd  $n$ .* We assume  $n$  is odd. By Theorem D, we have  $\bar{e}(V^0) = w_2(V^0)$  in  $H^2(K, \mathbb{Z}/2) = \{\pm 1\}$ . Since  $(a, b)_K = \text{inv}_K\{a, b\}$ , it is sufficient to show that  $w_2(V^0) = \{d, 2\} \in H^2(K, \mathbb{Z}/2)$ . Let  $W^0$  be the virtual quadratic  $K$ -module  $W - n(d)$ . We show Theorem S implies  $w_2(V^0) = w_2(W^0) + \{d, 2\}$ . In fact  $w(V^0) = w(V)w(n\delta)^{-1} = (w(W) + \{d, 2\})w(n(d))^{-1} = w(W^0) + \{d, 2\}$ . Note  $H^i(K, \mathbb{Z}/2) = 0$  for  $i > 2$ . Therefore it is sufficient to prove that  $w(W) = w(n(d))$ . By  $\{d, d\} = \{d, -1\}$ ,  $\{-1, -1\} = 0$  and  $n \equiv 1$ , we have

$$\begin{aligned} w(n(d)) &= (1 + \{d\})^n = 1 + \{d\} + \frac{n-1}{2} \{d, -1\} \\ &= (1 + \{-1\})^{(n-1)/2} (1 + \{(-1)^{(n-1)/2} d\}). \end{aligned}$$

Namely we have  $w(n(d)) = w(W')$  where  $W' = (\text{hyperbolic of dimension } n-1) \oplus ((-1)^{(n-1)/2} d)$  is the orthogonal direct sum. Hence it is sufficient to show that  $W \simeq W'$ . Since the discriminants are equal, it is sufficient to show that the index of  $W$  is also  $(n-1)/2$ . Let  $W_a$  be the quadratic  $K$ -module  $(L, \text{Tr}_{L/K}(ax^2))$  for  $a \in L^\times$ . The isomorphism class of  $W_a$  depends only on the class of  $a$  in  $L^\times/L^{\times 2}$ . If  $a \in K^\times$ , it is isomorphic to  $(a) \otimes W$ . Since  $K^\times/K^{\times 2} \rightarrow L^\times/L^{\times 2}$  is an injection of finite groups of the same order, it is an isomorphism. Hence for any  $a \in L^\times$ , there is some  $b \in K^\times$  such that  $W_a \simeq (b) \otimes W$  and the index of  $W_a$  is independent of  $a \in L^\times$ . Therefore it is sufficient to show that the index of  $W_a$  is  $(n-1)/2$  for single  $a$ . It follows from

**LEMMA 1.** *Let  $L$  be a separable extension of a field  $K$  of degree  $n$ ,  $t$  be a primitive element of  $L$  and  $g$  be the minimal polynomial of  $t$ . Put  $a = g'(t)^{-1}$ . Then the index of the quadratic  $K$ -module  $W_a = (L, \text{Tr}_{L/K}(ax^2))$  is equal to the integral part of half of  $n$*

$$\text{index } W_n = \left[ \frac{n}{2} \right].$$

*Proof.* We know  $\text{Tr}_{L/K}(t^i/g'(t)) = 0$  for  $0 \leq i \leq n-2$  and  $=1$  for  $i = n-1$  [S1] Chap. III Lemme 2. Hence the subspace spanned by  $(t^i)_{0 \leq i \leq [(n-2)/2]}$  is isotropic and is of dimension  $[n/2]$ . Since  $\dim \geq 2$  index, Lemma is proved.

Thus the proof of Theorem for odd  $n$  is completed.

*Remark.* By [S2] Chap. IV Théorème 7,  $w(W) = w(n(d))$  implies  $W \simeq n(d)$ .

## 2. $\text{Tr}_{L/K}(ax^2)$

In this section, let  $K$  be a complete discrete valuation field with residue field  $F$ . We do not assume that  $F$  is finite but keep the assumption that characteristic of  $F$  is not equal to 2. First we consider a totally ramified extension  $L$  of  $K$  and compute the quadratic  $K$ -module  $W_a = (L, \text{Tr}_{L/K}(ax^2))$  for  $a \in L^\times$ .

**PROPOSITION 1.** *Let  $L$  be a separable totally ramified extension of degree  $e$  of  $K$  and  $a \in L^\times$ . Let  $W_a$  be the quadratic  $K$ -module  $(L, \text{Tr}_{L/K}(ax^2))$  and  $d_a = N_{L/K}(a) \cdot d_{L/K} \in K^\times/K^{\times 2}$  be its discriminant.*

*A. Assume  $e = 2m + 1$  is odd. Then there is an orthogonal decomposition*

$$W_a \simeq ((-1)^m d_a) \oplus (\text{hyperbolic}).$$

*B. Assume  $e = 2m$  is even and let  $D$  be the valuation of the different  $D_{L/K}$ .*

*B1. If  $\text{ord}_L a \equiv D \pmod{2}$ , then  $W_a$  is hyperbolic.*

*B2. Assume  $\text{ord}_L a \equiv D + 1 \pmod{2}$ . Let  $\pi$  be a prime element of  $L$  and  $g$  be the minimal polynomial of  $\pi$  over  $K$ . Then the class  $\alpha_a = a \cdot g'(\pi)\pi^{-1} \in L^\times/L^{\times 2}$  is independent of  $\pi$  and is in  $F^\times/F^{\times 2}$ . Let  $\alpha_a$  also denote a unit of  $K$  whose class in  $K^\times/K^{\times 2}$  is  $\alpha_a$ . Then there is an orthogonal decomposition*

$$W_a \simeq (\alpha_a) \oplus ((-1)^{m-1} d_a/\alpha_a) \oplus (\text{hyperbolic}).$$

*Proof.*

*A.* Assume  $e$  is odd. Then  $K^\times/K^{\times 2} \rightarrow L^\times/L^{\times 2}$  is an isomorphism. Therefore by the same argument as in the proof of Theorem for odd  $n$  above, we see that the index is  $m$ . By comparing the discriminant, we obtain the result.

*B.* Assume  $e$  is even. Let  $\pi$  be a prime element of  $L$  and  $g$  be its minimal polynomial. First we prove the assertion for  $a_0 = g'(\pi)^{-1}$  and  $a_1 = \pi \cdot g'(\pi)^{-1}$ . Note that  $D = -\text{ord}_L a_0$  and the class of  $a_1$  in  $L^\times/L^{\times 2}$  is independent of choice of  $\pi$ . In fact, it is the image of the refined different  $\mathcal{D}(L/K) \in L^\times/1 + m_L$  [K] Section 2 by the equality 1.16 p. 322 loc. cit.. By Lemma 1, the quadratic module  $W_{a_0}$  is

hyperbolic. We show  $W_{a_1} \simeq (1) \oplus ((-1)^{m-1} d_{a_1}) \oplus (\text{hyperbolic})$ . By the formula in the proof of Lemma 1, the subspace spanned by  $\pi^{m-1}$  is isomorphic to (1) and the subspace spanned by  $(\pi^i)_{0 \leq i \leq m-2}$  is totally isotropic and perpendicular to  $\pi^{m-1}$ . Hence we have  $W_{a_1} \simeq (1) \oplus (\text{hyperbolic}) \oplus (\text{dimension } 1)$ . By comparing the discriminant, it is proved.

We consider general  $a \in L^\times$ . The image of  $K^\times/K^{\times 2} \rightarrow L^\times/L^{\times 2}$  is  $F^\times/F^{\times 2}$  and is of index 2. Hence the condition  $\text{ord}_L a \equiv D$  (resp.  $D+1$ ) is equivalent to that the class of  $a/a_0$  (resp.  $a/a_1$ ) in  $L^\times/L^{\times 2}$  is in the image of  $K^\times/K^{\times 2}$ . It further implies  $W_a \simeq (b) \otimes W_{a_0}$  (resp.  $W_a \simeq (b) \otimes W_{a_1}$ ) for some  $b \in K^\times$ . Therefore  $\text{ord}_L a \equiv D$  implies  $W_a$  is hyperbolic. Assume  $\text{ord}_L a \equiv D+1$ . Since  $a = \alpha_a \cdot a_1$  in  $L^\times/L^{\times 2}$  and  $\alpha_a \in K^\times$ , we have  $W_a \simeq (\alpha_a) \otimes W_{a_1}$ . Hence  $W_a \simeq (\alpha_a) \oplus (\text{hyperbolic}) \oplus (\text{dimension } 1)$  and comparing the discriminant, we obtain the assertion.

For a general extension, we compute the image of  $w_2(\text{Tr}_{L/K}(ax^2)) \in H^2(K, \mathbb{Z}/2)$  by the boundary map  $\partial: H^2(K, \mathbb{Z}/2) \rightarrow H^1(F, \mathbb{Z}/2)$ . The spectral sequence  $H^i(F, H^j(K^n, \mathbb{Z}/2)) \Rightarrow H^{i+j}(K, \mathbb{Z}/2)$  induces an exact sequence

$$0 \longrightarrow H^i(F, \mathbb{Z}/2) \longrightarrow H^i(K, \mathbb{Z}/2) \xrightarrow{\partial} H^{i-1}(F, \mathbb{Z}/2) \longrightarrow 0$$

for an integer  $i$ . The pairing with  $\{\pi\}$  for a prime element  $\pi$  gives a section of  $\partial$ . For  $i=2$ , we have a commutative diagram

$$\begin{array}{ccc} K^\times \times K^{\times 2} & \xrightarrow{\text{tame symbol}} & F^\times \\ \{\cdot, \cdot\} \downarrow & & \downarrow \{\cdot, \cdot\} \\ H^2(K, \mathbb{Z}/2) & \xrightarrow{\partial} & H^1(F, \mathbb{Z}/2). \end{array}$$

For an element  $c \in K^\times$  of even valuation, the class  $\{c\} \in K^\times/K^{\times 2} = H^1(K, \mathbb{Z}/2)$  is in the subgroup  $F^\times/F^{\times 2} = H^1(F, \mathbb{Z}/2)$ .

**PROPOSITION 2.** *Let  $L$  be a separable extension of  $K$  of degree  $n$ . We assume the extension of the residue field  $E$  over  $F$  is separable of degree  $f$  and the characteristic of  $F$  is  $> 2$ . Let  $a \in L^\times$  and  $W_a$  be the quadratic  $K$ -module  $(L, \text{Tr}_{L/K}(ax^2))$ .*

1. *If  $\text{ord}_L a \equiv D \pmod{2}$ , the boundary of the total Stiefel–Whitney class  $\partial w(W_a)$  is 0.*
2. *Assume  $\text{ord}_L a \equiv D+1 \pmod{2}$ . Let  $d_a = N_{L/K}(a) \cdot d_{L/K} \in K^\times/K^{\times 2}$  be the discriminant of  $W_a$ .*

2A. *If the ramification index  $e$  is odd, we have*

$$\partial w_2(W_a) = \binom{n}{2} \{-1\} + \begin{cases} 0 & f \text{ odd} \\ \{d_a\} & f \text{ even.} \end{cases}$$

2B. Assume  $e$  is even. Let  $K_1$  be the maximum unramified subextension of  $K$  in  $L$ ,  $\alpha_a \in E^\times/E^{\times 2}$  be as in Proposition 1 for the totally ramified extension  $L$  over  $K_1$  and  $d'_a = N_{E/F}(\alpha_a) \cdot d_{E/F} \in F^\times/F^{\times 2}$ . Then we have

$$\partial w_2(W_a) = \binom{f}{2} \{-1\} + \{d'_a\} + \begin{cases} \left(\frac{e}{2} - 1\right) \{-1\} & f \text{ odd} \\ \{d_a\} & f \text{ even.} \end{cases}$$

*Proof.* To show  $\partial w(W_a)$  is 0, it is enough to find a non-degenerate  $\mathcal{O}_K$ -lattice. In fact, then for an orthogonal basis over  $\mathcal{O}_K$ , the value of the quadratic form at each element of the basis is a unit. Let  $K_1$  be the maximum unramified subextension of  $K$  in  $L$ . For a quadratic  $K_1$ -module  $(W, Q)$ , let  $\text{Tr}_{K_1/K} W$  denote the quadratic  $K$ -module  $(W, \text{Tr}_{K_1/K} \circ Q)$ . Since  $\mathcal{O}_{K_1}$  is a non-degenerate  $\mathcal{O}_K$ -lattice of  $\text{Tr}_{K_1/K}((u))$  for a unit  $u$  of  $K_1$ , a non-degenerate  $\mathcal{O}_{K_1}$ -lattice of quadratic  $K_1$ -module  $W$  is a non-degenerate  $\mathcal{O}_K$ -lattice of  $\text{Tr}_{L/K} W$ .

1. Assume  $\text{ord}_L a \equiv D$ . By the argument above, it is enough to find a non-degenerate  $\mathcal{O}_K$ -lattice of  $W_a$  by assuming  $K = K_1$ . Namely we may assume  $L$  is totally ramified. If  $e$  is even,  $W_a$  is hyperbolic by Proposition 1 B1, and has a non-degenerate  $\mathcal{O}_K$ -lattice. Assume  $e$  is odd. By Proposition 1 A, we have  $W_a \simeq ((-1)^m d_a) \oplus (\text{hyperbolic})$ . Since the valuation  $\text{ord}_K d_a = \text{ord}_K (N_{L/K}(a) \cdot d_{L/K}) = \text{ord}_L(a) + D$  is even, the quadratic  $K$ -module  $W_a$  has a non-degenerate  $\mathcal{O}_K$ -lattice. Therefore we have  $\partial w(W_a) = 0$ .

2. We prove the case  $\text{ord}_L a \equiv D + 1 \pmod{2}$ , by using the following Lemma proved later.

LEMMA 2. Let  $W = W_1 \oplus W_2$  be the orthogonal direct sum of quadratic  $K$ -modules  $W_1$  and  $W_2$ . Assume  $W_1$  and  $(\pi) \otimes W_2$  have non-degenerate  $\mathcal{O}_K$ -lattices for a prime element  $\pi$ . Let  $d$  and  $d_i$  be the discriminants of  $W$  and  $W_i$  and  $r_2$  be the dimension of  $W_2$ . Then we have

$$\partial w_2(W) = \{d_1\} + \binom{r_2}{2} \{-1\} + \begin{cases} 0 & r_2 \text{ odd} \\ \{d\} & r_2 \text{ even.} \end{cases}$$

We complete the proof using Lemma. Assume first that  $e$  is odd. By Proposition 1A, we have  $W_a \simeq \text{Tr}_{K_1/K}(\text{hyperbolic}) \oplus \text{Tr}_{K_1/K}((-1)^{(e-1)/2} d_a^1)$  for  $d_a^1 = N_{L/K_1}(a) \cdot d_{L/K_1}$ . Here  $W_1 = \text{Tr}_{K_1/K}(\text{hyperbolic})$  is also hyperbolic and has a non-degenerate  $\mathcal{O}_K$ -lattice. Since the valuation  $\text{ord}_{K_1} d_a^1 = \text{ord}_L a + D$ , is odd, the quadratic  $K_1$ -module  $(\pi) \otimes ((-1)^{(e-1)/2} d_a^1)$  has a non-degenerate  $\mathcal{O}_{K_1}$ -lattice. Hence the quadratic  $K_1$ -module  $(\pi) \otimes W_2$  has a non-degenerate  $\mathcal{O}_K$ -lattice for  $W_2 = \text{Tr}_{K_1/K}((-1)^{(e-1)/2} d_a^1)$ . Therefore by applying Lemma 2 and using  $d_1 = (-1)^{(e-1)/2} f$ ,  $r_2 = f$  and  $d = d_a$ , we have

$$\partial w_2(W_a) = \left( \frac{(e-1)f}{2} + \binom{f}{2} \right) \{-1\} + \begin{cases} 0 & f \text{ odd} \\ \{d_a\} & f \text{ even.} \end{cases}$$

Since  $e$  is odd, we have

$$\frac{(e-1)f}{2} + \binom{f}{2} \equiv \frac{(e-1)f}{2} + e \binom{f}{2} = \frac{f(n-1)}{2} \equiv \binom{n}{2}$$

mod 2 and Proposition 2 is proved in this case.

Finally assume  $e$  is even. By Proposition 1 B2, we have an orthogonal decomposition  $W_a = W_1 \oplus W_2$  where  $W_1 = \text{Tr}_{K_1/K}((\alpha_a) \oplus \text{hyperbolic})$  and  $W_2 = \text{Tr}_{K_1/K}((-1)^{m-1} d_a^1 / \alpha_a)$ . Similarly as above, it is easily checked to satisfy the assumption of Lemma 2. The discriminant  $d_1 \in F^\times / F^{\times 2} \subset K^\times / K^{\times 2}$  is  $(-1)^{(e/2)-1} / \times \text{disc}(\text{Tr}_{K_1/K}(\alpha_a))$  and  $\text{disc}(\text{Tr}_{K_1/K}(\alpha_a)) = \text{disc}(\text{Tr}_{E/F}(\alpha_a)) = d'_a$ . Hence Lemma 2 gives us

$$\partial w_2(W_a) = \left( \left( \frac{e}{2} - 1 \right) f + \binom{f}{2} \right) \{-1\} + \{d'_a\} + \begin{cases} 0 & f \text{ odd} \\ \{d_a\} & f \text{ even.} \end{cases}$$

Thus Proposition 2 is proved.

*Proof of Lemma 2.* By the assumption that  $W_1$  has a non-degenerate lattice,  $w(W_1) \in H^*(K, \mathbb{Z}/2)$  is in  $H^*(F, \mathbb{Z}/2) \subset H^*(K, \mathbb{Z}/2)$ . We have  $\partial w(W) = w(W_1) \cdot \partial w(W_2) \in H^*(F, \mathbb{Z}/2)$ . We have  $w(W_1) \equiv 1 + \{d_1\} \pmod{(\text{degree} > 1)}$ . Let  $W'_2 = (\pi^{-1}) \otimes W_2$ . It has a non-degenerate  $\mathcal{O}_K$ -lattice and  $w(W'_2) \in H^*(F, \mathbb{Z}/2)$ . We have

$$w(W_2) = w((\pi) \otimes W'_2) = \sum_j w_j(W'_2) (1 + \{\pi\})^{r_2-j} = \sum_{i,j} \binom{r_2-j}{i} w_j(W'_2) \{\pi\}^i.$$

Since  $\{\pi, \pi\} = \{\pi, -1\}$  and  $\{-1, -1\} = 0$ , we have  $\{\pi\}^3 = 0$  and

$$\partial w_k(W_2) = (r_2 - k + 1) w_{k-1}(W'_2) + \binom{r_2 - k + 2}{2} \{w_{k-2}(W'_2), -1\}.$$

Hence we obtain

$$\begin{aligned} \partial w(W) &\equiv (1 + \{d_1\}) \left( r_2 + (r_2 - 1) w_1(W'_2) + \binom{r_2}{2} \{-1\} \right) \pmod{\text{degree} > 1} \\ &= r_2 + \{d_1\} + (r_2 - 1)(\{d_1\} + w_1(W'_2)) + \binom{r_2}{2} \{-1\}. \end{aligned}$$

By

$$(r_2 - 1)(\{d_1\} + w_1(W'_2)) = \begin{cases} 0 & r_2 \text{ odd} \\ \{d_1\} + \{d_2\} = \{d\} & r_2 \text{ even,} \end{cases}$$

Lemma 2 is proved.

*Proof of Theorem for even  $n$ .* Assume  $n$  is even. By Theorem D, we have  $\bar{e}(V^0) = w_2(V^0)$ . Further by  $w(V^0) = w(V)w(\delta)^{-1}(w(\delta) + w_2(V))w(\delta)^{-1} = 1 + w_2(V)$ , we have  $w_2(V^0) = w_2(V)$ . Hence by Theorem S, we have  $\bar{e}(V^0) = w_2(V) = w_2(W) + (d, 2)_K$ . Since  $\text{inv}_K = ({}_F) \circ \partial: H^2(K, \mathbb{Z}/2) \simeq \{\pm 1\}$ , it is enough to compute the boundary  $\partial w_2(W) \in H^1(F, \mathbb{Z}/2)$ . We check that Theorem is now a special case of Proposition 2 where  $F$  is finite and  $a = 1$ . In fact if  $f$  or  $D$  is even, the valuation of  $d$  is even and  $(d, 2)_K = 1$  and  $(\{d\}/F) = (d, \pi_K)_K$  for a prime element  $\pi_K$  of  $K$ . Thus the proof of Theorem is completed.

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