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Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **70 (1995)**

PDF erstellt am: **30.04.2024**

Persistenter Link: https://doi.org/10.5169/seals-53010

Comment. Math. Helvetici 70 (1995) 507–515

0010-2571/95/040507-09$1.50 + 0.20/0


# Local constant of $Ind_K^L 1$

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*Abstract.* Henniart has computed the local constant $\varepsilon$ $(Ind_K^L 1)$ for an extension $L$ over $K$ of local fields of odd degree in [H]. In this paper, we show that his formula is a consequence of results of Serre [S4] and of Deligne [D2]. Further we compute the local constant for an extension of even degree, assuming the residual characteristic is not equal to 2.

## 1. Local constant of $Ind_K^L 1$

Let $K$ be a local field. Namely $K$ is a complete discrete valuation field with finite residue field $F$ of order $q$. We assume that the characteristic of $F$ is not equal to 2. For a separable extension $L$ of degree $n$ over $K$, let $V = V_{L/K} = Ind_K^L 1$ be the induced representation of the absolute Galois gorup $G_K = \mathrm{Gal}\,(K^{sep}/K)$ from the unit representation 1 of $G_L$. Let $d = d_{L/K} \in K^\times/K^{\times 2}$ be the discriminant of $L$ over $K$ and $\delta = \delta_{L/K} = \det V_{L/K}$ be the character $G_K \to \mathbb{Z}/2$ corresponding to $d_{L/K} \in K^\times/K^{\times 2} = H^1(K, \mathbb{Z}/2)$. We consider the local constant ([D1])

$$\varepsilon(V^0) = \begin{cases} \varepsilon(V, \psi, \mu) \cdot \varepsilon(\delta, \psi, \mu)^{-n} & n \text{ odd} \\ \varepsilon(V, \psi, \mu) \cdot \varepsilon(1, \psi, \mu)^{-(n-1)} \cdot \varepsilon(\delta, \psi, \mu)^{-1} & n \text{ even} \end{cases}$$

for the virtual representation

$$V^0 = \begin{cases} V - n\,\delta & n \text{ odd} \\ V - ((n-1)1 + \delta) & n \text{ even.} \end{cases}$$

Since $\dim V^0 = 0$ and $\det V^0 = 1$, the local constant $\varepsilon(V^0)$ is independent of an additive character $\psi$ or a Haar measure $\mu$ and we drop them in the notation. Since $V$ is an orthogonal representaion, the Artin conductor

$$\alpha(V^0) = \begin{cases} a(V) - na(\delta) & n \text{ odd} \\ a(V) - ((n-1)a(1) + a(\delta)) & n \text{ even} \end{cases}$$

is an even integer by [S3] Théorème 1. We put

$$\bar{\varepsilon}(V^0) = \varepsilon(V^0)q^{-(a(V^0))/2}.$$

It is known to be $\pm 1$. As usual $(\overline{_F}) : F^\times/F^{\times 2} \to \{\pm 1\}$ denotes the Legendre symbol and $(,)_K : K^\times/K^{\times 2} \times K^\times/K^{\times 2} \to \{\pm 1\}$ denotes the Hilbert symbol.

THEOREM. *Let L be a separable extension of degree n of a local field K with residual characteristic* $\neq 2$.

I. ([H] Proposition 2) *If n is odd, for the virtual representation* $V^0_{L/K} = V_{L/K} - n\,\delta_{L/K}$, *we have*

$$\bar{\varepsilon}(V^0_{L/K}) = (d_{L/K}, 2)_K.$$

II. *Assume n is even. Let* $V^0_{L/K}$ *be the virtual representation* $V_{L/K} - ((n-1)1 + \delta_{L/K})$. *Let* $D_{L/K}$ *be the different of L over K and put* $D = \operatorname{ord}_L D_{L/K}$.

1. *If D is even, we have*

$$\bar{\varepsilon}(V^0_{L/K}) = 1.$$

2. *Assume D is odd. Let e be the ramification index,* $f = n/e$ *be the residual degree and* $\pi_K$ *be an arbitrary prime element of K.*

2A. *If e is odd, then f is even and we have*

$$\bar{\varepsilon}(V^0_{L/K}) = \left(\frac{-1}{F}\right)^{\binom{n}{2}} \times (d_{L/K}, \pi_K)_K.$$

2B. *Assume e is even. Let* $K_1$ *be the maximum unramified extension of K in L and E be the residue field of L. Let* $\pi_L$ *be a prime element of L and g be the minimal polynomial of* $\pi_L$ *over* $K_1$. *The class* $\alpha = g'(\pi_L)/\pi_L^D \in E^\times/E^{\times 2}$ *is independent of* $\pi_L$ *and we put* $d' = N_{E/F}(\alpha) \cdot d_{E/F} \in F^\times/F^{\times 2}$. *Then*

$$\bar{\varepsilon}(V^0_{L/K}) = \left(\frac{(-1)^{\binom{f}{2}} d'}{F}\right) \times \begin{cases} \left(\frac{-1}{F}\right)^{e/2-1} \cdot (d_{L/K}, 2)_K & f \text{ odd} \\ (d_{L/K}, \pi_K)_K & f \text{ even.} \end{cases}$$

Before starting the proof we briefly recall [D2] and [S4]. For a continuous orthogonal representation $V$ of the absolute Galois group $G_K$ of a field $K$, its Stiefel–Whitney class $w_i(V) \in H^i(K, \mathbb{Z}/2)$ is defined [D2] (1.3). The first one

$w_1(V) \in H^1(K, \mathbb{Z}/2)$ is the determinant character regarded as an element in $\mathrm{Hom}\,(G_K, \mathbb{Z}/2) = H^1(K, \mathbb{Z}/2)$. The total Stiefel–Whitney class $w(V) = 1 + w_1(V) + w_2(V) + \cdots \in 1 + H^1(K, \mathbb{Z}/2) + H^2(K, \mathbb{Z}/2) + \cdots$ satisfies the multiplicativity $w(V) = w(V_1) \cdot w(V_2)$ for the orthogonal direct sum $V = V_1 \oplus V_2$. Hence the Stiefel–Whitney class is defined for a virtual orthogonal representation $V = V_1 - V_2$ by $w(V) = w(V_1)w(V_2)^{-1}$. When $K$ is a local field of characteristic $\neq 2$, the second Stiefel–Whitney class is related to the local constant as follows. We identify $H^2(K, \mathbb{Z}/2)$ with $\{\pm\}$ by the isomorphism $\mathrm{inv}_K$.

THEOREM D. ([D2] Théorème (1.5), [S3] Théorème 1) *Let $K$ be a local field of characteristic $\neq 2$ with residue field of order $q$ and let $V^0$ be a virtual orthogonal representation of the absolute Galois group $G_K$. Assume that* $\dim V^0 = 0$ *and* $\det V^0 = 1$. *Then the Artin conductor $a(V^0)$ is an even integer and the local constant $\bar{\varepsilon}(V^0) = \varepsilon(V^0) \cdot q^{-a(V^0)/2}$ is $\pm 1$ and*

$$\bar{\varepsilon}(V^0) = w_2(V^0).$$

For a field $K$ of characteristic $\neq 2$, we call a quadratic $K$-module a $K$-vector space of finite dimension with a non-degenerate quadratic form. For a quadratic $K$-module $W$, its Stiefel–Whitney class $w_i(W) \in H^i(K, \mathbb{Z}/2)$ is defined ([S3] 1.2). The first one $w_1(W) \in H^1(K, \mathbb{Z}/2)$ is the discriminant regarded as an elelment in $K^\times/K^{\times 2} = H^1(K, \mathbb{Z}/2)$. The total Stiefel–Whitney class $w(W) = 1 + w_1(W) + w_2(W) + \cdots \in 1 + H^1(K, \mathbb{Z}/2) + H^2(K, \mathbb{Z}/2) + \cdots$ satisfies the multiplicativity $w(W) = w(W_1) \cdot w(W_2)$ for the orthogonal direct sum $W = W_1 \oplus W_2$. Hence the Stiefel–Whitney class is defined for a virtual orthogonal representation $W = W_1 - W_2$ by $w(W) = w(W_1)w(W_2)^{-1}$. For elements $a, b \in K^\times$, let $\{a\}$ denote the class of $a$ in $H^1(K, \mathbb{Z}/2) = \mathrm{K}^\times/\mathrm{K}^{\times 2}$ and let $\{a, b\} = \{a\} \cup \{b\} \in H^2(K, \mathbb{Z}/2)$ denote the cup-product. For a quadratic $K$-module $(W, Q)$ with an orthogonal basis $(e_i)_i$, its total Stiefel–Whitney class is $w(W) = \Pi_i(1 + \{Q(e_i)\})$ by definition. The Stiefel–Whitney class of the representation $V_{L/K}$ is related to the Stiefel–Whitney class of a quadratic form as follows.

THEOREM S. ([S4] Théorème 1) *Let $L$ be a finite separable extension of a field $K$ of characteristic $\neq 2$. Let $V = Ind_{G_K}^{G_L} 1$ be the induced orthogonal representation and $W$ be the quadratic $K$-module $(L, \mathrm{Tr}_{L/K}(x^2))$. Then by putting $d = d_{L/K} = w_1(W) \in H^1(K, \mathbb{Z}/2)$, we have*

$$w_2(V) = w_2(W) + \{d, 2\}$$

*in $H^2(K, \mathbb{Z}/2)$.*

In the rest of this section, we deduce Theorem for odd $n$ from Theorems D and S. The proof for even $n$ is more complicated and will be given at the end of the next section.

We prepare some terminology for quadratic $K$-modules. For $c \in K^\times$, let $(c)$ denote the quadratic $K$-module $(K, cx^2)$. The dimension of a maximal totally isotropic subspace of a quadratic $K$-module $W$ is called the index of $W$. If $\dim W = 2$ index $W$, the quadratic module $W$ is called hyperbolic.

*Proof of Theorem for odd n.* We assume $n$ is odd. By Theorem D, we have $\bar{\varepsilon}(V^0) = w_2(V^0)$ in $H^2(K, \mathbb{Z}/2) = \{\pm 1\}$. Since $(a, b)_K = \mathrm{inv}_K\{a, b\}$, it is sufficient to show that $w_2(V^0) = \{d, 2\} \in H^2(K, \mathbb{Z}/2)$. Let $W^0$ be the virtual quadratic $K$-module $W - n(d)$. We show Theorem S implies $w_2(V^0) = w_2(W^0) + \{d, 2\}$. In fact $w(V^0) = w(V)w(n\,\delta)^{-1} = (w(W) + \{d, 2\})w(n(d))^{-1} = w(W^0) + \{d, 2\}$. Note $H^i(K, \mathbb{Z}/2) = 0$ for $i > 2$. Therefore it is sufficient to prove that $w(W) = w(n(d))$. By $\{d, d\} = \{d, -1\}$, $\{-1, -1\} = 0$ and $n \equiv 1$, we have

$$w(n(d)) = (1 + \{d\})^n = 1 + \{d\} + \frac{n-1}{2}\{d, -1\}$$

$$= (1 + \{-1\})^{(n-1)/2}(1 + \{(-1)^{(n-1)/2}d\}).$$

Namely we have $w(n(d)) = w(W')$ where $W' =$ (hyperbolic of dimension $n - 1$) $\oplus$ $((-1)^{(n-1)/2}d)$ is the orthogonal direct sum. Hence it is sufficient to show that $W \simeq W'$. Since the discriminants are equal, it is sufficient to show that the index of $W$ is also $(n-1)/2$. Let $W_a$ be the quadratic $K$-module $(L, \mathrm{Tr}_{L/K}(ax^2))$ for $a \in L^\times$. The isomorphism class of $W_a$ depends only on the class of $a$ in $L^\times/L^{\times 2}$. If $a \in K^\times$, it is isomorphic to $(a) \otimes W$. Since $K^\times/K^{\times 2} \to L^\times/L^{\times 2}$ is an injection of finite groups of the same order, it is an isomorphism. Hence for any $a \in L^\times$, there is some $b \in K^\times$ such that $W_a \simeq (b) \otimes W$ and the index of $W_a$ is independent of $a \in L^\times$. Therefore it is sufficient to show that the index of $W_a$ is $(n-1)/2$ for single $a$. It follows from

LEMMA 1. *Let $L$ be a separable extension of a field $K$ of degree $n$, $t$ be a primitive element of $L$ and $g$ be the minimal polynomial of $t$. Put $a = g'(t)^{-1}$. Then the index of the quadratic $K$-module $W_a = (L, \mathrm{Tr}_{L/K}(ax^2))$ is equal to the integral part of half of $n$*

$$\text{index } W_n = \left[\frac{n}{2}\right].$$

*Proof.* We know $\mathrm{Tr}_{L/K}(t^i/g'(t)) = 0$ for $0 \le i \le n-2$ and $=1$ for $i = n-1$ [S1] Chap. III Lemme 2. Hence the subspace spaned by $(t^i)_{0 \le i \le [(n-2)/2]}$ is isotropic and is of dimension $[n/2]$. Since dim $\ge 2$ index, Lemma is proved.

Thus the proof of Theorem for odd $n$ is completed.

*Remark.* By [S2] Chap. IV Théorème 7, $w(W) = w(n(d))$ implies $W \simeq n(d)$.

## 2. $\mathrm{Tr}_{L/K}(ax^2)$

In this section, let $K$ be a complete discrete valuation field with residue filed $F$. We do not assume that $F$ is finite but keep the assumption that characteristic of $F$ is not equal to 2. First we consider a totally ramified extension $L$ of $K$ and compute the quadratic $K$-module $W_a = (L, \mathrm{Tr}_{L/K}(ax^2))$ for $a \in L^\times$.

PROPOSITION 1. *Let $L$ be a separable totally ramified extension of degree $e$ of $K$ and $a \in L^\times$. Let $W_a$ be the quadratic $K$-module $(L, \mathrm{Tr}_{L/K}(ax^2))$ and $d_a = N_{L/K}(a) \cdot d_{L/K} \in K^\times/K^{\times 2}$ be its discriminant.*

A. *Assume $e = 2m+1$ is odd. Then there is an orthogonal decomposition*

$$W_a \simeq ((-1)^m d_a) \oplus (\text{hyperbolic}).$$

B. *Assume $e = 2m$ is even and let $D$ be the valuation of the different $D_{L/K}$.*

*B*1. *If* $\mathrm{ord}_L\, a \equiv D \bmod 2$, *then $W_a$ is hyperbolic.*

B2. *Assume* $\mathrm{ord}_L\, a \equiv D + 1 \bmod 2$. *Let $\pi$ be a prime element of $L$ and $g$ be the minimal polynomial of $\pi$ over $K$. Then the class $\alpha_a = a \cdot g'(\pi)\pi^{-1} \in L^\times/L^{\times 2}$ is independent of $\pi$ and is in $F^\times/F^{\times 2}$. Let $\alpha_a$ also denote a unit of $K$ whose class in $K^\times/K^{\times 2}$ is $\alpha_a$. Then there is an orthogonal decomposition*

$$W_a \simeq (\alpha_a) \oplus ((-1)^{m-1} d_a/\alpha_a) \oplus (\text{hyperbolic}).$$

*Proof.*

A. Assume $e$ is odd. Then $K^\times/K^{\times 2} \to L^\times/L^{\times 2}$ is an isomorphism. Therefore by the same argument as in the proof of Theorem for odd $n$ above, we see that the index is $m$. By comparing the discriminant, we obtain the result.

B. Assume $e$ is even. Let $\pi$ be a prime element of $L$ and $g$ be its minimal polynomial. First we prove the assertion for $a_0 = g'(\pi)^{-1}$ and $a_1 = \pi \cdot g'(\pi)^{-1}$. Note that $D = -\mathrm{ord}_L\, a_0$ and the class of $a_1$ in $L^\times/L^{\times 2}$ is independent of choice of $\pi$. In fact, it is the image of the refined different $\mathscr{D}(L/K) \in L^\times/1 + m_L$ [K] Section 2 by the equality 1.16 p. 322 loc. cit.. By Lemma 1, the quadratic module $W_{a_0}$ is hyperbolic. We show $W_{a_1} \simeq (1) \oplus ((-1)^{m-1} d_{a_1}) \oplus$ (hyperbolic). By the formula in the proof of Lemma 1, the subspace spanned by $\pi^{m-1}$ is isomorphic to $(1)$ and the subspace spanned by $(\pi^i)_{0 \le i \le m-2}$ is totally isotropic and perpendicular to $\pi^{m-1}$. Hence we have $W_{a_1} \simeq (1) \oplus$ (hyperbolic) $\oplus$ (dimension 1). By comparing the discriminant, it is proved.

We consider general $a \in L^\times$. The image of $K^\times/K^{\times 2} \to L^\times/L^{\times 2}$ is $F^\times/F^{\times 2}$ and is of index 2. Hence the condition $\mathrm{ord}_L\, a \equiv D$ (resp. $D+1$) is equivalent to that the class of $a/a_0$ (resp. $a/a_1$) in $L^\times/L^{\times 2}$ is in the image of $K^\times/K^{\times 2}$. It further implies $W_a \simeq (b) \otimes W_{a_0}$ (resp. $W_a \simeq (b) \otimes W_{a_1}$) for some $b \in K^\times$. Therefore $\mathrm{ord}_L\, a \equiv D$ implies $W_a$ is hyperbolic. Assume $\mathrm{ord}_L\, a \equiv D+1$. Since $a = \alpha_a \cdot a_1$ in $L^\times/L^{\times 2}$ and $\alpha_a \in K^\times$, we have $W_a \simeq (\alpha_a) \otimes W_{a_1}$. Hence $W_a \simeq (\alpha_a) \oplus$ (hyperbolic) $\oplus$ (dimension 1) and comparing the discriminant, we obtain the assertion.

For a general extension, we compute the image of $w_2(\mathrm{Tr}_{L/K}(ax^2)) \in H^2(K, \mathbb{Z}/2)$ by the boundary map $\partial\colon H^2(K, \mathbb{Z}/2) \to H^1(F, \mathbb{Z}/2)$. The spectral sequence $H^i(F, H^j(K^{nr}, \mathbb{Z}/2)) \Rightarrow H^{i+j}(K, \mathbb{Z}/2)$ induces an exact sequence

$$0 \longrightarrow H^i(F, \mathbb{Z}/2) \longrightarrow H^i(K, \mathbb{Z}/2) \xrightarrow{\partial} H^{i-1}(F, \mathbb{Z}/2) \longrightarrow 0$$

for an integer $i$. The pairing with $\{\pi\}$ for a prime element $\pi$ gives a section of $\partial$. For $i=2$, we have a commutative diagram

$$\begin{array}{ccc} K^\times \times K^{\times 2} & \xrightarrow{\text{tame symbol}} & F^\times \\ \{,\}\big\downarrow & & \big\downarrow\{\} \\ H^2(K, \mathbb{Z}/2) & \xrightarrow[\partial]{} & H^1(F, \mathbb{Z}/2). \end{array}$$

For an element $c \in K^\times$ of even valuation, the class $\{c\} \in K^\times/K^{\times 2} = H^1(K, \mathbb{Z}/2)$ is in the subgroup $F^\times/F^{\times 2} = H^1(F, \mathbb{Z}/2)$.

PROPOSITION 2. *Let $L$ be a separable extension of $K$ of degree $n$. We assume the extension of the residue field $E$ over $F$ is separable of degree $f$ and the characteristic of $F$ is $> 2$. Let $a \in L^\times$ and $W_a$ be the quadratic $K$-module $(L, \mathrm{Tr}_{L/K}(ax^2))$.*

1. *If* $\mathrm{ord}_L\, a \equiv D \bmod 2$, *the boundary of the total Stiefel–Whitney class $\partial w(W_a)$ is* 0.
2. *Assume* $\mathrm{ord}_L\, a \equiv D + 1 \bmod 2$. *Let $d_a = N_{L/K}(a) \cdot d_{L/K} \in K^\times/K^{\times 2}$ be the discriminant of $W_a$.*

   2A. *If the ramification index $e$ is odd, we have*

$$\partial w_2(W_a) = \binom{n}{2}\{-1\} + \begin{cases} 0 & f \text{ odd} \\ \{d_a\} & f \text{ even.} \end{cases}$$

2B. *Assume e is even. Let $K_1$ be the maximum unramified subextension of K in L, $\alpha_a \in E^\times/E^{\times 2}$ be as in Proposition* 1 *for the totally ramified extension L over $K_1$ and $d'_a = N_{E/F}(\alpha_a) \cdot d_{E/F} \in F^\times/F^{\times 2}$*. Then we have

$$\partial w_2(W_a) = \binom{f}{2}\{-1\} + \{d'_a\} + \begin{cases} \left(\dfrac{e}{2} - 1\right)\{-1\} & f \text{ odd} \\ \{d_a\} & f \text{ even.} \end{cases}$$

*Proof.* To show $\partial w(W_a)$ is 0, it is enough to find a non-degenerate $\mathcal{O}_K$-lattice. In fact, then for an orthogonal basis over $\mathcal{O}_K$, the value of the quadratic form at each element of the basis is a unit. Let $K_1$ be the maximum unramified subextension of $K$ in $L$. For a quadratic $K_1$-module $(W, Q)$, let $\mathrm{Tr}_{K_1/K} W$ denote the quadratic $K$-module $(W, \mathrm{Tr}_{K_1/K} \circ Q)$. Since $\mathcal{O}_{K_1}$ is a non-degenerate $\mathcal{O}_K$-lattice of $\mathrm{Tr}_{K_1/K}((u))$ for a unit $u$ of $K_1$, a non-degenerate $\mathcal{O}_{K_1}$-lattice of quadratic $K_1$-module $W$ is a non-degenerate $\mathcal{O}_K$-lattice of $\mathrm{Tr}_{L/K} W$.

1. Assume $\mathrm{ord}_L\, a \equiv D$. By the argument above, it is enough to find a non-degenerate $\mathcal{O}_K$-lattice of $W_a$ by assuming $K = K_1$. Namely we may assume $L$ is totally ramified. If $e$ is even, $W_a$ is hyperbolic by Proposition 1 B1, and has a non-degenrate $\mathcal{O}_k$-lattice. Assume $e$ is odd. By Proposition 1 A, we have $W_a \simeq ((-1)^m d_a) \oplus$ (hyperbolic). Since the valuation $\mathrm{ord}_K d_a = \mathrm{ord}_K (N_{L/K}(a) \cdot d_{L/K}) = \mathrm{ord}_L(a) + D$ is even, the quadratic $K$-module $W_a$ has a non-degenerate $\mathcal{O}_K$-lattice. Therefore we have $\partial w(W_a) = 0$.

2. We prove the case $\mathrm{ord}_L\, a \equiv D + 1 \bmod 2$, by using the following Lemma proved later.

LEMMA 2. *Let $W = W_1 \oplus W_2$ be the orthogonal direct sum of quadratic K-modules $W_1$ and $W_2$. Assume $W_1$ and $(\pi) \otimes W_2$ have non-degenerate $\mathcal{O}_K$-lattices for a prime element $\pi$. Let d and $d_i$ be the discriminants of W and $W_i$ and $r_2$ be the dimension of $W_2$. Then we have*

$$\partial w_2(W) = \{d_1\} + \binom{r_2}{2}\{-1\} + \begin{cases} 0 & r_2 \text{ odd} \\ \{d\} & r_2 \text{ even.} \end{cases}$$

We complete the proof using Lemma. Assume first that $e$ is odd. By Proposition 1A, we have $W_a \simeq \mathrm{Tr}_{K_1/K}(\text{hyperbolic}) \oplus \mathrm{Tr}_{K_1/K}((-1)^{(e-1)/2} d_a^1)$ for $d_a^1 = N_{L/K_1}(a) \cdot d_{L/K_1}$. Here $W_1 = \mathrm{Tr}_{K_1/K}(\text{hyperbolic})$ is also hyperbolic and has a non-degenerate $\mathcal{O}_K$-lattice. Since the valuation $\mathrm{ord}_{K_1} d_a^1 = \mathrm{ord}_L\, a + D$, is odd, the quadratic $K_1$-module $(\pi) \otimes ((-1)^{(e-1)/2} d_a^1)$ has a non-degenerate $\mathcal{O}_{K_1}$-lattice. Hence the quadratic $K_1$-module $(\pi) \otimes W_2$ has a non-degenerate $\mathcal{O}_K$-lattice for $W_2 = \mathrm{Tr}_{K_1/K}((-1)^{(e-1)/2} d_a^1)$. Therefore by applying Lemma 2 and using $d_1 = (-1)^{(e-1)/2} f$, $r_2 = f$ and $d = d_a$, we have

$$\partial w_2(W_a) = \left(\frac{(e-1)f}{2} + \binom{f}{2}\right)\{-1\} + \begin{cases} 0 & f \text{ odd} \\ \{d_a\} & f \text{ even.} \end{cases}$$

Since $e$ is odd, we have

$$\frac{(e-1)f}{2} + \binom{f}{2} \equiv \frac{(e-1)f}{2} + e\binom{f}{2} = \frac{f(n-1)}{2} \equiv \binom{n}{2}$$

mod 2 and Proposition 2 is proved in this case.

Finally assume $e$ is even. By Proposition 1 B2, we have an orthogonal decomposition $W_a = W_1 \oplus W_2$ where $W_1 = \mathrm{Tr}_{K_1/K}((\alpha_a) \oplus \text{hyperbolic})$ and $W_2 = \mathrm{Tr}_{K_1/K}((-1)^{m-1} d_a^1/\alpha_a)$. Similarly as above, it is easily checked to satisfy the assumption of Lemma 2. The discriminant $d_1 \in F^\times/F^{\times 2} \subset K^\times/K^{\times 2}$ is $(-1)^{((e/2)-1)f} \times \mathrm{disc}\,(\mathrm{Tr}_{K_1/K}(\alpha_a))$ and $\mathrm{disc}\,(\mathrm{Tr}_{K_1/K}(\alpha_a)) = \mathrm{disc}\,(\mathrm{Tr}_{E/F}(\alpha_a)) = d'_a$. Hence Lemma 2 gives us

$$\partial w_2(W_a) = \left(\left(\frac{e}{2} - 1\right)f + \binom{f}{2}\right)\{-1\} + \{d'_a\} + \begin{cases} 0 & f \text{ odd} \\ \{d_a\} & f \text{ even.} \end{cases}$$

Thus Proposition 2 is proved.

*Proof of Lemma* 2. By the assumption that $W_1$ has a non-degenerate lattice, $w(W_1) \in H^*(K, \mathbb{Z}/2)$ is in $H^*(F, \mathbb{Z}/2) \subset H^*(K, \mathbb{Z}/2)$. We have $\partial w(W) = w(W_1) \cdot \partial w(W_2) \in H^*(F, \mathbb{Z}/2)$. We have $w(W_1) \equiv 1 + \{d_1\}$ mod (degree > 1). Let $W'_2 = (\pi^{-1}) \otimes W_2$. It has a non-degenerate $\mathcal{O}_K$-lattice and $w(W'_2) \in H^*(F, \mathbb{Z}/2)$. We have

$$w(W_2) = w((\pi) \otimes W'_2) = \sum_j w_j(W'_2)(1 + \{\pi\})^{r_2 - j} = \sum_{i,j} \binom{r_2 - j}{i} w_j(W'_2)\{\pi\}^i.$$

Since $\{\pi, \pi\} = \{\pi, -1\}$ and $\{-1, -1\} = 0$, we have $\{\pi\}^3 = 0$ and

$$\partial w_k(W_2) = (r_2 - k + 1)w_{k-1}(W'_2) + \binom{r_2 - k + 2}{2}\{w_{k-2}(W'_2), -1\}.$$

Hence we obtain

$$\partial w(W) \equiv (1 + \{d_1\})\left(r_2 + (r_2 - 1)w_1(W'_2) + \binom{r_2}{2}\{-1\}\right)(\text{mod degree} > 1)$$

$$= r_2 + \{d_1\} + (r_2 - 1)(\{d_1\} + w_1(W'_2)) + \binom{r_2}{2}\{-1\}.$$

By

$$(r_2 - 1)(\{d_1\} + w_1(W_2')) = \begin{cases} 0 & r_2 \text{ odd} \\ \{d_1\} + \{d_2\} = \{d\} & r_2 \text{ even,} \end{cases}$$

Lemma 2 is proved.

*Proof of Theorem for even n.* Assume $n$ is even. By Theorem D, we have $\bar{\varepsilon}(V^0) = w_2(V^0)$. Further by $w(V^0) = w(V)w(\delta)^{-1}(w(\delta) + w_2(V))w(\delta)^{-1} = 1 + w_2(V)$, we have $w_2(V^0) = w_2(V)$. Hence by Theorem S, we have $\bar{\varepsilon}(V^0) = w_2(V) = w_2(W) + (d, 2)_K$. Since $\mathrm{inv}_K = (_F) \circ \partial: H^2(K, \mathbb{Z}/2) \simeq \{\pm 1\}$, it is enough to compute the boundary $\partial w_2(W) \in H^1(F, \mathbb{Z}/2)$. We check that Theorem is now a special case of Proposition 2 where $F$ is finite and $a = 1$. In fact if $f$ or $D$ is even, the valuation of $d$ is even and $(d, 2)_K = 1$ and $(\{d\}/F) = (d, \pi_K)_K$ for a prime element $\pi_K$ of $K$. Thus the proof of Theorem is completed.

## REFERENCES


[D1] DELIGNE, P., *Les constantes des équations fonctionelles des fonctions L*, in Modular functions of one variable II, Lect. Notes in Math *349* (1972), 501–597, Springer-Verlag, Berlin–Heidelberg–New York.

[D2] DELIGNE, P., *Les constantes locales de l'équation fonctionelle de la fonction L d'Artin d'une représentation orthogonale*, Inventiones Math. *35* (1976), 299–316.

[H] HENNIART, G., *Galois ϵ-factors modulo roots of unity*, Inventiones Math. *78* (1984), 117–126.

[K] KATO, K., *Swan conductors with differential values*, Adv. Studies in Pure Math. *12* (1987), Kinokuniya, Tokyo, 315–342.

[S1] SERRE, J.-P., *Corps Locaux*, Hermann, Paris, 1968.

[S2] SERRE, J.-P., *Cours d'arithmétique*, Presses Universitaire de France, Paris, 1970.

[S2] SERRE, J.-P., *Conducteurs d'Artin des caractères réels*, Inventiones Math. *14* (1971), 173–183.

[S4] SERRE, J.-P., *L'invariant de Witt de la forme* $\mathrm{Tr}_{L/K}(x^2)$, Comm. Math. Helv. *59* (1984), 651–675.



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Received September 21, 1993; November 15, 1994