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Quadrilaterals and extremal quasiconformal extensions

J. M. Anderson and A. Hinkkanen

Abstract. We show that the smallest maximal dilatation for a quasiconformal extension of a quasisymmetric function of the unit circle may be larger than indicated by the change in the module of the quadrilaterals with vertices on the circle.

§1. Introduction

Let $\mathbb{D}$ denote the unit disk in the complex plane $\mathbb{C}$ and let f be a sense-preserving quasisymmetric homeomorphism of the unit circle $\partial\mathbb{D}$ onto itself. Consider quadrilaterals $Q = \mathbb{D}(z_1, z_2, z_3, z_4)$ whose domain is $\mathbb{D}$ and whose vertices z_1, z_2, z_3, z_4 follow each other in the positive (anticlockwise) direction on $\partial\mathbb{D}$. We denote the conformal module of Q by $M(Q)$ (for definitions, see [7, pp. 14–15]). The function f maps z_1, z_2, z_3, z_4 onto $f(z_1), f(z_2), f(z_3), f(z_4)$ and the corresponding quadrilateral with domain $\mathbb{D}$ is denoted by $f(Q)$. If the number $K \geq 1$ is such that f has a K-quasiconformal extension to a self-map of $\mathbb{D}$ then [7, p. 16]

$$\frac{1}{K} \leq \frac{M(f(Q))}{M(Q)} \leq K. \tag{1.1}$$

We now set

$$K_0 = K_0(f) = \sup\left\{\frac{M(f(Q))}{M(Q)} : Q \text{ has domain } \mathbb{D}\right\}, \tag{1.2}$$

so that K_0 is the smallest number K for which (1.1) holds for all quadrilaterals Q.

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We also set

$$K_1(f) = \inf\{K : f \text{ has a } K\text{-quasiconformal extension to a self-map of } \bar{\mathbb{D}}\}.$$

Then, as is well-known [6, p. 16]

$$K_0(f) \le K_1(f) \le [\lambda(K_0(f))]^{3/2},$$

where $\lambda(t)$ is the function determined by the Teichmüller ring [7, (6.4), p. 81]. The right hand quantity behaves asymptotically like $(1/64)\exp(3\pi K_0(f)/2)$ as $K_0(f) \to \infty$ [7, (6.10), p. 82]. However, it has been conjectured [14, Conjecture 3.21] that $K_0(f) = K_1(f)$ for all quasisymmetric f and it is the object of this note to show that this is not so (unless $K_0(f) = 1$, in which case it is easily seen that f is the restriction of a Möbius transformation and hence $K_1(f) = 1$).

THEOREM 1. *For each* $K > 1$, *there exists a sense-preserving quasisymmetric homeomorphism* f *of* $\partial\mathbb{D}$ *onto itself such that*

$$K_0(f) < K_1(f) = K.$$

This theorem is in contrast to a theorem of Jenkins [4, Theorem 1, p. 931] where general polygons are considered instead of quadrilaterals and a condition similar to (1.1) is given which is necessary and sufficient for f to have a K-quasiconformal extension to a map of $\bar{\mathbb{D}}$ onto itself. Thus Theorem 1 shows that, in general, it is not sufficient to consider the moduli of quadrilaterals alone to determine $K_1(f)$, though in [14], several examples are given where $K_0(f) = K_1(f)$. Hence, furthermore, any attempt to construct an extremal quasiconformal extension of f – an extension of f whose maximal dilatation is equal to $K_1(f)$ – by considering only the action of f on modules of quadrilaterals must necessarily fail.

Ever since Beurling and Ahlfors gave the necessary and sufficient condition for a homeomorphism of the unit circle to have a quasiconformal extension to the disk [2], the problem of characterizing such homeomorphisms, called quasisymmetric by Kelingos [5], and considering various relationships between the boundary map and its extensions, have been studied in the literature. A simple characterization of quasisymmetric maps of the extended real line onto itself fixing infinity was given in [2]. It is based on considering $M(f(Q))/M(Q)$ when $M(Q) = 1$ and one vertex of Q is at infinity. Agard and Kelingos [1, p. 448] considered a definition for quasisymmetric maps based on the requirement that $1/K \le M(f(Q))/M(Q) \le K$ for all Q with one vertex at infinity. They [1, p. 449] also mentioned the possibility of using the condition $1/K \leq M(f(Q))/M(Q) \leq K$ for all Q, particularly when f is not assumed to fix the point at infinity. This lead them to the quantity $K_0(f)$ defined above. The extremal quasiconformal extensions of a given quasisymmetric function f have been studied in great detail, particularly by Reich and by Strebel in their many papers, some of them joint, for example, [10], [11], [12]. The inequality $K_0(f) \leq K_1(f)$ being obvious from the definitions of these quantities and the geometric definition of quasiconformal mappings [7, p. 16], the question arises as to the exact nature of the relationship between $K_0(f)$ and $K_1(f)$. The paper by Jenkins [4] provides interesting insight into this problem in terms of the change of a suitable conformal module for polygons more general than quadrilaterals, and the connection between $K_0(f)$ and $K_1(f)$ is briefly discussed. The question of whether $K_0(f) = K_1(f)$ for all f, has probably been informally around since the 1960's, but we have not been able to find it in print except in [14].

§2. Parallelograms

We denote by V the closed parallelogram with vertices $\zeta_1 = 0$, $\zeta_2 = 1$, $\zeta_3 = \alpha + 1 + i\beta$, and $\zeta_4 = \alpha + i\beta$, where $\alpha > 0$ and $\beta > 0$. These vertices will also be called the geometrical vertices of V, to distinguish them from the vertices of some quadrilateral. Let $F_K(V)$ be the image of V under the horizontal affine stretch F_K that takes $x + iy$ onto $Kx + iy$, where $K > 1$, so that the vertices of $F_K(V)$ are $\tilde{\zeta}_1 = 0$, $\tilde{\zeta}_2 = K$, $\tilde{\zeta}_3 = K(\alpha + 1) + i\beta$, and $\tilde{\zeta}_4 = K\alpha + i\beta$. The function F_K is a K-quasiconformal mapping of V onto $F_K(V)$ with complex dilatation $\mu(F_K, z) \equiv (K-1)/(K+1)$. Moreover, F_K is uniquely extremal for its boundary values (see, e.g., [12]) so that $K_1(F_K \mid \partial V) = K$. Let Φ_j, for $j = 1, 2$, map V and $F_K(V)$, respectively, one-to-one conformally onto the unit disk $\mathbb{D}$. By conformal invariance the mapping $\tilde{F}_K = \Phi_2 \circ F_K \circ \Phi_1^{-1}$ of $\bar{\mathbb{D}}$ onto itself is uniquely extremal for its boundary values and $K_1(\tilde{F}_K \mid \partial \mathbb{D}) = K$, and, of course, $\tilde{F}_K \mid \partial \mathbb{D}$ is quasisymmetric. We shall show that $K_0(\tilde{F}_K \mid \partial \mathbb{D}) < K$. If z_1, z_2, z_3, z_4 are four distinct points on ∂V following each other in the positive direction, then we temporarily set $Z_j = \Phi_1(z_j)$, $w_j = F_K(z_j)$, and $W_j = \Phi_2(w_j)$ for $1 \leq j \leq 4$. However, $M(\mathbb{D}(Z_1, Z_2, Z_3, Z_4)) = M(V(z_1, z_2, z_3, z_4))$ and $M(\mathbb{D}(W_1, W_2, W_3, W_4)) = M(F_K(V)(w_1, w_2, w_3, w_4))$, and so we consider only moduli of quadrilaterals in V and $F_K(V)$.

We denote the (internal) angle of V ($F_K(V)$, respectively) with vertex at the origin by $\eta\pi$ ($\eta_1\pi$, respectively), so that $0 < \eta_1 < \eta < 1/2$ and

$$\tan \eta\pi = K \tan \eta_1 \pi. \tag{2.1}$$

Hence two opposite angles of V are equal to $\eta\pi$ and the two others are equal to

$(1-\eta)\pi$. The corresponding angles of $F_K(V)$ are $\eta_1\pi$ and $(1-\eta_1)\pi$. If $K>1$ and $\eta \in (0, 1/2)$ are given and $\eta_1 \in (0, \eta)$ is defined by (2.1) then

$$\frac{1}{K} < \frac{\eta_1}{\eta} < 1 < K, \tag{2.2}$$

$$\frac{1}{K} < 1 < \frac{1-\eta_1}{1-\eta} < K. \tag{2.3}$$

For if $x = \eta\pi$ and $h(x) = (K\eta_1 - \eta)\pi = K \arctan(K^{-1} \tan x) - x$ then $h(0) = 0$ and $h'(x) = (K^2 + \tan^2 x)^{-1}(K^2 - 1)\tan^2 x > 0$ so that $h(x) > 0$ for $0 < x < \pi/2$. This gives (2.2). Further, (2.2) implies (2.3) whenever $\eta, \eta_1 \in (0, 1/2)$.

The proof of Theorem 1 falls into two cases:

(i) when the supremum in (1.2) is attained for some quadrilateral Q; and
(ii) when

$$K_0(\tilde{F}_K \mid \partial \mathbb{D}) = K_0(F_K \mid \partial V) = \lim_{n\to\infty} \frac{M(F_K(Q_n))}{M(Q_n)}, \tag{2.4}$$

where the quadrilaterals Q_n with domain V and their images degenerate in some way. We consider the cases separately.

§3. The attained supremum

Suppose that for some non-degenerate quadrilateral Q with domain V we have

$$K_0(F_K) = K_0(F_K \mid \partial V) = \frac{M(F_K(Q))}{M(Q)}. \tag{3.1}$$

We show then that $K_0(F_K) < K = K_1(F_K)$. Suppose, on the contrary, that $K_0(F_K) = K$ and let ψ_1 and ψ_2 map Q and $F_K(Q)$ onto their respective canonical rectangles (for definitions, see [7, p. 15]). Thus $\psi_1(Q)$ can be taken to have vertices 0, $M(Q)$, $M(Q)+i$, i, and ψ_1 takes the vertices of the quadrilateral Q onto the geometrical vertices of $\psi_1(Q)$. Similarly, for $\psi_2(F_K(Q))$. But $M(F_K(Q)) = KM(Q)$ and hence both the functions F_K and $\psi_2 \circ F_K \circ \psi_1^{-1}$ are K-quasiconformal mappings of $\psi_1(Q)$ onto $\psi_2(F_K(Q))$ taking vertices onto vertices. But by [3, Beispiel 1] or [13,

p. 18], F_K is the *unique* K-quasiconformal mapping having this property. We conclude that $F_K = \psi_2 \circ F_K \circ \psi_1^{-1}$ or

$$F_K \circ \psi_1 = \psi_2 \circ F_K. \tag{3.2}$$

If we decompose ψ_1 and ψ_2 into their real and imaginary parts as $\psi_1 = u_1 + iv_1$ and $\psi_2 = u + iv$, then (3.2) becomes

$$Ku_1(x+iy) + iv_1(x+iy) = u(Kx+iy) + iv(Kx+iy).$$

Thus $u(Kx+iy)$ is a harmonic function of $x+iy$ for $x+iy \in V$ and, of course, $u(x+iy)$ is a harmonic function of $x+iy$ for $x+iy \in F_K(V)$. This implies that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 = K^2 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},$$

all functions evaluated at any $x+iy \in F_K(V)$. This yields $\partial^2 u/\partial x^2 = \partial^2 u/\partial y^2 = 0$. We deduce that the non-constant function $\psi_2 = u + iv$ is a polynomial in $w \in F_K(V)$ of degree 1 or 2. Since $F_K(V)$ is a parallelogram which is *not* a rectangle – here for the first time is this essential fact used – in either case ψ_2 cannot map $F_K(V)$ onto a rectangle since the angles at the geometrical vertices of $F_K(V)$ would have to be preserved, with at most one exceptional vertex when ψ_2 is a polynomial of degree 2. This contradiction shows that (3.1) cannot hold.

§4. The degenerate case; two-point degeneracy

Suppose that $\{Q_n\}$ is a sequence of quadrilaterals with domain V such that (2.4) holds. By passing to subsequences, if necessary, we may assume that the vertices $z_{j,n}$ for $1 \le j \le 4$ of Q_n tend to limit points $z_j \in \partial V$ for $1 \le j \le 4$ as $n \to \infty$ and that at least two of the points z_j coincide. Otherwise we have an attained supremum, in which case we have already shown that $K_0(F_K) < K$.

There are four possibilities, up to permutations.

Case I. $z_1 = z_2$ while z_1, z_3, and z_4 are distinct;
Case II. $z_1 = z_2 \neq z_3 = z_4$;
Case III. $z_1 = z_2 = z_3 \neq z_2$;
Case IV. $z_1 = z_2 = z_3 = z_4$.

To deal with possible permutations we may have to pass to conjugate quadrilaterals, obtained by replacing the ordering z_1, z_2, z_3, z_4 by z_2, z_3, z_4, z_1. These permutations, in effect, replace $M(Q)$ by $1/M(Q)$. Thus we must exclude also the possibility that

$$M(F_K(Q_n))/M(Q_n) \to 1/K \qquad \text{as } n \to \infty.$$

It will be evident below that our argument achieves this.

CASE I. Let φ_n map V conformally onto the upper half plane H taking $z_{1,n}, z_{2,n}, z_{3,n}, z_{4,n}$ onto a_n, ∞, 0, and 1, respectively. Thus $1 < a_n < \infty$ and $a_n \to \infty$ as $n \to \infty$. Similarly we set $w_{j,n} = F_K(z_{j,n})$ for $1 \le j \le 4$, and let $\tilde{\varphi}_n$ map $F_K(V)$ conformally onto H, taking $w_{1,n}, w_{2,n}, w_{3,n}, w_{4,n}$ onto b_n, ∞, 0, and 1, respectively. As before, $1 < b_n < \infty$ and $b_n \to \infty$ as $n \to \infty$.

If $1/m(a)$ denotes the module of the quadrilateral $H(a, \infty, 0, 1)$ when $1 < a < \infty$, then $M(Q_n) = 1/m(a_n)$ and $M(F_K(Q_n)) = 1/m(b_n)$. We estimate $m(b_n)/m(a_n)$ by using the explicit formula for $m(a)$ and then obtaining an asymptotic estimate for b_n in terms of a_n and K. By [7, pp. 59–60] we have, for $1 < a < \infty$, that

$$m(a) = M(H(\infty, 0, 1, a)) = M(H(\infty, 0, 1/\sqrt{a}, \sqrt{a})) = \frac{K(\sqrt{1 - r^2})}{K(r)},$$

where $K(t)$ denotes the complete elliptic integral

$$K(t) = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - t^2x^2)}}$$

and $r^2 = 1/a$. Since $K(0) = \pi/2$ and

$$K(t) \sim \frac{1}{2} \log \frac{1}{1 - t} \qquad \text{as } t \to 1-$$

(see, e.g., [8, Problem 90, p. 21]) we have

$$m(a) \sim \frac{1}{\pi} \log a \qquad \text{as } a \to \infty. \tag{4.1}$$

Let G_1 and G_2 be fixed conformal mappings of V and of $F_K(V)$ onto the upper half plane H taking the points z_j and w_j, respectively, for $1 \le j \le 4$, onto some finite points. Let L_n and $\tilde{L}_n$ be Möbius transformations taking the points $Z_{j,n} = G_1(z_{j,n})$

and $W_{j,n} = G_2(w_{j,n})$ for $1 \le j \le 4$ onto a_n, ∞, 0, 1 and b_n, ∞, 0, 1, respectively. Then $\varphi_n = L_n \circ G_1$ and $\tilde{\varphi}_n = \tilde{L}_n \circ G_2$. We may assume throughout that $-\infty < Z_{1,n} < Z_{2,n} < Z_{3,n} < Z_{4,n} < \infty$ and $-\infty < W_{1,n} < W_{2,n} < W_{3,n} < W_{4,n} < \infty$. We have

$$L_n(Z) = \frac{Z - Z_{3,n}}{Z - Z_{2,n}} \frac{Z_{4,n} - Z_{2,n}}{Z_{4,n} - Z_{3,n}} \tag{4.2}$$

so that

$$a_n = \varphi_n(z_{1,n}) = L_n(Z_{1,n}) = \frac{Z_{1,n} - Z_{3,n}}{Z_{1,n} - Z_{2,n}} \frac{Z_{4,n} - Z_{2,n}}{Z_{4,n} - Z_{3,n}}. \tag{4.3}$$

There are distinct real numbers Z_1, Z_3, and Z_4 so that

$$\lim_{n \to \infty} Z_{j,n} = Z_j \qquad \text{for } j = 1, 3, 4,$$

while $Z_{2,n} \to Z_1$. Thus, as $n \to \infty$,

$$a_n \sim C_1/(Z_{1,n} - Z_{2,n}),$$

where

$$C_1 = \frac{(Z_1 - Z_3)(Z_4 - Z_1)}{Z_4 - Z_3}$$

is a non-zero real number. Now if z_1 is not a geometrical vertex of V we have

$$G_1^{-1}(Z) = z_1 + C_2(Z - Z_1) + O((Z - Z_1)^2)$$

as $Z \to Z_1$ in $\bar{H}$ where $C_2 = (G_1^{-1})'(Z_1)$ is a non-zero complex number. But

$$G_1^{-1}(Z_{1,n}) = z_{1,n}, \qquad G_1^{-1}(Z_{2,n}) = z_{2,n},$$

and hence

$$a_n = |a_n| \sim \frac{C_3}{|z_{1,n} - z_{2,n}|}$$

as $n \to \infty$, where $C_3 > 0$. Similarly

$$b_n = |b_n| \sim \frac{C_4}{|w_{1,n} - w_{2,n}|}$$

as $n \to \infty$, where $C_4 > 0$. Now C_3 and C_4 are independent of n, depending, in fact, only on the auxiliary transformations G_1 and G_2 and the distinct points z_1, z_3, and z_4. Moreover, since $w_{j,n} = F_K(z_{j,n})$, we have

$$\frac{1}{K} \le \left| \frac{w_{1,n} - w_{2,n}}{z_{1,n} - z_{2,n}} \right| \le K$$

for all n. We conclude that

$$0 < C_5 < \frac{b_n}{a_n} < C_6 < \infty,$$

for suitable constants C_5 and C_6 independent of n. By (4.1),

$$\frac{M(F_K(Q_n))}{M(Q_n)} \sim \frac{\log a_n}{\log b_n} \to 1 \qquad \text{as } n \to \infty.$$

Since $K > 1$ we have $K_0(F_K) < K$ as required.

Suppose now that z_1 is a geometrical vertex of V and that V has the angle $\eta\pi$ at z_1, so that $F_K(V)$ has the angle $\eta_1\pi$ at w_1. Suppose that $W_{j,n} = G_2(w_{j,n}) \to W_j$ for $1 \le j \le 4$ as $n \to \infty$. To make notation easier we suppose that $Z_1 = W_1 = z_1 = w_1 = 0$. Now $G_1^{-1}(Z) \sim C_7 Z^\eta$ as $Z \to Z_1 = 0$ in $\bar{H}$, where $C_7 \ne 0$. By passing to a subsequence, if necessary, we may assume that $Z_{1,n}/Z_{2,n} \to \lambda$ and $W_{1,n}/W_{2,n} \to \tilde{\lambda}$ as $n \to \infty$, where $-\infty \le \lambda \le \infty$ and $-\infty \le \tilde{\lambda} \le \infty$. Since $w_{j,n} = F_K(z_{j,n})$ we have $\lambda = 0$ if and only if $\tilde{\lambda} = 0$, and $|\lambda| = \infty$ if and only if $|\tilde{\lambda}| = \infty$.

If $|\lambda| = |\tilde{\lambda}| = \infty$ then, as $n \to \infty$,

$$Z_{1,n} - Z_{2,n} \approx (z_{1,n}/C_7)^{1/\eta} - (z_{2,n}/C_7)^{1/\eta} \sim C_8 (z_{1,n})^{1/\eta},$$

$$W_{1,n} - W_{2,n} \sim C_9 (w_{1,n})^{1/\eta_1}.$$

Passing, if necessary, to a further subsequence, we may assume that

$$\left| \frac{w_{1,n}}{z_{1,n}} \right| \to \kappa \qquad \text{where } \frac{1}{K} \le \kappa \le K$$

as $n \to \infty$. Thus as $n \to \infty$

$$1/M(Q_n) = \frac{1}{\pi} \log a_n \sim -\frac{1}{\eta\pi} \log |z_{1,n}|,$$

$$\frac{1}{M(F_K(Q_n))} = \frac{1}{\pi}\log b_n \sim -\frac{1}{\eta_1 \pi}\log|w_{1,n}| \sim -\frac{1}{\eta_1 \pi}\log|z_{1,n}| \sim \frac{\eta}{\eta_1}\frac{1}{M(Q_n)}.$$

Since $1 < \eta/\eta_1 < K$ by (2.2), we again see that $K_0(F_K) < K$, as required. The case when λ and $\tilde{\lambda}$ are finite, possibly zero, are similar and so are omitted.

If, instead, V has the angle $(1-\eta)\pi$ at z_1, so that $F_K(V)$ has the angle $(1-\eta_1)\pi$ at w_1, the analysis is similar to the above and now

$$\frac{1}{M(F_K(Q_n))} \sim \frac{1-\eta}{1-\eta_1}\frac{1}{M(Q_n)}.$$

Thus

$$K_0(F_K) = \max\left\{\frac{1-\eta}{1-\eta_1}, \frac{1-\eta_1}{1-\eta}\right\} = \frac{1-\eta_1}{1-\eta} < K$$

from (2.3). Hence, in all subcases arising in Case I, we have $K_0(F_K) < K = K_1(F_K)$.

CASE II. This is similar to Case I. We perform the same preliminary transformations to find that

$$a_n = \frac{Z_{1,n}-Z_{3,n}}{Z_{1,n}-Z_{2,n}}\frac{Z_{4,n}-Z_{2,n}}{Z_{4,n}-Z_{3,n}} \sim \frac{C_1}{(Z_{1,n}-Z_{2,n})(Z_{4,n}-Z_{3,n})}$$

as $n \to \infty$, where $C_1 = -(Z_1 - Z_3)^2 \neq 0$. Thus, as before,

$$\pi/M(Q_n) \sim \log a_n \sim -\log|Z_{1,n}-Z_{2,n}| - \log|Z_{4,n}-Z_{3,n}|,$$

while in a similar fashion,

$$\pi/M(F_K(Q_n)) \sim \log b_n \sim -\log|W_{1,n}-W_{2,n}| - \log|W_{4,n}-W_{3,n}|.$$

As in Case I we find, again by passing to a subsequence if necessary, that each of the quotients

$$\frac{-\log|Z_{1,n}-Z_{2,n}|}{-\log|W_{1,n}-W_{2,n}|} \quad \text{and} \quad \frac{-\log|Z_{4,n}-Z_{3,n}|}{-\log|W_{4,n}-W_{3,n}|}$$

tends to a limit κ as $n \to \infty$, with $\kappa = 1$, or $\kappa = \eta/\eta_1$, or $\kappa = (1-\eta)/(1-\eta_1)$. If $\kappa_1 = \max\{1, \eta/\eta_1, (1-\eta_1)/(1-\eta)\}$ so that $1 < \kappa_1 < K$ then

$$-\log|Z_{1,n} - Z_{2,n}| \le (\kappa_1 + o(1))(-\log|W_{1,n} - W_{2,n}|)$$

and

$$-\log|Z_{3,n} - Z_{4,n}| \le (\kappa_1 + o(1))(-\log|W_{3,n} - W_{4,n}|)$$

and hence

$$\pi/M(Q_n) \le (\kappa_1 + o(1))\pi/M(F_K(Q_n)).$$

Similarly,

$$\pi/M(Q_n) \ge (\kappa_1^{-1} - o(1))\pi/M(F_K(Q_n))$$

and so $K_0(F_K) \le \kappa_1 < K = K_1(F_K)$ also in Case II.

§5. Quadruple degeneracy

CASE IV. Suppose that z_1 lies in the interior of an edge of V. For simplicity, we assume first that this edge is the lower horizontal edge of V. We shall frequently assert that various sequences tend to limits and this can always be achieved by passing to a subsequence if necessary. For all large n, the points $z_{j,n}$ are ordered from left to right along the edge. Any j could correspond to the leftmost point, and we may assume that it is the same j for all n. Let these points also be denoted by $\alpha_n < \beta_n < \gamma_n < \delta_n$. To be able to use definite notation, suppose that $\alpha_n = z_{3,n}$ for all n. Then $\beta_n = z_{4,n}$, $\gamma_n = z_{1,n}$, and $\delta_n = z_{2,n}$. Arguments similar to those presented below work in all the other three cases also. Let L_n be the Möbius transformation of the upper half plane H onto itself taking α_n, β_n, γ_n, and δ_n onto 0, 1, a_n, and ∞, respectively. Here $a_n \in (1, \infty)$ is determined by the cross ratio of the points $z_{j,n}$. Write $\tilde{\alpha}_n = F_K(\alpha_n)$ and so on, and let $\tilde{L}_n$ be the Möbius transformation of H onto itself taking $\tilde{\alpha}_n$, $\tilde{\beta}_n$, $\tilde{\gamma}_n$, and $\tilde{\delta}_n$ onto 0, 1, b_n, and ∞, respectively. We have $L_n(V) \subset H$ and $\tilde{L}_n(F_K(V)) \subset H$. Clearly, for all large n, the set $\partial L_n(V)$ contains $(-\infty, A_n] \cup [B_n, \infty]$ where $-\infty < A' < A_n < B_n < B' < 0$ and A' and B' are independent of n, and furthermore $B_n - A_n \to 0$ as $n \to \infty$. We may assume that $A_n \to A \in (-\infty, 0)$ so that $B_n \to A$ also. In fact, for any $\epsilon > 0$ there is an integer n_0 such for all $n \ge n_0$, the set $\bar{H} \backslash L_n(V)$ is contained in an ϵ-neighbourhood of A.

Similarly, we may assume that there is a point $\tilde{A} \in (-\infty, 0)$ such that $\bar{H}\backslash\tilde{L}_n(F_K(V))$ tends to $\tilde{A}$ in the above sense.

We may assume that $a_n \to a \in [1, \infty]$ and $b_n \to b \in [1, \infty]$. Now $M(Q_n) = M(L_n(V)(0, 1, a_n, \infty))$, and $M(F_K(Q_n)) = M(\tilde{L}_n(F_K(V))(0, 1, b_n, \infty))$. We shall show that

$$\frac{M(L_n(V)(0, 1, a_n, \infty))}{M(H(0, 1, a_n, \infty))} \to 1$$

and, for a similar reason, $M(\tilde{L}_n(F_K(V))(0, 1, b_n, \infty))/M(H(0, 1, b_n, \infty)) \to 1$ as $n \to \infty$. Now $M(H(0, 1, a_n, \infty)) = 1/m(a_n)$. We show below that $m(a_n)/m(b_n) \to 1$ as $n \to \infty$, which then implies that

$$K_0(F_K) = \lim_{n\to\infty} \frac{M(F_K(Q_n))}{M(Q_n)} = 1 < K = K_1(F_K),$$

as desired. (In the particular case considered now, it turns out that $a_n = b_n$. However, in other similar cases we need not have equality but something weaker.)

We first study the relationship between a_n and b_n. We have

$$L_n(z) = \frac{z - z_{3,n}}{z - z_{2,n}} \frac{z_{4,n} - z_{2,n}}{z_{4,n} - z_{3,n}} \quad \text{and} \quad \tilde{L}_n(w) = \frac{w - w_{3,n}}{w - w_{2,n}} \frac{w_{4,n} - w_{2,n}}{w_{4,n} - w_{3,n}},$$

and

$$L_n^{-1}(Z) = z_{2,n} + \frac{z_{2,n} - z_{3,n}}{Z - R_n} R_n,$$

where

$$R_n = \frac{z_{4,n} - z_{2,n}}{z_{4,n} - z_{3,n}}.$$

Since, in this particular case, $w_{j,n} = Kz_{j,n}$, we obtain that

$$b_n = (\tilde{L}_n \circ F_K \circ L_n^{-1})(a_n) = a_n,$$

as asserted above. Hence $m(a_n)/m(b_n) = 1$.

Consider then the relationship between $M(L_n(V)(0, 1, a_n, \infty))$ and $M(H(0, 1, a_n, \infty))$. If $a_n \to a \in (1, \infty)$ then it follows from the convergence properties of the conformal module [7, p. 27] that $M(L_n(V)(0, 1, a_n, \infty)) \to M(H(0, 1, a, \infty)) \in (0, \infty)$. Since, in any case, $1/K \le M(F_K(Q_n))/M(Q_n) \le K$ we further see that $a = 1$ if and only if $b = 1$, and $a = \infty$ if and only if $b = \infty$. So if $1 < a < \infty$ then $1 < b < \infty$, and similarly to the above, $M(\tilde{L}_n(F_K(V))(0, 1, b_n, \infty)) \to M(H(0, 1, b, \infty)) \in (0, \infty)$. Since $a_n = b_n$, we have $a = b$, and so

$$\frac{M(F_K(Q_n))}{M(Q_n)} = \frac{M(\tilde{L}_n(F_K(V))(0, 1, b_n, \infty))}{M(L_n(V)(0, 1, a_n, \infty))} \to \frac{M(H(0, 1, b, \infty))}{M(H(0, 1, a, \infty))} = 1,$$

as desired. We next consider the case $a = b = 1$. The case $a = b = \infty$ can either be dealt with in the same way, or reduced to the case $a = b = 1$ by passing to conjugate quadrilaterals, which does not affect the assumption of Case IV that all the z_j coincide.

Let ψ_n be the conformal mapping of $L_n(V)$ onto H fixing each of 0, 1, and ∞. If $\psi_n(a_n) = c_n$ then $M(L_n(V)(0, 1, a_n, \infty)) = M(H(0, 1, c_n, \infty))$. By the discussion in §4 before (4.1), we have

$$m(a) \sim \frac{\pi}{2K(1/\sqrt{a})} \sim \frac{\pi}{2 \log \dfrac{1}{a-1}} \qquad \text{as } a \to 1+.$$

Thus to show that

$$\frac{M(L_n(V)(0, 1, a_n, \infty))}{M(H(0, 1, a_n, \infty))} \to 1,$$

we need to demonstrate that

$$\log \frac{1}{a_n - 1} \sim \log \frac{1}{c_n - 1}$$

as $n \to \infty$.

Let $\omega(\gamma, z, D)$ denote the harmonic measure of the set $\gamma \subset \partial D$ at the point $z \in D$ with respect to the domain D. We have for $1 < a < \infty$,

$$\omega((1, a), i, H) = \frac{1}{\pi}(\arctan a - \arctan 1) = \frac{1}{\pi} \arctan \frac{a-1}{a+1} \sim \frac{a-1}{2\pi}$$

as $a \to 1+$. If $Z = X + iY$, $|Z - i| < 1/4$ and $1 < a < 2$ then

$$\begin{aligned}\omega((1, a), Z, H) &= \frac{1}{\pi}\left(\arctan\frac{a - X}{Y} - \arctan\frac{1 - X}{Y}\right) \\ &= \frac{1}{\pi}\arctan\frac{a - 1}{Y + (a - X)(1 - X)Y^{-1}}\end{aligned}$$

so that for all those Z and a,

$$\frac{1}{C} \leq \frac{\omega((1, a), Z, H)}{a - 1} \leq C$$

for some absolute constant $C > 1$. For every $\epsilon > 0$, there is an integer n_0 such that if $n \geq n_0$ then

$$\bar{H}\backslash\{Z : |Z - A| < \epsilon\} \subset L_n(V) \subset \bar{H}.$$

Let D_ϵ be the domain whose closure is $\bar{H}\backslash\{Z : |Z - A| < \epsilon\}$. It follows that for $n \geq n_0$,

$$\begin{aligned}&\omega((1, a_n), i, D_\epsilon) < \omega((1, c_n), \psi_n(i), H) \\ &\quad = \omega((1, a_n), i, L_n(V)) < \omega((1, a_n), i, H) \sim \frac{a_n - 1}{2\pi},\end{aligned}$$

as $n \to \infty$. Since $\psi_n(i) \to i$ as $n \to \infty$, we have

$$\frac{1}{C} \leq \frac{\omega((1, c_n), \psi_n(i), H)}{c_n - 1} \leq C$$

for all large n. We only need to show that

$$\omega((1, a_n), i, D_\epsilon) > C_1(a_n - 1)$$

for some fixed $C_1 > 0$, for all large n, to deduce, in view of all of the above, that $\log(1/(a_n - 1)) \sim \log(1/(c_n - 1))$ as $n \to \infty$.

The map

$$\Phi(Z) = \left(\frac{Z - (A + \epsilon)}{Z - (A - \epsilon)}\right)^2$$

takes D_ϵ conformally onto H, taking i, 1, and a_n onto

$$c_1 + ic_2 = \left(\frac{i-(A+\epsilon)}{i-(A-\epsilon)}\right)^2, \qquad c_3 = \left(\frac{1-(A+\epsilon)}{1-(A-\epsilon)}\right)^2, \qquad c_{4,n} = \left(\frac{a_n-(A+\epsilon)}{a_n-(A-\epsilon)}\right)^2,$$

respectively. Choose a small but fixed $\epsilon > 0$ so that $A \pm \epsilon \neq -1$. If I_n is the open interval with endpoints c_3 and $c_{4,n}$ then we obtain

$$\begin{aligned}
\omega((1, a_n), i, D_\epsilon) &= \omega(I_n, c_1 + ic_2, H) \\
&= \frac{1}{\pi}\left|\arctan\frac{c_1 - c_{4,n}}{c_2} - \arctan\frac{c_1 - c_3}{c_2}\right| \\
&= \frac{1}{\pi}\left|\arctan\frac{c_3 - c_{4,n}}{c_2 + (c_1 - c_{4,n})(c_1 - c_3)c_2^{-1}}\right| > C_1|c_3 - c_{4,n}| \\
&= C_1\left|\left(\frac{1-(A+\epsilon)}{1-(A-\epsilon)}\right)^2 - \left(\frac{a_n-(A+\epsilon)}{a_n-(A-\epsilon)}\right)^2\right| \\
&= \frac{4\epsilon C_1|1-a_n||a_n - A + A^2 - \epsilon^2 - a_n A|}{(1-(A-\epsilon))^2(a_n-(A-\epsilon))^2} > C_2|a_n - 1|
\end{aligned}$$

for some positive constants C_1 and C_2 that depend only on A, ϵ, and the distance of $A - \epsilon$ from -1. This completes the proof that $\log(1/(a_n - 1)) \sim \log(1/(c_n - 1))$ as $n \to \infty$, as desired.

We indicate briefly the changes to be made when z_1 lies in the interior of a non-horizontal side of V. We map V and $F_K(V)$ by rotations and translations so that this non-horizontal edge becomes a segment of the real axis and the images of V and $F_K(V)$ lie in $\bar{H}$. It only matters how the transformation of the map F_K looks like in a neighbourhood of the image of z_1. A calculation shows that in the case of the right non-horizontal side, the map corresponding to F_K is given by

$$x + iy \mapsto \frac{(K^2\alpha^2 + \beta^2)x + \alpha\beta(1-K^2)y + iK(\alpha^2+\beta^2)y}{\sqrt{(\alpha^2+\beta^2)(K^2\alpha^2+\beta^2)}}$$

which for $y = 0$ gives $x \mapsto K'x$ where

$$K' = \sqrt{\frac{K^2\alpha^2 + \beta^2}{\alpha^2 + \beta^2}} < K.$$

Recall that $\alpha + i\beta$ is one of the vertices of V. Since the change of the module $M(Q_n)$ only depends on the boundary mapping, we are reduced to considering an affine stretch by a factor not exceeding K. Hence all the previous arguments in the case of a horizontal side can now be followed. We leave any further details to the reader.

Suppose then that z_1 is a geometrical vertex of V. To fix ideas, we first consider the case when $z_1 = 0$. The map $P_1(z) = z^{1/\eta}$ takes V conformally onto a subset of H so that the points $P_1(z_{j,n})$ lie on the real axis $\mathbb{R}$ close to the origin. Let these points be $\alpha_n < \beta_n < \gamma_n < \delta_n$. One of these points may be equal to 0, and some may be positive and some negative. Without loss of generality, we suppose that $\alpha_n < \beta_n < 0 < \gamma_n < \delta_n$ and that $\alpha_n = P_1(z_{3,n})$ for all n. All other cases are similar. Let L_n be the Möbius transformation of the upper half plane H onto itself taking α_n, β_n, γ_n, and δ_n onto 0, 1, a_n, and ∞, respectively. Here $1 < L_n(0) \equiv d_n < a_n < \infty$. Then $L_n(P_1(V)) \subset H$. We perform the corresponding auxiliary maps on $F_K(V)$. In particular, we take $P_2(w) = w^{1/\eta_1}$ and choose the Möbius transformation $\tilde{L}_n$ of H in a suitable way. Then we consider $\chi_n = \tilde{L}_n \circ P_2 \circ F_K \circ P_1^{-1} \circ L_n^{-1}$, which is a $(K\eta/\eta_1)$-quasiconformal mapping of $L_n(P_1(V))$ onto $\tilde{L}_n(P_2(F_K(V)))$ fixing 0, 1, and ∞. We may assume that the maps χ_n tend to a $(K\eta/\eta_1)$-quasiconformal map χ of H onto itself, first locally uniformly in the spherical metric. We assume further that $d_n \to d \in [1, \infty]$ and that $\tilde{d}_n = \chi_n(d_n) \to \tilde{d} \in [1, \infty]$. Again there is $A \in (-\infty, 0)$ such that for any given $\epsilon > 0$, with D_ϵ defined as before, χ_n is defined in D_ϵ and tends to χ uniformly in the closure of D_ϵ. It is shown as above that in the limit it does not matter, for the purpose of determining $K_0(F_K)$, that χ_n is defined in a subset of H rather than in all of H. Thus the value of $\lim_{n\to\infty} M(F_K(Q_n))/M(Q_n)$ depends only on a_n and b_n, as before.

We set

$$R_n = \frac{z_{4/n}^{1/\eta} - z_{2,n}^{1/n}}{z_{4,n}^{1/\eta} - z_{3,n}^{1/\eta}} > 1$$

and

$$\tilde{R} = \frac{w_{4,n}^{1/\eta_1} - w_{2,n}^{1/\eta_1}}{w_{4,n}^{1/\eta_1} - w_{3,n}^{1/\eta_1}} > 1.$$

We may assume that $R_n \to R \in [1, \infty]$ and $\tilde{R}_n \to \tilde{R} \in [1, \infty]$ as $n \to \infty$. A calculation shows that

$$\chi_n(Z) = \tilde{R}_n \left[1 + \frac{w_{2,n}^{1/\eta_1} - w_{3,n}^{1/\eta_1}}{\left[F_K\left\{ \left(z_{2,n}^{1/\eta} + \dfrac{z_{2,n}^{1/\eta} - z_{3,n}^{1/\eta}}{Z - R_n} R_n \right)^{\eta} \right\} \right]^{1/\eta_1} - w_{2,n}^{1/\eta_1}} \right].$$

For $Z \in (-\infty, A - \epsilon) \cup (d_n, \infty]$, the application of F_K above amounts to multiplication by K. We assume that R and $\tilde{R}$ are finite. One can check that $R = \infty$ if and only if $\tilde{R} = \infty$, and this case can be reduced to the case when R is finite by passing to conjugate quadrilaterals. Now we obtain

$$\chi(Z) = \tilde{R}(1 + [\lambda_1 + (K(\lambda_2 + R\lambda_3/(Z-R))^{\eta})^{1/\eta_1}]^{-1}),$$

where

$$\lambda_1 = \lim_{n\to\infty} \frac{-w_{2,n}^{1/\eta_1}}{w_{2,n}^{1/\eta_1} - w_{3,n}^{1/\eta_1}}, \qquad \lambda_2 = \lim_{n\to\infty} \frac{z_{2,n}^{1/\eta}}{(w_{2,n}^{1/\eta_1} - w_{3,n}^{1/\eta_1})^{\eta_1/\eta}},$$

$$\lambda_3 = \lim_{n\to\infty} \frac{z_{2,n}^{1/\eta} - z_{3,n}^{1/\eta}}{(w_{2,n}^{1/\eta_1} - w_{3,n}^{1/\eta_1})^{\eta_1/\eta}}.$$

We may assume that these limits, possibly infinite, exist. We may further assume that $z_{3,n}/z_{2,n} \to \lambda_4 \in \bar{\mathbb{C}}\backslash\{1\}$, say (since $z_{3,n}/z_{2,n}$ has constant argument different from 0 modulo π). Now it is easily seen that λ_3 is a finite non-zero complex number. Next, clearly λ_1 and λ_2 are both zero, or both infinite, or both finite and non-zero. In the second case, χ will not be a homeomorphism, which is impossible. In the two other cases, the restriction of χ to $(-\infty, A-\epsilon) \cup (d_n, \infty]$ can be written as $P_3 \circ P \circ P_4$ where P_3 and P_4 are Möbius transformations while $P(\zeta) = K^{1/\eta_1}\zeta^{\eta/\eta_1}$, used here for $\zeta > 0$.

Note that $w_{2,n}^{1/\eta_1} - w_{3,n}^{1/\eta_1} > 0$ and that $\lambda_3/\lambda_2 = 1 - \lambda_4^{1/\eta} \in \mathbb{R}$. An analysis similar to the one above shows that for $A + \epsilon < Z < d_n$, we have

$$\chi(Z) = \tilde{R}(1 + [\lambda_1 + C_7|\lambda_2 + R\lambda_3/(Z-R)|^{\eta/\eta_1}]^{-1}),$$

where $C_7 = (F_K(e^{i\eta\pi}))^{1/\eta_1} = -[K\cos(\eta\pi)/\cos(\eta_1\pi)]^{1/\eta_1}$. This function χ has the same decomposition as above with the same P_3 and P_4 but with P replaced by

$$Q(\zeta) = -[K\cos(\eta\pi)/\cos(\eta_1\pi)]^{1/\eta_1}|\zeta|^{\eta/\eta_1},$$

used here for $\zeta < 0$. Since ϵ is arbitrary, we have found the boundary behaviour of χ.

The function χ changes the module of any quadrilateral with domain H by at most the same factor as the function h given by $h(x) = P(x)$ for $x \geq 0$ and $h(x) = Q(x)$ for $x < 0$. We compose h with conformal mappings of H onto the strip $S = \{x + iy : 0 < y < \pi\}$ and note that the function $g(z) = \log h(e^z)$ taking ∂S onto itself is given by $g(x) = px + q$, where $p = \eta/\eta_1$ and $q = \eta_1^{-1}\log K$, and by $g(x + i\pi) = px + q + \eta_1^{-1}\log(\cos(\eta\pi)/\cos(\eta_1\pi)) + i\pi$, for all real x. A calculation shows that g coincides with $g_1 \circ F_K \circ g_2^{-1}$ on ∂S, where $g_1(z) = e^{\eta z}$ and $g_2(z) = e^{\eta_1 z}$

are conformal maps of S onto the angles $S_1 = \{z : 0 < \arg z < \eta\pi\}$ and $\{z : 0 < \arg z < \eta_1\pi\}$, respectively. Reich [9, §III.3, p. 123] has shown that the affine stretch F_K is not extremal for its boundary values in S_1, and so there is $K_2 < K$ such that $F_K \mid \partial S_1$ has a K_2-quasiconformal extension to S_1. This can be lifted to a K_2-quasiconformal self-map of S with boundary values g. Therefore, the function h has a K_2-quasiconformal extension to H. It follows that h can change the module of any quadrilateral by the factor K_2 at most.

We now return to the numbers a_n and b_n. As before, we have $a = 1$ if and only if $b = 1$, and $a = \infty$ if and only if $b = \infty$. So if $1 < a < \infty$ then $1 < b < \infty$, then the above implies that $K_0(F_K) \leq K_2 < K = K_1(F_K)$. The cases $a = 1$ and $a = \infty$ are similar, so we only consider the case $a = 1$. Then also $d = \tilde{d} = b = 1$. We note that if $\epsilon_1 > 0$ then there is an integer n_0 such that for all $n \geq n_0$, the function χ_n restricted to the extended real axis apart from a small interval around the point A, and defined in a suitable way in this small interval, can be extended to a $(K_2 + \epsilon_1)$-quasiconformal mapping of H onto itself. Hence $M(F_K(Q_n))/M(Q_n) \leq K_2 + \epsilon_1$, and so, again, $K_0(F_K) \leq K_2 < K = K_1(F_K)$.

When z_1 is a geometrical vertex of V other than the origin, similar considerations can be followed, the only possible difference being that η and η_1 are replaced by $1 - \eta$ and $1 - \eta_1$. The only important thing about η and η_1 was that (2.2) holds, and now we use its counterpart (2.3) to get the desired conclusion. This completes our treatment of Case IV.

§6. Triple degeneracy

CASE III. We give only a sketch of the proof in this case, leaving the details to the reader. Consider first the case when z_1 lies in the interior of some edge of V, and suppose that it is the lower horizontal edge. We map V by a linear real polynomial onto a subset of H taking the leftmost and rightmost of the points $z_{1,n}$, $z_{2,n}$, and $z_{3,n}$ onto 0 and 1. For large n, $z_{4,n}$ will go to a point, possibly non-real but in $\bar{H}$, close to infinity in the spherical metric. We then map the image of V conformally onto H, fixing 0 and 1 and taking the image of $z_{4,n}$ to ∞. The sequence of these maps tends to the identity map. We pick a segment, S say, such as $[-1, 2]$, which contains $[0, 1]$ and is mapped by each such function onto a segment of $\mathbb{R}$ containing a fixed segment $[c, d]$ with $c < 0$ and $d > 1$. In S, each such map can be approximated by a linear mapping with non-zero derivative. We perform analogous transformations to $F_K(V)$. Then we proceed as in Case IV, the only difference being the use of the auxilliary conformal maps of a subset of H onto H. It can be verified by a straightforward, though tedious estimation that this does not make any essential difference.

When z_1 lies in the interior of some other edge of V or is a geometrical vertex of V, we again proceed as in Case IV, using power maps and linear maps, the only difference being that we again also use auxiliary conformal maps taking a suitable subset of H onto H, these conformal maps being close to the identity map and being almost linear in a neighbourhood of $[-1, 2]$, say. This concludes our sketch of the treatment of Case III.

§7. The affine stretch of other domains

It is clear that the above reasoning is valid for the affine stretch of a wide class of domains Δ, say. In the case of non-degenerating quadrilaterals when $K_0(F_K \mid \partial\Delta) = M(F_K(Q))/M(Q)$ for some quadrilateral Q we require two things:

(a) that the affine stretch of Δ is uniquely extremal for its boundary values. This is certainly the case when Δ has finite area [12];
(b) the mapping ψ_2 of $F_K(Q)$ (which, as a set, is the same as $F_K(\Delta)$) onto its canonical rectangle is not a polynomial of degree one or two. This is true for almost all domains Δ.

For degenerating quadrilaterals it is sufficient that the boundary of Δ consists of a finite number of straight line segments meeting in angles different from $\pi/2$. When, for example, the four vertices of Q_n degenerate to a single point z_1 then

(c) if z_1 is an interior point of a side of Δ then locally Δ looks like the upper half plane H where the affine stretch is not extremal for its boundary values;
(d) if z_1 is a geometrical vertex of Δ then locally Δ looks like an angle $\{z : 0 < \arg z < \alpha\}$. Once again the affine stretch is not extremal for its boundary values (see, for example, [9, p. 124]).

In items (c) and (d) it is the lack of extremality that is needed, rather than the lack of unique extremality.

It seems reasonable to suppose that the above considerations apply also to bounded domains Δ with sufficiently smooth boundary (possessing a tangent at every point) or to domains whose boundary consists of a finite number of smooth arcs intersecting in non-zero interior and exterior angles. The technical difficulties involved would make our proofs of these suggestions rather complicated. But we point out that if $\partial\Delta$ has a sharp enough cusp, at $z = 0$, say, then our method would certainly fail. An example of Reich [10, p. 82] is as follows. Let Δ_1 be the region

$$\{x + iy : y > \max\{C, |x|^\beta\}, x \in \mathbb{R}\}, \tag{7.1}$$

where $C > 0$ and $\beta \geq 1$ are constants. Then the affine stretch of Δ_1 is

(e) not extremal for its boundary values if $\beta = 1$;
(f) extremal but not uniquely extremal for its boundary values if $1 < \beta < 3$;
(g) uniquely extremal if $\beta \geq 3$.

Thus if the cusp of $\partial\Delta$ at $z = 0$ is sufficiently sharp then the mapping $w = 1/z$ might map Δ locally near $z = 0$ onto a region given by (7.1) with $\beta \geq 3$, say. Our method then fails, though it is now possible that $K_1(F_K \mid \partial\Delta) = K_0(F_K \mid \partial\Delta)$. Presumably in this case the supremum in $K_0(F_K \mid \partial\Delta)$ is nevertheless attained only in the limit as the vertices of the quadrilateral degenerate to the cusp.

What really matters in the degenerate case is that when one looks at the limiting functions obtained, after suitable renormalization, and the limiting domains obtained then these functions are, to within pre- and post-composition with Möbius transformations, equivalent to the affine stretch in domains where the affine stretch is not extremal for its boundary values. Call such functions Φ_K. Thus in such domains Δ_1 there is a number $K_2 < K$ such that $K_1(\Phi_K \mid \partial\Delta_1) \leq K_2 < K$ while in Δ itself we have $K_1(F_K \mid \partial\Delta) = K$. An example of this is the angle $\{z : 0 < \arg z < \alpha\}$ mentioned above where $K_2 = (1 + k_2)/(1 - k_2) < K$. Here $k_2 = k|\sin\alpha|/\alpha$ and $k = (K - 1)/(K + 1)$.

It also seems likely that the modules of the polygons introduced by Jenkins in [4] will not suffice to determine the minimal maximal dilatation $K_1(f)$ of a quasiconformal extension of a homeomorphism f of $\partial\mathbb{D}$ onto itself if the number of vertices of the permitted polygons remains bounded. Jenkins [4, Theorem 1, p. 931] has shown, however, that if arbitrarily many vertices are permitted then such modules will suffice (more precisely, instead of modules in the sense that we have discussed them, one considers solutions of suitable extremal problems for path families).

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