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Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **70 (1995)**

PDF erstellt am: **30.04.2024**

Persistenter Link: https://doi.org/10.5169/seals-53006

Comment. Math. Helvetici 70 (1995) 423–433 0010-2571/95/030423-11\$1.50 + 0.20/0

Counterexamples to the Kneser conjecture in dimension four

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Abstract. We construct a connected closed orientable smooth four-manifold whose fundamental group is the free product of two non-trivial groups such that it is not homotopy equivalent to $M_0 \# M_1$ unless M_0 or M_1 is homeomorphic to S^4. Let N be the nucleus of the minimal elliptic Enrique surface $V_1(2, 2)$ and put $M = N \cup_{\partial N} N$. The fundamental group of M splits as $\mathbb{Z}/2 * \mathbb{Z}/2$. We prove that $M \# k(S^2 \times S^2)$ is diffeomorphic to $M_0 \# M_1$ for non-simply connected closed smooth four-manifolds M_0 and M_1 if and only if $k \geq 8$. On the other hand we show that M is homeomorphic to $M_0 \# M_1$ for closed topological four-manifolds M_0 and M_1 with $\pi_1(M_i) = \mathbb{Z}/2$.

0. Introduction

If M is a closed connected three-manifold and $\alpha : \pi_1(M) \to \Gamma_0 * \Gamma_1$ is an isomorphism then there are closed connected three-manifolds M_0 and M_1 with $\Gamma_i = \pi_1(M_i)$ together with a diffeomorphism $f : M \to M_0 \# M_1$ inducing α on the fundamental groups. This theorem is known as Kneser's conjecture. It fails in dimension ≥ 5 by results of Cappell [1], [2]. Recently it has been shown that Kneser's conjecture holds in dimension four stably, i.e. if one allows additional connected sums with copies of $S^2 \times S^2$ [8], [11]. In this article we give counterexamples to the unstable version of Kneser's conjecture in dimension four. The first example does not split up to homotopy, the second splits topologically but not smoothly. We prove in section 1

THEOREM 0.1. *For distinct prime numbers p_0 and p_1 there exists a connected closed smooth orientable four-manifold M such that $\pi_1(M)$ is $(\mathbb{Z}/p_0 \times \mathbb{Z}/p_0) * (\mathbb{Z}/p_1 \times \mathbb{Z}/p_1)$ and if M is homotopy equivalent to a connected sum $M_0 \# M_1$, then M_0 or M_1 is homeomorphic to S^4.* □

In section 2 we assign to a closed oriented smooth four-manifold M together with an isomorphism $\alpha : \pi_1(M) \to \Gamma_0 * \Gamma_1$ an invariant $\sigma(M, \alpha) \in \mathbb{Z}/16 \times \mathbb{Z}/16$, provided that its universal covering is *Spin*. Namely, we split M as $M_0 \cup_S M_1$ according to α. Then S inherits a *Spin*-structure from $\tilde{M}$ and we define $\sigma(M, \alpha) = (\text{sign}\,(M_0) - R(S), \text{sign}\,(M_1) + R(S))$ for $\text{sign}\,(M_i)$ the signature and

$R(S)$ the Rohlin invariant. This invariant depends only on the stable oriented diffeomorphism type of M and we investigate its dependency on α.

Let N be the nucleus of the minimal elliptic Enriques surface $V_1(2, 2)$ in the notation of Gompf [7]. Put $M = N \cup_{\partial N} N^-$. The fundamental group of M is $\mathbb{Z}/2 * \mathbb{Z}/2$. In section 3 we show using Freedman's topological s-cobordism theorem in dimension four [6] and Donaldson's result about definite intersection forms of smooth four-manifolds [4] and the invariant of section 2

THEOREM 0.2. *M is homeomorphic to $M_0 \# M_1$ for two closed topological four-manifolds M_0 and M_1 with $\pi_1(M_i) = \mathbb{Z}/2$ but $M \# k(S^2 \times S^2)$ is diffeomorphic to $M_0 \# M_1$ for non-simply connected closed smooth four-manifolds M_0 and M_1 if and only if $k \geq 8$.* □

1. Examples not Splitting Homotopically

In this section we construct closed orientable four-manifolds whose fundamental group is a non-trivial free product and which are not homotopy equivalent to a non-trivial connected sum $M_0 \# M_1$ (see Theorem 1.4). As a preliminary we need the following Lemma which is taken from [9, Theorem 3 on page 162] whose proof we enclose for the reader's convenience.

LEMMA 1.3. *Suppose that m_i, r_i, n_i and q_i for $i = 0, 1$ are integers satisfying*

$$r_i > 1, \qquad r_i^{m_i} - 1 = n_i q_i, \qquad r_i \equiv \bmod n_i, \qquad (m_i, n_i) \neq 1 \qquad (q_0, q_1) = 1.$$

Then the group

$$\pi = (\mathbb{Z}/m_0 \times \mathbb{Z}/n_0) * (\mathbb{Z}/m_1 \times \mathbb{Z}/n_1)$$

has the presentation of deficiency -1

$$\pi = \langle a_0, b_0, a_1, b_1 \mid a_0^{m_0} = 1, [a_0, b_0] = b_0^{r_0 - 1}, a_1^{m_1} = 1,$$
$$[a_1, b_1] = b_1^{r_1 - 1}, b_0^{n_0} = b_1^{n_1} \rangle.$$

Proof. Obviously it suffices to show that the relation $b_0^{n_0} = 1$ follows from the other relations. We start with proving inductively for $k = 1, 2, \ldots$ the relation $a_i^k b_i a_i^{-k} = b_i^{-r_i^k}$ for $i = 0, 1$. The induction step follows from the computation

$$a_i^{k+1} b_i a_i^{-(k+1)} = a_i a_i^k b_i a_i^{-k} a_i^{-1} = a_i b_i^{r_i^k} a^{-1} = (a_i b_i a_i^{-1})^{r_i^k} = (b_i^{r_i})^{r_i^k} = b_i^{r_i^{k+1}}.$$

This implies for $k = m_i$ and $i = 0.1$

$$(b_i^{n_i})^{q_i} = b_i^{r_i^{m_i} - 1} = 1.$$

Since $b_0^{n_0} = b_1^{n_1}$ holds we conclude

$$(b_0^{n_0})^{q_0} = (b_0^{n_0})^{q_1} = 1.$$

Since q_0 and q_1 are prime, we get $b_0^{n_0} = 1$. □

We mention that for distinct primes p_0 and p_1 one can find the numbers m_i, r_i, n_i and q_i as required in Theorem 1.4 so that it applies to $\pi = (\mathbb{Z}/p_0 \times \mathbb{Z}/p_0) * (\mathbb{Z}/p_1 \times \mathbb{Z}/p_1)$ [9, page 163].

THEOREM 1.4. *Let M be the boundary of a regular neighborhood of an embedding into $\mathbb{R}^5$ of a 2-dimensional CW-complex X which realizes a presentation of*

$$\pi = (\mathbb{Z}/m_0 \times \mathbb{Z}/n_0) * (\mathbb{Z}/m_1 \times \mathbb{Z}/n_1)$$

of deficiency -1. Then M is not homotopy equivalent to a connected sum $M_0 \# M_1$ unless M_0 or M_1 is homeomorphic to S^4. □

For the proof we need the following well-known Lemma.

LEMMA 1.5. *Let M be a connected closed orientable four-manifold with fundamental group π and classifying map $f: M \to B\pi$. Denote by $b_p(\pi; F)$ the p-th Betti number of $B\pi$ with coefficients in the field F. If $f_*([M]) = 0$ in $H_4(B\pi; F)$, then*

$$2 \cdot (b_2(\pi; F) - b_1(\pi; F) + b_0(\pi; F)) \leq \chi(M).$$

Proof. Since the classifying map is 2-connected, the map $f^p: H^p(B\pi; F) \to H^p(M; F)$ is bijective for $p = 0, 1$ and injective for $p = 2$. Because of $f_*([M]) = 0$ its image for $p = 2$ is a totally isotropic subspace of $H^2(M; F)$ with respect to the intersection form. If we write the intersection form as an isomorphism $b: H^2(M; F) \to H^2(M; F)^*$, this is equivalent to the fact that composition $i^* \circ b \circ i$ for the inclusion $i: \mathrm{im}\,(f^2) \to H^2(M; F)$ is zero. Hence $H^2(M; F)$ contains a subspace which is isomorphic to the direct sum of two copies of $H^2(B\pi; F)$. This shows $b_p(M; F) = b_p(\pi; F)$ for $p = 0, 1$ and $b_2(M; F) \geq 2 \cdot b_2(\pi; F)$. From Poincaré duality, $\chi(M) = b_2(M; F) - 2 \cdot b_1(M; F) + 2 \cdot b_0(M; F)$ and the claim follows. □

Now we are ready to prove Theorem 1.4. We first explain the construction of M which depends on the presentation of π given in Lemma 1.3. Let X be a

2-dimensional CW-complex given by this presentation. Embed X into $\mathbb{R}^5$ and let M be the boundary of a regular neighborhood N of X. The resulting manifold M comes with a reference map f to $B\pi$ which induces an isomorphism on the fundamental groups. Obviously we have $[M,f] = 0$ in $\Omega_4(B\pi)$, a nullbordism is given by the regular neighborhood N. This implies $f_*([M]) = 0$ in $H_4(\pi; \mathbb{Z})$. One easily checks $\chi(M) = 2 \cdot \chi(N) = 2 \cdot \chi(X) = 4$.

Choose for $i = 0, 1$ a prime number p_i dividing both m_i and n_i. Let $\mathbb{F}_{p_i}$ be the field of p_i elements. One easily checks $b_k(\mathbb{Z}/p_i^l; \mathbb{F}_{p_i}) = 1$ for $k \geq 0$ and $l \geq 1$ and computes using the Künneth formula

$$b_2(\mathbb{Z}/m_i \times \mathbb{Z}/n_i; \mathbb{F}_{p_i}) - b_1(\mathbb{Z}/m_i \times \mathbb{Z}/n_i; \mathbb{F}_{p_i}) + b_0(\mathbb{Z}/m_i \times \mathbb{Z}/n_i; \mathbb{F}_{p_i}) = 2.$$

Assume that M is homotopy equivalent to $M_0 \# M_1$. By Kurosh subgroup theorem [12, Theorem 1.10 on page 178]) (and possible renumbering M_0 and M_1) it suffices to treat the two cases where $\pi_1(M_i) = \mathbb{Z}/m_i \times \mathbb{Z}/n_i$ for $i = 0, 1$ or where M_0 is simply connected. In the first case we get $\chi(M_i) \geq 4$ from Lemma 1.5 and hence $\chi(M_0 \# M_1) \geq 6$. This contradicts $\chi(M) = 4$. In the second case we have $\pi_1(M_1) = \pi_1(M)$ and again by Lemma 1.5 and the additivity of k-th Betti number $b_k(\pi; F)$ for $k \geq 1$ under free products we conclude $\chi(M_1) \geq 4$. This implies $\chi(M_0) \leq 2$. Hence M_0 is a homotopy sphere and by Freedman's result [5, Theorem 1.6, page 280] homeomorphic to S^4. This finishes the proof of Theorem 1.4.

2. A stable diffeomorphism invariant

We introduce a stable diffeomorphism invariant for a connected closed oriented smooth four-manifold M whose universal covering possesses a $Spin$-structure together with an isomorphism $\alpha : \pi_1(M) \to \Gamma_0 * \Gamma_1$. We will suppress base points in the context of fundamental groups since all the group theoretic conditions we will give are invariant under inner automorphisms. Let $K(\Gamma_0, 1) \cup [0, 1] \cup K(\Gamma_1, 1)$ be obtained by the disjoint union of the Eilenberg-MacLane spaces and $[0, 1]$ by identifying $\{i\}$ with the base point of $K(\Gamma_i, 1)$ for $i = 0, 1$. Choose a map

$$\bar{\alpha} : M \to K(\Gamma_0, 1) \cup [0, 1] \cup K(\Gamma_1, 1),$$

which is transversal to $1/2 \in [0, 1]$ and up to homotopy determined by the property that it induces on the fundamental groups the isomorphism α up to inner automorphisms if we identify the fundamental group of $K(\Gamma_0, 1) \cup [0, 1] \cup K(\Gamma_1, 1)$ for the base point $1/2$ with $\Gamma_0 * \Gamma_1$ in the obvious way. We orient $[0, 1]$ by the direction from 0 to 1. We get a trivialization of the normal bundle of $1/2$ in $[0, 1]$. This induces by transversality a trivialization of the normal bundle $\nu(S, M)$ of S in M where S is the preimage of $1/2$. In particular S splits M into the pieces M_0 and M_1, i.e. $M = M_0 \cup_S M_1$ where M_0 respectively M_1 is mapped by $\bar{\alpha}$ to $K(\Gamma_0, 1) \cup [0, 1/2]$ respectively $[1/2, 1] \cup K(\Gamma_1, 1)$. The inclusion $j : S \to M$ induces the trivial map on the fundamental groups and lifts to a map $\tilde{j} : S \to \tilde{M}$. The unique *Spin*-structure on $\tilde{M}$ restricts to a *Spin*-structure on $\tilde{j}^* T\tilde{M} = TS \oplus \nu(S, M)$. Since we have already fixed a trivialization of $\nu(S, M)$, this induces a *Spin*-structure on S. Denote by $R(S) \in \mathbb{Z}/16$ the Rohlin invariant of the closed three-dimensional *Spin*-manifold S which is the signature modulo 16 of any smooth *Spin*-nullbordism of S. Our invariant is defined by

DEFINITION 2.1. $\sigma(M, \alpha) = (\operatorname{sign}(M_0) - R(S), \operatorname{sign}(M_1) + R(S)) \in \mathbb{Z}/16 \times \mathbb{Z}/16$. □

Next we show that this invariant is well-defined and examine its dependency on α. Recall that a finitely generated group Γ is called *indecomposable* if Γ is non-trivial and $\Gamma \cong \Gamma' * \Gamma''$ implies that Γ' or Γ'' is trivial. Finite non-trivial groups are obviously indecomposable. We want to show

LEMMA 2.2. *Let M and M' be connected closed oriented smooth four-manifolds, whose universal coverings possess Spin-structures, together with isomorphisms $\alpha : \pi_1(M) \to \Gamma_0 * \Gamma_1$ and $\alpha' : \pi_1(M') \to \Gamma'_0 * \Gamma'_1$. Suppose Γ_0, Γ_1, Γ'_0 and Γ'_1 are indecomposable and not infinite cyclic. Assume that there is an oriented diffeomorphism*

$$f : M \# k(S^2 \times S^2) \to M' \# k'(S^2 \times S^2).$$

Then we get

$$\sigma(M, \alpha) = \sigma(M', \alpha').$$

where we may have to interchange the order of the summands $\mathbb{Z}/16 \times \mathbb{Z}/16$ in the case where Γ_0 and Γ_1 are isomorphic.

Proof. In the first step we show the existence of isomorphisms $\beta_i : \Gamma_i \to \Gamma'_i$ such that the composition $\alpha' \circ f_* \circ \alpha^{-1}$ is $(\beta_0 * \beta_1)$ up to inner automorphisms after possibly renumbering Γ_0 and Γ_1. By Kurosh Subgroup Theorem [12, Theorem 1.10 on page 178] and after possibly renumbering Γ_0 and Γ_1 the composition $\alpha' \circ f_* \circ \alpha^{-1}$ sends Γ_0 respectively Γ_1 to a conjugate of Γ'_0 respectively Γ'_1. Hence there are isomorphisms $\beta_i : \Gamma_i \to \Gamma'_i$ and an automorphism ε of $\Gamma_0 * \Gamma_1$ sending $\gamma_0 \in \Gamma_0$ to γ_0 and $\gamma_1 \in \Gamma_1$ to $\delta \gamma_1 \delta^{-1}$ for some $\delta \in \Gamma_0 * \Gamma_1$ such that $\alpha' \circ f_* \circ \alpha^{-1}$ is

$\varepsilon(\beta_0 * \beta_1)$ up to inner automorphisms. Without destroying this property one can change δ, β_0 and β_1 such that δ is trivial or δ begins with a non-trivial letter in Γ_1 and ends with a non-trivial letter in Γ_0. In the second case no element of Γ_1 can lie in the image of ε and hence the surjectivity of ε forces ε to be the identity and the claim follows.

In the next step we show that the choice of $\bar{\alpha}$ does not matter. Suppose we have two choices of maps $\bar{\alpha}$ and $\bar{\alpha}' : M \to K(\Gamma_0, 1) \cup [0, 1] \cup K(\Gamma_1, 1)$ which are transversal to 1/2. Let $M = M_0' \cup_{S'} M_1'$ be the splitting induced by $\bar{\alpha}'$. Since $\bar{\alpha}$ and $\bar{\alpha}'$ induce up to inner automorphisms the same homomorphism on the fundamental groups, they are homotopic. Hence there is a map $h : M \times [0, 1] \to K(\Gamma_0, 1) \cup [0, 1] \cup K(\Gamma_1, 1)$ which is transversal to 1/2 and $h_0 = \bar{\alpha}$ and $h_1 = \bar{\alpha}'$. As explained above h induces a splitting $M \times [0, 1] = W_0 \cup_Z W_1$ and Z inherits a Spin structure (and in particular an orientation) from h. The orientation of M induces orientations on $M \times [0, 1]$, W_0 and W_1. We use the convention for an oriented manifold V with boundary ∂V that ∂V inherits the orientation determined by the decomposition $TV|_{\partial V} = T\partial V \oplus \nu(\partial V, V)$ and the orientation on the normal bundle $\nu(\partial V, V)$ given by the outward normal field. Notice that the orientations on S and Z coming from the *Spin*-structures as described above agree with the ones coming from $S = \partial M_0$ and $Z = \partial W_0$ and are the opposites of the orientations coming from $S = \partial M_1$ and $Z = \partial W_1$. We claim that the orientation of $S \subset \partial Z$ agrees with the one coming from $S = \partial M_0$. Namely, the decompositions $TW_0|_Z = TZ \oplus \nu(Z, W_0)$ and $TZ|_S = TS \oplus \nu(S, Z)$ induce a decomposition

$$TW_0|_S = TS \oplus \nu(S, Z) \oplus \nu(S, M_0).$$

The orientation of S given by $S \subset \partial Z$ is compatible with this decomposition if one uses the outward normal fields on $\nu(S, Z)$ and $\nu(S, M_0)$. The decompositions $TM_0|_S = TS \oplus \nu(S, M_0)$ and $TW_0|_{M_0} = TM_0 \oplus \nu(M_0, W_0)$ yield

$$TW_0|_S = TS \oplus \nu(S, M_0) \oplus \nu(S, Z).$$

The orientation of S given by $S = \partial M_0$ is compatible with this decomposition if one uses the outward respectively inward normal field on $\nu(S, M_0)$ respectively $\nu(S, Z)$. One treats the other component S' similarly and gets $\partial Z = S \amalg (S')^-$. This implies

$$\operatorname{sign}(Z) = R(S) - R(S') \in \mathbb{Z}/16.$$

The boundary of W_0 is $M_0^- \cup Z \cup M_0'$ and of W_1 is $M_1^- \cup Z^- \cup M_1'$. This shows

$$-\operatorname{sign}(M_0) + \operatorname{sign}(Z) + \operatorname{sign}(M_0') = 0$$

and

$$-\operatorname{sign}(M_1) - \operatorname{sign}(Z) + \operatorname{sign}(M'_1) = 0.$$

and hence

$$\begin{aligned}&(\operatorname{sign}(M_0) - R(S), \operatorname{sign}(M_1) + R(S))\\&\quad = (\operatorname{sign}(M'_0) - R(S'), \operatorname{sign}(M'_1) + R(S')) \in \mathbb{Z}/16 \times \mathbb{Z}/16.\end{aligned}$$

In the final step we can assume that f is an oriented diffeomorphism from M to M'. Choose base point preserving maps $\bar{\beta}_i : K(\Gamma_i, 1) \to K(\Gamma'_i, 1)$ inducing β_i on the fundamental groups for $i = 0, 1$. By our first step $\bar{\alpha}'$ is homotopic to the composition

$$M' \xrightarrow{f^{-1}} M \xrightarrow{\bar{\alpha}} K(\Gamma_0, 1) \cup [0, 1] \cup K(\Gamma_1, 1) \xrightarrow{\bar{\beta}_0 \cup id \cup \bar{\beta}_1} K(\Gamma'_0, 1) \cup [0, 1] \cup K(\Gamma'_1, 1).$$

Obviously the invariant for the splitting of M' with respect to this composition is the same as the one for the splitting of M with respect to $\bar{\alpha}$ and the claim follows. □

3. Examples splitting topologically but not smoothly

In this section we give an example which splits topologically but not smoothly. Let us recall from [7] that every minimal elliptic surface $V_n(p_1, p_2, \ldots, p_k)$ (whose elliptic fibration with base $\mathbb{C}P^1$ has $6n$ cusp fibers and k multiple fibers with multiplicities p_i) can be decomposed as a union $N_n(p_1, p_2, \ldots, p_k) \cup_\Sigma \Phi_n$ along the Seifert fibered homology three-sphere $\Sigma(2, 3, 6n - 1)$ which as the link of a singularity bounds the Milnor fiber Φ_n of Σ_n. The piece $N_n(p_1, p_2, \ldots, p_k)$ is called *nucleus* of $V_n(p_1, p_2, \ldots, p_k)$.

THEOREM 3.1. *Let $N = N_1(2, 2)$ be the nucleus of the Enriques surface $V = V_1(2, 2)$. Put $M = N \cup_{\partial N} N^-$.*

(1) *M is homeomorphic to $M_0 \# M_1$ for two closed topological four-manifolds M_0 and M_1 with $\pi_1(M_i) = \mathbb{Z}/2$ for $i = 0, 1$.*
(2) *$M \# k(S^2 \times S^2)$ is diffeomorphic to $M_0 \# M_1$ for non-simply connected closed smooth four-manifolds M_0 and M_1 if and only if $k \geq 8$. In fact, $M \# 8(S^2 \times S^2)$ is diffeomorphic to $V \# V^-$.*

Proof. First recall from [7] that the nucleus of a minimal elliptic surface is constructed by taking a regular neighborhood of one cusp fiber and a section of the elliptic fibration. This gives a four-manifold in the homotopy type of $S^2 \vee S^2$. Then one performs all the logarithmic transformations inside this neighborhood. The following properties of N are easy consequences from this construction, for details see [7, Section 3]. We also remind the reader that the Enriques surface V has even intersection from $E_8 \oplus H$ and its universal covering is the Kummer surface which is *Spin*.

(1) $\operatorname{sign}(N) = 0$ and $\chi(N) = 3$.

(2) The inclusion $N \to V$ induces an isomorphism on the fundamental groups and $\pi_1(N) = \mathbb{Z}/2$. Since the Milnor fiber Φ_n is simply connected this implies that Σ is π_1-null in N, i.e. the inclusion of $\Sigma = \Sigma(2, 3, 5) = \partial N$ into N induces the trivial map on the fundamental groups.

(3) The intersection form of N is even and $\tilde{N}$ is *Spin*. □

The first assertion of our theorem follows directly from the following lemma by setting $X = X' = N$.

LEMMA 3.2. *Let X and X' be two topological four-manifolds with the same boundary Σ, a homology three-sphere. Assume that Σ is a π_1-null in X and that X has a good fundamental group. Let C be a contractible four-manifold with boundary Σ. Then there exists a homeomorphism*

$$X \cup_\Sigma X' \approx (X \cup_\Sigma C) \# (X' \cup_\Sigma C).$$

Proof. Recall that a *good* fundamental group is one for which the topological s-cobordism theorem holds. By [6] examples of good groups are poly-finite or -cyclic groups. Also, by [5, Theorem 1.4′ on page 367] a manifold C as in the lemma exists.

By taking the connected sum inside the contractible parts C, we obtain a homeomorphism

$$(X \cup_\Sigma C) \# (X' \cup_\Sigma C) \approx (X \cup_\Sigma (C \# C)) \cup_\Sigma X'.$$

Hence it suffices to show that X and $(X \cup_\Sigma (C \# C))$ are homeomorphic relative boundary. By assumption, the topological s-cobordism theorem holds for this fundamental group and thus it remains to construct an s-cobordism between X and $X \cup_\Sigma (C \# C)$ relative boundary.

Note that $C \cup_\Sigma C$ is a simply connected closed topological four-manifold with the same integral homology as S^4 and hence by [5, Theorem 1.6 page 371] is homeomorphic to S^4. Let C_0 be a complement of the interior of an embedded disk $D^4 \subset \operatorname{int}(C)$. Then we obtain a homeomorphism

$$(C \# C) \cup_{\Sigma \times \{0,1\}} \Sigma \times [0, 1] \to C_0 \cup_\Sigma C_0 \cup_{S^3 \times \{0,1\}} S^3 \times [0, 1] \to S^3 \times S^1.$$

This gives an embedding $j: \Sigma \times [0, 1] \to \partial(S^3 \times D^2)$. Let $f: \Sigma \times [0, 1] \to X \times [0, 1]$ be the inclusion. Define W by $S^3 \times D^2 \cup_f X \times [0, 1]$, i.e. by the push out

$$\begin{array}{ccc} \Sigma \times [0,1] & \xrightarrow{j} & S^3 \times D^2 \\ {\scriptstyle f}\downarrow & & \downarrow \\ X \times [0,1] & \xrightarrow{\hat{j}} & W \end{array}$$

We want to show that $\hat{j}: X \times [0, 1] \to W$ is a simple homotopy equivalence. Since the simple homotopy type of W relative $X \times [0, 1]$ depends only on the homotopy class of f [3, II.5.5] and Σ is π_1-null in X, we can assume that j factorizes as

$$f: \Sigma \times [0, 1] \xrightarrow{f_1} Z \xrightarrow{f_2} X \times [0, 1],$$

where Z is obtained from $\Sigma \times [0, 1]$ by collapsing the 1-skeleton to a point. Define Y by the push out $Y = S^3 \times D^2 \cup_{f_1} Z$. Then W is also the push out

$$\begin{array}{ccc} Z & \xrightarrow{\bar{j}} & Y \\ {\scriptstyle f_2}\downarrow & & \downarrow \\ X \times [0,1] & \xrightarrow{\hat{j}} & W \end{array}$$

The map $\bar{j}: Z \to Y$ is a homology equivalence as $j: \Sigma \times [0, 1] \to S^3 \times D^2$ is. Since Z and $S^3 \times D^2$ are simply connected, $\bar{j}$ and hence $\hat{j}$ are simple homotopy equivalences [3, II.8.5. and I.5.9]. This shows that the inclusion of X into W is simple homotopy equivalence. Similarly, one verifies that the inclusion of the other part $X \cup_\Sigma (C \# C)$ of the boundary of W into W is a homotopy equivalence. Hence W is a s-cobordism. This finishes the proof of the lemma and thus also of the first assertion.

Suppose that $f: M \# k(S^2 \times S^2) \to M_0 \# M_1$ is a diffeomorphism for connected smooth four-manifolds M_0 and M_1 which are non-simply connected. By Kurosh Subgroup Theorem [12, Theorem 1.10 on page 178] $\pi_1(M_i) = \mathbb{Z}/2$ for $i = 0, 1$. There is an obvious choice of isomorphisms $\alpha: \pi_1(M) \to \mathbb{Z}/2 * \mathbb{Z}/2$ and $\alpha': \pi_1(M_0 \# M_1) \to \mathbb{Z}/2 * \mathbb{Z}/2$ such that

$$\sigma(M, \alpha) = (\operatorname{sign}(N) - R(\Sigma), \operatorname{sign}(N^-) + R(\Sigma)) = (8, 8) \in \mathbb{Z}/16 \times \mathbb{Z}/16$$

and

$$\sigma(M_0 \# M_1, \alpha) = (\operatorname{sign}(M_0), \operatorname{sign}(M_1)) \in \mathbb{Z}/16 \times \mathbb{Z}/16.$$

From Lemma 2.2 we get $\sigma(M, \alpha) = \sigma(M_0 \# M_1, \alpha')$. This shows for $i = 0, 1$

$$|\mathrm{sign}\,(M_i)| \geq 8.$$

The intersection form of N is even and hence its signature is divisible by eight and its rank is even [13, Corollary 1 on page 53]. Suppose that $b_2(M_i) \leq 9$. Then the rank of the intersection form must be eight and its signature must be ± 8. Hence we can find an orientation such that the intersection form on the smooth oriented closed four-manifold M_i is the definite form E_8. This is impossible by Donaldson's result [4, Theorem 1 on page 397] that a definite intersection form of a smooth closed oriented 4-manifold is equivalent up to sign to the standard Euclidean form. Therefore $b_2(M_i) \geq 10$ for $i = 0, 1$. Since $\pi_1(M_i)$ is finite, we conclude

$$\chi(M_i) = 2 + b_2(M_i) \geq 12.$$

Since $\chi(N) = 3$, we have $\chi(M) = 6$. Now we get

$$\begin{aligned} 6 + 2k = \chi(M \# k(S^2 \times S^2)) &= \chi(M_0 \# M_1) \\ &= \chi(M_0) + \chi(M_1) - 2 \geq 12 + 12 - 2 \geq 22 \end{aligned}$$

and hence

$$k \geq 8.$$

It remains to prove that $M \# 8(S^2 \times S^2)$ is a diffeomorphic to $V \# V^-$. Since $M = N \cup_\Sigma N^-$, $V = N \cup_\Sigma \Phi$ and the connected sum of V and V^- may be taken inside the Milnor fibers Φ, it suffices to show that $N \# 8(S^2 \times S^2)$ is diffeomorphic (relative boundary) to $N \cup_\Sigma (\Phi \# \Phi^-)$. Here we take the connected sum always in the interior of the manifolds.

It was shown in [7, Fig. 27] (we are in the case $n = 1$) that Φ has a handle decomposition with one 0-handle and eight 2-handles. Therefore, inside $\Phi \# \Phi^-$ we find eight disjointly embedded 2-spheres with trivial normal bundle. These are given by gluing together in pairs the cores of corresponding 2-handles. It is easy to check that after doing 2-surgeries on these eight 2-spheres, i.e. cutting out $S^2 \times D^2$ and replacing it by $D^3 \times S^1$, one gets the product $\partial\Phi \times [0, 1] = \Sigma \times [0, 1]$.

Reversing this procedure, we see that one can do eight 1-surgeries on (the collar of) $N = N \cup_\Sigma (\Sigma \times [0, 1])$ to obtain $N \cup_\Sigma (\Phi \# \Phi^-)$. We point out that by [7, Fig. 27] all framings for the 2-handles in Φ are even and thus $N \cup_\Sigma (\Phi \# \Phi^-)$ has an even intersection form.

Changing slightly our point of view, we see that since all the surgered circles are nullhomotopic in N, each of these 1-surgeries has the effect of taking a connected sum with an oriented S^2-bundle over S^2. But the nontrivial bundle cannot occur because the resulting manifold must have an even intersection form. Hence we do end up with $N \# 8(S^2 \times S^2)$ which finishes the proof of our last claim in Theorem 3.1. □

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Received May 25, 1994; December 3, 1994