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Rational points of bounded height on Del Pezzo surfaces of degree six

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Abstract. Let K be a number field. Denote by V_3 a split Del Pezzo surface of degree six over K and by ω its canonical divisor. Denote by W_3 the open complement of the exceptional lines in V_3. Let $N_{W_3}(-\omega, X)$ be the number of K-rational points on W_3 whose anticanonical height $H_{-\omega}$ is bounded by X. Manin has conjectured that asymptotically $N_{W_3}(-\omega, X)$ tends to $cX(\log X)^3$, where c is a constant depending only on the number field and on the normalization of the height. Our goal is to prove the following theorem: For each number field K there exists a constant c_K such that $N_{W_3}(-\omega, X) \le c_K X(\log X)^{3+2r}$, where r is the rank of the group of units of O_K. The constant c_K is far from being optimal. However, if K is a purely imaginary quadratic field, this proves an upper bound with a correct power of $\log X$. The proof of Manin's conjecture for arbitrary number fields and a precise treatment of the constants would require a more sophisticated setting, like the one used by [Peyre] to prove Manin's conjecture and to compute the correct asymptotic constant (in some normalization) in the case $K = \mathbb{Q}$. Up to now the best result for arbitrary K goes back, as far as we know, to [Manin-Tschinkel], who gives an upper bound $N_{W_3}(-\omega, X) \le cX^{1+\varepsilon}$.

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1. Introduction

1.1. *Del Pezzo surfaces*

Let K be an algebraic number field and $\bar{K}$ an algebraic closure.

DEFINITION 1.1.1. A Del Pezzo surface over K is defined to be a smooth projective surface defined over K whose anticanonical divisor $-\omega_V$ is ample and whose field of rational functions over $\bar{K}$ is a purely transcendental extension of $\bar{K}$.

The self-intersection number $d = \omega_V \cdot \omega_V$ is called the degree of the Del Pezzo surface. If the anticanonical divisor is very ample it coincides with the projective degree of the anticanonical embedding of V.

Essentially we need only the following classical result about Del Pezzo surfaces.

THEOREM 1.1.2. *Let V be a Del Pezzo surface over K of degree 6, then $V \otimes_K \bar{K}$ is isomorphic to the blowing up of three distinct points P_0, P_1, P_2 in $\mathbb{P}^2_{\bar{K}}$ in general position, i.e. non collinear.*

This explains the standard notation V_3 for Del Pezzo surfaces of degree 6 over K, and $\bar{V}_3$ for $V_3 \otimes_K \bar{K}$. We denote by $\pi : \bar{V}_3 \to \mathbb{P}^2_{\bar{K}}$ the birational morphism induced by the blowing up of three points.

LEMMA 1.1.3. *The only exceptional divisors on $\bar{V}_3$ are the inverse images $E_i = \pi^{-1}(P_i)$ of the blown up points and the strict transforms L_{ij} of the lines l_{ij} passing through the points P_i and P_j.*

The six effective divisors E_i and L_{ij} coincide with the straight lines on $\bar{V}_3$ in the anticanonical embedding. That is why we shall refer to them as the "lines" on $\bar{V}_3$.

PROPOSITION 1.1.4. *The divisor $E_0 + 2L_{01} + 2E_1 + L_{12}$ and those derived from it by the action of the symmetric group S_3 on the subscripts are members of the anticanonical linear system.*

For proofs and additional information consult e.g. [Manin, Chap. 4]. Observe that the divisor $E_0 + L_{01} + E_1 + L_{12} + E_2 + L_{02}$, as a weighted sum of these divisors, also belongs to the anticanonical system.

An exceptional divisor may not be defined over the groundfield K. We exclude this situation from further investigation and assume throughout this paper that our surfaces are **split**, i.e. that all exceptional divisors are defined over K.

1.2. *Heights*

We give a brief summary of the theory of local and global heights or, in another terminology, of Weil functions and associated heights needed in the sequel. For proofs of the statements mentioned in this subsection we refer to the standard literature, e.g. [Lang1, Chap. 10] or [Serre, Chap. 6].

We fix once for all a complete set of embeddings, up to conjugation, of the field K in $\mathbb{R}$ or $\mathbb{C}$. We denote the real ones by $\sigma_1, \ldots, \sigma_{r_1}$, and the complex ones by $\tau_1, \ldots, \tau_{r_2}$. We put $r = r_1 + r_2 - 1$.

Let $| \ |$ be the ordinary absolute value on $\mathbb{R}$ or $\mathbb{C}$. To each embedding we attach an Archimedean absolute value v_i of K, given by $v_i(\xi) = |\sigma_i(\xi)|$ in the real case, and by $v_i(\xi) = |\tau_i(\xi)|^2$ in the complex one.

We denote by P_K the set of prime ideals of O_K. For a prime $\mathfrak{p} \in P_K$ lying over a prime $p \in \mathbb{Z}$ with ramification index $e_\mathfrak{p}$, local degree $d_\mathfrak{p}$, and residue class degree $f_\mathfrak{p}$ we introduce an ultrametric absolute value $v_\mathfrak{p}(\xi)$ for $\xi \in K$ by writing the ideal ξO_K as $\Pi_{\mathfrak{p} \in P_K} \mathfrak{p}^{\nu_\mathfrak{p}}$ and putting

$$v_\mathfrak{p}(\xi) = p^{-(\nu_\mathfrak{p}/e_\mathfrak{p})d_\mathfrak{p}} = p^{-\nu_\mathfrak{p} f_\mathfrak{p}}.$$

We denote by M_K the set of all these absolute values. The chosen normalization ensures that for every $\xi \in K^*$ the *product formula* holds:

$$\prod_{v \in M_K} v(\xi) = 1.$$

Let V be a smooth projective variety and D and effective divisor of V. We embed V in $\mathbb{P}^n_K$ with projective coordinates $(x_0 : \ldots : x_n)$. Suppose that D is defined by an homogeneous ideal, generated by a system $f_j(x) = f_j(x_0 : \ldots : x_n)$ of homogeneous equations of degree d_j.

Since we do not consider the most general setting, the following definitions will be sufficient for our purposes:

DEFINITION 1.2.1. Let $v \in M_K$ be an absolute value on K and x a K-rational point on V with projective coordinates $(x_0 : \ldots : x_n)$. The local height (or Weil function) λ_v associated with the divisor D is defined to be

$$\lambda_v(x) = \inf_j \sup_i v(x_i^{d_j}/f_j(x)).$$

This is defined as in [Serre, Chap. 6], but note that the function λ appearing in *Example* 5 of §6.2 is the logarithm of the present height.

PROPOSITION 1.2.2. *For each absolute value $v \in M_K$ there exists a constant $c_v > 0$ such that for all K-rational points x on V we have $\lambda_v(x) \geq c_v$.*

DEFINITION 1.2.3. The (global) height associated with the divisor D is defined to be

$$H_D(x) = \prod_{v \in M_K} \lambda_v(x).$$

DEFINITION 1.2.4. The finite height D associated with the divisor D is defined to be

$$h_D(x) = \prod_{\mathfrak{p} \in P_K} \lambda_{v_\mathfrak{p}}(x).$$

Remark. In view of our normalizations the height of a point x depends on the choice of the field K. However, one checks immediately that multiplying the x_i by a constant leaves the local, the global and the finite height invariant. All heights are defined outside $\operatorname{Supp} D$.

We say that two global heights H_D and $H_{D'}$ are equivalent, and we write $H_D \sim H_{D'}$, if they differ only by a multiplicative function, bounded from above and from below, i.e. if there exists a constant $c > 0$ such that $(1/c)H_D(x) < H_{D'}(x) < cH_D(x)$ for any point $x \in V(K)$ outside $\operatorname{Supp} D$.

THEOREM 1.2.5. *If D and D' are two linearly equivalent divisors then $H_D \sim H_{D'}$.*

This theorem implies that, up to equivalence, the global height is independent of the choice of the generators f_j and of the choice of an embedding. Note that this need not be the case for finite heights. Thus, given a smooth projective variety V and an effective divisor D, we shall further on speak about *the* height H_D.

PROPOSITION 1.2.6. *Let $\phi : V' \to V$ be a morphism between two smooth projective varieties. Let D be an effective divisor on V and $D' = \phi^*(D)$ its pullback divisor on V'. Then the identity $H_D \circ \phi \sim H_{D'}$ holds.*

This identity is often called the *morphism formula.* The morphism formula implies: if h_D is a finite height associated with the divisor D, then $h_D \circ \phi$ is a finite height associated with the divisor D'.

PROPOSITION 1.2.7. *Let D and D' be two effective divisors on a smooth projective variety V, then $H_D H_{D'} \sim H_{D+D'}$.*

The above definition of global height is linked with the better known one by the following proposition:

PROPOSITION 1.2.8. *Let V be a smooth variety embedded in projective space by means of a morphism associated with the linear system $\mathscr{L}(D)$ of a very ample divisor D, and x a K-rational point on V with projective coordinates $(x_0 : \ldots : x_n)$. Define*

$$H_{\mathscr{L}(D)}(x) = \prod_{v \in M_K} \sup_i v(x_i).$$

Then we have $H_{\mathscr{L}(D)} \sim H_D$.

1.3. *Counting problems*

Let X be a large positive number, K a number field, V a smooth projective algebraic surface over K, and W an open subset of V. Let D be an effective divisor on V.

DEFINITION 1.3.1. The counting function $N_W(D, X)$ is defined to be the number of rational points x in $W(K)$ whose height $H_D(x)$ does not exceed X.

THEOREM 1.3.2. (*Schanuel*) *Let* V *be* $\mathbb{P}^1$ *and* $\mathscr{L}(D) = O(2)$. *Then for* X *tending to infinity*

$$N_{\mathbb{P}^1}(D, X) = cX^2 + o(X^2).$$

The constant c depends only on the number field and on the normalization of the height. For a proof and a more precise statement see [Schanuel].

Our aim is to investigate the asymptotic behaviour of $N_{V_3}(-\omega, X)$ as X goes to infinity. As V_3 contains six copies of $\mathbb{P}^1$ the leading term will certainly be cH^2. That is why in what follows we consider only the open complement W_3 of the six exceptional lines in V_3. We are interested in the asymptotics of

$$N_{W_3}(-\omega, X) = \operatorname{card}\{x \in W_3(K) \mid H_{-\omega}(x) < X\}.$$

Previous results about the asymptotic behaviour of counting functions on Del Pezzo surfaces have appeared in [Batyrev-Manin], [Franke-Manin-Tschinkel] and especially in [Manin-Tschinkel] and [Tschinkel].

A special remark should be made about the beautiful work of [Peyre]. Peyre calculates not only the asymptotics of the rational points on V_3 in the case $K = \mathbb{Q}$, but he succeeds also in the difficult task of giving an interpretation of the exact constant by means of Tamagawa numbers.

In our investigation we restrict our interest to the task of finding upper bounds for $N_{W_3}(-\omega, X)$ over arbitrary number fields without trying to determine the precise constants.

2. Counting rational points on V_3

2.1. *Finite heights on* V_3

Let V be a smooth projective variety embedded in $\mathbb{P}^n_K$ with coordinates $(x_0: \ldots : x_n)$ and D an effective divisor defined by m homogeneous elements $f_j(x) = f_j(x_0: \ldots : x_n)$ of degree d_j. Let x be a K-rational point of V with integer coordinates. We write $\mathfrak{N}$ for the absolute norm $\mathfrak{N}_{K|\mathbb{Q}}$. Denote by $\mathfrak{a}$ the integral ideal $(x_0, \ldots, x_n)$ and by $\mathfrak{b}$ the ideal $(f_1(x), \ldots, f_m(x))$.

LEMMA 2.1.1. *Suppose that for all homogeneous polynomials f_j we have $d_j = 1$, then the finite height of x with respect to D satisfies the identity*

$$h_D(x) = \mathfrak{N}(\mathfrak{a}^{-1}\mathfrak{b}).$$

Proof. Let $v_\mathfrak{p}(\mathfrak{c})$ be the exponent of $\mathfrak{p}$ in the factorization of the (fractional) ideal $\mathfrak{c}$ into prime ideals. Observe that $v_\mathfrak{p}(\mathfrak{a}) = \inf_i v_\mathfrak{p}(x_i O_K)$ and $v_\mathfrak{p}(\mathfrak{b}) = \inf_j v_\mathfrak{p}(f_j(x)O_K)$. Hence, as the subscripts i and j are independent, we have

$$\begin{aligned} h_D(x) &= \prod_{\mathfrak{p}\in P_K} \inf_j \sup_i v_\mathfrak{p}(x_i/f_j(x)) \\ &= \prod_{\mathfrak{p}\in P_K} \sup_i v_\mathfrak{p}(x_i)/\sup_j v_\mathfrak{p}(f_j(x)) \\ &= \prod_{\mathfrak{p}\in P_K} \sup_i \mathfrak{p}^{-v_\mathfrak{p}(x_i O_K)f_\mathfrak{p}}/\sup_j \mathfrak{p}^{-v_\mathfrak{p}(f_j(x)O_K)f_\mathfrak{p}} \\ &= \prod_{\mathfrak{p}\in P_K} \mathfrak{p}^{-v_\mathfrak{p}(\mathfrak{a})f_\mathfrak{p}}/\mathfrak{p}^{-v_\mathfrak{p}(\mathfrak{b})f_\mathfrak{p}} \\ &= \prod_{\mathfrak{p}\in P_K} \mathfrak{p}^{-v_\mathfrak{p}(\mathfrak{a}\mathfrak{b}^{-1})f_\mathfrak{p}} = \mathfrak{N}(\mathfrak{a}^{-1}\mathfrak{b}). \qquad \square \end{aligned}$$

All split V_3-surfaces are isomorphic over K. Hence the choice of one model will be sufficient. V_3 may for example be viewed as the subvariety of $\mathbb{P}^2 \times \mathbb{P}^2$, with bihomogeneous coordinates $(x_0 : x_1 : x_2) \times (y_0 : y_1 : y_2)$, given by the equations $x_0 y_0 = x_1 y_1 = x_2 y_2$. This model comes naturally along with two projections π_x and π_y of V_3 into $\mathbb{P}^2$. The exceptional divisors can be described as $E_0 = \pi_x^{-1}(1:0:0)$ or $L_{12} = \pi_y^{-1}(1:0:0)$, or by homogeneous equations of degree one in $\mathbb{P}^2 \times \mathbb{P}^2$, as follows: $E_0 = \{x_1 = 0, x_2 = 0\}$, $L_{12} = \{y_1 = 0, y_2 = 0\}$. We refer for further details to [Hartshorne, Chap. 5].

LEMMA 2.1.2. *Let $x = (a_0 : a_1 : a_2) \times (b_0 : b_1 : b_2)$ be a K-rational point on V_3 with integer coordinates. A set of finite heights with respect to the exceptional divisors is given by the following expressions and by those derived from these two by the action of the symmetric group S_3 on the subscripts:*

$$h_{E_0}(x) = \mathfrak{N}((a_1, a_2)(a_0, a_1, a_2)^{-1}) = \mathfrak{N}(b_0(b_0, b_1, b_2)(b_0, b_1)^{-1}(b_0, b_2)^{-1}),$$

$$h_{L_{12}}(x) = \mathfrak{N}(a_0(a_0, a_1, a_2)(a_0, a_1)^{-1}(a_0, a_2)^{-1}) = \mathfrak{N}((b_1, b_2)(b_0, b_1, b_2)^{-1}).$$

Proof. By symmetry it is enough to compute the finite height with respect to the divisor E_0. Embed $\mathbb{P}^2 \times \mathbb{P}^2$ by the Segre map ψ in $\mathbb{P}^8$. Thus $(x_0 : x_1 : x_2) \times (y_0 : y_1 : y_2)$ is mapped to $(x_0 y_0 : \ldots : x_i y_j : \ldots : x_2 y_2)$. We introduce new projective coordinates by putting $z_{ij} = x_i y_j$. The image of V_3 under ψ is given by the identities $z_{00} = z_{11} = z_{22}$. The image E_0' of the divisor E_0 is defined by the system of homogeneous equations $z_{00} = z_{10} = z_{11} = z_{12} = z_{20} = z_{21} = z_{22} = 0$. The finite height of a point z now follows immediately from Lemma 2.1.1:

$$h_{E_0'}(z) = \mathfrak{N}\left(\frac{(z_{00}, z_{10}, z_{11}, z_{12}, z_{20}, z_{21}, z_{22})}{(z_{00}, z_{01}, z_{02}, z_{10}, z_{11}, z_{12}, z_{20}, z_{21}, a_{22})}\right).$$

By the morphism formula we have that $h_{E_0'}(\psi(x))$ is a finite height associated with the divisor E_0. Thus by the foregoing argument we have to compute

$$h_{E_0}(x) = \frac{\mathfrak{N}((a_0 b_0, a_1 b_0, a_1 b_1, a_1 b_2, a_2 b_0, a_2 b_1, a_2 b_2))}{\mathfrak{N}((a_0 b_0, a_0 b_1, a_0 b_2, a_1 b_0, a_1 b_1, a_1 b_2, a_2 b_0, a_2 b_1, a_2 b_2))}.$$

But as $a_0 b_0 = a_1 b_1 = a_2 b_2$, we have $a_0 b_0 \in (a_1, a_2)(b_1, b_2)$. Hence

$$\begin{aligned} h_{E_0}(x) &= \mathfrak{N}((a_0 b_0) + (a_1, a_2)(b_0 + (b_1, b_2)))/\mathfrak{N}((a_0 + (a_1, a_2))(b_0 + (b_1, b_2))) \\ &= \mathfrak{N}((a_1, a_2)(b_0 + (b_1, b_2)))/\mathfrak{N}((a_0 + (a_1, a_2))(b_0 + (b_1, b_2))) \\ &= \mathfrak{N}((a_1, a_2))/\mathfrak{N}(a_0 + (a_1, a_2)) \\ &= \mathfrak{N}((a_1, a_2)(a_0, a_1, a_2)^{-1}). \end{aligned}$$

□

Notation. We write $\mathfrak{d} = (a_0, a_1, a_2)$, $\mathfrak{c}_0' = (a_1, a_2)\mathfrak{d}^{-1}$, $\mathfrak{c}_{01}' = \mathfrak{d}(a_2)(a_0, a_2)^{-1} \times (a_1, a_2)^{-1}$. Similarly we define the ideals $\mathfrak{c}_1'$, $\mathfrak{c}_{12}'$, $\mathfrak{c}_2'$ and $\mathfrak{c}_{02}'$. Observe that these ideals satisfy the identities $(a_0) = \mathfrak{d}\mathfrak{c}_1'\mathfrak{c}_{12}'\mathfrak{c}_2'$, $(a_1) = \mathfrak{d}\mathfrak{c}_0'\mathfrak{c}_{02}'\mathfrak{c}_2'$, $(a_2) = \mathfrak{d}\mathfrak{c}_0'\mathfrak{c}_{01}'\mathfrak{c}_1'$.

2.2. *The idea of Manin and Tschinkel*

Let U be the group of units in O_K. Note that the subgroup of $U^3 \times U^3$, $(u_0, u_1, u_2) \times (v_0, v_1, v_2)$, defined by $u_0 v_0 = u_1 v_1 = u_2 v_2$ acts transitively on W_3. Since finite heights are invariant under this action, it makes sense to write $h_D(\bar{x})$ for the orbit $\bar{x}$ of any K-rational point x in W_3. On the other hand, global heights are not invariant, whence the following definition.

DEFINITION 2.2.1. $n(X)$ is the number of orbits γ on W_3 which contain at least one rational point $x \in \gamma$ such that $H_{-\omega}(x) \leq X$.

Let $b(X)$ be an upper bound for the number of rational points x in an orbit γ which satisfy $H_{-\omega}(x) < X$. Then $N_{W_3}(-\omega, X)$ is by definition smaller than $b(X)n(X)$. We may thus say that any upper bound for $n(X)$ will yield an upper bound for $N_{W_3}(-\omega, X)$ up to the action of units.

By the functorial properties of heights (Proposition 1.2.7) and by Proposition 1.2.2 (see also [Tschinkel]) there arise constants $c, c', c'' > 0$ such that

$$\begin{aligned} H_{-\omega}(x) &\geq cH_{E_0+2L_{01}+2E_1+L_{12}}(x) \\ &\geq c'H_{E_0}(x)H_{L_{01}}(x)^2H_{E_1}(x)^2H_{L_{12}}(x) \\ &\geq c''h_{E_0}(\bar{x})h_{01}(\bar{x})^2h_{E_1}(\bar{x})^2h_{L_{12}}(\bar{x}). \end{aligned}$$

Choosing another representation of the anticanonical divisor leads to another inequality. This motivates the introduction of finite heights in [Manin-Tschinkel] and the following definition.

DEFINITION 2.2.2. $\nu(X)$ is the number of orbits of K-rational points x in W_3 that satisfy the six simultaneous inequalities:

$$h_{E_0}(\bar{x})h^2_{L_{01}}(\bar{x})h^2_{E_1}(\bar{x})h_{L_{12}}(\bar{x}) < X,$$

and those derived from it by the action of the symmetric group S_3 on the subscripts.

Since we are not interested in determining the exact constant we can do as if the constants c, c' etc. were equal to 1. Thus by definition we have $n(X) \leq \nu(X)$. Hence an upper bound for $\nu(X)$ will yield an upper bound for $N_{W_3}(-\omega, X)$ up to the action of units.

2.3. *Transforming the problem*

As pointed out by [Tschinkel] the following idea can be considered an application of Weil's theory of distributions.

DEFINITION 2.3.1. $\mu(X)$ is the number of sextuplets $(\mathfrak{c}'_0, \mathfrak{c}'_{01}, \mathfrak{c}'_1, \mathfrak{c}'_{12}, \mathfrak{c}'_2, \mathfrak{c}'_{02})$ of nonzero ideals in O_K that satisfy the six simultaneous inequalities:

$$\mathfrak{N}(\mathfrak{c}'_0)\mathfrak{N}(\mathfrak{c}'_{01})^2\mathfrak{N}(\mathfrak{c}'_1)^2\mathfrak{N}(\mathfrak{c}'_{12}) < X,$$

and those derived from it by the action of the symmetric group S_3 on the subscripts.

Remark. The non-triviality may also be expressed as $1 \le \mathfrak{N}(\mathfrak{c}'_i)$, $1 \le \mathfrak{N}(\mathfrak{c}'_{ij})$. As the number of ideals in O_K with bounded norm is finite, the numbers $\nu(X)$, $\mu(X)$ and later on $\mu(\mathfrak{K}_{i_0}, \ldots, \mathfrak{K}_{i_5}, X)$ and $\mu(\mathfrak{b}_{i_0}, \ldots, \mathfrak{b}_{i_5}, X)$ will be finite.

Let x be a K-rational point on V_3. We can represent the point x with relatively prime integer coordinates. This means that we fix once for all a family of ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_h$ representing the h classes of ideals $\mathfrak{K}_i$ in O_K and additionally require from our coordinates to satisfy $(a_0, a_1, a_2) = \mathfrak{a}_i$ respectively $(b_0, b_1, b_2) = \mathfrak{a}_j$ for some i, j.

LEMMA 2.3.2. *We have* $\nu(X) \le \mu(X)$.

Proof. Represent x by relatively prime integer coordinates and define the ideals $\mathfrak{d}$, $\mathfrak{c}'_0$, $\mathfrak{c}'_{01}$, etc. as in the preceding subsection. Since the integral ideals are non-trivial we have $1 \le \mathfrak{N}(\mathfrak{c}'_i)$ and $1 \le \mathfrak{N}(\mathfrak{c}'_{ij})$. Moreover our calculations show e.g. that

$$h_{E_0}(\bar{x})h^2_{L_{01}}(\bar{x})h^2_{E_1}(\bar{x})h_{L_{12}}(\bar{x}) = \mathfrak{N}(\mathfrak{c}'_0)\mathfrak{N}(\mathfrak{c}'_{01})^2\mathfrak{N}(\mathfrak{c}'_1)^2\mathfrak{N}(\mathfrak{c}'_{12}).$$

Hence the first inequality is satisfied. Similarly we check the other inequalities.

Remark that by identity $(a_0) = \mathfrak{d}\mathfrak{c}'_1\mathfrak{c}'_{12}\mathfrak{c}'_2$ the ideal $\mathfrak{d}$ has to belong to the inverse class of $\mathfrak{c}'_1\mathfrak{c}'_{12}\mathfrak{c}'_2$. Since the coordinates are relatively prime, $\mathfrak{d}$ has to be equal to the corresponding representative $\mathfrak{a}_i$. Thus $\mathfrak{d}$ is uniquely determined by the sextuplet $\mathfrak{c}'_0$, $\mathfrak{c}'_{01}$, $\mathfrak{c}'_1$, $\mathfrak{c}'_{12}$, $\mathfrak{c}'_2$ and $\mathfrak{c}'_{02}$.

Let $x' = (a'_0 : a'_1 : a'_2) \times (b'_0 : b'_1 : b'_2)$ be a second rational point on W_3 with relatively prime integer coordinates and with the same associated set of ideals $\mathfrak{d}$, $\mathfrak{c}'_0$, $\mathfrak{c}'_{01}$, etc. as x. Then the three identities $(a_0) = \mathfrak{d}\mathfrak{c}'_1\mathfrak{c}'_{12}\mathfrak{c}'_2 = (a'_0)$, $(a_1) = \mathfrak{d}\mathfrak{c}'_0\mathfrak{c}'_{02}\mathfrak{c}'_2 = (a'_1)$ and $(a_2) = \mathfrak{d}\mathfrak{c}'_0\mathfrak{c}'_{01}\mathfrak{c}'_1 = (a'_2)$ imply that x and x' belong to the same orbit. □

Thus any upper bound for $\mu(X)$ will also be an upper bound for $\nu(X)$. For a further more sophisticated (and more powerful) development of this idea we refer to [Peyre].

3. Counting ideals in number fields

In this section we generalize in a suitable manner the classical theorem about the number of ideals with bounded norm in a given number field (for a survey we refer to [Lang2, Chap. 6]). One of the main obstructions to an asymptotic formula is given by the difficulty to establish precise error terms for the volume of a certain fundamental domain.

3.1. *Lattices*

To simplify the notation we put $\mathfrak{c}_0 = \mathfrak{c}_0'$, $\mathfrak{c}_1 = \mathfrak{c}_{01}'$, $\mathfrak{c}_2 = \mathfrak{c}_1'$, $\mathfrak{c}_3 = \mathfrak{c}_{12}'$, $\mathfrak{c}_4 = \mathfrak{c}_2'$, $\mathfrak{c}_5 = \mathfrak{c}_{02}'$ and choose the set of subscripts m in $\mathbb{Z}/6\mathbb{Z}$. We observe that the above six inequalities can now be written as

$$\mathfrak{N}(\mathfrak{c}_m)\mathfrak{N}(\mathfrak{c}_{m+1})^2\mathfrak{N}(\mathfrak{c}_{m+2})^2\mathfrak{N}(\mathfrak{c}_{m+3}) < X,$$

for $m = 0, \ldots, 5$. In particular we remark that the system of inequalities is mapped into itself by a translation of the subscripts modulo 6. In this sense the six inequalities are equivalent.

DEFINITION 3.1.1. Let $i_0, \ldots, i_5$ be six positive integers such that $i_j \leq h$. Then $\mu(\mathfrak{K}_{i_0}, \ldots, \mathfrak{K}_{i_5}, X)$ is the number of sextuplets $(\mathfrak{c}_0, \ldots, \mathfrak{c}_5)$ of nonzero ideals of O_K with the property that $\mathfrak{c}_j \in \mathfrak{K}_{i_j}$ and that for every $m \in \mathbb{Z}/6\mathbb{Z}$

$$\mathfrak{N}(\mathfrak{c}_m \cdot \mathfrak{c}_{m+1}^2 \cdot \mathfrak{c}_{m+2}^2 \cdot \mathfrak{c}_{m+3}) < X.$$

Fix once for all a set of representatives $\mathfrak{b}_i \subset O_K$, $1 \leq i \leq h$, for the inverse classes $\mathfrak{K}_i^{-1}$. We write $\bar{\xi}$ for the class $\xi \cdot U$ and $\bar{\mathfrak{b}}_i$ for the set of classes $\{\bar{\beta} \mid \beta \in \mathfrak{b}_i\}$.

DEFINITION 3.1.2. Let $i_0, \ldots, i_5$ be six positive integers such that $i_j \leq h$. Then $\mu(\mathfrak{b}_{i_0}, \ldots, \mathfrak{b}_{i_5}, X)$ is the number of sextuplets $(\bar{\xi}_0, \ldots, \bar{\xi}_5)$ of classes of integers modulo U with the property that $\bar{\xi}_j \in \bar{\mathfrak{b}}_{i_j}$, that $\mathfrak{N}(\mathfrak{b}_{i_j}) \leq \mathfrak{N}(\xi_i)$, and that for every $m \in \mathbb{Z}/6\mathbb{Z}$

$$\mathfrak{N}(\xi_m \cdot \xi_{m+1}^2 \cdot \xi_{m+2}^2 \cdot \xi_{m+3}) < X\mathfrak{N}(\mathfrak{b}_{i_m} \cdot \mathfrak{b}_{i_{m+1}}^2 \cdot \mathfrak{b}_{i_{m+2}}^2 \cdot \mathfrak{b}_{i_{m+3}}).$$

Notation. We write $\mathfrak{e}_m$ for $\mathfrak{b}_{i_m} \cdot \mathfrak{b}_{i_{m+1}}^2 \cdot \mathfrak{b}_{i_{m+2}}^2 \cdot \mathfrak{b}_{i_{m+3}}$.

LEMMA 3.1.3. *We have*

$$\mu(X) = \sum \mu(\mathfrak{K}_{i_0}, \ldots, \mathfrak{K}_{i_5}, X) = \sum \mu(\mathfrak{b}_{i_0}, \ldots, \mathfrak{b}_{i_5}, X).$$

The sums are taken over all sextuplets of classes of ideals, respectively over all sextuplets of representatives $\mathfrak{b}_{i_j}$.

Proof. It suffices to prove that $\mu(\mathfrak{K}_{i_0}, \ldots, \mathfrak{K}_{i_5}, X) = \mu(\mathfrak{b}_{i_0}, \ldots, \mathfrak{b}_{i_5}, X)$. Fix six classes of ideals $\mathfrak{K}_{i_0}, \ldots, \mathfrak{K}_{i_5}$ and let $(\mathfrak{c}_0, \ldots, \mathfrak{c}_5)$ be a sextuplet of non-zero ideals of O_K contained in $\mathfrak{K}_{i_j}$, $0 \leq j \leq 5$, satisfying the corresponding six inequalities. Each product $\mathfrak{c}_j \cdot \mathfrak{b}_{i_j}$ is equal to a principal ideal (ξ_j), for some $\xi_j \in O_K$. Hence we can attach to each sextuplet of ideals a different sextuplet $(\xi_0, \ldots, \xi_5)$ of algebraic integers contained in the ideals $\mathfrak{b}_{i_j}$. Moreover, we obtain $\mathfrak{N}(\mathfrak{b}_{i_j}) \leq \mathfrak{N}(\mathfrak{b}_{i_j} \cdot c_j) = \mathfrak{N}(\xi_j)$ and the six inequalities

$$\mathfrak{N}(\xi_m \cdot \xi_{m+1}^2 \cdot \xi_{m+2}^2 \cdot \xi_{m+3}) < X\mathfrak{N}(\mathfrak{e}_m).$$

Conversely, fix six representatives $\mathfrak{b}_{i_j}$ and define the six fractional ideals $\mathfrak{e}_m$ as before. Suppose that six numbers $\xi_0, \ldots, \xi_5$ in O_K contained in the $\mathfrak{b}_{i_j}$ are given in such a way that the corresponding six inequalities are fulfilled. Then it suffices to set $\mathfrak{c}_j = \xi_j \mathfrak{b}_{i_j}^{-1}$ to get back six ideals satisfying the required inequalities. This map is not yet one-to-one. If we multiply the ξ_i with units $u_i \in U$ we get the same set of ideals. However this is the only obstruction to bijectivity. We get rid of this obstruction by going over to classes modulo U. □

Suppose that for all choices of sextuplets $\mathfrak{b}_{j_0}, \ldots, \mathfrak{b}_{j_5}$ the integers $\mu(\mathfrak{b}_{j_0}, \ldots, \mathfrak{b}_{j_5}, X)$ have the same upper bound $m(X)$. Then we have $\mu(X) \leq ch^6 m(X)$. Hence, as we are not interested in multiplicative constants, an upper bound for one of the $\mu(\mathfrak{b}_{i_0}, \ldots, \mathfrak{b}_{i_5}, X)$ will also do for $\mu(X)$.

3.2. *Fundamental domains*

Denote by A the product $\mathbb{R} \times \cdots \times \mathbb{R} \times \mathbb{C} \times \cdots \times \mathbb{C}$, the first r_1 factors being real and the next r_2 being complex. Denote by J the subset of A consisting of those elements all of whose coordinates are nontrivial, and by W the subgroup of U of roots of unity.

Write an element of A^6 as

$$x = (x_{01}, \ldots x_{0r_1}, z_{01}, \ldots z_{0r_2}, \ldots, x_{51}, \ldots x_{5r_1}, z_{51}, \ldots z_{5r_2})$$

with $x_{ij} \in \mathbb{R}$ and $z_{ik} \in \mathbb{C}$. Then an element $u = (u_0, \ldots, u_5)$ of the group U^6 acts on A^6 and J^6 as follows: $ux = (\sigma_1(u_0)x_{01}, \ldots, \sigma_{r_1}(u_0)x_{0r_1}, \tau_1(u_0)z_{01}, \ldots, \tau_{r_2}(u_0)z_{0r_2}, \ldots)$.

DEFINITION 3.2.1. A fundamental domain of J^6 for U^6/W^6 is a subset D of J_6 with the following three properties: D is stable under the action of W^6, $U^6 D = J^6$ and $yD \cap D = \varnothing$ for $y \notin W^6$, $y \in U^6$.

DEFINITION 3.2.2. For x an element in A^6 with coordinates x_{ij} and z_{ik}, we consider the partial norms $N_i(x)$, defined by

$$N_i(x) = \prod_{j=1}^{r_1} \prod_{k=1}^{r_2} |x_{ij}|\,|z_{ik}|^2,$$

and define the norm $N(x)$ as

$$N(x) = \prod_{i=0}^{5} N_i(x) = \prod_{i=0}^{5} \prod_{j=1}^{r_1} \prod_{k=1}^{r_2} |x_{ij}|\,|z_{ik}|^2.$$

Let x be an element of J^6 with coordinates $x_{ij} \in \mathbb{R}$ and $z_{ik} \in \mathbb{C}$. Introducing polar coordinates (r_{ij}, ϑ_{ij}) with $0 < r_{ij}$ and $\vartheta_{ij} = \pm 1$ in the real case, and $(\varrho_{ik}, \varphi_{ik})$ with $0 < \varrho_{ik}$ and $0 \leq \varphi_{ik} < 2\pi$ in the complex case, we can write $x_{ij} = \vartheta_{ij} r_{ij}$ and $z_{ik} = \varrho_{ik}\, e^{\sqrt{-1}\varphi_{ik}}$,

LEMMA 3.2.3. *Let $\eta_1, \ldots, \eta_r$ be a basis for U modulo roots of unity. A fundamental domain D of J^6 for U^6/W^6 is given in polar coordinates by the following $6(r+1)$ conditions*:

$$\log(r_{ij}) - \frac{1}{d}\log\left(\prod_{j=1}^{r_1}\prod_{k=1}^{r_2} r_{ij}\varrho_{ik}^2\right) = \sum_{l=1}^{r} c_{il}\log(|\sigma_j(\eta_l)|),$$

$$\log(\varrho_{ik}) - \frac{1}{d}\log\left(\prod_{j=1}^{r_1}\prod_{k=1}^{r_2} r_{ij}\varrho_{ik}^2\right) = \sum_{l=1}^{r} c_{il}\log(|\tau_k(\eta_l)|),$$

with $i = 0, \ldots, 5$, $j = 1, \ldots, r_1$, $k = 1, \ldots, r_2$, and $0 \leq c_{il} < 1$.

Proof. Let x be an element of J^6 with coordinates $x_{ij} \neq 0$ and $z_{ik} \neq 0$. We define a map $\Phi : J^6 \to \mathbb{R}^{6(r+1)}$ as follows:

$$x \mapsto \left(\log\left(\frac{|x_{01}|}{N_0(x)^{1/d}}\right), \ldots, \log\left(\frac{|x_{0r_1}|}{N_0(x)^{1/d}}\right), 2\log\left(\frac{|z_{01}|}{N_0(x)^{1/d}}\right), \ldots, 2\log\left(\frac{|z_{0r_2}|}{N_0(x)^{1/d}}\right), \ldots\right).$$

The image $\Phi(J^6)$ is contained in the linear subspace H of $\mathbb{R}^{6(r+1)}$ determined by the six equations $y_{i1} + \cdots + y_{i,r+1} = 0$, for $i = 0, \ldots, 5$.

Now the embedding $\Psi : (\xi_0, \ldots, \xi_5) \mapsto (\sigma_1(\xi_0), \ldots, \sigma_{r_1}(\xi_0), \tau_1(\xi_0), \ldots, \tau_{r_2}(\xi_0), \ldots)$ of K^6 into A^6 allows us to view U^6 as a subset of J^6, whose image under Φ is a lattice Λ of maximal rank in H, spanned by the $6r$ vectors

$\omega_{il} = \Phi \circ \Psi(0, \ldots, 0, \eta_l, 0, \ldots, 0)$, for $l = 1, \ldots, r$. The subscript $i = 0, \ldots, 5$ indicates the position of η_l in the corresponding vector. It follows from classical theory that the kernel of $\Phi \circ \Psi$ is W^6 (see [Samuel, Chap. 4]). Hence there is an additive action of U^6/W^6 on H. Thus, given a fundamantal domain F for the lattice Λ, we obtain a fundamental domain of J^6 for U^6/W^6 as $D = \Phi^{-1}(F)$.

We now choose a fundamental domain F for Λ the set of all linear combinations $\Sigma_{i=0}^{5} \Sigma_{l=1}^{r} c_{il}\omega_{il}$, where $0 \leq c_{il} < 1$. The result is then immediate. □

Remark. The domain D is a star-body, i.e., it satisfies $tD = D$ for all $t > 0$.

3.3. *Geometry of numbers*

A sextuplet of ideals $(\mathfrak{b}_{i_0}, \ldots, \mathfrak{b}_{i_5})$, viewed as a free $\mathbb{Z}$-module in K^6, is mapped by Ψ into a lattice B in A^6 (see [Lang2, Chap. 5]) with discriminant

$$\Delta(B) = (\sqrt{d})^6 2^{-6r_2} \mathfrak{N}(\mathfrak{b}_{i_0} \cdots \mathfrak{b}_{i_5}).$$

DEFINITION 3.3.1. $D(X)$ is the subset of the fundemental domain D consisting of those points x which satisfy $\mathfrak{N}(\mathfrak{b}_{ij}) \leq N_j(x)$ and the six inequalities

$$N_m(x)N_{m+1}(x)^2 N_{m+2}(x)^2 N_{m+3}(x) < X\mathfrak{N}(\mathfrak{e}_m).$$

DEFINITION 3.3.2. $\mu(B, D, X)$ is the number of points of the lattice B contained in the domain $D(X)$.

LEMMA 3.3.3. *Let* $\mathfrak{b}_{i_0}, \ldots, \mathfrak{b}_{i_5}$ *be six fixed representatives of the inverse classes* $\mathfrak{K}_{i_j}^{-1}$ *and* B *the corresponding lattice in* A^6. *Then we have* $\mu(\mathfrak{b}_{i_0}, \ldots, \mathfrak{b}_{i_5}, X) = \mu(B, D, X)$.

Proof. The actions of U^6 on K^{*6} and on J^6 commute with Ψ. Hence the elements of a given U^6-orbit in K^{*6} are mapped by Ψ into the elements of one and the same U^6-orbit in J^6. Thus, to each sextuplet $(\bar{\xi}_0, \ldots, \bar{\xi}_5)$ of classes modulo U, such that $\bar{\xi}_j \in \bar{\mathfrak{b}}_{i_j}$, we can attach a well-defined element x of $D \cap B$. By definition we have $\mathfrak{N}(\xi_m) = N_m(x)$, and the inequalities follow immediately. □

By identifying $\mathbb{C}$ with $\mathbb{R}^2$ in the usual manner, A^6 may be identified with $\mathbb{R}^{6d}$. Thus it makes sense to talk about the volume $V(S)$ of certain subsets S of A^6. As we are only interested in upper bounds modulo multiplicative constants, the volume of $D(X)$ should provide a satisfactory upper bound for $\mu(B, D, X)$. More precisely:

PROPOSITION 3.3.4. *There exist two constants c and c', depending only on the number field, such that*

$$\mu(B, D, X) \leq c V(D(c'X)).$$

Proof. Let X be as large as necessary. Fix a cell $\mathscr{C}$ of B. Throughout this proof a "cell" will always be a translate of $\mathscr{C}$. Denote by δ the length of the longest diagonal of $\mathscr{C}$. As D is a star-body, there exists a constant $c_1 > \delta$ such that a c_1-neighbourhood of any point $P \in B \cap D$ includes a cell which is completely contained in the interior of D. Let $x \in D(X)$. For $0 \leq c_{il} < 1$ let c_2 be a constant larger than all the values

$$\exp\left(\sum_{l=1}^{r} c_{il} \log(|\sigma_j(\eta_l)|)\right) \quad \text{and} \quad \exp\left(\sum_{l=1}^{r} c_{il} \log(|\tau_k(\eta_l)|)\right).$$

Then from the definition of our fundamental domain in Lemma 3.2.2 we obtain for $i = 0, \dots, 5$ the estimates

$$r_{ij} < c_2 N_i(x)^{1/d} \quad \text{and} \quad \varrho_{ik} < c_2 N_i(x)^{1/d}.$$

Let Δ be a vector in $\mathbb{R}^{6d}$ of maximal length δ. An immediate verification shows that for $m = 0, \dots, 5$:

$$N_m(x+\Delta)N_{m+1}(x+\Delta)^2 N_{m+2}(x+\Delta)^2 N_{m+3}(x+\Delta) < \prod_{i \in I_m} \prod_{j=1}^{r_1} \prod_{k=1}^{r_2} (r_{ij}+\delta)^{\varepsilon_i}(\varrho_{ik}+\delta)^{2\varepsilon_i},$$

where $I_m = \{m, m+1, m+2, m+3\} \subset \mathbb{Z}/6\mathbb{Z}$, and $\varepsilon_i = 1$ if $i = m$ or $m+3$ and $\varepsilon_i = 2$ if $i = m+1$ or $m+2$. We compute the product on the right. The factors of r_{ij} and ϱ_{ik} that will appear have for total exponent at most $r_1 + 2r_2$ for the subscripts m or $m+3$ and $2(r_1 + 2r_2)$ for the subscripts $m+1$ or $m+2$. As $d = r_1 + 2r_2$ we can bound each term up to a constant by $N_m(x)N_{m+1}(x)^2 M_{m+2}(x)^2 N_{m+3}(x)$. Thus the right hand side is smaller than $c'\mathfrak{N}(\mathfrak{e}_m)X$, with c' a constant that does not depend on m, and any c_1-neighbourhood of a point $P \in \mu(B, D, X)$ includes a cell which is completely contained in the interior of $D(c'X)$.

Denote by c the maximum number of cells that can intersect a c_1-ball, by n' the number of cells that are completely contained in the interior of $D(c'X)$ and by V' the volume of $\mathscr{C}$. We now define a map from the set of lattice points contained in $D(X)$ to the set of cells which are completely contained in $D(c'X)$ as follows: We attach to each $P \in \mu(B, D, X)$ any one of the cells that are completely contained in a c_1-neighbourhood of P and that are at the same time completely contained in the interior of $D(c'X)$. In the image of this map the same cell will appear at most c times. Hence we have the estimates

$$\mu(B, D, X) \leq cn'V' \leq cV(c'X). \qquad \square$$

3.4. *Volume computations*

Let b_i and e_i be positive constants. S is the subset of $\mathbb{R}^6$ given by $b_i < s_i$ and by the six inequalities

$$s_m s_{m+1}^2 s_{m+2}^2 s_{m+3} < e_m X'.$$

LEMMA 3.4.1. *There exists a constant c such that* $V(S) < cX'(\log X')^3$.

Proof. Choose X' large enough. Since we are not interested in multiplicative constants we are allowed to set $b_i = e_m = 1$. Moreover, it will suffice to determine the leading term of

$$I(S) = \int \cdots \int ds_0 \cdots ds_5,$$

where the integral is taken over S.

By symmetry we are free to assume that one expression, say $s_0 s_1^2 s_2^2 s_3$, is larger than the five others. This amounts to splitting the domain of integration into six parts. Then, by comparing these expressions, we are led to the inequalities $s_0 s_5 < s_2 s_3$ and $s_3 s_4 < s_0 s_1$. Define S' to be the subset of $\mathbb{R}^6$ given by $1 < s_i$, for every i, and by $s_0 s_1^2 s_2^2 s_3 < X'$, $s_0 s_5 < s_2 s_3$ and $s_3 s_4 < s_0 s_1$. Replacing the integration domain S by S' will enlarge our integral, up to a fixed multiplicative constant c, i.e. $I(S) < cI(S')$. Integrating $I(S')$ over s_4 and s_5 leads to

$$I(S') \leq \int \cdots \int \frac{s_0 s_1}{s_3} \frac{s_2 s_3}{s_0} \, ds_0 \, ds_1 \, ds_2 \, ds_3,$$

where the integral is taken over the set $S'' \subset \mathbb{R}^4$, given by $1 < s_i$, for every i, $s_0 s_1^2 s_2^2 s_3 < X'$. Integrating over s_0 we obtain

$$I(S'') \leq \int \cdots \int (s_1 s_2) \frac{X'}{s_1^2 s_2^2 s_3} \, ds_1 \, ds_2 \, ds_3,$$

where the second integral can be taken over the cube $1 < s_1 < X'$, $1 < s_2 < X'$ and $1 < s_3 < X'$. Of course this integral is equal to $X'(\log X')^3$. □

LEMMA 3.4.2. *There exists a constant c such that* $V(D(X')) \leq cX'(\log X')^3$.

Proof. Let $S' \subset \mathbb{R}^{6(r+1)}$ be the set of norms $0 < r_{ij}$ and $0 < \varrho_{ij}$ of the points of $D(X')$. Working in polar coordinates the volume $V(D(X'))$ can be computed up to a multiplicative constant as

$$\int \cdots \int \varrho_{01} \cdots \varrho_{0r_2} \cdots \varrho_{51} \cdots \varrho_{5r_2} dr_{01} \cdots dr_{0r_1} \cdots dr_{51} \cdots dr_{5r_1} d\varrho_{01} \cdots d\varrho_{0r_2} \cdots d\varrho_{51} \cdots d\varrho_{5r_2},$$

where the integral is taken over S'.

Let $(s_0, \ldots, s_5, c_{01}, \ldots, c_{0r}, \ldots, c_{51}, \ldots, c_{5r})$ be new variables for $\mathbb{R}^{6(r+1)}$, and let S'' be the subset defined by the inequalities $\mathfrak{N}(\mathfrak{b}_{ij}) < s_j$ as well as by the six inequalities

$$s_m s_{m+1}^2 s_{m+2}^2 s_{m+3} < \mathfrak{N}(\mathfrak{e}_m) X'.$$

A diffeomorphism from S'' to S' is given as follows:

$$r_{ij} = s_i^{1/d} \exp\left(\sum_{l=1}^{r} c_{il} \log\left(|\sigma_j(\eta_l)|\right)\right), \qquad \varrho_{ik} = s_i^{1/d} \exp\left(\sum_{l=1}^{r} c_{il} \log\left(|\tau_k(\eta_l)|\right)\right).$$

In the other direction we have $s_i = \Pi_{j=1}^{r_1} \Pi_{k=1}^{r_2} r_{ij} \varrho_{ik}^2$, and the numbers c_{iq} are uniquely determined by the r_{ij} and ϱ_{ij}. Indeed it is well known that the determinant

$$\begin{vmatrix} 1 & \log(|\sigma_1(\eta_1)|) & \cdots & \log(|\sigma_1(\eta_r)|) \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \log(|\sigma_{r_1}(\eta_1)|) & \cdots & \log(|\sigma_{r_1}(\eta_r)|) \\ 1 & \log(|\tau_1(\eta_1)|) & \cdots & \log(|\tau_1(\eta_r)|) \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \log(\tau_{r_2}(\eta_1)|) & \cdots & \log(|\tau_{r_2}(\eta_r)|) \end{vmatrix}$$

does not vanish. In fact, it is equal to $\pm d2^{-r_2}R$, where R is the regulator of K (see [Lang2, Chap. 5]). As in [Lang2, Chap. 5] the Jacobian determinant of the diffeomorphism is equal to the product of the determinants

$$\begin{vmatrix} \frac{1}{d}\frac{r_{i1}}{s_i} & r_{i1}\log(|\sigma_1(\eta_1)|) & \cdots & r_{i1}\log(|\sigma_1(\eta_r)|) \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{d}\frac{r_{ir_1}}{s_i} & r_{ir_1}\log(|\sigma_{r_1}\eta_1)|) & \cdots & r_{ir_1}\log(|\sigma_{r_1}(\eta_r)|) \\ \frac{1}{d}\frac{\varrho_{i1}}{s_i} & \varrho_{i1}\log(|\tau_1(\eta_1)|) & \cdots & \varrho_{i1}\log(|\tau_1(\eta_r)|) \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{d}\frac{\varrho_{ir_2}}{s_i} & \varrho_{ir_2}\log(|\tau_{r_2}(\eta_1)|) & \cdots & \varrho_{ir_2}\log(|\tau_{r_2}(\eta_r)|) \end{vmatrix}$$

for $i = 0, \ldots, 5$, and hence equal to

$$c\frac{\prod_{i=0}^{5}\prod_{j=1}^{r_1} r_{ij}\prod_{k=1}^{r_2}\varrho_{ik}}{s_0\cdots s_5} = c\frac{1}{\prod_{i=0}^{5}\prod_{k=1}^{r_2}\varrho_{ik}}.$$

Thus, up to a constant which depends only on the field, our integral becomes

$$\int\cdots\int ds_0\cdots ds_5\,dc_{01}\cdots dc_{0r}\cdots dc_{51}\cdots dc_{5r},$$

where integration runs over S''. On setting $b_j = \mathfrak{N}(\mathfrak{b}_{i_j})$ and $e_i = \mathfrak{N}(\mathfrak{e}_i)$, we see that $V(S'') = V(S)$. By Lemma 3.4.1, this is at most a constant times $X'(\log X')^3$. □

COROLLARY 3.4.3. *There is a constant c such that $\mu(B, D, X) < cX(\log X)^3$.*

Proof. This is a consequence of Proposition 3.3.4 and Lemma 3.4.2. □

The final result

4.1. *The units*

We make use of some ideas of [Manin-Tschinkel]. Let a be a K-rational point in $\mathbb{P}^2$ with integer, nonzero coordinates (a_0, a_1, a_2). Define

$$H_{\mathcal{O}(1)}(a) = \prod_{v\in M_K}\sup_i v(a_i).$$

For $j = 0, 1, 2$ we have by the product formula $H_{\mathcal{O}(1)}(a) = \Pi_{v\in M_K}\sup_i v(a_i/a_j)$.

DEFINITION 4.1.1. Assume $H_{\mathcal{O}(1)}(a) \leq X$. Then $b'(a, X)$ is the number of K-rational points $a' = (a_0 : u_1 a_1 : u_s a_2)$, $u_i \in U$, which satisfy $H_{\mathcal{O}(1)}(a') \leq X$.

LEMMA 4.1.2. *There exists a positive constant c, which does not depend on a, such that $b'(a, X) \leq c(\log X)^{2r}$.*

Proof. Let X be as large as necessary. The assumption $\Pi_{v \in M_K} \sup_i v(a_i/a_j) \leq X$ and the obvious fact that $\sup_i v(a_i/a_j) \geq 1$ imply $\sup_i v(a_i/a_j) \leq X$. Consequently $1/X \leq v(a_i/a_j) \leq X$. These inequalities do not depend on the choice of v or on the choice of the subscripts i and j.

Similarly, for $i = 1, 2$ we obtain $1/X \leq v(u_i a_i/a_0) \leq X$. On combining these two inequalities with $1/X \leq v(a_i/a_0) \leq X$, we get the inequalities $1/X^2 \leq v(u_i) \leq X^2$. Observe that these inequalities no longer depend on a. From the Dirichlet theorem it follows that there are no more than $O((\log X)^r) \times O((\log X)^r)$ units with this property. This implies $b'(a, X) \leq c(\log X)^{2r}$, as required. □

We go back to our model for V_3. Fix x a K-rational point on W_3 with integer bihomogeneous coordinates $(a_0 : a_1 : a_2) \times (b_0 : b_1 : b_2)$. Let π_x and π_y be the standard projections of V_3 into $\mathbb{P}^2$.

LEMMA 4.1.3. $(H_{\mathcal{O}(1)}(\pi_x(x)))^{1/2} \leq H_{\mathcal{O}(1)}(\pi_y(x))$, *and* $(H_{\mathcal{O}(1)}(\pi_y(x)))^{1/2} \leq H_{\mathcal{O}(1)}(\pi_x(x))$.

Proof. Remember that $a_0 b_0 = a_1 b_1 = a_2 b_2$. Since the coordinates are nonzero the result follows from the trivial inequality $\sup_{i \neq j} v(b_i b_j) \leq \sup_i v(b_i^2)$, together with the product formula. Indeed,

$$\prod_{v \in M_K} \sup_i v(a_i) = \prod_{v \in M_K} \sup_i v(b_1 b_2 a_i) = \prod_{v \in M_K} \sup_{i \neq j} v(b_i b_j a_0) \leq \left(\prod_{v \in M_K} \sup_i v(b_i) \right)^2 = (H_{\mathcal{O}(1)}(\pi_y(x)))^2.$$

□

LEMMA 4.1.4. *Assume that $1 < X$ and $H_{-\omega}(x) \leq X$. Then there exists a positive constant c, which does not depend on x, such that $H_{\mathcal{O}(1)}(\pi_x(x)) \leq cX$ and $H_{\mathcal{O}(1)}(\pi_y(x)) \leq cX$.*

Proof. Without loss of generality we may assume that $1 \leq H_{\mathcal{O}(1)}(\pi_x(x))$. Let D be the divisor of $\mathbb{P}^2$ given in projective coordinates $(x_0 : x_1 : x_2)$ by the homogeneous equation $\{x_0 = 0\}$. By Proposition 1.2.8 we have $H_D \sim H_{\mathcal{O}(1)}$. Moreover we have for the corresponding pullbacks: $\pi_x^*(D) = E_1 + L_{12} + E_2$ and $\pi_y^*(D) = L_{01} + E_0 + L_{02}$. Thus, by the morphism formula, the functorial properties of heights, and Lemma 4.1.3 there exist positive constants c, c', etc. such that

$$\begin{aligned}
H_{-\omega}(x) &\geq cH_{E_0+L_{01}+E_1+L_{12}+E_2+L_{02}}(x)\\
&\geq c'H_{E_0+L_{01}+E_1}(x)H_{L_{12}+E_2+L_{02}}(x)\\
&\geq c''H_D(\pi_x(x))H_D(\pi_y(x))\\
&\geq c'''H_{\mathcal{O}(1)}(\pi_x(x))H_{\mathcal{O}(1)}(\pi_y(x))\\
&\geq c'''H_{\mathcal{O}(1)}(\pi_x(x))(H_{\mathcal{O}(1)}(\pi_x(x)))^{1/2}\\
&\geq c'''H_{\mathcal{O}(1)}(\pi_x(x)).
\end{aligned}$$
□

DEFINITION 4.1.5. Assume $H_{-\omega}(x) \leq X$. Then $b(x, X)$ is the number of K-rational points $x' = (a_0 : u_1 a_1 : u_2 a_2) \times (b_0 : u'_1 b_1 : u'_2 b_2)$, $u_i \in U$, which satisfy $H_{-\omega}(x') \leq X$.

COROLLARY 4.1.6. *There exists a positive constant c, which does not depend on x, such that* $b(x, X) \leq c(\log X)^{2r}$.

Proof. Observe that π_x induces a bijection between points x on W_3 and points a in $\mathbb{P}^2$ with nonzero coordinates. By Lemma 4.1.4 a K-rational point x' with coordinates $(a_0 : u_1 a_1 : u_2 a_2) \times (b_0 : u'_1 b_1 : u'_2 b_2)$ satisfying $H_{-\omega}(x') \leq X$ is mapped into a K-rational point $a' = (a_0 : u_1 a_1 : u_2 a_2)$ which satisfies $H_{\mathcal{O}(1)}(a) < c'X$. Hence $b(x, X) < b(a, c'X)$, and we conclude with Lemma 4.1.2. □

4.2. *Conclusion*

In subsection 2.2 we have seen that $N_{W_3}(-\omega, X)$ is bounded by $b(X)n(X)$, where $n(X)$ denotes the number of orbits γ of rational points under the action of units, containing a rational point x satisfying $H_{-\omega}(x) \leq X$, and $b(X)$ denotes an upper bound for the number of rational points $x' \in \gamma$ satisfying the same inequality. Since $n(X)$ is bounded, up to a multiplicative constant, by $\mu(B, D, X)$, and $b(X)$ can be taken equal, up to a multiplicative constant, to the upper bound of $b(x, X)$, the following theorem is an immediate consequence of Corollary 3.4.3 and Corollary 4.1.6.

THEOREM 4.2.1. *For each number field K there exists a constant* c_K *such that*

$$N_{W_3}(-\omega, X) \leq c_K X(\log X)^{3+2r}.$$

The theorem proves an upper bound with a correct power of $\log X$ in two cases:

COROLLARY 4.2.2. *Let $K = \mathbb{Q}$ or let K be a purely imaginary quadratic field. Then there exists a constant c_K such that*

$$N_{W_3}(-\omega, X) \leq c_K X(\log X)^3.$$

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