Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 70 (1995)

Artikel: Uniqueness for the harmonic map flow from surface to general targets.
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DOI: https://doi.org/10.5169/seals-53000

Comment. Math. Helvetici 70 (1995) 310–338

0010-2571/95/020310-29$1.50 + 0.20/0

Uniqueness for the harmonic map flow from surfaces to general targets*

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Abstract. Let M be a two-dimensional compact Riemannian manifold with smooth (possibly empty) boundary, N an arbitrary compact manifold. If u and v are weak solutions of the harmonic map flow in $H^1(M \times [0, T]; N)$ whose energy is non-increasing in time and having the same initial data $u_0 \in H^1(M, N)$ (and same boundary values if $\partial M \neq \emptyset$) then $u = v$. Combined with a result of M. Struwe, this shows any such u is smooth in the complement of a finite subset of $M \times (0, T]$.

1. Introduction

Let M be a compact two-dimensional Riemannian manifold with smooth (possibly empty) boundary ∂M, N an arbitrary compact Riemannian manifold of dimension k, which we assume isometrically embedded in $\mathbb{R}^p$. In this paper we obtain a uniqueness result for solutions of the 'harmonic map flow' of maps from M to N:

$$\begin{cases} u_t - \Delta u = \operatorname{tr}_M u^* A & \text{on } M \times (0, T) \\ u(x, t) = \gamma(x) & \text{for } t \geq 0, x \in \partial M \\ u(x, 0) = u_0(x), & x \in M \end{cases} \tag{1.1}$$

where $u(x, t)$ takes values in $N \subset \mathbb{R}^p$, A is the second fundamental form of N in $\mathbb{R}^p$ and the superscript (*) denotes pullback to M. Time-independent solutions of (1.1) correspond to harmonic maps from M to N.

By the well-known theorem of J. Eells and J. Sampson [12] (extended by R. Hamilton [11] to the case of manifolds with boundary), (1.1) has a smooth solution defined for all time and converging to a harmonic map from M to N, under the assumption that N has non-positive sectional curvatures; this solution is essentially unique. If no curvature assumptions are made on N, this is no longer true. For two-dimensional domains, the existence of a global weak solution with finite singular set for arbitrary targets (and $u_0 \in H^1(M, N)$) was obtained by M. Struwe in

* Mathematics subject classification (1991): 35K55, 58E20, 58G11.

1984 [1], when $\partial M = \emptyset$. He also showed that this solution is unique in the class of solutions with finite singular set (we recall the precise statements below). Similar results were obtained by K. C. Chang [2] in the case of non-empty boundary, with boundary data $\gamma \in H^{3/2}(M; N)$. We will refer to this solution as the 'almost regular' solution with initial data u_0. The main result of this paper extends the uniqueness results in [1] and [2] to the general class of H^1 weak solutions of (1.1) whose energy is non-increasing as a function of time.

In order to state our main result precisely, we introduce a space which played an important role in [1]. Define:

$$V^T = H^1(M \times [0, T]; N) \cap L^\infty([0, T]; H^1(M, N)) \cap L^2([0, T]; H^2(M, N)).$$

Denote by $E_u(t) = \int_{M \times \{t\}} |\nabla u|^2 \, dx$ the total energy of the map u at time t. Our main result is:

THEOREM 1.1. *Let $u \in H^1(M \times [0, T]; N)$ be a weak solution of the harmonic map flow* (1.1) *with initial conditions $u_0 \in H^1(M, N)$. Assume $E_u(t) \le E_{u_0}$ a.e. in $I = [0, T]$. Then there exists $T' \in (0, T)$ such that $u \in V^{T'}$.*

Combined with the results of M. Struwe and K. C. Chang, this theorem immediately implies a uniqueness and partial regularity statement for solutions of (1.1) in $H^1(M \times [0, T]; N)$ whose energy is monotone in time. Prior to stating the corollary, we recall the main results of [1] and [2] (stated in the case $\partial M = \emptyset$ for simplicity).

THEOREM 1.2 (M. Struwe, [1]). *Assume $\partial M = \emptyset$. For any initial value $u_0 \in H^1(M; N)$ there exists a number $T_0 = T_0(u_0) > 0$ and a solution $v \in \bigcap_{T < T_0} V^T$ of* (1.1) *with $u(\cdot, 0) = u_0$. Moreover,*

(i) *v is smooth in $M \times (0, T_0]$ with the exception of finitely many points (x_i, T_0), $1 \le i \le K$;*
(ii) *v is the unique solution of* (1.1) *in the space $\bigcap_{T < T_0} V^T$ with initial data u_0;*
(iii) *The energy $E_v(t)$ is finite for all $t \in [0, T_0]$ and nonincreasing in t.*

The authors of [1] and [2] also showed that the solution can be continued to a weak solution of (1.1) in $M \times [0, \infty)$ whose singular set is finite.

We will refer to T_0 as the 'first singular time' of u_0. Combining theorems 1.1 and 1.2 one obtains the following corollary.

COROLLARY 1.3. *Let M be a two-dimensional compact Riemannian manifold with smooth (possibly empty) boundary, N an arbitrary compact Riemannian mani-*

fold. If u is a weak solution of (1.1) *in $H^1(M \times [0, T]; N)$ with initial data $u_0 \in H^1(M, N)$ and satisfying $E_u(t) \leq E_u(s)$ for $t \geq s$ (and with boundary values $\gamma \in H^{3/2}(M; N)$ if $\partial M \neq \emptyset$), then u coincides with the 'almost regular' solution with initial data u_0. In particular, u is smooth in $M \times (0, T]$ away from finitely many points.*

By the main result of [10], this is the optimal regularity for weak solutions of the harmonic map flow in two dimensions, even if the initial map is smooth. Corollary 1.3 may be regarded as a parabolic version of F. Hélein's theorem on smoothness of weakly harmonic maps in $H^1(M; N)$ when M is two-dimensional [16]. We remark that it is well-known that, for regular solutions, the total energy is non-increasing as a function of time (see e.g. [1, lemma 3.4]). In addition, monotonicity of the energy (weighted by the heat kernel of M) is presumably necessary (perhaps sufficient) for uniqueness in higher dimensions.

These results had previously been obtained for spherical targets, by the author [14], following work of T. Rivière under the assumption that the initial map has small energy (and also for spherical targets, [3]).

Proof of corollary 1.3. We may assume $T' < T_0$, the first singular time of u_0. By the uniqueness statement (ii) in theorem 1.2, u coincides with the 'almost regular' solution in $[0, T']$. We now repeat the argument starting at T' (and using the energy monotonicity hypothesis) to obtain the first condition of the corollary. The partial regularity statement follows immediately from uniqueness and (i) in theorem 1.2.

Outline of the proof of theorem 1.1. As in [3] and [14], the main idea is to use Wente's two-dimensional 'compensation lemma'.

LEMMA 1.4 (Wente [8] and Brézis-Coron [9]). *Let M be a compact two-dimensional Riemannian manifold with (possibly empty) smooth boundary. If $\eta \in H^1\Omega^2(M)$ and $\theta \in H^1(M)$, then $\delta\eta \cdot d\theta \in H^{-1}(M)$ and*

$$\|\delta\eta \cdot d\theta\|_{H^{-1}} \leq c_3 \|\delta\eta\|_{L^2} \|d\theta\|_{L^2},$$

for some $c = c(M) > 0$.

REMARK. When M is a bounded domain in $\mathbb{R}^2$ (with the Euclidean metric) and $\eta = \eta_1 dx \wedge dy$, $d\theta = \theta_x dx + \theta_y dy$, we have:

$$\delta\eta \cdot d\theta = \theta_x(\eta_1)_y - \theta_y(\eta_1)_x.$$

This was the case dealt with in [8] and [9]. In the general case one takes local conformal coordinates in which the metric is written as $g_{ij} = e^{-2v}\delta_{ij}$, $1 \le i, j \le 2$. This implies $\delta_g \eta = e^{-2v}\delta_{eucl}\eta$, so locally we are back in the Euclidean case, and we may globalize with a simple partitions-of-unity argument.

In section 2.1 we show the harmonic map flow equation may be written as:

$$u_t - \Delta u = -\sum_{i,a} \langle du \cdot \omega_{ia}, e_i \rangle e_a, \tag{1.2}$$

where $\{e_r\}$ is an arbitrary global orthonormal frame on N, whose first k vectors $\{e_i\}$ are tangent to N and whose last $p-k$ vectors $\{e_a\}$ are normal to N; such a frame may be assumed to exist, by an observation of Hélein [16]. The ω_{ia} are connection 1-forms ($\omega_{ia} = \langle de_i, e_a \rangle$).

The main new technical ingredient in this paper is the construction of a time-dependent orthonormal frame 'adapted to u' (in the sense that $e_i(x,t) \in T_{u(x,t)}N$ for $i = 1, \ldots, k$ and a.e. (x,t)), which has the additional property that $\delta\omega_{ia} \in L^2(M \times I)$ (theorem 3.1). Since the frame is time-dependent the direct minimization argument of [15] and [16] does not apply; instead we use time-discretization (as in [13]) to solve a non-linear parabolic system for the $\{e_r\}$. Our argument works in all dimensions. The frame $\{e_r(x,t)\}$ we construct satisfies the equation:

$$\delta\omega_{rs} + \left\langle \frac{\partial e_r}{\partial t}, e_s \right\rangle = \phi_{rs},$$

for some $\phi_{rs} \in L^2(M \times I)$.

In dimension two, the condition $\delta\omega_{ia} \in L^2(M \times I)$ implies higher regularity of the frame for a short time; specifically, we show that $de_r \in L^2(I_1, L^4)$ (theorem 3.2), $I_1 = [0, T_1]$. If we consider the Hodge decomposition of the connection 1-forms for this frame:

$$\omega_{ia} = dA_{ia} + \delta B_{ia} + H_{ia},$$

it turns out that (1.2) may be written as:

$$u_t - \Delta u = -\sum_{i,a} \langle du \cdot \delta B_{ia} \rangle e_a + f(x,t), \tag{1.3}$$

where $f(x,t) \in L^4([0,T], L^{4/3}(M))$. Note that the main term in (1.3) is now (essentially) in a form to which Wente's lemma 1.4 applies (due to the presence of the e_a in (1.3), the actual argument is different from that used in the case of spherical targets – see section 2.4).

We regard (1.3) as a perturbation of the linear non-homogeneous heat equation with source term in $L^4(I, L^{4/3})$ (actually, we first subtract the solution of the linear homogeneous heat equation to absorb initial and boundary values). For this it is necessary to assume that δB_{ia} is small in L^2 norm, uniformly in T. This is possible for a short time, by an argument similar to that used in [14]. This shows

$$u \in L^4([0, T'], W^{2,4/3}),$$

for a short time T' (depending on u). Since $W^{2,3/4} \subset W^{1,4}$ in dimension 2, this implies $u \in L^4(I'; W^{1,4})$ (where $I' = [0, T']$). But then (1.1) may be regarded as a linear non-homogeneous heat equation with 'source term' in $L^2(M \times I')$. Thus by linear existence and uniqueness (e.g. [17, p. 243]) we have $u \in L^2(I'; H^2(M))$; hence u is in the space V^T.

We try to adhere to self-explanatory notation; the abbreviations used more often are listed at the end of the paper.

2. Proof of Theorem 1.1

In this section M is a compact two-dimensional manifold, with or without boundary. Our goal is to prove the following theorem.

THEOREM 1.1. *Let $u \in H^1(M \times [0, T]; N)$ be a weak solution of the harmonic map flow* (1.1) *with initial conditions $u_0 \in H^1(M; N)$. Assume $E_u(t) \leq E_{u_0}$ a.e. in $I = [0, T]$. Let T_0 be the first singular time for the 'almost-regular' solution with initial data u_0. Then there exists $T' < \min\{T, T_0\}$ such that*:

$$u \in L^2([0, T'], H^2(M; N)).$$

In particular, $u \in V^T$.

Proof. We follow, broadly speaking, the 'perturbation' argument used in [3], [4] and [14]. Our first goal is to rewrite the harmonic map flow (1.1) in the form

$$u_t - \Delta u = \text{'compensation terms'} + \text{terms in } L^4(I, L^{4/3}),$$

where by 'compensation term' we mean, loosely speaking, an expression of the form $da \cdot \delta b$, where a and b are, respectively, an H^1 function and an H^1 2-form on M.

Doing this for general targets will require the introduction of time-dependent 'H^1 othonormal frames adapted to $u(x, t)$' satisfying a regularity property (2.1(vi,

vii) below). By an argument of F. Hélein [16], we may assume the existence on the target $N^k \subset \mathbb{R}^p$ of an orthonormal p-frame, the first k vectors of which are tangent to N. Composing such a frame with $u(x, t)$ we obtain an orthonormal frame $\bar{e}_r \in H^1(M \times I, \mathbb{R}^p)$, $r = 1, \ldots, p$. In section 3 we show (Theorem 3.1) that it is possible to change $(\bar{e}_r)$ to a frame (e_r) satisfying:

$$\begin{aligned}
&\text{(i)}\ \langle e_r, e_s\rangle = \delta_{rs} \qquad a.e.(x,t);\\
&\text{(ii)}\ e_i(x,t) \in TN(u(x,t)), \qquad i = 1, \ldots, k \qquad a.e.(x,t);\\
&\text{(iii)}\ \int_{M_t} |\nabla e_r|^2\, dx \le cE_0 \qquad a.e.(t);\\
&\text{(iv)}\ \int_0^T \int_{M_t} \left|\frac{\partial e_r}{\partial t}\right|^2 dx\, dt = K < \infty;\\
&\text{(v)}\ \int_M |\nabla e_r^0|^2\, dx \le cE_0
\end{aligned} \tag{2.1}$$

(conditions (2.1 (i–v)) are also satisfied by $\bar{e}_r$) and with the additional property:

$$\delta\omega_{rs} \in L^2(M \times I), \tag{2.1(vi)}$$

where we define $\omega_{rs} = \langle de_r, e_s\rangle \in L^\infty(I, L^2\Omega^1)$. We assume henceforth indices q, r, s range from 1 to p, i, j from 1 to k and a, b from $k+1$ to p. In theorem 3.2 we show that in the two-dimensional case one obtains, in addition:

$$de_r \in L^2(I_1, L^4), \tag{2.1(vii)}$$

where $I_1 = [0, T_1]$ and $T_1 < \min\{T, T_0\}$ depends on u and on the e_r.

2.1. *Rewriting the equation for u*

We fix for the rest of this section an orthonormal frame (e_r) satisfying conditions (2.1(i–vi, vii)) above. Let

$$u_i = \langle du, e_i\rangle \in L^\infty(I, L^2).$$

Then, since equation (1.1) implies $\langle u_t - \Delta u, e_i\rangle = 0$, we have:

$$\begin{aligned}
\delta u_i &= \langle -\Delta u, e_i\rangle - \langle du \cdot de_i\rangle\\
&= \langle -u_t, e_i\rangle - \sum_j u_j \cdot \omega_{ij},
\end{aligned} \tag{2.2}$$

where the fact that (e_r) is adapted (condition (2.1(ii))) was used. In addition, since the same condition implies:

$$du = \sum_i u_i e_i,$$

we have:

$$\begin{aligned} -\Delta u = \delta\, du &= \sum_i \delta u_i e_i - \sum_i u_i \cdot de_i \\ &= \sum_i \delta u_i e_i - \sum_{i,r} (u_i \cdot \omega_{ir}) e_r. \end{aligned} \tag{2.3}$$

From (2.2) and (2.3) one easily computes:

$$\begin{aligned} u_t - \Delta u &= -\sum_{i,j} (u_j \cdot \omega_{ij}) e_i - \sum_{i,r} (u_i \cdot \omega_{ir}) e_r \\ &= -\sum_{i,a} (u_i \cdot \omega_{ia}) e_a, \\ &= -\sum_{i,a} \langle du \cdot \omega_{ia}, e_i \rangle e_a, \end{aligned} \tag{2.4a}$$

where we used the fact that $\omega_{ij} = -\omega_{ji}$ for all $i, j = 1, \ldots, k$.

2.2. *Use of the Hodge decomposition*

Hodge decomposition theorem. We recall the following standard result (see e.g. [5, ch. 4]).

2.2.1 ($\partial M = \varnothing$). Denote by $\mathscr{H}^p$ $(0 \le p \le n)$ the space of harmonic forms in M of degree p. We have the orthogonal Hilbert space decomposition:

$$L^2\Omega^p(M) = d\Omega^{p-1}H^1(M) \oplus \delta\Omega^{p+1}H^1(M) \oplus \mathscr{H}^p.$$

2.2.2 ($\partial M \neq \varnothing$). Let $\theta \in \Omega^1(M)_{|\partial M}$ be the metric dual to the unit normal ν. Any p-form $\omega \in \Omega^p(M)$ has a unique orthogonal decomposition at points of ∂M: $\omega = \omega_t + \theta \wedge \omega_n$, where $i_\nu \omega_n = 0$. (We may assume ν and the decomposition have been extended to a tubular neighborhood of ∂M.) Denote by:

$\Omega^p H^1_0(M)$ – the H^1 closure of the space of smooth p-forms ω in M with compact support in int (M); by

$\mathscr{H}^p_N$ – the space of (smooth) p-forms ω in M such that $d\omega = \delta\omega = 0$ and $\omega_t = 0$ on ∂M. This is a finite dimensional vector space, isomorphic to the relative cohomology space $H^p(M, \partial M, \mathbb{R})$. We have:

$$L^2\Omega^p(M) = d\Omega^{p-1}H10(M) \oplus \delta\Omega^{p+1}H^1(M) \oplus \mathscr{H}^p_N.$$

In the unique decomposition

$$\omega = d\alpha + \delta\beta + h \qquad (\delta\alpha = d\beta = 0)$$

corresponding to either of the two splittings above, one has the bounds:

$$\|\alpha\|_{H^1} + \|\beta\|_{H^1} \leq c\|\omega\|_{L^2},$$

for some $c = c(M) > 0$. (In the case $n = 2$, $p = 1$ we normalize β by requiring $\int_M \beta = 0$, and if $\partial M = \varnothing$ we also set $\int_M * \alpha = 0$.)

We write the Hodge decomposition for the 1-forms $\omega_{ia} \in L^\infty(I, L^2\Omega^1)$ as

$$\omega_{ia} = dA_{ia} + \delta B_{ia} + H_{ia}, \tag{2.4b}$$

where

$$\|A_{ia}\|_{L^\infty(I,H^1)} + \|B_{ia}\|_{L^\infty(I,H^1)} + \|H_{ia}\|_{L^\infty(I,L^2)} \leq c\|\omega_{ia}\|_{L^\infty(I,L^2)} \leq cE_0^{1/2} \tag{2.5}$$

and

$$\sup_{M_t} |H_{ia}| \leq c\|H_{ia}\|_{L^2(M_t)} \leq cE_0^{1/2} \qquad a.e.(t),$$

given that the space of harmonic 1-forms on M is finite-dimensional. In particular, $H_{ia} \cdot u_i \in L^4(I, L^{4/3})$, since:

$$\|H_{ia} \cdot u_i\|_{L^{4/3}(M_t)} \leq c\left(\sup_{M_t} |H_{ia}|\right)\|u_i\|_{L^2(M_t)} \leq cE_0 \qquad a.e.(t).$$

Condition (2.1(vi)) implies that ω_{ia} is 'co-exact modulo more regular terms', in the following sense: since

$$\begin{cases} -\Delta A_{ia} = \delta\omega_{ia} \in L^2(M \times I) \\ \displaystyle\int_M * A_{ia} = 0 \qquad (A_{ia} \in H_0^1 \quad \text{if } \partial M \neq \varnothing), \end{cases} \tag{2.6}$$

we have

$$A_{ia} \in L^2(I, H^2(M)).$$

To proceed we appeal to the following interpolation inequality.

LEMMA 2.1. *Assume M is two-dimensional. There exists $c = c(M) > 0$ such that if $f \in H^1(M)$,*

$$\int_M |f|^4 \, dx \le c \|f\|^2_{H^1(M)} \|f\|^2_{L^2(M)}$$

For bounded open sets in $\mathbb{R}^2$, this is classical (see e.g. [17, p. 63f]). For closed surfaces, see [1, lemma 3.1] for a proof.

Applying lemma 2.1 to the dA_{ia}, we obtain:

$$\int_{Mt} |dA_{ia}|^4 \, dx \le \|dA_{ia}\|^2_{H^1(M_t)} \|dA_{ia}\|^2_{L^2(M_t)} \qquad a.e.(t).$$

This implies:

$$\begin{aligned} \|u_i \cdot dA_{ia}\|^4_{L^{4/3}(M_t)} &\le c \|u_i\|^4_{L^2(M_t)} \|dA_{ia}\|^4_{L^4(M_t)} \\ &\le c E_0^2 \|A_{ia}\|^2_{H^2(M_t)} \|dA_{ia}\|^2_{L^2(M_t)} \qquad a.e.(t), \end{aligned}$$

so (by (2.5), (2.6) and the Calderón–Zygmund inequality):

$$\begin{aligned} \|u_i \cdot dA_{ia}\|^4_{L^4(I, L^{4/3})} &\le c E_0^3 \|A_{ia}\|^2_{L^2(I, H^2)} \\ &\le c E_0^3 \|\delta\omega_{ia}\|^2_{L^2(M \times I)}. \end{aligned}$$

Thus we may rewrite (2.4a) in the form:

$$u_t - \Delta u = -\sum_{i,a} \langle du \cdot \delta B_{ia}, e_i \rangle e_a + \sum_{i,a} (u_i \cdot C_{ia}) e_a, \tag{2.7}$$

where by definition $u_i \cdot C_{ia} = -u_i \cdot (dA_{ia} + H_{ia}) \in L^4(I, L^{4/3})$.

2.3. *Rewriting the equation for $w = u - v$*

Now let $v : M \times I \to \mathbb{R}^p$ be the solution of the linear homogeneous heat equation on M with initial data u_0 (and boundary data γ if $\partial M \ne \varnothing$). From (2.7) we obtain for $w = u - v$:

$$w_t - \Delta w = -\sum_{i,a} < dw \cdot \delta B_{ia}, e_i > e_a + f(x, t),$$

where:

$$f(x,t) = \sum_{i,a} (u_i \cdot C_{ia})e_a - \sum_{i,a} < dv \cdot \delta B_{ia}, e_i > e_a.$$

We *claim* $f \in L^4(I, L^{4/3})$. This has already been verified for the first term appearing in the definition of f. For the second term, we have (using (2.5) and lemma 2.1):

$$\begin{aligned} \|dv \cdot \delta B_{ia}\|_{L^{4/3}(M_t)} &\le c\|dv\|_{L^4(M_t)} \|\delta B_{ia}\|_{L^2(M_t)} \\ &\le c\|dv\|^{1/2}_{L^2(M_t)} \|v\|^{1/2}_{H^2(M_t)} E_0^{1/2}, \end{aligned}$$

which implies:

$$\|dv \cdot \delta B_{ia}\|^4_{L^4(I,\, L^{4/3})} \le cE_0^3 \|v\|^2_{L^2(I,\, H^2)},$$

proving the claim.

2.4. *Regularity of solutions of linear equations*

In this subsection we consider the general non-homogeneous linear system of the form:

$$\begin{cases} \Phi_t - \Delta\Phi = -(d\Phi \cdot \delta B)e + f(x,t) & \text{on } M \times I \\ \Phi(x,.) = 0 \quad x \in \partial M \\ \Phi(.\,,0) = 0 \quad \text{in } M, \end{cases} \tag{2.9a}$$

where $B \in L^\infty(I, \Omega^2 H^1), f \in L^4(I, L^{4/3})$ and $e \in L^2(I, W^{1,4}) \cap L^\infty(M \times I)$. If $\|B\|_{L^\infty(I,H^1)}$ is small, one may prove the following regularity result.

LEMMA 2.2. *There exists $\varepsilon > 0$ (depending on M and T) with the following property. Let $I' = [0, T']$, where $T' \in (0, T]$ is arbitrary. Let $f \in L^4(I', L^{4/3})$, $B \in L^\infty(I', \Omega^2 H^1)$ and $e \in L^2(I', W^{1,4}) \cap L^\infty(M \times I')$. Assume $\|B\|_{L^\infty(I',H^1)} < \varepsilon$. Let Φ be a solution of (2.9a) in $(H^1 \cap L^\infty)(M \times I')$, such that $\|\nabla\Phi\|_{L^2(M)} \in L^\infty(I')$. Then $\Phi \in L^4(I', W^{2,4/3})$.*

Remark. It is important to observe (to avoid circularity in the proof of theorem 1.1) that ε depends on M and T, but does not depend on T', f or e.

Proof. Fix an arbitrary $T' \in (0, T]$ throughout the proof; set $I' = [0, T']$. The idea is to regard (2.9a) as a perturbation of the non-homogeneous linear heat equation. We recall the relevant linear theory. Consider the equation:

$$\begin{cases} \Phi_t - \Delta\Phi = g & \text{in } M \times (0, T) \\ \Phi(x, .) = 0 & \text{on } \partial M \\ \Phi(. , 0) = 0 & \text{in } M \end{cases} \tag{2.10}$$

THEOREM 2.1. (P. Grisvard [6], theorem 9.3 and remark 9.15). *Assume $g \in L^p(I, L^q(M))$, where $1 < p, q < \infty$ are arbitrary. Problem* (2.10) *has a unique solution in the space*:

$$L^p_0(I, W^{2,q}) = \{\Phi \in L^p(I, W^{2,q}) \mid \Phi_t \in L^p(I, L^q), \Phi_{|\partial M} = \Phi, \Phi(. , 0) = 0 \text{ in } M\}.$$

Moreover the map $\mathscr{L} : L^p(I, L^q) \to L^p_0(I, W^{2,q})$, $\mathscr{L}(g) = 0$ is an isomorphism with inverse $L\Phi = \Phi_t - \Delta\Phi$.

We now use theorem 2.1 (with $q = 4/3$ and $1 < p < \infty$ arbitrary) to show that, for small enough $\varepsilon > 0$, (2.9a) has a unique solution $\Phi_1 \in L^p_0(I', W^{2,4/3})$ (assuming $f \in L^p(I', L^{4/3})$). Observe that (2.9a) may be written as

$$\Phi + \mathscr{L}((d\Phi \cdot \delta B)e) = \mathscr{L}(f), \qquad \Phi \in L^p_0(I', W^{2,4/3}).$$

Showing that $\Phi \mapsto \Phi + \mathscr{L}(d\Phi \cdot \delta B^\varepsilon)$ is an isomorphism of $L^p_0(I'; W^{2,4/3})$ will establish the existence and uniqueness claimed for (2.9a). Let $\Phi \in L^p_0(I'; W^{2,4/3})$. The Sobolev embedding $W^{1,4/3}(M) \hookrightarrow L^4(M)$ implies $d\Phi \in L^p(I'; L^4)$. By Hölder's inequality we have, for almost every $t \in I'$:

$$\|(d\Phi \cdot \delta B)e\|_{L^{4/3}(M_t)} \leq c\,\|dB\|_{L^2(M_t)}\,\|d\Phi\|_{L^4(M_t)} \leq c\varepsilon\,\|d\Phi\|_{L^4(M_t)},$$

and hence:

$$\|(d\Phi \cdot \delta B)e\|_{L^p(I', L^{4/3})} \leq c\varepsilon\,\|\Phi\|_{L^p(I, W^{2,4/3})}.$$

Since $\mathscr{L} : L_p(I'; L^{4/3}) \to L^p_0(I'; W^{2,4/3})$ is bounded (with the bound depending on T, but not on T'), choosing $\varepsilon = \varepsilon_0$ sufficiently small establishes the claim.

Our next goal is to show that the solution Φ_1 of (2.9a) obtained in the previous paragraph coincides with the solution $\Phi_2 = \Phi$ whose existence is assumed in the lemma. This will follow from the claim below. For this part of the argument, we *fix* $p = 4$.

CLAIM. *Let $\Phi_1 \in L^4(I'; W^{2,4/3})$ and $\Phi_2 \in (H^1 \cap L^\infty)(M \times I')$ be solutions of* (2.9a), *with $\|\nabla\Phi_2\|_{L^2} \in L^\infty(I')$. Then $\Phi_1 = \Phi_2$ (assuming ε is sufficiently small).*

Proof. Let $\Psi = \Phi_1 - \Phi_2$. Note $W^{2,4/3} \subset C(M)$, so $\Phi_1 \in L^4(I; L^\infty)$, hence $\Psi \in L^4(I; L^\infty)$. Also, the assumptions on Φ_1 and Φ_2 clearly imply $\nabla\Psi \in L^4(I; L^2)$. We will show that $\Psi \equiv 0$. We have:

$$\Psi_t - \Delta\Psi = (d\Psi \cdot \delta B)e,$$

so taking inner products with Ψ and integrating,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{M_t} |\Psi|^2 + \int_{M_t} |\nabla\Psi|^2 \\
&\quad = \int_{M_t} \langle (d\Psi \cdot \delta B)e, \Psi\rangle \\
&\quad \le c\|\nabla\Psi\|_2\|\nabla B\|_2(\|\nabla\Psi\|_2 + \|\Psi\|_4\|\nabla e\|_4) \\
&\quad \le c\varepsilon\|\nabla\Psi\|_2^2 + c\|\Psi\|_4\|\nabla e\|_4\|\nabla\Psi\|_2\|\nabla B\|_2 \\
&\quad \le c\varepsilon\|\nabla\Psi\|_2^2 + c\|\nabla e\|_4\left(\frac{1}{\lambda}\|\Psi\|_4^4 + \lambda\|\nabla\Psi\|_2^{4/3}\|\nabla B\|_2^{4/3}\right),
\end{aligned}$$

where in the first inequality Wente's lemma 1.4 was used and in the third Young's inequality (for some positive λ to be chosen below).

For $\varepsilon < \varepsilon_0$ sufficiently small (depending only on c) this implies:

$$\frac{1}{2}\frac{d}{dt}\int_{M_t} |\Psi|^2 + \frac{1}{2}\int_{M_t} |\nabla\Psi|^2 \le \frac{c}{\lambda}\|\nabla e\|_4\|\Psi\|_\infty^2\|\Psi\|_2^2 + c\lambda\|\nabla e\|_4\|\nabla\Psi\|_2^{4/3}.$$

It is easy to verify the integrability of the last term in $[0, T']$:

$$\int_0^T \|\nabla\Psi\|_2^{4/3}\|\nabla e\|_4\,dt \le c\left(\int_0^T \|\nabla\Psi\|_2^4\right)^{1/3}\left(\int_0^T \|\nabla e\|_4^{3/2}\right)^{2/3},$$

which is finite, given the assumptions on Ψ and e. We now choose λ so that

$$\frac{1}{2}\int_0^T\int_M |\nabla\Psi|^2\,dx\,dt > c\lambda\left(\int_0^T \|\nabla\Psi\|_2^4\right)^{1/3}\left(\int_0^T \|\nabla e\|_4^{3/2}\right)^{2/3}. \tag{2.11}$$

Then $f(t) = \|\Psi\|_{L^2(M_t)}$ satisfies a differential inequality of the form:

$$f' + h \le \frac{c}{\lambda} g f,$$

where $\int_0^{T'} h\,dt > 0$ and $\int_0^{T'} g\,dt < \infty$. The first inequality follows from (2.11); the second from $\|\nabla e\|_4 \in L^2(I')$ and $\|\Psi\|_\infty^2 \in L^2(I')$. Gronwall's lemma (or a simple direct argument) then shows that $f(0) = 0$ implies $f(t)$ vanishes for all $t \in I'$. This concludes the proof of the claim, and with it the proof of lemma 2.2.

We will also need to consider the analogous regularity result for an equation slightly different from (2.9a). This will be very similar to lemma 2.2, but simpler. The equation is:

$$\begin{cases} \Phi_t - \Delta\Phi = d\Phi \cdot \delta B + f(x,t) & \text{on } M \times (0,T) \\ \Phi(x,.) = 0 & \text{on } \partial M \\ \Phi(.\,,0) = 0 & \text{on } M. \end{cases} \tag{2.9b}$$

LEMMA 2.3. *Fix $2 \leq p < \infty$. There exists an $\varepsilon > 0$ (depending on M, p and T) with the following property. Let Φ be a solution of* (2.9b) *in $H^1(M \times [0, T'])$, where $T' \in (0, T]$ is arbitrary. Assume $f \in L^p(I', L^{4/3})$ and $B \in L^\infty(I', \Omega^2 H^1)$, with $\|B\|_{L^\infty(I', H^1)} < \varepsilon$. Then $\Phi \in L^p(I', W^{2,4/3})$.*

Proof. The first part of the proof of lemma 2.2 (with $e \equiv 1$) guarantees the existence of a solution $\Phi_1 \in L^p(I', W^{2,4/3})$ (for $\varepsilon < \varepsilon_0$ small enough). Recall the following classical result:

THEOREM 2.2. (J. L. Lions and E. Magenes [7], p. 89). *Assume $g \in L^2(I, H^{-1}(\mathrm{M}))$. Then problem* (2.10) *has a unique solution in the space*:

$$W_0 = \{\Phi \in L^2(I, H^1) | \Phi_t \in L^2(I, H^{-1}), \Phi_{|\partial M} = 0, \Phi(.\,,0) = 0 \text{ in } M\}.$$

Moreover the map $\mathcal{U} : L^2(I, H^{-1}) \to W_0$, $\mathcal{U}(g) = \Phi$ is an isomorphism with inverse $L\Phi = \Phi_t - \Delta\Phi$.

We use this to show (2.9b) has a unique solution in W_0' (defined as W_0 above, with T' in place of T); this is implied by the statement that $\Phi \mapsto \Phi - \mathcal{U}(\delta B \cdot d\Phi)$ is an isomorphism of W_0' (note that $f \in L^2(I', H^{-1})$, since $p \geq 2$). For this we use Wente's lemma 1.4, which implies that, if $\Phi \in L^2(I', H^1)$:

$$\|\delta B \cdot d\Phi\|_{H^{-1}(M_t)} \leq c\|\Phi\|_{H^1(M_t)}\|B\|_{H^1(M_t)},$$

and hence:

$$\|\delta B \cdot d\Phi\|_{L^2(I', H^{-1})} \leq c\varepsilon\|\Phi\|_{L^2(I'; H^1)}.$$

Since $\mathcal{U}$ is bounded, this implies the map above is an isomorphism of W_0' for ε small enough, and (2.9b) has a unique solution in W_0'. Since both the solution Φ whose experience is assumed in the lemma and the Φ_1 obtained in the previous paragraph are in W_0' (note $(\Phi_1)_t \in L^p(I', L^{4/3}) \hookrightarrow L^2(I', H^{-1})$, since $p \geq 2$), they must coincide. This concludes the proof of lemma 2.3.

2.5. *Conclusion of the proof of Theorem* 1.1

We apply lemma 2.2 to equation (2.8) and a time interval $[0, T_2]$, where $T_2 < T_1 < T_0$ (T_0 is the first singular time of u_0 and the choice of T_1 guarantees that $\|\nabla e\|_4 \in L^2([0, T_1])$—see (2.1(vii))). *Fix* the $\varepsilon > 0$ given by lemma 2.2.

Let $B = (B_{ia})$ be the two-forms defined in terms of the map $u(x, t)$ by the Hodge decomposition (2.4b); recall that $B \in L^\infty(I, H^1)$. We *claim* that one may find a decomposition:

$$B = B_1 + B_2$$

such that $B_2 \in C^\infty((0, T'), \Omega^2(M))$ and for a short time (that is, in $M \times [0, T']$ where T' depends on ε, u and $\{e_r\}$) we have:

$$\|B_1\|_{L^\infty(I', H^1)} < \varepsilon$$

($I' = [0, T']$.) Moreover,

$$\sup_{M \times [0, T']} |\delta B_2| \leq c E_0^{1/2}.$$

In particular, $\langle \delta B_2 \cdot dw \rangle \in L^4(I', L^{4/3})$ and

$$\|\langle \delta B_2 \cdot dw \rangle\|_{L^4(I'; L^{4/3})} \leq c T'^{1/4} E_0. \tag{2.12}$$

To obtain this decomposition it suffices to write at time zero:

$$B(0) = B_1(0) + B_2(0),$$

with $\|B_1(0)\|_{H^1} < \varepsilon/2$ and $B_2(0) \in C^\infty \Omega^2(M)$. In the next section we show that the improved adapted frame $\{e_r\}$ given by theorem 3.1 attains its initial data strongly in $H^1(M)$ (see lemma 3.1). Thus $\omega_{ia}(t) \to \omega_{ia}(0)$ strongly in $L^2(M)$. Since the linear operator that assigns B_{ia} to ω_{ia} is bounded (from $\Omega^1 L^2(M)$ to $\Omega^2 H^1(M)$), this implies:

$$\|B(t) - B(0)\|_{H^1(M_t)} < \varepsilon/2 \qquad \text{for } t \in [0, T'], \quad T' = T'(\varepsilon, u, \{e_r\}) < T_2,$$

hence setting $B_2(t) = B_2(0)$ and $B_1(t) = B(t) - B_2(0)$ we obtain a decomposition of B with the desired properties, proving the claim. (A simpler version of this argument was already used in [14], following a suggestion of M. Struwe.) We fix this T' for the remainder of the proof.

We now apply the decomposition obtained in the previous paragraph to re-write (2.8) as:

$$\begin{cases} w_t - \Delta w = \sum_{ia} \langle dw \cdot \delta(B_1)_{ia}, e_i \rangle e_a + f_1 \quad \text{in } M \times (0, T'), \\ w(x, .) = 0, \quad x \in \partial M, \\ w(. , 0) = 0 \quad \text{in } M, \end{cases} \tag{2.13}$$

where $f_1 = f + \langle \delta B_2, dw \rangle \in L^4(I', L^{4/3})$ (by (2.12)). By lemma 2.2 (applied to (2.13)) we have $w \in L^4(I', W^{2,4/3})$, for the T' found in the previous paragraph. Since clearly $v \in L^4(I', W^{2,4/3})$, we have $u \in L^4(I', W^{2,4/3})$. As explained in the outline of the proof, it follows from this and (1.1) that $u \in L^2(I', H^2(M))$, hence $u \in V^T$. This concludes the proof of theorem 1.1.

3. Existence of adapted frames

In this section M is a compact Riemannian manifold (with or without boundary), whose dimension is arbitrary until further notice.

Let $I = [0, T]$ (where we allow T to be infinite) and consider a map

$$u = u(x, t) : M \times I \to N^k \subset \mathbb{R}^p \quad (N \text{ compact})$$

such that

$$u \in H^1(M \times I; N);$$

$$u_0 = u(. , 0) \in H^1(M, N), \qquad \int_M |\nabla u_0|^2 \leq E_0;$$

$$\int_{M_t} |\nabla u|^2 \, dx \leq E_0 \quad a.e.(t) \tag{3.1}$$

We assume the existence of an 'H^1 orthonormal frame in $\mathbb{R}^p$ adapted to u', that is, of maps:

$$\bar{e}_r \in H^1(M \times I; \mathbb{R}^p), \qquad r = 1, \ldots, p$$

such that:

(i) $\langle \bar{e}_r, \bar{e}_s \rangle = \delta_{rs}$ $a.e.(x, t)$;

(ii) $\bar{e}_i(x, t) \in TN(u(x, t))$, $\quad i = 1, \dots, k$ $a.e.(x, t)$;

(iii) $\displaystyle\int_{M_t} |\nabla \bar{e}_r|^2 \, dx \le cE_0$ $a.e.(t)$;

(iv) $\displaystyle\int_0^T \int_{M_t} \left| \frac{\partial \bar{e}_r}{\partial t} \right|^2 dx \, dt = K < \infty$

(v) $\displaystyle\int_M |\nabla \bar{e}_r^0|^2 \, dx \le cE_0$,

where $\bar{e}_r^0 = \bar{e}_r(\,.\,, 0)$. Henceforth we assume indices q, r, s range from 1 to p, indices i from 1 to k and a from $k+1$ to p. We denote by $\mathbb{F}^p$ the manifold of orthonormal frames in $\mathbb{R}^p$, so $(\bar{e}_r) \in H^1(M; \mathbb{F}^p)$. An adapted frame satisfying (i)–(v) may be obtained by composing an existing global o.n. p-frame in $N \subset \mathbb{R}^p$ whose first k vectors are tangent to N with a map $u(x, t)$ satisfying (3.1) (it is enough to assume the existence of a frame in the image of u).

Given $(\bar{e}_r)$ satisfying (i)–(v) we define the 1-forms:

$$\bar{\omega}_{rs} = \langle d\bar{e}_r, \bar{e}_s \rangle \in L^\infty(I, L^2\Omega^1(M)).$$

In general we only have $\delta\bar{\omega}_{rs} \in H^{-1}(M_t)$ $a.e.(t)$. The main result of this section is that by changing the frame this can be improved.

THEOREM 3.1. *Assume the existence of an adapted frame* $(\bar{e}_r)$ *satisfying* (i)–(v), *where* $u(x, t)$ *is a time-dependent map satisfying* (3.1). *Then we may find an adapted frame* $(e_r) \in H^1(M \times I; \mathbb{F}^p)$ *satisfying* (i)–(v) (*with different constants on the right-hand sides of* (iii) *and* (iv)), *coinciding with* $(\bar{e}_r^0)$ *for* $t = 0$ *and with the additional property*:

$$\delta\omega_{rs} \in L^2(M \times I)$$

(*where by definition* $\omega_{rs} = \langle de_r, e_s \rangle$). *The* $e_r(x, t)$ *are weak solutions of the system*:

$$\left\langle \frac{\partial e_r}{\partial t}, e_s \right\rangle + \delta\omega_{rs} = \phi_{rs},$$

for some functions $\phi_{rs} \in L^2(M \times I)$, with boundary conditions:

$$\frac{\partial e_r}{\partial v}_{|\partial M} = 0$$

if $\partial M \neq \emptyset$.

Proof. The idea is to obtain the (e_r) by solving a parabolic equation on $M \times I$ with initial data $(\bar{e}_r^0)$. We will do this following the time-discretization method used in [13, proof of Theorem 1]. The main difficulty is defining the functional in variation ($F_n(e)$ below) so as to obtain adapted frames in the limit.

The 'background frame' $(\bar{e}_r)$ defines pointwise isometries $\Pi_{(x,t)} : \mathbb{R}^p \to \mathbb{R}^p$ by:

$$\Pi_{(x,t)} v = (\langle v, \bar{e}_1(x, t)\rangle, \ldots, \langle v, \bar{e}_p(x, t)\rangle).$$

Recall that we may assume the maps $t \mapsto \bar{e}_r(.\,, t)$ are in $C^0(I, L^2(M, \mathbb{R}^p))$. Thus for each $t \in I$ we have a bounded linear map $\Pi_t : L^\infty(M; \mathbb{R}^p) \to L^\infty(M; \mathbb{R}^p)$ given by:

$$(\Pi_t v)(x) = (\langle v(x), \bar{e}_1(x, t)\rangle, \ldots, \langle v(x), \bar{e}_p(x, t)\rangle).$$

The maps Π_t will be used below in the definition of a functional on frames.

STEP 1. Given $h \in (0, T)$ we define the sequence of times:

$$t_n = nh; \qquad n = 0, 1, \ldots, [T/h].$$

Here $[x]$ denotes the largest integer less than or equal to x. We further fix the notation $I_n = [t_n, t_{n+1})$, $\Pi_n = \Pi_{t_n}$, $\bar{e}_r^n = \bar{e}_r(.\,, t_n)$. When T is finite, we adopt the following endpoint conventions: $I_n = [h[T/h], T)$, $\bar{e}_r^{n+1} = \bar{e}_r(.\,, T) \in L^2(M; \mathbb{R}^p)$ if $n = [T/h]$ (we may assume T/h is not an integer). All the intervals I_n have length at most h.

We define inductively a sequence of frames $(e_r^n) \in H^1(M, \mathbb{F}^p)$ (not necessarily adapted), $n = 0, 1, \ldots, [T/h] + 1$ as follows. Let $e_r^0 = \bar{e}_r^0$, and given (e_r^n) solve the variational problem:

$$(e_r^{n+1}) \text{ minimizes } F_n \text{ in } H^1(M; \mathbb{F}^p),$$

where

$$F_n(e) = \sum_r \int_M |\nabla e_r|^2\, dx + \frac{1}{h} \sum_r \int_M |\Pi_{n+1}(e_r) - \Pi_n(e_r^n)|^2\, dx.$$

$F_n(e)$ is clearly sequentially weakly lower semi-continuous in $H^1(M; \mathbb{F}^p)$, so a minimizing frame exists in $H^1(M; \mathbb{F}^p)$ and (e_r^{n+1}) is well-defined.

The Euler-Lagrange equation for F_n is:

$$\begin{cases} \delta\omega_{rs} + \dfrac{1}{h}\langle e_r - e_r^n, e_s\rangle = \phi_{rs}^n, \\ \left\langle \dfrac{\partial e_r}{\partial \nu}, e_s \right\rangle = 0 \quad \text{on } \partial M \end{cases} \tag{3.2a}$$

for all $r, s = 1, \dots, p$, where $\omega_{rs} = \langle de_r, e_s\rangle$ and

$$\begin{cases} \phi_{rs}^n = \dfrac{1}{h}\{\langle e_r - e_r^n, e_s\rangle\}_+ + \{\psi_{rs}^n\}_-, \\ \psi_{rs}^n = \dfrac{1}{h}\sum\limits_q \langle e_r^n, \bar{e}_q^n\rangle\langle e_s, \bar{e}_q^{n+1} - \bar{e}_q^n\rangle. \end{cases} \tag{3.2b}$$

(Here for a $p \times p$ matrix $A = (a_{rs})$ we set $\{a_{rs}\}_+ = \frac{1}{2}(a_{rs} + a_{sr})$, $\{a_{rs}\}_- = \frac{1}{2}(a_{rs} - a_{sr})$). The (standard) derivation of (3.2) is included in an appendix to this section. We need some estimates for the minimizers $(e_r^n), n \geq 1$. First observe that if $v \in (H^1 \cap L^\infty)(M; \mathbb{R}^p)$,

$$\int_M \langle v(x), \bar{e}_r(x, t+h) - \bar{e}_r(x, t)\rangle^2\, dx \leq ch \int_M |v|^2 \left(\int_t^{t+h} \left|\frac{\partial \bar{e}_r}{\partial t}\right|^2 ds \right) dx, \tag{3.3}$$

for all $t \in [0, T-h]$ (by Cauchy-Schwarz and $|\bar{e}_r(x, t+h) - \bar{e}_r(x, t)|^2 \leq h \int_t^{t+h} |\partial \bar{e}_r/\partial t|^2\, ds$). Thus, for $v(x) \in L^\infty(M; \mathbb{R}^p)$,

$$\begin{aligned} \int_M |(\Pi_{n+1} - \Pi_n)v|^2\, dx &= \sum_r \int_M \langle v, \bar{e}_r^{n+1} - \bar{e}_r^n\rangle^2\, dx \\ &\leq h \sum_r \int_M |v|^2 \left(\int_{I_n} \left|\frac{\partial \bar{e}_r}{\partial t}\right|^2 dt \right) dx, \end{aligned} \tag{3.4}$$

in particular:

$$\int_M |(\Pi_{n+1} - \Pi_n)e_r^n|^2\, dx \leq ch \sum_s \int_M \int_{I_n} \left|\frac{\partial \bar{e}_s}{\partial t}\right|^2 dt\, dx \tag{3.5}$$

Since $F_n(e^{n+1}) \leq F_n(e^n)$, we obtain for $n \geq 0$:

$$\sum_r \int_M |\nabla e_r^{n+1}|^2\,dx + \frac{1}{h}\sum_r \int_M |\Pi_{n+1}e_r^{n+1} - \Pi_n(e_r^n)|^2\,dx$$
$$\leq \sum_r \int_M |\nabla e_r^n|^2\,dx + c\sum_r \int_M \int_{I_n} \left|\frac{\partial \bar{e}_r}{\partial t}\right|^2 dt\,dx.$$

Adding these inequalities for varying n starting at 0, we obtain:

$$\sum_r \int_M |\nabla e_r^{n+1}|^2\,dx + \frac{1}{h}\sum_{m=0}^{n}\sum_r \int_M |\Pi_{m+1}e_r^{m+1} - \Pi_m e_r^m|^2\,dx$$
$$\leq \sum_r \int_M |\nabla \bar{e}_r^0|^2\,dx + c\sum_r \int_M \int_0^{t_{n+1}} \left|\frac{\partial \bar{e}_r}{\partial t}\right|^2 dt\,dx \leq c(E_0 + K). \tag{3.6}$$

We also have, since Π_t is a pointwise isometry for each t:

$$|e_r^{n+1} - e_r^n| = |\Pi_{n+1}e_r^{n+1} - \Pi_{n+1}e_r^n|$$
$$\leq |\Pi_{n+1}e_r^{n+1} - \Pi_n e_r^n| + |\Pi_{n+1}e_r^n - \Pi_n e_r^n|,$$

which combined with (3.5) gives for each n:

$$\frac{1}{h}\int_M |e_r^{n+1} - e_r^n|^2\,dx \leq \frac{c}{h}\int_M |\Pi_{n+1}e_r^{n+1} - \Pi_n e_r^n|^2\,dx$$
$$+ c\sum_s \int_M \int_{I_n} \left|\frac{\partial \bar{e}_s}{\partial t}\right|^2 dt\,dx. \tag{3.7}$$

The form of the functional F_n allows us to estimate how far (e^{n+1}) is from being an adapted frame. Clearly, a frame is 'adapted' if, and only if, $\langle e_i, \bar{e}_a\rangle = 0$ *a.e.* for each $i = 1, \ldots, k$ and $a = k+1, \ldots, p$. But again from the fact that Π_n and Π_{n+1} are pointwise isometries, we obtain for each i and a in these ranges, pointwise on M:

$$\langle e_i^{n+1}, \bar{e}_a^{n+1}\rangle = \langle \Pi_{n+1}e_i^{n+1}, \Pi_{n+1}\bar{e}_a^{n+1}\rangle$$
$$= \langle \Pi_{n+1}e_i^{n+1} - \Pi_n e_i^n, \Pi_{n+1}\bar{e}_a^{n+1}\rangle + \langle \Pi_n e_i^n, \Pi_n \bar{e}_a^n\rangle$$
$$= \langle \Pi_{n+1}e_i^{n+1} - \Pi_n e_i^n, \Pi_{n+1}\bar{e}_a^{n+1}\rangle + \langle e_i^n, \bar{e}_s^n\rangle$$
$$= \cdots = \sum_{m=0}^{n} \langle \Pi_{m+1}e_i^{m+1} - \Pi_m e_i^m, \Pi_{n+1}\bar{e}_a^{n+1}\rangle,$$

where in the second equality we used the fact that $\Pi_{n+1}\bar{e}_a^{n+1} = \Pi_n \bar{e}_a^n$ and in the last that $\langle e_i^0, \bar{e}_a^0\rangle = \langle \bar{e}_i^0, \bar{e}_a^0\rangle = 0$.

Combined with (3.6), this implies for each i, a and n:

$$\int_M \langle e_i^{n+1}, \bar{e}_a^{n+1}\rangle^2\,dx \le c\sum_{m=0}^{n}\int_M |\Pi_{m+1}r_i^{m+1} = \Pi_m e_i^m|^2\,dx$$

$$\le ch(E_0 + K). \tag{3.8}$$

STEP 2. We now define on $M \times [0, T]$ two 'frames',

$$e_r^h(x, t), \qquad \tilde{e}_r^h(x, t),$$

where neither is adapted and the e_r^h are in $L^\infty(M \times I)$ but are *not* orthonormal. For $t \in I_n$, $n = 1, \ldots, [T/h]$, set:

$$e_r^h(x, t) = \frac{t - nh}{h}e_r^{n+1}(x) + \frac{(n+1)h - t}{h}e_r^n(x),$$

$$\tilde{e}_r^h(x, t) = e_r^{n+1}(x, t).$$

It is easy to see that both (e_r^h) and $(\tilde{e}_r^h)$ are in $L^\infty(I; H^1(M; \mathbb{R}^p))$, and moreover that $e_r^h \in H^1(M \times I; \mathbb{F}^p)$, uniformly in h in each case. In fact,

$$\frac{\partial e_r^h}{\partial t} = \frac{1}{h}(e_r^{n+1} - e_r^n)$$

on $M \times I_n$, so using (3.6) and (3.7) we obtain:

$$\begin{aligned}\int_0^T\int_M \left|\frac{\partial e_r^h}{\partial t}\right|^2 dx\,dt &= \sum_{m=0}^{[T/h]}\int_{I_n}\int_M \left|\frac{\partial e_r^h}{\partial t}\right|^2 dx\,dt\\ &= \sum_{m=0}^{[T/h]}\frac{1}{h}\int_M |e_r^{m+1} - e_r^m|^2\,dx\\ &\le \frac{c}{h}\sum_{m=0}^{[T/h]}\int_M |\Pi_{m+1}e_r^{m+1} - \Pi_m e_r^m|^2\,dx\\ &\quad + c\sum_s\sum_{m=0}^{[T/h]}\int_M\int_{I_m}\left|\frac{\partial \bar{e}_s}{\partial t}\right|^2 dt\,dx\\ &\le c(E_0 + K) + c\sum_s\int_0^T\int_M\left|\frac{\partial \bar{e}_s}{\partial t}\right|^2 dx\,dt \le c(E_0 + 2K).\end{aligned} \tag{3.9}$$

We also easily obtain from (3.6):

$$\sum_r \int_{M_t} |\nabla \tilde{e}_r^h|^2 \, dx \le \sum_r \int_M |\nabla e_r^0|^2 \, dx + c \sum_r \int_0^t \int_M \left|\frac{\partial \bar{e}_r}{\partial t}\right|^2 dx \, dt \le c(E_0 + K), \tag{3.10}$$

for all $t \in [0, T]$. Exactly the same bound holds for $\Sigma_r \int_{M_t} |\nabla e_r^h|^2 \, dx$.

From the Euler-Lagrange equations (3.2) for the e_r^n, one sees that the $\tilde{e}_r^h, e_r^h$ satisfy weakly in $M \times I$ the equations:

$$\delta \tilde{\omega}_{rs}^h + \left\langle \frac{\partial e_r^h}{\partial t}, \tilde{e}_s^h \right\rangle = \phi_{rs}^h, \tag{3.11}$$

where $\tilde{\omega}_{rs}^h = \langle d\tilde{e}_r^h, \tilde{e}_s^h \rangle$ and

$$\phi_{rs}^h = \left\{ \left\langle \frac{\partial e_r^h}{\partial t}, \tilde{e}_s \right\rangle \right\}_+ + \{\psi_{rs}^h\}_-,$$

$$\psi_{rs}^h = \frac{1}{h} \sum_q \langle e_r^n, \bar{e}_q^n \rangle \langle e_s^{n+1}, \bar{e}_q^{n+1} - \bar{e}_q^n \rangle, \tag{3.12}$$

for $(x, t) \in M \times I_n$. The family $\{\phi_{rs}^h\}_{n>0}$ is uniformly bounded in $L^2(M \times I)$; we have:

$$\begin{aligned}
\int_0^T \int_M (\psi_{rs}^h)^2 \, dx \, dt &\le c \sum_{m=0}^{[T/h]} \sum_q \int_{I_n} \int_M \frac{1}{h^2} |\bar{e}_q^{m+1} - \bar{e}_q^m|^2 \, dx \, dt \\
&\le c \sum_{m=0}^{[T/h]} \sum_q \int_{I_m} \int_M \left|\frac{\partial \bar{e}_q}{\partial t}\right|^2 dx \, dt \\
&= c \sum_q \int_0^T \int_M \left|\frac{\partial \bar{e}_q}{\partial t}\right|^2 dx \, dt,
\end{aligned}$$

where (3.3) was used. This and (3.9) imply:

$$\begin{aligned}
\int_0^T \int_M (\phi_{rs}^h)^2 \, dx \, dt &\le c \int_0^T \int_M \left|\frac{\partial e_r^h}{\partial t}\right|^2 dx \, dt + c \int_0^T \int_M (\psi_{rs}^h)^2 \, dx \, dt \\
&\le c(E_0 + 3K).
\end{aligned} \tag{3.13}$$

Our last estimate for the $(\tilde{e}_r^h)$ is obtained by observing that, for $t \in I_n$:

$$\langle \tilde{e}_i^h(x, t), \bar{e}_a(x, t) \rangle = \langle e_i^{n+1}(x), \bar{e}_a^{n+1}(x) \rangle + \langle \bar{e}_a(x, t) - \bar{e}_a(x, (n+1)h), e_i^{n+1}(x) \rangle,$$

so using (3.3) again,

$$\int_{M_t} \langle \tilde{e}_i^h, \bar{e}_a \rangle^2 \, dx \leq c \int_M \langle e_i^{n+1}(x), \bar{e}_a^{n+1}(x) \rangle^2 \, dx + ch \int_M \int_{I_n} \left| \frac{\partial \bar{e}_a}{\partial t} \right|^2 dt \, dx.$$

Recalling (3.8), this implies:

$$\int_{M_t} \langle \tilde{e}_i^h, \bar{e}_a \rangle^2 \, dx \leq ch(E_0 + 2K) \tag{3.14}$$

for all $t \in [0, T]$.

STEP 3. The frame (e_r) whose existence is claimed in the theorem is obtained by taking limits of the (e_r^h) as $h \to 0$. The uniform estimates (3.9) and (3.10) obtained in step 2 imply that (after possible passage to a subsequence):

$$\begin{aligned} e_r^h &\rightharpoonup e_r, && \text{weakly in } H^1(M \times I; \mathbb{R}^p) \\ e_r^h &\longrightarrow e_r, && \text{strongly in } L^2_{loc}(M \times I; \mathbb{R}^p) \text{ and } a.e. \end{aligned} \tag{3.15}$$

(we don't know at this point if (e_r) is an o.n. frame). Moreover, since (by (3.10)) $d\tilde{e}_r^h$ is uniformly bounded (in h and t) in $L^2\Omega^1(M_t)$ for $t \in [0, T]$, we have:

$$d\tilde{e}_r^h \rightharpoonup \theta_r \in L^2\Omega^1(M \times I),$$

weakly in L^2 (in fact, $d\tilde{e}_r^h(t) \rightharpoonup \theta_r(t)$ weakly in $L^2\Omega^1(M_t)$ for each $t \in I$). Now observe that for $t \in I_n$:

$$|e_r^h - \tilde{e}_r^h|^2 = \frac{1}{h^2} |(n+1)h - t|^2 |e_r^{n+1} - e_r^n|^2 \leq |e_r^{n+1} - e_r^n|^2,$$

so

$$\int_{M_t} |e_r^h - \tilde{e}_r^h|^2 \, dx \leq \int_M |e_r^{n+1} - e_r^n|^2 \, dx = h^2 \int_M \left| \frac{\partial e_r^h}{\partial t} \right|^2 dx,$$

$$\int_0^T \int_M |e_r^h - \tilde{e}_r^h|^2 \, dx \, dt \leq h^2 \int_0^T \int_M \left| \frac{\partial e_r^h}{\partial t} \right|^2 dx \, dt \leq ch^2(E_0 + 2K),$$

using (3.9). This implies $\|e_r^h - \tilde{e}_r^h\|_{L^2(M \times I; \mathbb{R}^p)} \to 0$ as $h \to 0$, which has the following consequences:

(i) $\tilde{e}_r^h \to e_r$ strongly in $L^2(M \times I; \mathbb{R}^p)$ and (we may assume) almost everywhere. In particular, (e_r) is an orthonormal frame.
(ii) $d\tilde{e}_r^h \rightharpoonup de_r$ (weakly in $L^2\Omega^1(M \times I)$), hence $\theta_r = de_r$; in particular, $d\tilde{e}_r^h(t) \rightharpoonup de_r(t)$ weakly in $L^2(M_t)$, for each $t \in I$.
(iii) From (3.14) we have that:

$$\int_{M_t} \langle \tilde{e}_i^h, \bar{e}_a \rangle^2 \, dx \longrightarrow 0 \qquad \text{as } h \to 0,$$

uniformly in $t \in [0, T]$, for all i, a in the appropriate ranges. Hence $\langle e_i, \bar{e}_a \rangle = 0$ $a.e.(x, t)$, which means (e_r) is adapted. From (ii) and (3.10) it follows that

$$\int_{M_t} |\nabla e_r|^2 \, dx \le c(E_0 + K) \qquad a.e.(t). \tag{3.16}$$

The last issue to consider is 'convergence of the Euler-Lagrange equation.' This is not hard to verify. Indeed from (ii) above:

$$\tilde{\omega}_{rs} \rightharpoonup \langle de_r, e_s \rangle = \omega_{rs}$$

(weakly in $L^2\Omega^1(M \times I)$), and from (3.15) and (i) above,

$$\left\langle \frac{\partial e_r^h}{\partial t}, \tilde{e}_s^h \right\rangle \rightharpoonup \left\langle \frac{\partial e_r}{\partial t}, e_s \right\rangle,$$

weakly in $L^2(M \times I)$. In addition, by the uniform bound (3.13) we may assume:

$$\phi_{rs}^h \rightharpoonup \phi_{rs} \in L^2(M \times I),$$

weakly in $L^2(M \times I)$. Thus we may take limits in (3.11) as $h \to 0$ and obtain:

$$\delta\omega_{rs} + \left\langle \frac{\partial e_r}{\partial t}, e_s \right\rangle = \phi_{rs}. \tag{3.17}$$

Since $\langle \partial e_r / \partial t, e_s \rangle \in L^2(M \times I)$, so does $\delta\omega_{rs}$, concluding the proof of theorem 3.1.

In the proof of theorem 1.1 (section 2) we used the observation that the $e_r(x, t)$ attain their initial values in the strong H^1 sense. This is verified in the following lemma:

LEMMA 3.1. *$\Sigma_r \|de_r(t) - d\bar{e}_r^0\|^2_{L^2(M)} \to 0$ as $t \downarrow 0$.*

Proof. First observe that, by the calculation in (3.9):

$$\int_M |e_r^h(x,t) - \bar{e}_r^0(x)|^2\,dx \le t\int_0^T\int_M \left|\frac{\partial e_r^h}{\partial t}\right|^2 dx\,dt \le ct(E_0 + 2K),$$

hence by (3.15):

$$\|e_r(t) - \bar{e}_r^0\|^2_{L^2(M)} \le ct(E_0 + 2K),$$

which shows that $e_r(t) \to \bar{e}_r^0$ strongly in $L^2(M)$ as $t \to 0$. Combined with (3.15), this shows that $de_r(t) \rightharpoonup d\bar{e}_r^0$ as $t \to 0$ (weakly in $L^2(M)$).

For any $t \in I$, we have:

$$\begin{aligned}\sum_r \|de_r(t) - d\bar{e}_r^0\|^2_{L^2} &= \sum_r \|de_r(t)\|^2_{L^2} - \sum_r \|d\bar{e}_r^0\|^2_{L^2} - 2\sum_r \langle de_r(t) - d\bar{e}_r^0, d\bar{e}_r^0\rangle_{L^2(M)} \\ &\le \liminf_{h\to 0}\left[\sum_r \|d\bar{e}_r^h(t)\|^2_{L^2} - \sum_r \|d\bar{e}_r^0\|^2_{L^2}\right] \\ &\quad - 2\sum_r \langle de_r(t) - d\bar{e}_r^0, d\bar{e}_r^0\rangle_{L^2(M)},\end{aligned}$$

where the inequality follows from (ii) above. Recall (3.10):

$$\sum_r \|d\bar{e}_r^h(t)\|^t_{L^2} \le \sum_r \|d\bar{e}_r^0\|^2_{L^2} + c\sum_r \int_0^t\int_M \left|\frac{\partial \bar{e}_r}{\partial t}\right|^2 dx\,dt,$$

for each $h \in (0, T)$ and $t \in (0, T)$. Combining the two preceding inequalities, we obtain:

$$\sum_r \|de_r(t) - d\bar{e}_r^0\|^2_{L^2} \le c\sum_r \int_0^t\int_M \left|\frac{\partial \bar{e}_r}{\partial t}\right|^2 dx\,dt - 2\sum_r \langle de_r(t) - d\bar{e}_r^0, d\bar{e}_r^0\rangle_{L^2(M)}.$$

Since (as observed in the first paragraph of the proof) $de_r(t) \rightharpoonup d\bar{e}_r^0$ as $t \to 0$, we obtain the conclusion of the lemma.

Remark. One may readily identify the weak limit ϕ_{rs} in (3.17). Defining $\hat{e}_r^h(x,t) = e_r^n(x)$ for $(x,t) \in M \times I_n$, we may write (3.12) as

$$\phi_{rs}^h = \left\{\left\langle \frac{\partial e_r^h}{\partial t}, \tilde{e}_s\right\rangle\right\}_+ + \{\psi_{rs}^h\}_-,$$

$$\psi_{rs}^h = \sum_q \langle \hat{e}_r^h, \bar{e}_q^{n_h(t)}\rangle\left\langle \tilde{e}_s^h, \frac{1}{h}(\bar{e}_q^{n_h(t)+1} - \bar{e}_q^{n_h(t)})\right\rangle,$$

with $n_h(t) = n$ defined by $t \in I_n$. It is not hard to see that, as $h \to 0$,

$$\bar{e}_q^{n_h(t)} \longrightarrow \bar{e}_q, \qquad \hat{e}_r^h \longrightarrow e_r,$$

strongly in $L^2(M \times I)$ in each case, and that:

$$\frac{1}{h}(\bar{e}_q^{n_h(t)+1} - \bar{e}_q^{h_h(t)}) \rightharpoonup \frac{\partial \bar{e}_q}{\partial t},$$

weakly in $L^2(M \times I)$. Hence $\psi_{rs}^h \rightharpoonup \psi_{rs}$, where

$$\psi_{rs} = \sum_q \langle e_r, \bar{e}_q \rangle \left\langle e_s, \frac{\partial \bar{e}_q}{\partial t} \right\rangle.$$

Thus $\phi_{rs} = \{\langle \partial e_r/\partial t, e_s \rangle\}_+ + \{\psi_{rs}\}_-$. This fact is not needed in the proof of the main theorem. (It follows that $\phi_{ia} = 0$.)

Our last result is a short-time 'higher regularity' statement for the improved adapted frame $\{e_r\}$ obtained in theorem 3.1, which holds in the two-dimensional case. This is needed in the proof of the main theorem 1.1.

THEOREM 3.2. *Assume* $\dim M = 2$. *Let* $\{e_r\}$ *be an adapted orthonormal frame satisfying properties* (3.1(i–vi)), *and satisfying equation* (3.17), *for some* $\phi_{rs} \in L^2(M \times I)$. *There exists* $T_1 \in (0, T)$ *such that*

$$e_r \in L^2([0, T_1], W^{2,4/3}), \qquad \textit{for } r = 1, \ldots, p;$$

in particular, $de_r \in L^2([0, T_1], \Omega^1 L^4(M))$.

Proof. The argument parallels closely the proof of theorem 1.1 in section 2. From $de_r = \Sigma_s \omega_{rs} e_s$, we derive:

$$-\Delta e_r = \sum_s [\delta\omega_{rs} e_s - \omega_{rs} \cdot de_s]; \tag{3.18}$$

let $\omega_{rs} = dA_{rs} + \delta B_{rs} + H_{rs}$ be the Hodge decomposition of ω_{rs}, with $A_{rs} \in H_0^1$ and H_{rs} harmonic (and in $\mathcal{H}_N$ if $\partial M \neq \varnothing$). Let $C_{rs} = dA_{rs} + H_{rs}$. Since $\Delta A_{rs} = -\delta\omega_{rs} \in L^2(M \times I)$, it follows *exactly* as in section 2.2 that $de_s \cdot C_{rs} \in L^4(I, L^{4/3})$. Taking into account (3.17), (3.18) and $(e_r)_t = \Sigma_s \langle (e_r)_t, e_s \rangle e_s$, we obtain:

$$(e_r)_t - \Delta e_r = -\sum_s de_s \cdot \delta B_{rs} - \sum_s de_s \cdot C_{rs} + \sum_s \phi_{rs} e_s.$$

Clearly $\Sigma_s \phi_{rs} e_s \in L^2(M \times I) \subset L^2(I, L^{4/3})$. Let $\{g_r\}, r = 1, \ldots, p$ solve the linear heat equation with the same initial and boundary data as e_r, and set $f_r = e_r - g_r$. We have:

$$\begin{cases} (f_r)_t - \Delta f_r = -\sum_s df_s \cdot \delta B_{rs} - \sum_s de_s \cdot C_{rs} + \sum_s \phi_{rs} e_s - \sum_s dg_s \cdot \delta B_{rs} \\ (f_r)_{|t=0} = \dfrac{\partial f_r}{\partial \nu}_{|\partial M} = 0. \end{cases}$$

Given that $dg_s \in L^4(I, L^4)$ and $\delta B_{rs} \in L^\infty(I, L^2)$, it is clear that $dg_s \cdot \delta B_{rs} \in L^4(I, L^{4/3})$. By the argument in section 2.5, given any $\varepsilon > 0$ we may find a decomposition of $B = (B_{rs})$:

$$B(t) = B^1(t) + B^2(t),$$

valid for a short time interval $[0, T_1]$, such that $\|B^1\|_{H^1(M_t)} < \varepsilon$ for all t in $[0, T_1]$, and $B^2 \in C^\infty(M \times (0, T_1))$ satisfies $\delta B^2_{rs} \cdot de_s \in L^4(I_1, L^{4/3})$. We conclude the f_r are solutions of a system of the form:

$$(\Phi_r)_t - \Delta \Phi_r = -\sum_s d\Phi_s \cdot \delta B^1_{rs} + \chi_r(x, t),$$

with zero initial and boundary data, where $\chi_r \in L^2(I, L^{4/3})$ and $\|\delta B^1_{rs}\| < \varepsilon$. Choosing the ε given by lemma 2.3 (applied to $p = 2$ on $[0, T]$), we conclude $e_r \in L^2(I_1, L^{4/3})$; this concludes the proof of theorem 3.2. (Strictly speaking, f_r satisfies homogeneous *Neumann* conditions if $\partial M \neq \emptyset$; but clearly lemma 2.3 is valid for Neumann conditions as well.)

Appendix

We include here a derivation of the Euler-Lagrange equations (3.2) for the functional $F_n(e)$ in $H^1(M; \mathbb{F}^p)$. Recall:

$$F_n(e) = \sum_r \int_M |\nabla e_r|^2\, dx + \frac{1}{h} \sum_r \int_M |\Pi_{n+1}(e_r) - \Pi_n(e_r^n)|^2\, dx,$$

where

$$\Pi_n(v) = (\langle v, \bar{e}_1^n \rangle, \ldots, \langle v, \bar{e}_p^n \rangle) \in \mathbb{R}^p.$$

Consider a variation $e_r^\varepsilon = \Sigma_s\, [exp(\varepsilon a)]_{rs} e_s$ of the e_r, where $a \in so(p)$ and exp denotes the exponential map in $SO(p)$. Defining v_r by $v_r = \Sigma_q\, a_{rq} e_q$ (so that $a_{rs} = \langle v_r, e_s \rangle$), we have:

$$
\begin{aligned}
&\frac{1}{2}\left(\frac{d}{d\varepsilon}\right)_{|\varepsilon=0} \int_M \sum_r |\Pi_{n+1}(e_r^\varepsilon) - \Pi_n(e_r^n)|^2 \\
&= \frac{1}{2}\left(\frac{d}{d\varepsilon}\right)_{|\varepsilon=0} \int_M \sum_{r,s} (\langle e_r^\varepsilon, \bar{e}_s^{n+1} \rangle - \langle e_r^n, \bar{e}_s^n \rangle)^2 \\
&= \int_M \left[\sum_r \langle v_r, e_r \rangle - \sum_{r,s} \langle v_r, \bar{e}_s^{n+1} \rangle \langle e_r^n, \bar{e}_s^n \rangle \right] \\
&= \int_M \left[\sum_r \langle v_r, e_r - e_r^n \rangle - \sum_{r,s} \langle v_r, \bar{e}_s^{n+1} - \bar{e}_s^n \rangle \langle e_r^n, \bar{e}_s^n \rangle \right] \\
&= \int_M \left[\sum_{r,s} \langle e_r - e_r^n, e_s \rangle \langle v_r, e_s \rangle - \sum_{r,s,q} \langle e_s, \bar{e}_q^{n+1} - \bar{e}_q^n \rangle \langle e_r^n, \bar{e}_q^n \rangle \langle v_r, e_s \rangle \right].
\end{aligned}
$$

For the first term in $F_n(e^\varepsilon)$, the variational derivative is:

$$
\begin{aligned}
\frac{1}{2}\frac{d}{d\varepsilon}_{|\varepsilon=0} \int_M \sum_r |\nabla e_r^\varepsilon|^2 &= \int_M \sum_r \langle de_r \cdot dv_r \rangle = \sum_{r,s} \int_M \langle de_r, e_s \rangle \cdot \langle dv_r, e_s \rangle \\
&= \sum_{r,s} \int_M \langle de_r, e_s \rangle \cdot d(\langle v_r, e_s \rangle) \\
&\quad - \sum_{r,s} \int_M \langle de_r, e_s \rangle \cdot \langle v_r. de_s \rangle.
\end{aligned}
$$

The second term in the previous expression vanishes, since it may be written as:

$$
\int_M \sum_{r,s,q} (\omega_{rs} \cdot \omega_{sq}) \langle v_r, e_q \rangle = 0,
$$

given that $\Sigma_s\, \omega_{rs} \cdot \omega_{sq}$ is symmetric in r and q, while $a_{rq} = \langle v_r, e_q \rangle$ is skew-symmetric in r and q. Thus we have:

$$
\frac{1}{2}\left(\frac{d}{d\varepsilon}\right)_{|\varepsilon=0} \int_M \sum_r |\nabla e_r^\varepsilon|^2 = \int_M \sum_{r,s} (\delta\omega_{rs}) \langle v_r, e_s \rangle + \int_{\partial M} (i_\nu \omega_{rs}) \langle v_r, e_s \rangle.
$$

Combining this with the preceding calculation, we obtain:

$$\frac{1}{2}\frac{d}{d\varepsilon}F_n(e^\varepsilon)_{|\varepsilon=0} = \int_M \sum_{r,s}\left\{\delta\omega_{rs} + \frac{1}{h}\langle e_r - e_r^n, e_s\rangle - \psi_{rs}^n\right\}a_{rs} + \int_{\partial M}(i_v\omega_{rs})a_{rs},$$

with ψ_{rs}^n defined in (3.2b). Thus we have the following equation for the skew-symmetric components:

$$\delta\omega_{rs} + \frac{1}{h}\{\langle e_r - e_r^n, e_s\rangle\}_- = \{\psi_{rs}^n\}_-,$$

or:

$$\delta\omega_{rs} + \frac{1}{h}\langle e_r - e_r^n, e_s\rangle = \phi_{rs}^n,$$

where $\phi_{rs}^n = 1/h\{\langle e_r - e_r^n, e_s\rangle\}_+ + \{\psi_{rs}^n\}_-$. This concludes the proof of (3.2). (For the boundary condition, observe that $i_v\omega_{rs} = \langle\partial e_r/\partial v, e_s\rangle$).

NOTATION. We try to adhere to self-explanatory notation; the following abbreviations are often used: $I = [0, T]$; $M_t = M \times \{t\}$.

$W^{k,q}(M)$ is the Sobolev space of functions (or maps to N) which have k distributional derivatives in L^q; H^s, $s \in \mathbb{R}$, denotes the scale of Hilbert spaces with $h^k = W^{k,2}$ for $k \in \mathbb{N}$. The domain M and target (N or $\mathbb{R}^p$) are usually omitted from the notation, with the understanding that, as usual:

$$W^{k,q}(M, N) = \{u \in W^{k,q}(M, \mathbb{R}^p) \mid u(x) \in N\ a.e.(x)\}.$$

$$L^p(I, W^{k,q}) = L^p([0, T]; W^{k,q}(M, N)).$$

$\Omega^p W^{k,q}$, $\Omega^p H^s$, etc. denote spaces of differential forms of degree p with coefficients in the corresponding Sobolev spaces (smooth forms if no space is indicated); δ denotes the co-differential in the metric of M.

c denotes a generic positive constant whose value depends only on M and N. For ω, η in $\Omega^1(M) \otimes \mathbb{R}^p$, $\alpha \in \Omega^1(M)$, $v : M \to \mathbb{R}^p$, we denote by $\omega \cdot \alpha$ the inner product on M (a function from M to $\mathbb{R}^p$), by $\langle\omega, v\rangle \in \Omega^1(M)$ the inner product in $\mathbb{R}^p$ and by $\langle\omega \cdot \eta\rangle$ the (real-valued) inner product of ω and η.

For a $p \times p$ matrix $A = (a_{rs})$, we denote by $\{a_{rs}\}_+ = \frac{1}{2}(a_{rs} + a_{sr})$, $\{a_{rs}\}_- = \frac{1}{2}(a_{rs} - a_{sr})$ the symmetric and skew-symmetric components of A.

Acknowledgements

The author's work on this problem was initiated during a visit to the ETH Zürich (January–July 1993). It is a pleasure to thank Jürgen Moser for the invitation and the Forschungsinstitut for its hospitality and financial support, as well as Michael Struwe for encouragement and helpful conversations on this subject. The author is also indebted to Russell Brown for providing useful information regarding Theorem 2.1.

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Received June 20, revised July 7, 1994