

**Zeitschrift:** Commentarii Mathematici Helvetici  
**Herausgeber:** Schweizerische Mathematische Gesellschaft  
**Band:** 70 (1995)

**Artikel:** Remarks on approximate harmonic maps.  
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**DOI:** <https://doi.org/10.5169/seals-52993>

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## Remarks on approximate harmonic maps

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### §1. Introduction

The analytical difficulties in the study of harmonic maps come from the fact that the maps take their values in a curved, compact Riemannian manifold  $N$ . One natural way to tackle such a problem is to use a so called penalty approximation, that is, to relax this nonlinear, nonconvex constraint. Roughly speaking, one studies, instead of the standard Dirichlet integrals, the following variational integral

$$\int_M \left[ |\nabla U|^2 + \frac{1}{\varepsilon^2} d^2(U, N) \right] dx, \quad (1.1)$$

where  $M$  is a compact, Riemann manifold with (or without boundary)  $\partial M$ , and  $U : M \rightarrow \mathbb{R}^k$ . Here we view, via Nash's isometric embedding,  $N$  as a compact submanifold of  $\mathbb{R}^k$ , and  $d(U, N)$  denotes the distance from  $U$  to  $N$ .

The above approach has been employed successfully by Chen and Struwe [CS] in establishing the global existence of weak solutions to the heat flow of harmonic maps. Moreover, to study such approximate energy functional (1.1) may also be natural in the Ginzburg–Laudau's approach to various physical problems, see, e.g., [BBH] and references therein.

The present note is bought out by our previous work [CL] on the evolution of harmonic maps with Dirichlet boundary conditions. We shall establish here first the Schoen–Uhlenbeck's Theorem, or “small energy regularity theorem”, for energy minimizing maps. The problem is essentially reduces to obtain an a priori estimate for a family of smooth approximate solutions with small energy. As an application of our method we shall also prove one of the main results of [BBH2] concerning asymptotic limits for the Ginzburg–Landau model of scalar fields.

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\* The research is partially supported by the NSF-grant DMS #9123532.

\*\* The research is partially supported by the NSF-grant DMS #9149555.

Next we shall consider the uniqueness of suitable weak solutions to the heat flow of harmonic maps. As a first step we show that the weak solutions obtained in [CS] and [CL] coincide with the classical solution whenever the latter exist. This result is rather similar to the uniqueness theorem of J. Serrin [S] for the Navier–Stokes equations.

Uniqueness of weak solutions to the heat flow of harmonic maps fails in general as was shown by J. M. Coron et al. [C]. It remains an interesting open question whether certain suitable weak solutions (such as those constructed in [CS] and [CL]) will be unique.

## §2. Regularity of energy minimizing maps

Let  $N$  be a compact smooth Riemannian submanifold of  $\mathbb{R}^k$ . As we shall discuss only the regularity of energy minimizing maps into  $N$ , the metric on the domain manifold does not play an important role (as long as one assumes a certain minimal smoothness). To simplify the presentation we therefore assume that the domain of our maps is the unit ball  $B_1^m = \{x \in \mathbb{R}^m = |x| < 1\}$  in  $\mathbb{R}^m$ ,  $m \geq 2$ , with the standard Euclidean metric.

The well-known Schoen-Uhlenbeck Theorem states that

**THEOREM. [SU].** *Let  $U : B_1^m \rightarrow N$  be an energy minimizing map. Then there is an  $\varepsilon_0 = \varepsilon_0(m, N) > 0$  such that, if  $\int_{B_1} |\nabla U|^2 dx = \varepsilon \leq \varepsilon_0$ , then*

$$\sup_{B_{1/2}(0)} |\nabla U|^2 \leq C(m, N)\varepsilon. \quad (2.1)$$

Here we shall give an alternative proof of the above statement. To do so we consider a family of approximate solutions  $U_\delta$ ,  $\delta \in (0, 1)$  where  $U_\delta = B_1^m \rightarrow \mathbb{R}^K$  be defined as follows

$$U_\delta(x) = \begin{cases} U(x) & \text{if } x \in B_{1/2}^m(0) \\ V_\delta(x) & \text{if } \frac{1}{2} \leq |x| \leq 1 \end{cases} \quad (2.2)$$

where  $V_\delta$  minimizes

$$I_\delta(V) = \int_{B_1 \setminus B_{1/2}} \left\{ |\nabla V|^2 + \frac{1}{\delta^2} d^2(V, N) \right\} dx \quad (2.3)$$

subject to the Dirichlet boundary conditions  $V = U$  on  $\partial(B_1 \setminus B_{1/2})$ .

Since  $I_\delta(V_\delta) \leq I_\delta(U) = \int_{B_1 \setminus B_{1/2}} |\nabla U|^2 dx$ , we may choose a sequence of  $\delta_i \downarrow 0$  such that  $U_i = U_{\delta_i}$  converges to some  $U_*$  weakly in  $H^1(B_1, N)$  and converges strongly to  $U_*$  in  $L^2(B_1, N)$ . Moreover  $\int_{B_1} |\nabla U_*|^2 dx \leq \int_{B_1} |\nabla U|^2 dx$  by lower semicontinuity of the energy.

Since  $U$  is an energy minimizing map, we see  $U_*$  must also be energy minimizing. Moreover,  $U_i \rightarrow U_*$  in  $H^1(B_1, N)$  by Fatou's Lemma. Finally, for all  $i$  sufficiently large,  $I_{\delta_i}(V_{\delta_i}) \leq \varepsilon \leq \varepsilon_0$ .

Next, we let  $e_\delta(V) = |\nabla V|^2 + 1/\delta^2 d^2(V, N)$ ,  $0 < \delta < 1$ . Then from [CS] (see also [CL]) we have the monotonicity inequality for  $V_\delta$ :

$$\Phi(r, x) \leq \Phi(\rho, x) \quad (2.4)$$

for all  $x \in B_{15/16} \setminus B_{9/16}$ ,  $0 < r < \rho < \text{dist}(x, \partial(B_1 \setminus B_{1/2}))$ . Here  $\Phi(r, x) = r^{2-n} \int_{B_r(x)} e_\delta(V_\delta) dx \cdot \exp(cr)$ . Also we have the Bochner-type inequality:

$$\Delta e \geq -ce(1+e) \quad \text{in } B_1 \setminus B_{1/2} \quad (2.5)$$

where  $e = e_\delta(V_\delta)$ .

By (2.4) and (2.5) and since  $\int_{B_1 \setminus B_{1/2}} e_{\delta_i}(V_{\delta_i}) dx \leq \varepsilon$ , we have the following estimate as in [CS]:

$$\sup_{\frac{5}{8} \leq |x| \leq \frac{5}{6}} e_{\delta_i}(V_{\delta_i})(x) \leq C(m, N)\varepsilon \quad (2.6)$$

provided  $\varepsilon \leq \varepsilon_0(m, N)$ .

Let  $i \rightarrow \infty$ . we obtain, in particular, that

$$\text{OSC}_{\partial B_{3/4}} U_* \leq C\sqrt{\varepsilon} \leq C\sqrt{\varepsilon_0} \quad (2.7)$$

Since  $U_*$  is an energy minimizing map,  $U_*(B_{3/4})$  is contained in ball  $B_{2c\sqrt{\varepsilon}}(p)$  for some  $p \in N$ .

To see this, we let  $p_0 = U_*(x_0)$ , for some  $x_0 \in \partial B_{3/4}$ . Let  $B_{2c\sqrt{\varepsilon}}(p_0)$  be the ball of radius  $2c\sqrt{\varepsilon}$  in  $\mathbb{R}^K$  so that  $U_*(\partial B_{3/4}) \subset B_{c\sqrt{\varepsilon}}(p_0)$ . Let  $\pi : \mathbb{R}^K \rightarrow B_{2c\sqrt{\varepsilon}}(p_0)$  be the retraction map, i.e.,  $\pi(x) = x$  if  $x \in B_{2c\sqrt{\varepsilon}}(p_0)$ , and  $\pi(x) = 2c\sqrt{\varepsilon}(x - p_0)/|x - p_0|$  if  $x \notin B_{2c\sqrt{\varepsilon}}(p_0)$ . Since  $\sqrt{\varepsilon} \leq \sqrt{\varepsilon_0}$  is very small, and  $N$  is a smooth submanifold,  $p_0 \in N$ , we see the nearest point projection map  $\partial B_{2c\sqrt{\varepsilon}}(p_0) \xrightarrow{\pi_N} N \cap B_{\partial c\sqrt{\varepsilon}}(p_0)$  is well-defined and is distance decreasing map from  $\partial B_{2c\sqrt{\varepsilon}}(p_0)$  to  $N \cap B_{\partial c\sqrt{\varepsilon}}(p_0)$ .

Now, if  $U_*$  is energy minimizing with  $U_*(\partial B_{3/4}) \subset B_{c\sqrt{\varepsilon}}(p_0)$ , then  $\tilde{U} = \pi_N \circ \pi \circ U_* : B_{3/4} \rightarrow N$  is also energy minimizing with  $\tilde{U} = U_*$  on  $\partial B_{3/4}$ . In fact,  $\int_{B_{3/4}} |\nabla \tilde{U}|^2 dx \leq \int_{B_{3/4}} |\nabla U_*|^2 dx$  and the equality is valid if and only if  $U_*(B_{3/4}) \subset B_{2c\sqrt{\varepsilon}}(p_0) \cap N$ .

Having seen  $U_*(B_{3/4}) \subset B_{2c\sqrt{\varepsilon}}(p_0) \cap N$ , the regularity of  $U_*$  on  $B_{1/2}$  follows from the standard elliptic theory (See e.g. [J]). By our definition of  $U_*$ ,  $U_* = U$  on  $B_{1/2}$ , we, in particular, obtain that

$$\sup_{B_{1/2}} |\nabla U|^2 \leq C \int_{B_1} |\nabla U|^2 = c\varepsilon. \quad (2.8)$$

Q.E.D.

### §3. A theorem of Bethuel–Brezis–Helein

Let  $\Omega \subseteq \mathbb{R}^2$  be a smooth bounded connected domain. Consider the functional

$$E_\varepsilon(U) = \int_{\Omega} |\nabla U|^2 + \frac{1}{2\varepsilon^2} \int_{\Omega} (|U|^2 - 1)^2 \quad (3.1)$$

which is defined for maps  $U \in H^1(\Omega, \mathbb{C})$ , we let  $g : \partial\Omega \rightarrow \mathbb{C}$  be a smooth map with  $|g(x)| = 1, \forall x \in \partial\Omega$ . We also assume that  $\deg(g, \partial\Omega) = 0$  and hence there is a smooth extension of  $g^* : \Omega \rightarrow S^1$  with  $g^* = g$  on  $\partial\Omega$ .

By a theorem of C. B. Morrey, there is a map  $U_0 : \Omega \rightarrow S^1$  which minimizes  $\int_{\Omega} |\nabla U|^2 dx$  over the set  $H_g^1(\Omega, S^1) = \{U \in H^1(\Omega, \mathbb{C}) : u = g \text{ on } \partial\Omega\}$ . Moreover,  $U_0$  is smooth. When  $\Omega$  is simply connected, a simple lifting argument shows that  $U_0 = e^{i\phi_0}$ . Here  $\phi_0$  is the harmonic extension of  $\phi$ ,  $e^{i\phi} = g$  on  $\partial\Omega$ .

Here we want to show a uniform estimate for the minimizers  $U_\varepsilon$  of (3.1) for  $0 < \varepsilon < 1$  under the hypothesis that  $\deg(g, \partial\Omega) = 0$ .

To do so, we note first that

$$E_\varepsilon(U_\varepsilon) \leq \int_{\Omega} |\nabla U_0|^2 dx, \quad \text{for all } 0 < \varepsilon < 1. \quad (3.2)$$

For any sequence  $\varepsilon_i \rightarrow 0$ , there is a subsequence of  $U_{\varepsilon_i}$  which converges weakly in  $H^1$  and strongly in  $L^2$  to some  $U_* \in H^1(\Omega, S^1)$ . Moreover,  $U_* = g$  on  $\partial\Omega$ , and  $\int_{\Omega} |\nabla U_*|^2 dx \leq \int_{\Omega} |\nabla U_0|^2 dx$ . By the minimizing property of  $U_0$ , we see  $U_*$  again is a minimizer of  $\int_{\Omega} |\nabla U|^2 dx$  over  $H_g^1(\Omega, S^1)$ . In particular,  $U_*$  is smooth. Moreover, it follows from (3.2) that  $U_{\varepsilon_i}$  converges strongly to  $U_*$ . We therefore obtain the following

**LEMMA.** *For any  $\varepsilon_0 > 0$ , there is an  $r_0 > 0$  depending only on  $\partial\Omega$  and  $g$  such that if  $U_\varepsilon$  is a minimizer of (3.1) then*

$$\int_{\Omega \cap B(x, r_0)} \left[ |\nabla U_\varepsilon|^2 + \frac{1}{2\varepsilon^2} (|U_\varepsilon|^2 - 1)^2 \right] dx \leq \varepsilon_0 \quad (3.3)$$

for all  $x \in \bar{\Omega}$  provided  $0 < \varepsilon \leq \varepsilon_*(r_0, \varepsilon_0)$ .

*Proof.* Let  $F$  denote the set of energy minimizing maps over the set  $H_g^1(\Omega, S^1)$ . Then it is easy to see that  $F$  is compact in  $H^1(\Omega, S^1)$ . Moreover, by Morrey's theorem, one has for any  $\varepsilon_0 > 0$ ,  $U_* \in F$

$$\int_{\Omega \cap B(x, r)} |\nabla U_*|^2 dx \leq \varepsilon_0/2 \quad (3.4)$$

for all  $x \in \bar{\Omega}$  and  $0 < r \leq r_0$  provided that  $r_0$  is chosen to be suitably small.

Now we apply the convergence argument above to conclude that (3.3) is valid for all minimizers  $U_\varepsilon$  whenever  $0 < \varepsilon \leq \varepsilon_*$ . Note that  $1/\varepsilon^2 \int_{\Omega} (|U_\varepsilon|^2 - 1)^2 dx \rightarrow 0$  is  $\varepsilon \rightarrow 0^+$ .

**THEOREM.** *Let  $U_\varepsilon$  be a minimizer of (3.1) over the set  $H_g^1(\Omega, \mathbb{C})$  with  $\deg(g, \partial\Omega) = 0$ . Then*

$$\sup_{\Omega} \left[ |\nabla U_\varepsilon|^2 + \frac{1}{2\varepsilon^2} (|U_\varepsilon|^2 - 1)^2 \right] \leq C(g), \quad (3.5)$$

for all  $0 < \varepsilon < 1$ .

*Proof.* It is obvious, by the maximum-principle, that  $|U_\varepsilon| \leq 1$  on  $\Omega$ . Thus  $|U_\varepsilon|(1 - |U_\varepsilon|^2)1/\varepsilon^2 \leq 1/\varepsilon^2 \leq 1/\varepsilon_*^2$  whenever  $\varepsilon \geq \varepsilon_* = \varepsilon_*(g) > 0$ . It follows that (3.5) is true whenever  $\varepsilon \geq \varepsilon_*$ .

For  $0 < \varepsilon < \varepsilon_*$ , we use the estimate (3.3). It follows from the identical arguments as in the previous section, that one obtains the interior estimate

$$\sup_{\Omega'} \left[ |\nabla U_\varepsilon|^2 + \frac{1}{2\varepsilon^2} (1 - |U_\varepsilon|^2)^2 \right] \leq C(g, \Omega') \quad (3.6)$$

for all  $0 < \varepsilon \leq \varepsilon_*$ .

For the estimate near the boundary of  $\Omega$ , we refer to [CL]. We should point out that the monotonicity inequality in the present situation is automatically valid. Combining (3.6) with the boundary estimate one concludes

$$\sup_{\Omega} \left[ |\nabla U_\varepsilon|^2 + \frac{1}{2\varepsilon^2} (1 - |U_\varepsilon|^2)^2 \right] \leq C(\varepsilon_0, r_0) \quad (3.7)$$

for all  $0 < \varepsilon \leq \varepsilon_*$ .

*Remark.* When  $\Omega$  is, in addition, simply connected, the energy-minimizer over  $H_g^1(\Omega, S^1)$  is also unique. In this case, one can show that  $U_\varepsilon \rightarrow U_*$  energy-minim-

izer uniformly and strongly in  $H^1(\Omega, \mathbb{C})$ . Moreover, as in [BBH2], one has, for  $\psi_\varepsilon = 1/\varepsilon^2(1 - |U_\varepsilon|^2) \geq 0$ ,

$$\begin{cases} 2\varepsilon^2 \Delta \psi_\varepsilon - \psi_\varepsilon \geq -4|\nabla U_\varepsilon|^2 \geq -C_1, & \text{for all } 0 < \varepsilon \leq \varepsilon_*, \\ \psi_\varepsilon|_{\partial\Omega} = 0 \end{cases} \quad (3.8)$$

Let  $x_0 \in \Omega$  be such that  $\psi_\varepsilon(x_0) = \max_{x \in \bar{\Omega}} \psi_\varepsilon(x) > 0$ , then  $\Delta \psi_\varepsilon(x_0) \leq 0$ , and thus

$$\psi_\varepsilon(x_0) \leq C_1. \quad (3.9)$$

From (3.9) we obtain  $\|\Delta U_\varepsilon\|_{L^\infty(\Omega)} \leq C_1$ , and  $U_\varepsilon \rightarrow U_0$  in  $C^{1,\alpha}(\Omega)$ ,  $\forall \alpha < 1$ , follows.

#### §4. Uniqueness problems

Let  $M$  be a compact Riemannian manifold with possible nonempty boundary  $\partial M$ , and let  $N$  be as before. The equation for harmonic maps  $U : M \rightarrow N$  can be written as

$$\Delta U + A(U)(\nabla U, \nabla U) = 0 \quad (4.1)$$

where  $\Delta$  is the Beltrami-operator on  $M$ , and  $A(U)$  is the second fundamental form of  $N$  at  $U$ . Thus the corresponding equations for the heat flow are

$$\frac{\partial U}{\partial t} - \Delta U = A(U)(\nabla U, \nabla U), \quad (x, t) \in M \times (0, \infty). \quad (4.2)$$

Given initial data  $U_0 : M \rightarrow N$ , one then is interested in solving (4.2) with

$$U(x, t) = U_0(x) \quad \text{for } (x, t) \in (M \times \{0\}) \cup (\partial M \times (0, \infty)). \quad (4.3)$$

Suppose  $U_0, M, \partial M$  and  $N$  are smooth. It is well-known (see, e.g. [J]) that the problem (4.2) (4.3) has a unique smooth solution  $U(x, t)$ ,  $(x, t) \in M \times [0, T]$ , for some  $T > 0$  which may depend on the various data mentioned. On the other hand, it was shown in [CS] (for the case  $\partial M = \emptyset$ ) and [CL] that (4.2), (4.3) has a global weak-solution which is smooth off a relatively small closed subset of  $M \times (0, \infty)$ . One of the natural question is that whether such suitable weak solutions obtained in [CS] and [CL] are unique.

Here we want to show that the weak solution obtained in [CS] and [CL] must coincide with the classical solution on the time interval  $[0, T^*]$ , here  $0 < T^* \leq \infty$  is

the first time of blow-up for the classical solution. The latter means that there is a smooth solution  $\tilde{U}$  of (4.2), (4.3) on the time interval  $[0, T^*)$  and that  $\lim_{t \rightarrow T^*} \|\nabla \tilde{U}\|_{L^\infty(M)}(t) = +\infty$ .

For this purpose, we adopt the same notations as that in [CS] and [CL]. Consider a sequence of approximate solutions  $U^k$  such that

$$\frac{\partial}{\partial t} U^k - \Delta U^k + k\chi'(d^2(U^k, N)) \frac{d}{dU} \left( \frac{d^2(U^k, N)}{2} \right) = 0 \quad (4.4)$$

in  $M \times (0, \infty)$ ,

$$U^k(x, t) = (U_0(x), \quad \text{on } (M \times \{0\}) \cup (\partial M \times (0, \infty))). \quad (4.5)$$

We claim there are positive constant  $C_0, T_0$  depending only on  $U_0, \partial M, M, N$  such that

$$\sup_{M \times [0, T_0]} e(U^k) \leq C_0, \quad \text{for } k = 1, 2, \dots \quad (4.6)$$

where  $e(U^k) = |\nabla U^k|^2 + (k/2)\chi(d^2(U^k, N))$ .

In fact, for any  $x_0 \in M, 0 < t_0 < R_M$  ( $R_M$  is the injectivity radius of  $M$ ) and  $r = \sqrt{t_0}$ , one has that (we adopt the same notations as that in [CS] and [CL])

$$\begin{aligned} \Phi(r, U^k, (x_0, t_0)) &= \frac{t_0}{2} \int_{R^m} e(U^k) G_{(x_0, t_0)}|_{t=0} \phi^2(|x - x_0|) dx \\ &\leq \frac{t_0}{2} \left( \int_{B_{t_0^{(1-2)/2}}} + \int_{R^m \setminus B_{t_0^{(1-2)/2}}} \right) e(U^k) G_{(x_0, t_0)}|_{t=0} \phi^2(|x - x_0|) dx \\ &\leq C t_0^{1-(\varepsilon m/2)} \|U_0\|_{C^1(M)} + C t_0^{1-(m/2)} e^{-(1/t_0^\varepsilon)} E(U_0) \leq \varepsilon_0, \end{aligned} \quad (4.7)$$

if  $\varepsilon < 2/m$  and  $t_0$  is suitably small. Then, (4.6) follows from the small energy regularity theorems in [CS] and [CL].

By the definition of  $\chi(d^2)$  and (4.6) (cf. [CL]), one has  $\chi(d^2(U^k, N)) = d^2(U^k, N)$  for all large  $k$ 's. We claim that (4.6) implies that  $U^k$  converges to the classical solution  $\tilde{U}$  in  $W_p^{2,1}(M \times (0, T_0))$  as  $k \rightarrow \infty$ . Here

$$W_p^{2,1}(M \times (0, T_0)) = \{V : V, D_x V, D_x^2 V, V_t \in L^p(M \times (0, T_0))\}, \quad 1 < p < \infty.$$

Suppose, for the moment, that the above claim is true. Then we want to show the weak solution obtained in [CS] and [CL] coincides with the classical solution on

$[0, T^*]$ . To do so, we let  $0 < T_0^* \leq T^*$  be such that

$$T_0^* = \sup\{t \in [0, T) : \lim_{k \rightarrow \infty} U^k = \tilde{U}\}. \quad (4.8)$$

Here the limit is taking in  $W_p^{2,1}(M \times (0, t))$ , ( $p \geq m + 1$ ). If  $T_0^* < T^*$ , then for  $\varepsilon_0 > 0$ , there is an  $r_0 > 0$ , such that  $C_m r^2 \int_{B_{2r}(x_0)} |\nabla \tilde{U}|^2(t_0) dx < \varepsilon_0$ , for all  $0 < r \leq r_0$ , and  $(x_0, t_0) \in \bar{M} \times [0, T_0^*]$ . We let  $t_1 < T_0^*$  be such that  $T_0^* - t_1 \ll r_0^2$ . Then since  $U^k(\cdot, t_1) \rightarrow \tilde{U}(\cdot, t_1)$  in  $W^{2,p}(M)$  as  $k \rightarrow \infty$ , we may assume, for all large  $k$ 's, that

$$C_m r_0^{2-m} \oint_{B_{2r_0}(x_0)} |\nabla U^k|^2(t_1) dx < \varepsilon_0.$$

Then by small energy regularity theorem of [CS] and [CL], one has, as above,  $U^k \rightarrow \tilde{U}$  in  $W_p^{2,1}(t_1, t_1 + r_0)$ . This contradicts the definition of  $T_0^*$ .

Finally we would like to prove the above claim.

Let  $\psi_k = d(U^k, N)$ , then by a simple calculation (cf. [CL], (4.18)), one has, by (4.5)–(4.6), that

$$\begin{cases} \frac{\partial}{\partial t} \psi_k - \Delta \psi_k \leq -k \psi_k + |\nabla U^k|^2 & \text{in } M \times [0, T_0] \\ \psi_k = 0 & \text{on } (M \times \{0\}) \cup (\partial M \times [0, T_0]). \end{cases} \quad (4.9)$$

In deriving (4.9), we have used the fact that  $d(U^k, N) \rightarrow 0$  uniformly as  $k \rightarrow \infty$ . (cf. (4.6)). Again, by the maximum principle, one has

$$\max_{(x,t) \in M \times [0, T_0]} \psi_k \leq \frac{1}{k} \max_{(x,t) \in M \times [0, T_0]} |\nabla U^k|^2 \leq \frac{1}{k} C_0. \quad (4.10)$$

Hence from (4.4),  $(\partial/\partial t)U^k - \Delta U^k \in L^\infty(M \times [0, T_0])$ , and our claim follows from the standard  $L^p$ -theory for parabolic systems [LSU].

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Received March 3, 1994

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Preface – Contributors – Accessible Rotation Numbers of Chaotic States, Kathleen T. Alligood and Timothy Sauer – Prime End Rotation Numbers Associated with the Hénon Maps, Marcy Barge – On the Rotation Shadowing property for Annulus Maps, Fernanda Botelho and Liang Chen – A Nielsen-Type theorem for Area-Preserving Homeomorphisms of the Two Disc, Kenneth Boucher, Morton Brown, and Edward Slaminka – Irrational Rotations on Simply Connected Domains, Beverly L. Brechner – The Rational Dynamic of Cofrontiers, Beverly L. Brechner, Merle D. Guay and John C. Mayer – A Periodic Homeomorphism of the Plane, Morton Brown – Dynamical Connections between a Continuous Map and Its Inverse Limit Space, Liang Chen and Shihai Li – Horseshoelike Mappings and Chainability, James F. Davis – Iterated Function System, Compact Semigroups and Topological Contractions, P. F. Duvall, Jr., John Wesley Emert, and Laurence S. Husch – The Forced Damped Pendulum and the Wada Property, Judy Kennedy and James A. Yorke – Denjoy Meets Rotation and an Indecomposable Cofrontier; John C. Mayer and Lex G. Oversteegen – New Problems in Continuum Theory, Sam B. Nadler, Jr., and Gary A. Seldomridge – Dense Embeddings into Cubes and Manifolds, J. Nikiel, H. M. Tuncali, and E. E. Tymchatyn – An Example Concerning Disconnection Numbers, Robert Pierce – Indecomposable Continua, Prima Ends, and Julia Sets, James T. Rogers, Jr – Homeomorphisms of Cofrontiers with Unique Rotation Numbers, Mark H. Turpin – Self-Homeomorphism Star Figures, Włodzimierz J. Charatonik, Anne Dilks Dye, and James F. Reed – Index.

HENNING STICHTENOTH. **Algebraic Function Fields and Codes**, Springer Verlag 1993, 260 pp., DM. 48.-.

I. Foundations of the Theory of Algebraic Function Fields – II. Geometric Goppa Codes – III. Extensions of Algebraic Function Fields – IV. Differentials of Algebraic Function Fields – V. Algebraic Function Fields over Finite Constant Fields – VI. Examples of Algebraic Function Fields – VII. More about Geometric Goppa Codes – VIII. Subfield Subcodes and Trace Codes – Appendix A. Field Theory – Appendix B. Algebraic Curves and Algebraic Function Fields.

LENNART CARLESON, THEODORE C. GAMELIN. **Complex Dynamics**, Springer-Verlag 1993, 174 pp., DM. 55.-.

Preface – I. Conformal and Quasiconformal Mappings – II. Fixed Points and Conjugations – III. Basic Rational Iteration – IV. Classification of Periodic Components – V. Critical Points and Expanding Maps – VI. Application of Quasiconformal Mappings – Local Geometry of the Fatou Set – VII. Quadratic Polynomials – Epilogue – References – Index – Symbol Index.

WELINGTON DE MELO, SEBASTIAN VAN STRIEN. **One-Dimensional Dynamics**, Springer-Verlag 1993, 605 pp., DM. 148.-.

Introduction – I. Circle Diffeomorphism – 1. The Combinatorial Theory of Poincaré – 2. The Topological Theory of Denjoy – 2.a The Denjoy Inequality – 2.b Ergodicity – 3. Smooth Conjugacy Results – 4. Families of Circle Diffeomorphisms; Arnol'd Tongues – 5. Counter-Examples to Smooth Linearizability – 6. Frequency of Smooth Linearizability – 7. Some Historical Comments and Further Remarks – II. The Combinatorics of One-Dimensional Endomorphisms – 1. The Theorem of Sarkovskii – 2. Coverings Maps of the Circle as Dynamical Systems – 3. The Kneading Theory and Combinatorial Equivalence – 3.a Examples – 3.b Hofbauer's Tower Construction – 4. Full Families and Realization of Maps – 5. Families of Maps and Renormalization – 6. Piecewise Monotone Maps can be Modelled by Polynomial Maps – 7. The Topological Entropy – 8. The Piecewise Linear Model – 9. Continuity of the Topological Entropy – 10. Monotonicity of the Kneading Invariant for the Quadratic