Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 69 (1994)

Artikel: The geometric invariants of direct products of virtually free groups.
Autor: Meinert, Holger
DOI: https://doi.org/10.5169/seals-52247

Comment. Math. Helvetici 69 (1994) 39–48

0010-2571/94/010039-10$1.50 + 0.20/0

The geometric invariants of direct products of virtually free groups

HOLGER MEINERT

1. Introduction

1.1. *Summary.* The purpose of this paper is to compute the homological and the homotopical geometric invariants of [Bi-Re] and [Re 88] for direct products $G = G_1 \times G_2 \times \cdots \times G_l$ of finitely generated virtually free groups. As an application we determine the finiteness properties "type FP_m" and "type F_m" for all subgroups of G above the commutator subgroup G'.

1.2. Recall that a group (or a monoid) G is said to be of type FP_m, where $m \in \mathbf{N}_0$, if the trivial G-module $\mathbf{Z}$ admits a projective $\mathbf{Z}G$-resolution, which is finitely generated in all dimensions $\leq m$ [Bi 76/81]. Moreover, a group G is of type F_m if an Eilenberg–McLane complex $K(G, 1)$ for G with finite m-skeleton exists [Wa]. Type F_m always implies type FP_m, but it's not known whether the converse is true. More details can be found in [Bi 76/81], [Br], [Rat].

The homological invariants $\Sigma^m(G; \mathbf{Z})$ and the homotopical invariants $\Sigma^m(G)$ referred to above are conical subsets of the real vector space $V(G) := \mathrm{Hom}\,(G; \mathbf{R})$. They can be defined in terms of FP_m-properties of certain submonoids of G in the homological case and in terms of connectivity properties of pieces of universal coverings of certain $K(G, 1)$-complexes in the homotopical case. We will give the definitions in Section 2; for a survey the reader is referred to [Bi 93], [Bi-Str].

1.3. *The result.* Let $G = G_1 \times G_2 \times \cdots \times G_l$ be the direct product of l finitely generated virtually free groups. We denote by $\mathscr{L}$ the lattice of all subsets of

$$\mathscr{I} := \{j \in \{1, \ldots, l\} \mid G_j/G_j' \text{ infinite and } G_j \text{ virtually (free of rank } \geq 2)\}$$

and if $\sigma \in \mathscr{L}$ we write $|\sigma|$ for its cardinality. For $\sigma \in \mathscr{L}$ we consider the subgroup $H_\sigma \leq G$ generated by the union of all G_i, $i \in \sigma$. If ω is the complement of σ in $\mathscr{I}$, then G is the direct product $H_\sigma \times H_\omega \times H$, where H is the subgroup of G generated by all G_i with $i \notin \mathscr{I}$. Now, the canonical projection $\pi_\sigma : G \twoheadrightarrow H_\sigma$ induces an injective $\mathbf{R}$-linear map $\pi_\sigma^* : V(H_\sigma) \rightarrowtail V(G)$, and we can state our main result.

THEOREM. *Let $G = G_1 \times \cdots \times G_l$ be the direct product of l finitely generated virtually free groups. Then the homological and the homotopical geometric invariants of G coincide and their complements in $V(G)$ are given by the formula*

$$\Sigma^m(G;\mathbf{Z})^c = \Sigma^m(G)^c = \left(\bigcup_{\sigma \in \mathscr{L}, |\sigma| \le m} \pi_\sigma^* V(H_\sigma) \right) - \{0\}. \qquad (*)$$

Note that $\Sigma^m(G;\mathbf{Z})^c = \Sigma^m(G)^c$ are equal to $\pi_{\mathscr{I}}^* V(H_{\mathscr{I}}) - \{0\}$ if $m \ge |\mathscr{I}|$. Moreover, the theorem says, in other words, that *a non-zero homomorphism $\chi : G \to \mathbf{R}$ is in $\Sigma^m(G;\mathbf{Z})^c = \Sigma^m(G)^c$ if and only if its kernel contains $H_\omega \times H$ for some $\omega \in \mathscr{L}$ with $|\omega| \ge |\mathscr{I}| - m$.*

The three inclusions which are necessary to prove the theorem will be established in Paragraph 2.3, Proposition 3.7 and Proposition 4.3.

1.4. *Remarks.* 1) Sometimes it might be convenient to replace $\mathscr{I}$ by the set of all j such that G_j is virtually (free of rank ≥ 2). This yields the same result because groups with finite Abelianization do not admit any non-zero homomorphism into the reals.

2) The homological part of the theorem is essentially contained in the author's diploma thesis [Mei 90]. However, all proofs given here are new.

1.5. The problem of how to compute the invariants of a direct product in terms of the invariants of the factors is still open. It is conceivable that the answer is given by the

CONJECTURE. *If $G = G_1 \times G_2$ is of type F_m then*

$$\Sigma^m(G_1 \times G_2)^c = \bigcup_{p+q=m} (\pi_1^* \Sigma^p(G_1)^c + \pi_2^* \Sigma^q(G_2)^c),$$

where $\pi_i^* : V(G_i) \rightarrowtail V(G)$ is induced by the projection $\pi_i : G \twoheadrightarrow G_i$ and $+$ denotes the complex-sum in the real vector space $V(G)$.

The conjecture is true for $m = 1$ [Bi-Neu-Str] (also see [Bi-Str]) and $m = 2$ [Geh]; the inclusion $\subseteq$ holds for arbitrary m [Geh]. Gehrke's method also gives a formula for $\Sigma^m(G)^c$ if G is the direct product of l groups $G_1, G_2, \ldots, G_l$ of type F_m with the property that $\Sigma^1(G_i) = \Sigma^m(G_i)$ for all $1 \le i \le l$. For example, f.g. virtually free groups, 1-relator groups, polycyclic groups or fundamental groups of compact 3-manifolds are of that type for all m. In this case $\Sigma^m(G)^c$ is the union of all subsets $\pi_{i_1}^* \Sigma^1(G_{i_1})^c + \cdots + \pi_{i_k}^* \Sigma^1(G_{i_k})^c$ of $V(G)$ with $1 \le i_1 < \cdots < i_k \le l$ and $k \le m$. Our theorem follows from Gehrke's result, but his proof is much longer and needs totally different techniques.

1.6. *Normal subgroups with Abelian quotient.* Let N be a normal subgroup of $G = G_1 \times \cdots \times G_l$ with Abelian quotient G/N. We define the *depth* $\vartheta(N)$ *of* N by

$$\vartheta(N) := \min \{d \in \mathbf{N}_0 \mid NHH_\omega \text{ has finite index in } G \text{ for every } \omega \in \mathscr{L} \text{ with } |\omega| = d\}.$$

Note that $0 \le \vartheta(N) \le |\mathscr{I}|$, that $\vartheta(N) = 0$ if and only if G/NH is finite, that $\vartheta(N)$ is equal to $1 + \#\{j \in \mathscr{I} \mid |G_j : G_j \cap N| < \infty\}$ if G/N has torsion free rank 1 and G/NH is infinite and that $\vartheta(G') = |\mathscr{I}|$. We say that a group is of type F_∞ if it is of type F_m for all m and note that G has this property. Now, the finiteness properties of N can be read off from the depth $\vartheta(N)$.

COROLLARY. *Let N be a normal subgroup of the direct product $G = G_1 \times \cdots \times G_l$ of l finitely generated virtually free groups and assume that G/N is Abelian. If $\vartheta(N) = 0$ then N is of type F_∞, and if $\vartheta(N) > 0$ then N is of type F_m and not of type FP_{m+1}, where $m = |\mathscr{I}| - \vartheta(N)$.*

Proof. The linear subspace of $V(G)$ consisting of all homomorphisms $\chi : G \to \mathbf{R}$ which vanish on N will be denoted by $V(G; N)$. Then we use the following result of R. Bieri and B. Renz ([Bi-Re], [Re 88]; see also [Bi 93] or [Bi-Str]): N is of type FP_m (resp. F_m) if and only if $V(G; N) \subseteq \Sigma^m(G; \mathbf{Z})$ (resp. $V(G; N) \subseteq \Sigma^m(G)$).

Now, by formula $(*)$ a non-zero homomorphism $\chi \in V(G)$ is an element of $\Sigma^m := \Sigma^m(G; \mathbf{Z}) = \Sigma^m(G)$ if and only if its kernel does not contain any $H_\omega \times H$ with $|\omega| \ge |\mathscr{I}| - m$. Next, we observe that the existence of a non-zero homomorphism $\chi : G \to \mathbf{R}$ whose kernel contains N and $H_\omega \times H$ for some $\omega \in \mathscr{L}$ is equivalent with the assertion that the Abelian group G/NHH_ω be infinite. From this we infer that $V(G; N) \subseteq \Sigma^m$ if and only if NHH_ω has finite index in G for all $\omega \in \mathscr{L}$ with $|\omega| \ge |\mathscr{I}| - m$.

Now, $\vartheta(N) = 0$ implies $V(G; N) \subseteq \Sigma^m$ for all $m \in \mathbf{N}_0$, so N is of type F_∞ by the result quoted above. If we assume $\vartheta(N) > 0$, it follows that $V(G; N) \subseteq \Sigma^m$ if and only if $\vartheta(N) \le |\mathscr{I}| - m$. In other words, N is of type FP_m if and only if N is of type F_m if and only if $m \le |\mathscr{I}| - \vartheta(N)$. □

1.7. A concrete example is given as follows. Let $D_m := \langle x_1, y_1 \mid - \rangle \times \cdots \times \langle x_m, y_m \mid - \rangle$, define a D_m-action on F, the free group on generators $\{a_k \mid k \in \mathbf{Z}\}$, by $x_i \cdot a_k := a_{k+1} =: y_i \cdot a_k$ and put $A_m := F \rtimes D_m$. If G is the direct product of $m + 1$ free groups of rank 2 consider the homomorphism $\chi : G \twoheadrightarrow \mathbf{Z}$ which sends each basis element of each free factor of G onto 1. Then A_m is isomorphic to the kernel

N of χ and the depth of N is $\vartheta(N) = 1$. Hence A_m is of type F_m and not of type FP_{m+1} by our corollary.

The groups A_m were introduced in [Bi 76] to establish the existence of groups of type FP_m which are not of type FP_{m+1} for $m \in \mathbb{N}$, where the case $m = 2$ is due to J. R. Stallings [Sta].

1.8. Recently, S. M. Gersten proved that each of the groups A_m, $m \geq 2$, satisfies a fifth degree polynomial isoperimetric inequality [Ger]. On the other hand these groups are neither combable nor asynchronously automatic (see [ECHLPT]) since groups with one of these properties are of type F_∞ ([Al], [ECHLPT], [Ger]). No examples of groups with sub-exponential isoperimetric function which are not combable were known before.

Now, one can use the corollary above to characterize all combable normal subgroups N with Abelian quotient of a direct product G of finitely many free groups of finite rank ≥ 2. Using [Al], [ECHLPT], [Ger] and our result that N is of type F_∞ if and only if N has finite index in G, one can conclude: N is combable (automatic, asynchronously automatic, biautomatic) if and only if N has finite index in G.

1.9. There is a slight overlap with work of G. Baumslag and J. E. Roseblade [Bau-Ro]. One of their main theorems states that every finitely presented subgroup S of a direct product of two free groups is a finite extension of a direct product of two free groups (of finite rank). If S contains the derived subgroup G', then we recover their result from our corollary. In fact, if G is a direct product of l free groups of finite rank ≥ 2, then every normal subgroup N of type FP_l with $G' \leq N$ has finite index in G. In particular, N is a finite extension of a direct product of l free groups (of finite rank). Hence we have enough examples to ask:

QUESTION. *Let G be the direct product of l free groups of finite rank* ≥ 2. *Is every subgroup of type* FP_l *in G a finite extension of a direct product of l free groups* (*of finite rank*)?

1.10. *Acknowledgements.* It is a pleasure to thank Professor Robert Bieri for the introduction into the theory and for his encouragement in both finding the results and presenting the material. I am also indebted to Ralf Gehrke for many fruitful discussions on the subject. Lastly I would like to express my thanks to the "Arbeitsgruppe 8.2 des Fachbereichs Mathematik der Universität Frankfurt" for the hospitality and the stimulating atmosphere during the last two years.

2. The geometric invariants

2.1. *The homological invariants.* Let G be a group and $\chi : G \to \mathbf{R}$ a homomorphism. Then we consider the submonoid $G_\chi := \{g \in G \mid \chi(g) \geq 0\}$ of G and put for $m \in \mathbf{N}_0$

$$\Sigma^m(G; \mathbf{Z}) := \{\chi \in V(G) \mid G_\chi \textit{ is of type } \mathrm{FP}_m\} \subseteq V(G).$$

The complement of $\Sigma^m(G; \mathbf{Z})$ in $V(G)$ will be denoted by $\Sigma^m(G; \mathbf{Z})^c$. It follows from [Bi-Re] that $\Sigma^m(G; \mathbf{Z}) \neq \emptyset$ if and only if $0 \in \Sigma^m(G; \mathbf{Z})$ if and only if G is of type FP_m.

2.2. *The homotopical invariants.* Let G be a group of type F_m and X the universal cover complex of a $K(G, 1)$-complex with finite m-skeleton. If $\chi \in V(G)$, then G acts via χ on $\mathbf{R}$ and any continuous G-equivariant map $h = h_\chi : X \to \mathbf{R}$ shall be called a *height function* (*with respect to* χ). For a real number r we denote by $X_h^{[r,\infty)}$ the maximal subcomplex of X contained in $h^{-1}([r, \infty))$. $X_h^{[r,\infty)}$ is called *essentially k-connected in X* for some $k \geq -1$, if there is a $d \geq 0$ with the property that the map $\pi_i(X_h^{[r,\infty)}) \to \pi_i(X_h^{[r-d,\infty)})$ induced by inclusion is trivial for all $i \leq k$. Then we define

$$\Sigma^m(G) := \{\chi \in V(G) \mid X_h^{[0,\infty)} \textit{ is essentially } (m-1)\textit{-connected in } X\} \subseteq V(G)$$

and $\Sigma^m(G)^c := V(G) - \Sigma^m(G)$. This definition does not depend on the choice of X and h [Bi-Str], and we always have $0 \in \Sigma^m(G)$.

2.3. It is an open problem as to whether the two invariants coincide if both are defined. However, $\Sigma^0(G) = \Sigma^0(G; \mathbf{Z}) = V(G)$ for all groups, $\Sigma^1(G) = \Sigma^1(G; \mathbf{Z})$ for all finitely generated groups and by a result of Renz (see [Bi 93] or [Bi-Str]) $\Sigma^m(G) = \Sigma^2(G) \cap \Sigma^m(G; \mathbf{Z})$ holds for every group G of type F_m if $m \geq 2$. This proves the first inclusion, $\Sigma^m(G; \mathbf{Z})^c \subseteq \Sigma^m(G)^c$, of our theorem.

3. The homotopical part of the theorem

The aim of this section is to prove that $\Sigma^m(G)^c$ is contained in the right hand side of formula $(*)$. However, we start with two easy results on arbitrary groups. Recall that the subspace of $V(G)$ consisting of all homomorphisms which vanish on a subgroup $S \leq G$ is denoted by $V(G; S)$.

3.1. LEMMA. *Let $Z = Z(G)$ be the centre of a group G of type F_m. Then $\Sigma^m(G)$ contains the complement of the subspace $V(G; Z)$.*

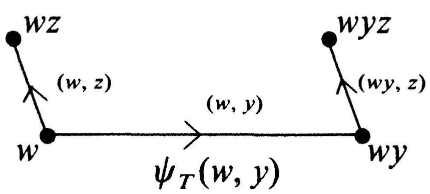
Proof. Exactly as in the homological case ([Bi-Re], Lemma 5.2) using the homotopical version of the Σ^m-criterion ([Bi 93], Theorem A; [Bi-Str]). □

3.2. LEMMA. *Let G be a group of type F_m and let $S \leq G$ be a subgroup of finite index. If $\chi : G \to \mathbf{R}$ is a homomorphism, then $\chi \in \Sigma^m(G)$ if and only if $\chi|_S \in \Sigma^m(S)$.*

Proof. Let X be the universal cover of a $K(G, 1)$-complex with finite m-skeleton and let $h : X \to \mathbf{R}$ be a height function with respect to $\chi : G \to \mathbf{R}$. Then X is the universal cover of a $K(S, 1)$ with finite m-skeleton and h is also a height function with respect to $\chi|_S : S \to \mathbf{R}$. Now the claim is obvious by the definition of $\Sigma^m(-)$. □

3.3. *A construction.* We now turn to free groups F of finite rank. Let $\mathscr{Y} \subseteq F$ be a finite set of free generators and consider the Cayley graph $T := \Gamma(F; \mathscr{Y})$ of F with respect to $\mathscr{Y}$. This is a combinatorial F-tree with set of vertices V the elements of F, with set of oriented edges E the pairs $e = (w, y) \in F \times \mathscr{Y}$, the origin of e given by w and the terminus given by wy (cf. [Serre]). By the inverse edge e^- we mean e with the opposite orientation and by $P(T)$ we denote the set of all edge paths of T.

Now, let $\chi : F \to \mathbf{R}$ be a non-zero homomorphism. Without loss of generality we may assume that there is an element $z \in \mathscr{Y}$ with $\chi(z) > 0$. Then we define F-maps $\psi_T : V \to V$ and $\psi_T : E \to P(T)$ by putting $\psi_T(w) := wz$ for $w \in V$, $\psi_T(w, z) := (wz, z)$ and $\psi_T(w, y) := (w, z)^-(w, y)(wy, z)$ for $(w, y) \in E$ with $y \neq z$. Moreover, we define a combinatorial height function $h_T : V \to \mathbf{R}$ by $h_T(w) := \chi(w)$ for $w \in V$.



The geometric realisation X of T is a contractible 1-dimensional CW-complex, on which F acts freely by permuting the cells, i.e. X is the universal cover of a finite 1-dimensional $K(F, 1)$. By linear extension of h_T we equip X with a height function $h : X \to \mathbf{R}$ with respect to χ. Now, by a suitable realisation of ψ_T we obtain for every $\varepsilon > 0$ a continuous cellular F-equivariant map $\psi : X \to X$ with $h(\psi(x)) \geq h(x) - \varepsilon$ for all $x \in X$ and $h(\psi(x^0)) = h(x^0) + \chi(z)$ for all 0-cells $x^0 \in X^0$. □

3.4. Let $G = F_1 \times \cdots \times F_l$ be the direct product of l free groups of finite rank. Then $\mathscr{J} = \{j \mid \operatorname{rk} F_j \geq 2\}$ and the subgroup H generated by all F_i with $i \notin \mathscr{J}$ is equal to the centre $Z = Z(G)$ of G. Let $\chi : G \to \mathbf{R}$ be a non-zero homomorphism and recall that $\mathcal{L}$ is the lattice of all subsets of $\mathcal{I}$. Then the crucial step is the following:

3.5. PROPOSITION. *Suppose there is an element $\sigma \in \mathcal{L}$ with the properties that $|\sigma| > m$ and that $\chi(F_i) \neq \{0\}$ for all $i \in \sigma$. Then $\chi \in \Sigma^m(G)$.*

Proof. Put $\chi_i := \chi|_{F_i}$ for $i = 1, \ldots, l$ and choose the universal covering X_i of a finite 1-dimensional $K(F_i, 1)$-complex together with the height function $h_i : X_i \to \mathbf{R}$ as in 3.3. Then $X := X_1 \times \cdots \times X_l$ is the universal cover of a finite l-dimensional $K(G, 1)$-complex and $h : X \to \mathbf{R}$ defined by $h := h_1 p_1 + \cdots + h_l p_l$ is a height function with respect to χ if p_i is the projection $X \twoheadrightarrow X_i$. Now, by 3.3 again there is a $\delta > 0$ and there are continuous cellular F_i-equivariant maps $\psi_i : X_i \to X_i$ for all $i \in \sigma$ with the property that $h_i(\psi_i(x_i)) \geq h_i(x_i) - \delta/l$ for all $x_i \in X_i$ and $h_i(\psi_i(x_i^0)) \geq h_i(x_i^0) + \delta$ for all 0-cells $x_i^0 \in X_i^0$ (recall that the definition of ψ_i depends on a *non-zero* homomorphism χ_i whereas the definition of X_i and h_i does not).

Next, we put $\varphi : X \to X$ to be the product map $\varphi := \prod_{i=1}^{l} \varphi_i$, where $\varphi_i := \psi_i$ if $i \in \sigma$ and $\varphi_i := \mathrm{Id}_{X_i}$ otherwise. Then φ is a continuous cellular G-equivariant map with $h(\varphi(x)) \geq h(x) + \delta/l$ for all $x \in X^m$. To see this let $x = (x_1, \ldots, x_l) \in X^m$ and note that the number of x_k with $x_k \notin X_k^0$ is at most $m < |\sigma| \leq l$. Hence there is at least one $i \in \sigma$ such that $x_i \in X_i^0$. Consequently $h(\varphi(x)) \geq h(x) + \delta - m \cdot \delta/l \geq h(x) + \delta/l$.

Using the homotopical version of the Σ^m-criterion ([Bi 93], Theorem A; [Bi-Str]) we see that $\chi \in \Sigma^m(G)$. □

3.6. *Remarks.* 1) Note that the height functions h_i and h used above are valuations in the sense of [Re 87] (Remark on p. 468) and [Re 88].

2) One can prove that the following assertion is valid for arbitrary groups G_1 and G_2 of type F_m, where $m = m_1 + m_2 + 1$ with $m_i \in \mathbf{N}_0$. If $\chi_i \in \Sigma^{m_i}(G_i) - \{0\}$, then $\chi_1 \times \chi_2 \in \Sigma^m(G_1 \times G_2)$ (see [Geh]). A similar result holds for the homological invariants.

Now we are ready to prove the homotopical part of our theorem.

3.7. PROPOSITION. *Let $G = G_1 \times \cdots \times G_l$ be the direct product of l finitely generated virtually free groups. Then*

$$V(G) - \left(\bigcup_{\sigma \in \mathcal{L}, |\sigma| \leq m} \pi_\sigma^* V(H_\sigma) \right) \subseteq \Sigma^m(G).$$

Proof. Let $\chi : G \to \mathbf{R}$ be a homomorphism in the left hand side. Then either (i) χ does not vanish on the subgroup $H \leq G$ generated by all G_i with $i \notin \mathcal{I}$, where $\mathcal{I}$

is the set of all j with G_j/G_j' infinite and G_j virtually (free of rank ≥ 2), or (ii) there exists a $\sigma \in \mathscr{L}$, the lattice of all subsets of $\mathscr{I}$, with $|\sigma| > m$ and $\chi(G_i) \neq \{0\}$ for all $i \in \sigma$.

Next, we consider a subgroup $S = F_1 \times \cdots \times F_l$ of finite index in G with $F_i \leq G_i$ free of finite rank. By Lemma 3.2 we have $\chi \in \Sigma^m(G)$ if and only if $\chi|_S \in \Sigma^m(S)$. Now, in case (i) χ does not vanish on the subgroup of G generated by all virtually (infinite cyclic) factors G_i. Hence $\chi|_S$ is non-trivial on the centre $Z(S)$ of S so the result follows from Lemma 3.1, and case (ii) is obviously covered by Proposition 3.5. □

4. The homological part of the theorem

In this section we prove the remaining inclusion of formula (*). As in Section 3 we begin with a result on the Σ's of arbitrary groups.

4.1. PROPOSITION. *Suppose that* $N \rightarrowtail G \overset{\pi}{\twoheadrightarrow} Q$ *is a short exact sequence of groups of type* FP_m *and let* $\psi : Q \to \mathbf{R}$ *be a homomorphism. Then* $\psi \in \Sigma^m(Q; \mathbf{Z})$ *if and only if* $\psi \circ \pi \in \Sigma^m(G; \mathbf{Z})$.

Proof. We may assume that $m \geq 1$ and we put $\chi := \psi \circ \pi$, so that N is contained in the kernel of χ. The obvious ring homomorphism $\pi_* : \mathbf{Z}G_\chi \twoheadrightarrow \mathbf{Z}Q_\psi$ induces spectral sequences

$$\mathrm{Tor}_p^{\mathbf{Z}Q_\psi}(\mathrm{Tor}_q^{\mathbf{Z}G_\chi}(\prod \mathbf{Z}G_\chi; \mathbf{Z}Q_\psi); \mathbf{Z}) \underset{p}{\Rightarrow} \mathrm{Tor}_{p+q}^{\mathbf{Z}G_\chi}(\prod \mathbf{Z}G_\chi; \mathbf{Z})$$

for arbitrary direct products $\Pi\, \mathbf{Z}G_\chi$ of copies of $\mathbf{Z}G_\chi$ ([Rot], Theorem 11.62).

Since $\mathbf{Z}G_\chi$ is a free $\mathbf{Z}N$-module and $\mathbf{Z}G_\chi \otimes_{\mathbf{Z}N} \mathbf{Z} \cong \mathbf{Z}Q_\psi$ as G_χ-modules with the obvious actions, a change-of-ring isomorphism ([Rot], Theorem 11.64) yields $\mathrm{Tor}_q^{\mathbf{Z}G_\chi}(\Pi\, \mathbf{Z}G_\chi; \mathbf{Z}Q_\psi) \cong \mathrm{Tor}_q^{\mathbf{Z}N}(\Pi\, \mathbf{Z}G_\chi; \mathbf{Z})$. Now, N is of type FP_m, hence $\mathrm{Tor}_q^{\mathbf{Z}N}(-; \mathbf{Z})$ commutes with direct products for $q < m$ ([Bi 76/81], Theorem 1.3), and we obtain $\mathrm{Tor}_q^{\mathbf{Z}N}(\Pi\, \mathbf{Z}G_\chi; \mathbf{Z}) = 0$ if $1 \leq q < m$ and $\cong \Pi\, (\mathbf{Z}Q_\psi)$ if $q = 0$.

We find that the above spectral sequence has enough collapsing to yield isomorphisms $\mathrm{Tor}_n^{\mathbf{Z}Q_\psi}(\Pi\, \mathbf{Z}Q_\psi; \mathbf{Z}) \cong \mathrm{Tor}_n^{\mathbf{Z}G_\chi}(\Pi\, \mathbf{Z}G_\chi; \mathbf{Z})$ for $n < m$ and arbitrary direct products Π. Another appeal to Theorem 1.3 of [Bi 76/81] now gives the result by the definition of $\Sigma^m(-; \mathbf{Z})$. □

4.2. *Remarks.* 1) A similar result holds for the homotopical geometric invariants [Mei 93].

2) If N satisfies the weaker condition that the Abelian groups $H_i(N; \mathbf{Z})$ are finitely generated for $1 \le i \le m-1$, and G is of type FP_m, then $\psi \circ \pi \in \Sigma^m(G; \mathbf{Z})$ implies $\psi \in \Sigma^m(Q; \mathbf{Z})$.

Now everything is present to complete the proof of our theorem.

4.3. PROPOSITION. *Let $G = G_1 \times \cdots \times G_l$ be the direct product of l finitely generated virtually free groups. Then*

$$\left(\bigcup_{\sigma \in \mathscr{L}, |\sigma| \le m} \pi_\sigma^* V(H_\sigma) \right) - \{0\} \subseteq \Sigma^m(G; \mathbf{Z})^c.$$

Proof. Let $m > 0$ and let $\chi : G \to \mathbf{R}$ be a non-zero homomorphism with $\chi \in \pi_\sigma^* V(H_\sigma)$ for some $\sigma \in \mathscr{L}$ with $|\sigma| \le m$. Then there is a non-zero $\chi_\sigma \in V(H_\sigma)$ such that $\chi = \chi_\sigma \circ \pi_\sigma$.

Let ω be the complement of σ in $\mathscr{I}$. Then $G \cong H_\sigma \times H_\omega \times H$ and Proposition 4.1 asserts that $\chi \in \Sigma^m(G; \mathbf{Z})^c$ if and only if $\chi_\sigma \in \Sigma^m(H_\sigma; \mathbf{Z})^c$ since $H_\omega \times H$ is of type F_∞. Now, H_σ has a subgroup $S = F_1 \times \cdots \times F_{|\sigma|}$ of finite index which is a direct product of $|\sigma|$ free groups of finite rank ≥ 2. By the analogue of Lemma 3.1, the homological finite index result [Bi-Str], we find that $\chi_\sigma \in \Sigma^m(H_\sigma; \mathbf{Z})^c$ if and only if $\chi_\sigma|_S \in \Sigma^m(S; \mathbf{Z})^c$. In view of the inequality $|\sigma| \le m$ the result follows once we have established the next lemma. □

4.4. LEMMA. *Let $S = F_1 \times \cdots \times F_s$ be the direct product of s free groups of finite rank ≥ 2. Then $\Sigma^s(S; \mathbf{Z}) = V(S) - \{0\}$.*

Proof. For each $i = 1, \dots, s$ there is a free F_i-resolution $\mathbf{E}_i \twoheadrightarrow \mathbf{Z}$ of the form $0 \to (\mathbf{Z}F_i)^{r_i} \to \mathbf{Z}F_i \to \mathbf{Z} \to 0$, where $r_i \ge 2$ is the rank of F_i. Putting $\mathbf{E} := \mathbf{E}_1 \otimes_{\mathbf{Z}} \cdots \otimes_{\mathbf{Z}} \mathbf{E}_s$ yields a free S-resolution $\mathbf{E} \twoheadrightarrow \mathbf{Z}$ with $E_n \cong (\mathbf{Z}S)^{k_n}$ and $k_n = 0$ if $n > s$. Moreover, $\mathbf{E}$ has the additional property that $k_{s+1} - k_s + k_{s-1} - \cdots \pm k_0 = -(r_1 - 1)(r_2 - 1) \cdots (r_s - 1) < 0$ as is easily seen by induction on $s \in \mathbf{N}$. Now, a result on the partial Euler characteristics [Bi-Str] asserts that $\Sigma^s(S; \mathbf{Z}) - \{0\} = \emptyset$. □

REFERENCES

[Al] ALONSO, J. M.: *Combings of groups.* In: *Algorithms and Classification in Combinatorial Group Theory*, G. Baumslag and C. F. Miller III (eds.), MSRI Publ. *23*, Springer-Verlag, New York (1992).

[Bau-Ro] BAUMSLAG, G. and ROSEBLADE, J. E.: *Subgroups of direct products of free groups.* J. London Math. Soc. (2) *30*, 44–52 (1984).

[Bi 76] BIERI, R.: *Normal subgroups in duality groups and in groups of cohomological dimension two.* J. Pure and Appl. Alg. *7*, 35–52 (1976).

[Bi 76/81] BIERI, R.: *Homological dimension of discrete groups.* Queen Mary College Mathematics Notes, London (1976, 2nd ed. 1981).

[Bi 93] BIERI, R.: *The geometric invariants of a group – a survey with emphasis on the homotopical approach.* In: *Geometric Group Theory, Vol.* 1 (Edited by G. A. Niblo & M. A. Roller). London Math. Soc. Lecture Note Series *181*, 24–36 (1993).

[Bi-Neu-Str] BIERI, R., NEUMANN, W. D. and STREBEL, R.: *A geometric invariant for discrete groups.* Invent. Math. *90*, 451–477 (1987).

[Bi-Re] BIERI, R. and RENZ, B.: *Valuations on free resolutions and higher geometric invariants of groups.* Comment. Math. Helvetici *63*, 464–497 (1988).

[Bi-Str] BIERI, R. and STREBEL, R.: *Geometric invariants for discrete groups.* Preprint of a book (to be published by W. de Gruyter), Universität Frankfurt a.M. (1992).

[Br] BROWN, K. S.: *Cohomology of groups.* Grad. Texts in Math. *87*, Springer-Verlag, New York (1982).

[ECHLPT] EPSTEIN, D. B. A., CANNON, J. W., HOLT, D. F., LEVY, S. V. F., PATTERSON, M. S. and THURSTON, W. P.: *Word processing in groups.* Jones and Bartlett, Boston (1992).

[Geh] GEHRKE, R.: *Die höheren geometrischen Invarianten für Gruppen mit Kommutatorrelationen.* Dissertation, Universität Frankfurt a.M. (1992).

[Ger] GERSTEN, S. M.: *Finiteness properties of asynchronously automatic groups.* Preprint, University of Utah (1992).

[Mei 90] MEINERT, H.: *Über die höheren geometrischen Invarianten für endliche direkte Produkte von Gruppen.* Diplomarbeit, Universität Frankfurt a.M. (1990).

[Mei 93] MEINERT, H.: *Die höheren geometrischen Invarianten Σ^m von Gruppen via Operationen auf CW-Komplexen und der Beweis der Σ^m-Vermutung für metabelsche Gruppen endlichen Prüfer-Ranges.* Preprint, Universität Frankfurt a.M. (1993).

[Rat] RATCLIFFE, J. G.: *Finiteness conditions for groups.* J. Pure and Appl. Alg. *27*, 173–185 (1983).

[Re 87] RENZ, B.: *Geometric invariants and HNN-extensions.* In: *Group Theory* (Proceedings of the Singapore Group Theory Conference, June 8–19, 1987; ed. by Kai Nah Cheng and Yu Kiang Leong), 465–484, W. de Gruyter, Berlin–New York (1989).

[Re 88] RENZ, B.: *Geometrische Invarianten und Endlichkeitseigenschaften von Gruppen.* Dissertation, Universität Frankfurt a.M. (1988).

[Rot] ROTMAN, J. J.: *An introduction to homological algebra.* Academic Press, New York–San Francisco–London (1979).

[Serre] SERRE, J.-P.: *Trees.* Springer-Verlag, Berlin–Heidelberg–New York (1980).

[Sta] STALLINGS, J. R.: *A finitely presented group whose 3-dimensional integral homology is not finitely generated.* Amer. J. Math. *85*, 541–543 (1963).

[Wa] WALL, C. T. C.: *Finiteness conditions for CW-complexes.* Ann. Math *81*, 56–69 (1965).

J. W. Goethe-Universität
Fachbereich Mathematik
Robert-Mayer-Str. 6-10
D-60054 Frankfurt a.M.

Received April 22, 1992.