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On the classification of constant mean curvature tori in $\mathbb{R}^3$

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1. Introduction

Let S be a compact oriented surface and $i : S \to \mathbb{R}^3$ an immersion with constant mean curvature. Hopf [6] investigated such immersions, and for genus $(S) = 0$ he showed that $i : S \to \mathbb{R}^3$ must be an embedding of a round sphere. Conversely, the genus of the surface S is 0, if i is an embedding. This statement was proved by Alexandrov [1]. Only a few years ago Wente [10] and Kapouleas [7] proved the existence of constant mean curvature immersions for genus $(S) = 1$ and genus $(S) \geq 2$, respectively. In this work we will only look at constant mean curvature immersions with genus $(S) = 1$.

First the relation of hyperelliptic curves and constant mean curvature immersions is sketched. For a rigorous formulation see Bobenko [3].

Let u be a solution of the elliptic-sinh Gordon equation

$$u_{w\bar{w}} + \sinh u = 0 \tag{1}$$

on a simply-connected domain $\Omega \subset \mathbb{C}$. There is an algorithm that associates an immersion $i : \Omega \to \mathbb{R}^3$ to u with constant mean curvature $\frac{1}{2}$. Conversely, every constant mean curvature immersion yields a solution u of equation (1).

On the other hand we can associate quasi-periodic solutions of equation (1) on $\mathbb{R}^2$ to hyperelliptic curves

$$X : y^2 = x \prod_{i=1}^{2g} (x - e_i) \tag{2}$$

where the branch points are distinct and satisfy

$$e_{i+g} = \frac{1}{\bar{e}_i}, \qquad i = 1, \ldots, g. \tag{3}$$

We first have to fix some notation to write down an explicit formula for solutions

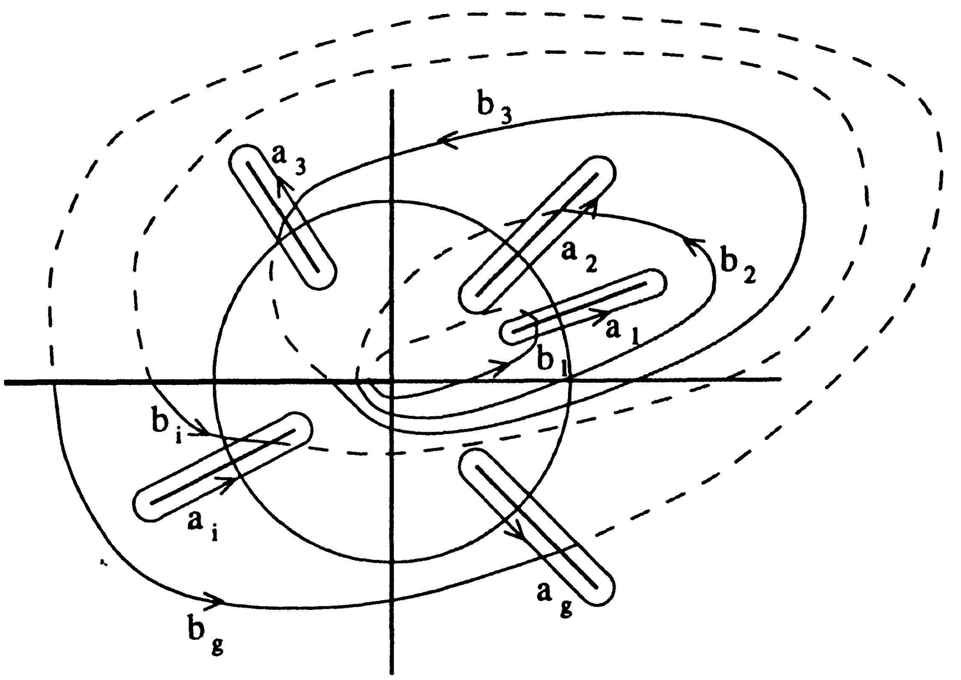

Figure 1

of equation (1). In figure (1) a canonical basis $a_1, \ldots, a_g, b_1, \ldots, b_g$ of $H_1(X, \mathbb{Z})$ with intersection numbers

$$a_i b_j = \delta_{ij}, \qquad a_i a_j = 0, \qquad b_i b_j = 0, \qquad i, j = 1, \ldots, g \tag{4}$$

is introduced. Let Ω_0 and Ω_∞ be meromorphic differentials on X, holomorphic outside 0 and ∞, respectively, which satisfy the conditions

$$\int_{a_i} \Omega_0 = \int_{a_i} \Omega_\infty = 0, \qquad i = 1, \ldots, g \tag{5}$$

and

$$\begin{aligned} &\Omega_0 \text{ has a pole of second order at } 0,\\ &\Omega_\infty \text{ has a pole of second order at } \infty. \end{aligned} \tag{6}$$

Define the vectors μ_0, μ_∞ by

$$\mu_0 = \left(\int_{b_1} \Omega_0, \ldots, \int_{b_g} \Omega_0\right)$$

$$\mu_\infty = \left(\int_{b_1} \Omega_\infty, \ldots, \int_{b_g} \Omega_\infty\right),$$

and for $\zeta \in \mathbb{C}^g$ put

$$u(\zeta) = 2 \log \frac{\theta\left(\zeta + \left(\frac{1}{2}, \ldots, \frac{1}{2}\right)\right)}{\theta(\zeta)}$$

where θ is the Riemann theta function of X for the given homology basis. The function

$$u(\zeta + w\mu_0 + \bar{w}\mu_\infty) \tag{7}$$

is a real quasi-periodic solution of equation (1) for every $\zeta \in \mathbb{R}^g$.

The question arises, whether it is possible to choose X in a way, such that X yields constant mean curvature tori. The answer to this question was given by Bobenko [4] and Pinkall-Sterling [9].

THEOREM 1.1.

(1) *Under the correspondence mentioned above X yields constant mean curvature tori in $\mathbb{R}^3$ if and only if*
  (a) *Ω_∞ has a root $p = (x_0, y_0)$ with $|x_0| = 1$;*
  (b) *Let γ be a path that connects the two points (x_0, y_0) and $(x_0, -y_0)$. Then the span of the vectors*

$$v_0 = \left(\int_\gamma \Omega_0, \int_{b_1} \Omega_0, \ldots, \int_{b_g} \Omega_0\right)$$
$$v_\infty = \left(\int_\gamma \Omega_\infty, \int_{b_1} \Omega_\infty, \ldots, \int_{b_g} \Omega_\infty\right)$$

  in $\mathbb{C}^{g+1}$ must contain two linearly independent rational vectors. In this case one gets a $(g-2)$-parameter family of constant mean curvature tori.

(2) *Every constant mean curvature torus arises in such a way.*

It is known that there are no curves satisfying the condition (a) for genus $(X) = 1$. Wente found constant mean curvature tori which are known to correspond to curves with genus $(X) = 2$ or genus $(X) = 3$. In 1991 Ercolani–Knörrer–Trubowitz [5] proved the existence of such curves for even genus (X). All curves constructed there have the additional property, that the set of branch points is invariant under the map $x \mapsto 1/x$. In this paper the existence of curves X fulfilling the conditions (a) and (b) is proved for genus (X) arbitrary.

2. Preliminaries

The map $\sigma : X \to X$

$$(x, y) \mapsto \left(\frac{1}{\bar{x}}, \frac{\left(\prod_{i=1}^{2g} e_i\right)^{1/2} \bar{y}}{\bar{x}^{g+1}} \right)$$

is an antiholomorphic involution of X. The sign of $(\Pi_{i=1}^{2g} e_i)^{1/2}$ is chosen in such a way, that the points lying over S^1 are fixed points of σ. Then σ_* acts as follows on the cycles:

$$\begin{aligned} \sigma_*(a_i) &= -a_i, \qquad i = 1, \dots, g \\ \sigma_*(b_i) &= b_i + \sum_{j=1}^{g} \lambda_{ij} a_j, \qquad i = 1, \dots, g \end{aligned} \tag{8}$$

with $\lambda_{ij} \in \mathbb{Z}$; $i, j = 1, \dots, g$, and

$$\gamma - \sigma_* \gamma = \sum_{j=1}^{g} \mu_j a_j, \tag{9}$$

with $\mu_j \in \mathbb{Z}$; $j = 1, \dots, g$.
It is possible to choose Ω_0, Ω_∞ in a way, such that

$$\sigma^* \Omega_0 = \bar{\Omega}_\infty$$

holds. It follows that the vectors v_0, v_∞ are complex conjugate. The new vectors

$$\begin{aligned} v &:= v_0 + v_\infty \\ w &:= i(v_\infty - v_0) \end{aligned}$$

are elements of $\mathbb{R}^{g+1}$.
Now consider the map $f : \mathbb{C}^g \to \mathbb{C} \times Gr(2, \mathbb{R}^{g+1})$

$$(e_1, \dots, e_g) \mapsto (\text{root of } \Omega_\infty, \langle v, w \rangle).$$

f is a multivalued function and one should restrict the domain of definition of f to the open subset $U \subset \mathbb{C}^g$, where all the branch points are distinct. $Gr(2, \mathbb{R}^{g+1})$

denotes the Grassmannian of 2-dimensional subspaces of $\mathbb{R}^{g+1}$. The vectors v and w are linearly independent and $\langle v, w\rangle$ is a welldefined element of $Gr(2, \mathbb{R}^{g+1})$.

It is interesting to look at this map, because if one finds a root $p = (x_0, y_0)$ with $|x_0| = 1$ and if $\langle v, w\rangle$ contains two linearly independent rational vectors, the existence of constant mean curvature tori is guaranteed. In section 3 the following theorem will be proved.

THEOREM 2.1. *Let $e = (e_1, \ldots, e_g)$ be in U. Assume that the differentials Ω_0, Ω_∞ on the hyperelliptic curve*

$$X : y^2 = x \prod_{i=1}^{2g} (x - e_i)$$

fulfill the following conditions:

(1) *Ω_0, Ω_∞ have a common root α over $x = 1$,*

(2) *Ω_0, Ω_∞ don't have any other common roots,*

(3) *$(\Omega_\infty - \Omega_0)(e_m) \neq 0$ for $m = 1, \ldots, 2g$, and $\Omega_\infty - \Omega_0$ has a root of order* 1 *at α.*

Then $df(e)$ is invertible.

We denote X_e as the hyperelliptic curve associated to the point $e \in U$. Due to this theorem it follows, that arbitrarily close to e there are points, such that the corresponding curves X_e fulfill conditions (a) and (b). In section 4 we will finally show

THEOREM 2.2. *For every $g \geq 2$ there are curves $X_e, e \in U$, satisfying the conditions* (1), (2), (3) *above.*

This theorem will be proved by induction on g.

3. Simplification

Proof of Theorem 2.1. Since dimensions are equal it is enough to show that $df(e)$ is injective. The strategy is due to Krichever [8], Bikbaev and Kuksin [2].

Let $e(\tau)$, $\tau \in \mathbb{R}$, be an arbitrary differentiable curve passing through e, such that $f(e(\tau))$ changes only in order τ^2, in other words

$$\begin{pmatrix} v(\tau) \\ w(\tau) \end{pmatrix} = A(\tau) \begin{pmatrix} v(0) \\ w(0) \end{pmatrix} + \mathcal{O}(\tau^2), \qquad \text{with } A(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{10}$$

$$\alpha(\tau) = 1 + \mathcal{O}(\tau^2). \tag{11}$$

We want to conclude that

$$\frac{d}{d\tau} e(\tau)\bigg|_{\tau=0} = 0$$

holds. This implies that $df(e)$ is injective. Put $B(\tau) := A(\tau)^{-1}$, clearly

$$B(\tau)\begin{pmatrix} v(\tau) \\ w(\tau) \end{pmatrix} = \begin{pmatrix} v(0) \\ w(0) \end{pmatrix} + \mathcal{O}(\tau^2)$$

and after differentiation

$$\dot{B}(0)\begin{pmatrix} v(0) \\ w(0) \end{pmatrix} + \begin{pmatrix} \dot{v}(0) \\ \dot{w}(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{12}$$

These are $2g+2$ equations, $2g$ of them describe relations among period integrals.

Define differentials ω_1, ω_2 by

$$\begin{pmatrix} \omega_1(\tau) \\ \omega_2(\tau) \end{pmatrix} := B(\tau)\begin{pmatrix} \Omega_0(\tau) + \Omega_\infty(\tau) \\ i(\Omega_\infty(\tau) - \Omega_0(\tau)) \end{pmatrix}. \tag{13}$$

By integration of $\omega_1(\tau), \omega_2(\tau)$ one get's multivalued meromorphic functions on $X_{e(\tau)}$:

$$\Omega_i(P, \tau) := \int_{J(P)}^{P} \omega_i(\tau), \qquad i = 1, 2 \tag{14}$$

where J denotes the hyperelliptic involution.

LEMMA 3.1. *The functions*

$$\frac{\partial}{\partial\tau} \Omega_i(P, \tau)\bigg|_{\tau=0}$$

are single-valued meromorphic functions on X_e. At the points $e_1, \ldots, e_{2g}, 0, \infty$ they have first order poles. Furthermore there are non-zero complex numbers $c_1, \ldots, c_{2g}$ such that

$$res_{P=e_m}\left(\frac{\partial}{\partial\tau} \Omega_2(P, \tau)\bigg|_{\tau=0}\right) = c_m \frac{\partial}{\partial\tau} e_m\bigg|_{\tau=0}, \qquad m = 1, \ldots, 2g. \tag{15}$$

Due to this lemma it is enough to show that

$$\frac{\partial}{\partial \tau} \Omega_2(P, \tau) \bigg|_{\tau=0} \equiv 0.$$

This will prove the theorem. We first prove this lemma, before we continue the proof of the theorem.

Proof. To see that the functions $(\partial/\partial\tau)\Omega_i(P, \tau)\,|_{\tau=0}$ are single-valued, we have to look at the corresponding b-periods:

$$\begin{aligned}\frac{\partial}{\partial \tau} \int_{b_j} \omega_1 \bigg|_{\tau=0} &= \frac{\partial}{\partial \tau} \int_{b_j} (b_{11}(\Omega_0 + \Omega_\infty) + b_{12} i(\Omega_\infty - \Omega_0)) \bigg|_{\tau=0} \\ &= \int_{b_j} (\dot{b}_{11}(0)(\Omega_0 + \Omega_\infty) + \dot{b}_{12}(0) i(\Omega_\infty - \Omega_0) + \dot{\Omega}_0 + \dot{\Omega}_\infty) \\ &= 0.\end{aligned}$$

The last identity is true due to equation (12). The same is true for ω_2 and the first statement is proved.

Expand $\omega_i(\tau)$ at $e_m(\tau)$ in the local coordinate $(x - e_m(\tau))^{1/2}$:

$$\omega_i(x, \tau) = \sum_{k=-1}^{\infty} (x - e_m(\tau))^{k/2} x_k^{i,m}(e(\tau))\, dx.$$

Put $P = (x, y)$, then we get

$$\begin{aligned}\frac{\partial}{\partial \tau} \Omega_i(P, \tau) \bigg|_{\tau=0} &= \int_{J(P)}^{P} \frac{\partial}{\partial \tau} \omega_i(x, \tau) \bigg|_{\tau=0} \\ &= 2 \sum_{k=-1}^{\infty} \bigg(-(x - e_m(0))^{k/2} \frac{\partial}{\partial \tau} e_m(0) x_k^{i,m}(e(0)) \\ &\quad + \frac{2}{2+k} (x - e_m)^{1+k/2} \frac{\partial}{\partial \tau} x_k^{i,m}(e(\tau)) \bigg|_{\tau=0} \bigg).\end{aligned}$$

It follows that the functions $(\partial/\partial\tau)\Omega_i(P, \tau)\,|_{\tau=0}$ have first order poles at the points $e_1, \ldots, e_{2g}$ and the same is true for 0 and ∞ by a similar calculation. Due to the assumption (3) in Theorem 2.1 the claim about the numbers c_m is obvious. □

Let's continue the proof of the theorem. Take $P \in X_e$ with $\omega_2(P) \neq 0$. The implicit function theorem yields a curve $P(\tau)$ with

$$\Omega_2(P(\tau), \tau) = \Omega_2(P, 0) \tag{16}$$

and after differentiation

$$\frac{d}{d\tau}\Omega_2(P(\tau), \tau)\Big|_{\tau=0} = \omega_2(P)\frac{d}{d\tau}P(\tau)\Big|_{\tau=0} + \frac{\partial}{\partial\tau}\Omega_2(P, \tau)\Big|_{\tau=0} = 0. \tag{17}$$

Define a new function

$$\dot{\Omega}_1(P) := \frac{d}{d\tau}\Omega_1(P(\tau), \tau)\Big|_{\tau=0}. \tag{18}$$

The function $\dot{\Omega}_1$ is welldefined and by the equation (17) above one gets

$$\dot{\Omega}_1(P) = \frac{\partial}{\partial\tau}\Omega_1(P, \tau)\Big|_{\tau=0} - \frac{\partial}{\partial\tau}\Omega_2(P, \tau)\Big|_{\tau=0} \cdot \frac{\omega_1(P)}{\omega_2(P)}. \tag{19}$$

It follows that $\dot{\Omega}_1$ is a meromorphic function on X. To finish the proof of the theorem we need the following lemma:

LEMMA 3.2. *The functions $(\partial/\partial\tau)\Omega_i(P, \tau)|_{\tau=0}$ have a root of order 2 at α.*

Proof. By equation (11) the differentials $(\partial/\partial\tau)\omega_i(P, \tau)|_{\tau=0}$ have a root of order 1 at α. The functions

$$h_i(P) := \int_\alpha^P \frac{\partial}{\partial\tau}\omega_i(P, \tau)\Big|_{\tau=0}$$

have a root or order 2 at α. Now look at

$$\begin{aligned}\frac{\partial}{\partial\tau}\Omega_i(P, \tau)\Big|_{\tau=0} &= \int_{J(P)}^P \frac{\partial}{\partial\tau}\omega_i(P, \tau)\Big|_{\tau=0}\\ &= \int_{J(P)}^{J(\alpha)} \frac{\partial}{\partial\tau}\omega_i(P, \tau)\Big|_{\tau=0} + \int_{J(\alpha)}^{\alpha} \frac{\partial}{\partial\tau}\omega_i(P, \tau)\Big|_{\tau=0}\\ &\quad + \int_\alpha^P \frac{\partial}{\partial\tau}\omega_i(P, \tau)\Big|_{\tau=0}.\end{aligned}$$

With equation (12) one gets

$$\int_{J(\alpha)}^{\alpha} \frac{\partial}{\partial \tau} \omega_i(P, \tau) \bigg|_{\tau=0} = 0$$

and this implies

$$\frac{\partial}{\partial \tau} \Omega_i(P, \tau) \bigg|_{\tau=0} = 2h_i(P). \qquad \square$$

$\dot{\Omega}_1$ has $2g$ roots at the branch points $e_1, \ldots, e_{2g}$ and another 4 roots over $x = 1$. The roots of ω_2 lying outside the set $\{\alpha, J(\alpha)\}$ yield $2g$ poles of $\dot{\Omega}_1$, together with the simple poles at 0 and ∞ we see that $\dot{\Omega}_1$ has at most $2g + 2$ poles. Consequently $\dot{\Omega}_1$ is the zero-function and one gets the following equation:

$$\frac{\partial}{\partial \tau} \Omega_2(P, \tau) \bigg|_{\tau=0} \cdot \omega_1(P) = \frac{\partial}{\partial \tau} \Omega_1(P, \tau) \bigg|_{\tau=0} \cdot \omega_2(P). \tag{20}$$

There are $2g$ roots of ω_2 outside the set $\{\alpha, J(\alpha)\}$, which can't coincide with roots of ω_1 due to the assumption (2). These $2g$ roots of ω_2 must be roots of $(\partial/\partial\tau)\Omega_2(P, \tau)\,|_{\tau=0}$. Together with the 4 roots lying over $x = 1$ we conclude that $(\partial/\partial\tau)\Omega_2(P, \tau)\,|_{\tau=0}$ has at least $2g + 4$ roots. But $(\partial/\partial\tau)\Omega_2(P, \tau)\,|_{\tau=0}$ has at most $2g + 2$ poles at the branch points. We get $(\partial/\partial\tau)\Omega_2(P, \tau)\,|_{\tau=0} \equiv 0$ and by Lemma 3.1 $(d/d\tau)e(\tau)\,|_{\tau=0} = 0$ follows. This proves the theorem. $\square$

4. Induction

Theorem 2.2 will be proved by induction on g. We will see that a good configuration of branch points for genus g yields a good configuration of branch points for genus $g + 1$. Let's first prepare the induction step.

Take a point $e = (e_1, \ldots, e_g)$ for which the conditions (1), (2), (3) are fulfilled. The corresponding curve X_e and differentials $\Omega_0^g, \Omega_\infty^g, \Omega_0^g + \Omega_\infty^g$ look like

$$X_e : y_0^2 = x \prod_{i=1}^{g} (x - e_i)\left(x - \frac{1}{\bar{e}_i}\right)$$

$$\Omega_0^g = \frac{c_g \prod_{i=1}^{g} (x - \beta_i)}{x y_0}\, dx, \qquad \beta_1 = 1, \quad c_g \in \mathbb{C},$$

$$\Omega^g_\infty = \frac{\prod_{i=1}^{g}(x-\alpha_i)}{y_0}\,dx, \qquad \alpha_1 = 1,$$

$$\Omega^g_\infty - \Omega^g_0 = \frac{d_g \prod_{i=1}^{g+1}(x-\xi_i)}{xy_0}\,dx, \qquad \xi_1 = 1, \quad d_g \in \mathbb{C}.$$

For $(e_1, \ldots, e_g, a, t) \in U \times S^1 \times (-\varepsilon, \varepsilon)$, $\varepsilon > 0$ we define

$$X_{(e,a,t)} : y^2 = x(x - ae^{t})(x - ae^{-t}) \prod_{i=1}^{g}(x-e_i)\left(x - \frac{1}{\bar{e}_i}\right),$$

and corresponding normalized differentials

$$\Omega^{g+1}_0 = \frac{c_{g+1} \prod_{i=1}^{g+1}(x-\beta_i^{g+1})}{xy}\,dx, \qquad c_{g+1} \in \mathbb{C},$$

$$\Omega^{g+1}_\infty = \frac{\prod_{i=1}^{g+1}(x-\alpha_i^{g+1})}{y}\,dx,$$

$$\Omega^{g+1}_\infty - \Omega^{g+1}_0 = \frac{d_{g+1} \prod_{i=1}^{g+2}(x-\xi_i^{g+1})}{xy}\,dx, \qquad d_{g+1} \in \mathbb{C}.$$

Due to the compactness of $X_{(e,a,t)}$, the normalization conditions and the residue theorem one has the following equations

$$\begin{aligned} &\alpha_i^{g+1}(e, a, 0) = \alpha_i, && i = 1, \ldots, g, \\ &\alpha_{g+1}^{g+1}(e, a, 0) = a \\ &\xi_i^{g+1}(e, a, 0) = \xi_i, && i = 1, \ldots, g+1, \\ &\xi_{g+2}^{g+1}(e, a, 0) = a \end{aligned} \tag{21}$$

and

$$\Omega^{g+1}_\infty(e, a, 0) = \Omega^g_\infty. \tag{22}$$

Due to the reduction (21) we delete the superscript $g+1$ from $\alpha_i^{g+1}, \xi_i^{g+1}$. Now put

$$\alpha_1 = u_1 + iu_2, \qquad e_i = x_i + iy_i, \qquad i = 1, \ldots, g$$

and let's impose the further conditions on X_e

(4) $\operatorname{rank}\left(\dfrac{\partial u_r}{\partial x_i \partial y_j}\right) = 2, \qquad r = 1, 2,$

(5) the real part of the meromorphic function

$$k(x) = 1 + x \frac{\dfrac{\partial}{\partial x}\dfrac{\Omega_\infty^g}{dx}}{\dfrac{\Omega_\infty^g}{dx}}$$

doesn't vanish identically on S^1.

The conditions (4) and (5) are used to prove the following lemma:

LEMMA 4.1. *The map* $h : U \times S^1 \times (-\varepsilon, \varepsilon) \to \mathbb{C} \times \mathbb{R}$

$(e, a, t^2) \mapsto (\alpha_1, |\alpha_{g+1}|)$

has maximal rank in a point $P = (e, a, 0)$, *where* $Re(k(a)) \neq 0$.

REMARK. This lemma together with the property

$$\left.\frac{\partial}{\partial \tau} \xi_{g+2}\right|_{t=0} = 0$$

yields the existence of curves $X_{(e,a,t)}$ of genus $g+1$, which satisfy the conditions (1), (2), (3). Taking t small enough the conditions (4) and (5) are also fulfilled.

Proof. Due to the reduction (21) and condition (4) we have

$$\operatorname{rank}\left(\frac{\partial u_r}{\partial x_i \partial y_j}\right) = 2, \qquad \left(\left.\frac{\partial |\alpha_{g+1}|}{\partial x_i \partial y_j}\right|_P\right) = 0, \qquad r = 1, 2; \quad i, j = 1, \ldots, g.$$

It remains to prove that

$$\frac{\partial}{\partial t^2}|\alpha_{g+1}|\Big|_P = \mathrm{Re}\left(\frac{\partial}{\partial t^2}\alpha_{g+1}\bar{\alpha}_{g+1}\right)\Big|_P \neq 0.$$

For this we will deduce an equation for $(\partial/\partial t^2)\alpha_{g+1}|_P$. Differentiation of Ω_∞^{g+1} yields

$$\frac{\partial}{\partial t^2}\Omega_\infty^{g+1}\Big|_P = \frac{\left(-\sum_{i=1}^{g+1}\frac{\partial}{\partial t^2}\alpha_i\Big|_P\frac{1}{x-\alpha_i}\right)\prod_{i=1}^{g}(x-\alpha_i)}{y_0}dx + \frac{ax\prod_{i=1}^{g}(x-\alpha_i)}{2(x-a)^2y_0}dx.$$

Since

$$res_{x=a}\left(\frac{\partial}{\partial t^2}\Omega_\infty^{g+1}\Big|_P\right) = 0$$

we get the equation

$$res_{x=a}\left(\frac{\partial}{\partial t^2}\alpha_{g+1}\Big|_P\Omega_\infty^g\right) = res_{x=a}\left(\frac{ax}{2(x-a)^2}\Omega_\infty^g\right),$$

and

$$\frac{\partial}{\partial t^2}\alpha_{g+1}\Big|_P\cdot\bar{a} = \frac{1}{2}+\frac{1}{2}x\frac{\frac{\partial}{\partial x}\frac{\Omega_\infty^g}{dx}}{\frac{\Omega_\infty^g}{dx}}\Bigg|_{x=a}.$$

Since $Re(k(a)) \neq 0$ we have

$$\frac{\partial}{\partial t^2}|\alpha_{g+1}|\Big|_P \neq 0,$$

and the lemma is proved. □

Finally, we have to prove the existence of curves X_ϵ of genus $g = 2$ which satisfy the conditions (1) up to (5). For the beginning of the induction results of Bobenko [4] and Ercolani–Knörrer–Trubowitz [5] are used.

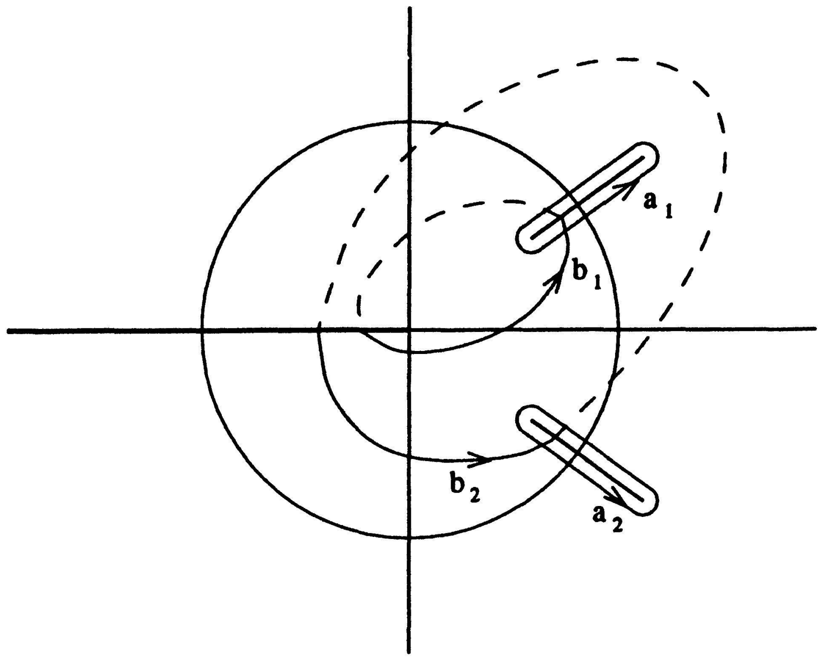

Figure 2

Let X_e be the hyperelliptic curve (figure (2))

$$X_e : y^2 = x(x-\mu)\left(x - \frac{1}{\bar{\mu}}\right)(x-\bar{\mu})\left(x-\frac{1}{\mu}\right) \tag{23}$$

with normalized differentials

$$\Omega_0 = \frac{\bar{\alpha}_1\bar{\alpha}_2(x-\beta_1)(x-\beta_2)}{xy}\,dx,$$

$$\Omega_\infty = \frac{(x-\alpha_1)(x-\alpha_2)}{y}\,dx.$$

Let C_1, C_2 be the elliptic curves

$$C_1 : y^2 = (z-2)(z-\lambda)(z-\bar{\lambda}), \qquad \lambda = \mu + \frac{1}{\mu}$$

$$C_2 : y^2 = (z+2)(z-\lambda)(z-\bar{\lambda})$$

and

$$\varphi_v = \frac{(z-\zeta_v)}{y}\,dz$$

meromorphic differentials on C_v with vanishing a-periods (see figure (3)).

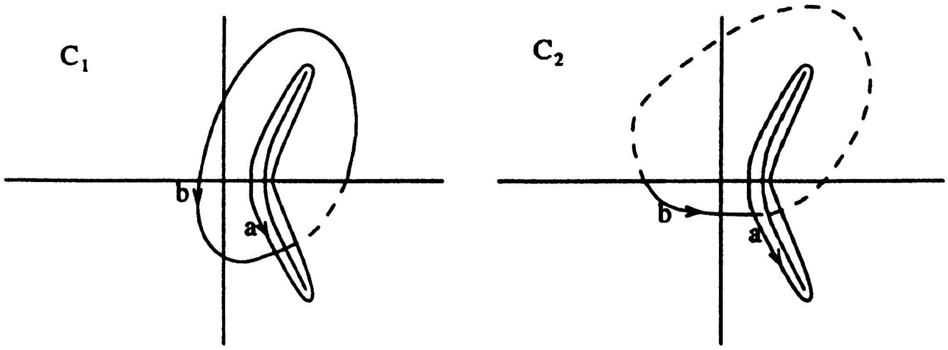

Figure 3

There are maps $\tau_\nu : X_e \to C_\nu$ given by

$$(x, y) \mapsto \left(x + \frac{1}{x}, \frac{x + (-1)^\nu}{x^2} y \right).$$

The pullback of φ_ν with respect to τ_ν is given by

$$\tau_1^* \varphi_1 = \frac{(x^2 - \zeta_1 x + 1)(x + 1)}{xy} dx,$$

$$\tau_2^* \varphi_2 = \frac{(x^2 - \zeta_2 x + 1)(x - 1)}{xy} dx.$$

Taking the sum and the difference one gets

$$\tau_1^* \varphi_1 + \tau_2^* \varphi_2 = 2\Omega_\infty,$$

$$\tau_1^* \varphi_1 - \tau_2^* \varphi_2 = 2\Omega_0.$$

Introduce new parameters r, θ by the equation

$$\lambda = 2 + re^{i\theta}. \tag{24}$$

Now, look at the following lemma:

LEMMA 4.2.

(i) *There is a unique* $\theta = \theta_0 \in (0, \pi/2)$, *such that* $\xi_1(r, \theta_0) = 2$ *holds for arbitrary* r,

(ii) $\dfrac{\partial \xi_1}{\partial \theta}(r, \theta_0) = \dfrac{-r}{2 \sin(\theta_0)}$,

(iii) $\xi_2(r, \theta) = 2 + r \cos(\theta) + \mathcal{O}(r^2)$.

Proof. Let's make the change of variables $\xi = z - 2$ and let's define

$$Z(r, \theta) := \xi_1(r, \theta) - 2.$$

The curve C_1 is given by

$$y^2 = \xi(\xi^2 - 2r\xi \cos\theta + r^2)$$

and for the differential φ_1 we have

$$\varphi_1 = \frac{\xi - Z(r, \theta)}{y}\, d\xi.$$

Following Bobenko [4] one has

$$\int_a \frac{\xi\, d\xi}{y} = \sqrt{8r} \int_\theta^\pi \frac{\cos t\, dt}{\sqrt{\cos\theta - \cos t}},$$

and there is a unique $\theta = \theta_0 \in (0, \pi/2)$ for which

$$\int_\theta^\pi \frac{\cos t\, dt}{\sqrt{\cos\theta - \cos t}} = 0.$$

Consequently, we have the equation

$$Z(r, \theta) = 0 \Leftrightarrow \theta = \theta_0.$$

To prove (ii) we first observe that $Z(r, \theta) = rZ(1, \theta)$. Differentiation of φ_1 yields

$$\frac{\partial}{\partial\theta} \varphi_1(1, \theta)\bigg|_{\theta=\theta_0} = \left(-\frac{\partial Z}{\partial\theta}(1, \theta)\right)\bigg|_{\theta=\theta_0} \cdot \frac{d\xi}{y} - \sin\theta_0 \frac{\xi^3\, d\xi}{y^3} \tag{25}$$

and

$$d\left(\frac{-\xi^2 \cos\theta_0 + \xi}{y}\right) = -\sin^2\theta_0 \frac{\xi^3\, d\xi}{y^3} - \frac{1}{2}\cos\theta_0 \frac{\xi\, d\xi}{y} + \frac{1}{2}\frac{d\xi}{y}. \tag{26}$$

Due to

$$\int_a \frac{\partial}{\partial\theta} \varphi_1(1, \theta)\bigg|_{\theta=\theta_0} = 0$$

equation (25) gives rise to

$$\frac{\partial Z}{\partial \theta}(1, \theta)\bigg|_{\theta=\theta_0} \cdot \int_a \frac{d\xi}{y} = -\sin\theta_0 \int_a \frac{\xi^3\, d\xi}{y^3}.$$

Integration of equation (26) yields

$$-\sin\theta_0 \int_a \frac{\xi^3\, d\xi}{y^3} = \frac{1}{2}\frac{\cos\theta_0}{\sin\theta_0}\int_a \frac{\xi\, d\xi}{y} - \frac{1}{2\sin\theta_0}\int_a \frac{d\xi}{y}.$$

The first expression on the right is zero and we get

$$\frac{\partial Z}{\partial \theta}(1, \theta)\bigg|_{\theta=\theta_0} = -\frac{1}{2\sin\theta_0},$$

which proves (ii).

The curve C_2 is given by

$$y^2 = (\xi + 4)(\xi^2 - 2r\xi\cos\theta + r^2)$$

and the differential φ_2 looks like

$$\varphi_2 = \frac{\xi - (\xi_2 - 2)}{y}\, d\xi.$$

Put

$$Q(r, \theta) := \frac{1}{2\pi i}\int_a \frac{d\xi}{y},$$

and we have

$$Q(0, \theta) = res_{\xi=0}\left(\frac{d\xi}{\xi\sqrt{\xi+4}}\right) = \frac{1}{2},$$

$$\begin{aligned}\frac{\partial}{\partial r} Q(r, \theta)\bigg|_{r=0} &= res_{\xi=0}\left(\frac{\partial}{\partial r}\frac{d\xi}{y}\bigg|_{r=0}\right)\\ &= res_{\xi=0}\left(\frac{\cos\theta\, d\xi}{\xi^2\sqrt{\xi+4}}\right) = -\frac{1}{16}\cos\theta.\end{aligned}$$

Consequently,

$$Q(r, \theta) = \frac{1}{2} - \frac{1}{16} r \cos\theta + \mathcal{O}(r^2). \tag{27}$$

Similarly we put

$$P(r, \theta) := \frac{1}{2\pi i} \int_a \frac{\xi \, d\xi}{y},$$

and this yields

$$P(r, \theta) = \frac{1}{2} r \cos\theta + \mathcal{O}(r^2). \tag{28}$$

Since the integral of φ_2 over a is identically zero, (iii) follows from the equations (27) and (28). □

We use this lemma to prove the final step:

PROPOSITION 4.3. *There are curves X_e of genus $g = 2$ which satisfy the conditions* (1), . . . , (5).

Proof. For $\theta = \theta_0$ the differential φ_1 has a root over $z = 2$. Put $\zeta_1 = 2$. Then Ω_0 and Ω_∞ have a common root α over $x = 1$ and condition (1) is fulfilled.

For condition (2) we have to look at α_2 and β_2. They satisfy the equations

$$\zeta_2 \beta_2 = 2, \qquad 2\alpha_2 = \zeta_2.$$

Suppose $\alpha_2 = \beta_2$ holds, then we have $\zeta_2^2 = 4$, but for ζ_2 we know

$$\zeta_2(r, \theta) = 2 + r\cos\theta + \mathcal{O}(r^2).$$

For condition (3) we have to examine the roots of $\Omega_\infty - \Omega_0 = \tau_2^* \varphi_2$. Due to the equation above for ξ_2 the roots of the polynomial

$$p(x) = (x^2 - \zeta_2 x + 1)(x - 1)$$

don't lie in the branch points and $\Omega_\infty - \Omega_0$ has a root of order 1 at α. For small r the conditions (1), (2), (3) are satisfied.

Now look at the condition (4). We want to show that the matrix

$$\left(\frac{\partial u_r}{\partial x_i \partial y_j}\right)$$

with $e_1 = \mu$ and $e_2 = \bar{\mu}$ has rank 2. If we rotate the configuration of branch points around the origin, also α_1 is rotated. Moreover, if we move θ for fixed r, the root α_1 can only move on the real axis. Now look at the equations

$$\alpha_1 + \alpha_2 = \frac{1}{2}(\zeta_1 + \zeta_2),$$

$$\alpha_1 \alpha_2 = \frac{1}{2}(\zeta_2 - \zeta_1) + 1.$$

Suppose we have

$$\left.\frac{d\alpha_1}{d\theta}\right|_{\theta = \theta_0} = 0,$$

then we can conclude

$$\left.\frac{d\zeta_1}{d\theta}\right|_{\theta = \theta_0} = 0,$$

but

$$\left.\frac{d\zeta_1}{d\theta}\right|_{\theta = \theta_0} = \frac{-r}{2 \sin \theta_0}.$$

So, the assumption was false and we get the desired result.

For condition (5) we take the limit $r \to 0$ and we get $k(a) = 1/2$ (using the identities $\mu = 1, \alpha_2 = 1$). Thus the proof of the theorem is complete. □

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