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On the classification of constant mean curvature tori in \mathbb{R}^3

CHRISTIAN JAGGY

1. Introduction

Let S be a compact oriented surface and $i: S \to \mathbb{R}^3$ an immersion with constant mean curvature. Hopf [6] investigated such immersions, and for genus (S) = 0 he showed that $i: S \to \mathbb{R}^3$ must be an embedding of a round sphere. Conversely, the genus of the surface S is 0, if *i* is an embedding. This statement was proved by Alexandrov [1]. Only a few years ago Wente [10] and Kapouleas [7] proved the existence of constant mean curvature immersions for genus (S) = 1 and genus $(S) \ge 2$, respectively. In this work we will only look at constant mean curvature immersions with genus (S) = 1.

First the relation of hyperelliptic curves and constant mean curvature immersions is sketched. For a rigorous formulation see Bobenko [3].

Let u be a solution of the elliptic-sinh Gordon equation

$$u_{w\bar{w}} + \sinh u = 0 \tag{1}$$

on a simply-connected domain $\Omega \subset \mathbb{C}$. There is an algorithm that associates an immersion $i: \Omega \to \mathbb{R}^3$ to u with constant mean curvature $\frac{1}{2}$. Conversely, every constant mean curvature immersion yields a solution u of equation (1).

On the other hand we can associate quasi-periodic solutions of equation (1) on \mathbb{R}^2 to hyperelliptic curves

$$X: y^{2} = x \prod_{i=1}^{2g} (x - e_{i})$$
⁽²⁾

where the branch points are distinct and satisfy

$$e_{i+g} = \frac{1}{\bar{e}_i}, \qquad i = 1, \dots, g.$$
 (3)

We first have to fix some notation to write down an explicit formula for solutions



Figure 1

of equation (1). In figure (1) a canonical basis $a_1, \ldots, a_g, b_1, \ldots, b_g$ of $H_1(X, \mathbb{Z})$ with intersection numbers

$$a_i b_j = \delta_{ij}, \qquad a_i a_j = 0, \qquad b_i b_j = 0, \qquad i, j = 1, \dots, g$$
 (4)

is introduced. Let Ω_0 and Ω_{∞} be meromorphic differentials on X, holomorphic outside 0 and ∞ , respectively, which satisfy the conditions

$$\int_{a_i} \Omega_0 = \int_{a_i} \Omega_\infty = 0, \qquad i = 1, \dots, g$$
(5)

and

 Ω_0 has a pole of second order at 0,

 Ω_{∞} has a pole of second order at ∞ .

Define the vectors μ_0, μ_∞ by

$$\mu_0 = \left(\int_{b_1} \Omega_0, \ldots, \int_{b_g} \Omega_0\right)$$
$$\mu_\infty = \left(\int_{b_1} \Omega_\infty, \ldots, \int_{b_g} \Omega_\infty\right),$$

(6)

and for $\zeta \in \mathbb{C}^g$ put

$$u(\zeta) = 2\log \frac{\theta\left(\zeta + \left(\frac{1}{2}, \ldots, \frac{1}{2}\right)\right)}{\theta(\zeta)}$$

where θ is the Riemann theta function of X for the given homology basis. The function

$$u(\zeta + w\mu_0 + \bar{w}\mu_\infty) \tag{7}$$

is a real quasi-periodic solution of equation (1) for every $\zeta \in \mathbb{R}^{g}$.

The question arises, whether it is possible to choose X in a way, such that X yields constant mean curvature tori. The answer to this question was given by Bobenko [4] and Pinkall-Sterling [9].

THEOREM 1.1.

- (1) Under the correspondence mentioned above X yields constant mean curvature tori in \mathbb{R}^3 if and only if
 - (a) Ω_{∞} has a root $p = (x_0, y_0)$ with $|x_0| = 1$;
 - (b) Let γ be a path that connects the two points (x_0, y_0) and $(x_0, -y_0)$. Then the span of the vectors

$$v_0 = \left(\int_{\gamma} \Omega_0, \int_{b_1} \Omega_0, \dots, \int_{b_g} \Omega_0\right)$$
$$v_{\infty} = \left(\int_{\gamma} \Omega_{\infty}, \int_{b_1} \Omega_{\infty}, \dots, \int_{b_g} \Omega_{\infty}\right)$$

in C^{g+1} must contain two linearly independent rational vectors. In this case one gets a (g − 2)-parameter family of constant mean curvature tori.
 (2) Every constant mean curvature torus arises in such a way.

It is known that there are no curves satisfying the condition (a) for genus (X) = 1. Wente found constant mean curvature tori which are known to correspond to curves with genus (X) = 2 or genus (X) = 3. In 1991 Ercolani-Knörrer-Trubowitz [5] proved the existence of such curves for even genus (X). All curves constructed there have the additional property, that the set of branch points is invariant under the map $x \mapsto 1/x$. In this paper the existence of curves X fulfilling the conditions (a) and (b) is proved for genus (X) arbitrary.

2. Preliminaries

The map $\sigma: X \to X$

$$(x, y) \mapsto \left(\frac{1}{\bar{x}}, \frac{\left(\prod_{i=1}^{2g} e_i\right)^{1/2} \bar{y}}{\bar{x}^{g+1}}\right)$$

is an antiholomorphic involution of X. The sign of $(\prod_{i=1}^{2g} e_i)^{1/2}$ is chosen in such a way, that the points lying over S^1 are fixed points of σ . Then σ_* acts as follows on the cycles:

$$\sigma_{*}(a_{i}) = -a_{i}, \qquad i = 1, \dots, g$$

$$\sigma_{*}(b_{i}) = b_{i} + \sum_{j=1}^{g} \lambda_{ij}a_{j}, \qquad i = 1, \dots, g$$
(8)

with $\lambda_{ij} \in \mathbb{Z}$; $i, j = 1, \ldots, g$, and

$$\gamma - \sigma_* \gamma = \sum_{j=1}^g \mu_j a_j, \tag{9}$$

with $\mu_j \in \mathbb{Z}$; j = 1, ..., g. It is possible to choose Ω_0, Ω_∞ in a way, such that

$$\sigma^*\Omega_0 = \bar{\Omega}_\infty$$

holds. It follows that the vectors v_0 , v_{∞} are complex conjugate. The new vectors

$$v := v_0 + v_\infty$$
$$w := i(v_\infty - v_0)$$

are elements of \mathbb{R}^{g+1} . Now consider the map $f: \mathbb{C}^g \to \mathbb{C} \times Gr(2, \mathbb{R}^{g+1})$

$$(e_1,\ldots,e_g)\mapsto$$
 (root of $\Omega_{\infty},\langle v,w\rangle$).

f is a multivalued function and one should restrict the domain of definition of f to the open subset $U \subset \mathbb{C}^g$, where all the branch points are distinct. $Gr(2, \mathbb{R}^{g+1})$

denotes the Grassmannian of 2-dimensional subspaces of \mathbb{R}^{g+1} . The vectors v and w are linearly independent and $\langle v, w \rangle$ is a welldefined element of $Gr(2, \mathbb{R}^{g+1})$.

It is interesting to look at this map, because if one finds a root $p = (x_0, y_0)$ with $|x_0| = 1$ and if $\langle v, w \rangle$ contains two linearly independent rational vectors, the existence of constant mean curvature tori is guaranteed. In section 3 the following theorem will be proved.

THEOREM 2.1. Let $e = (e_1, \ldots, e_g)$ be in U. Assume that the differentials Ω_0, Ω_∞ on the hyperelliptic curve

$$X: y^2 = x \prod_{i=1}^{2g} (x - e_i)$$

fulfill the following conditions:

(1) $\Omega_0, \Omega_{\infty}$ have a common root α over x = 1,

(2) Ω_0, Ω_∞ don't have any other common roots,

(3) $(\Omega_{\infty} - \Omega_0)(e_m) \neq 0$ for m = 1, ..., 2g, and $\Omega_{\infty} - \Omega_0$ has a root of order 1 at α . Then df (e) is invertible.

We denote X_e as the hyperelliptic curve associated to the point $e \in U$. Due to this theorem it follows, that arbitrarily close to e there are points, such that the corresponding curves X_e fulfill conditions (a) and (b). In section 4 we will finally show

THEOREM 2.2. For every $g \ge 2$ there are curves $X_e, e \in U$, satisfying the conditions (1), (2), (3) above.

This theorem will be proved by induction on g.

3. Simplification

Proof of Theorem 2.1. Since dimensions are equal it is enough to show that df(e) is injective. The strategy is due to Krichever [8], Bikbaev and Kuksin [2].

Let $e(\tau)$, $\tau \in \mathbb{R}$, be an arbitrary differentiable curve passing through e, such that $f(e(\tau))$ changes only in order τ^2 , in other words

$$\binom{v(\tau)}{w(\tau)} = A(\tau) \binom{v(0)}{w(0)} + \mathcal{O}(\tau^2), \quad \text{with } A(0) = \binom{1 \quad 0}{0 \quad 1}$$
(10)

$$\alpha(\tau) = 1 + \mathcal{O}(\tau^2). \tag{11}$$

We want to conclude that

$$\left.\frac{d}{d\tau}\,e(\tau)\,\right|_{\tau\,=\,0}=0$$

holds. This implies that df(e) is injective. Put $B(\tau) := A(\tau)^{-1}$, clearly

$$B(\tau) \begin{pmatrix} v(\tau) \\ w(\tau) \end{pmatrix} = \begin{pmatrix} v(0) \\ w(0) \end{pmatrix} + \mathcal{O}(\tau^2)$$

and after differentiation

$$\dot{B}(0) \begin{pmatrix} v(0) \\ w(0) \end{pmatrix} + \begin{pmatrix} \dot{v}(0) \\ \dot{w}(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$
(12)

These are 2g + 2 equations, 2g of them describe relations among period integrals. Define differentials ω_1, ω_2 by

$$\begin{pmatrix} \omega_1(\tau) \\ \omega_2(\tau) \end{pmatrix} := B(\tau) \begin{pmatrix} \Omega_0(\tau) + \Omega_\infty(\tau) \\ i(\Omega_\infty(\tau) - \Omega_0(\tau)) \end{pmatrix}.$$
 (13)

By integration of $\omega_1(\tau)$, $\omega_2(\tau)$ one get's multivalued meromorphic functions on $X_{e(\tau)}$:

$$\Omega_i(P,\tau) \coloneqq \int_{J(P)}^P \omega_i(\tau), \qquad i = 1, 2$$
(14)

where J denotes the hyperelliptic involution.

LEMMA 3.1. The functions

$$\left. \frac{\partial}{\partial \tau} \Omega_i(P, \tau) \right|_{\tau = 0}$$

are single-valued meromorphic functions on X_e . At the points $e_1, \ldots, e_{2g}, 0, \infty$ they have first order poles. Furthermore there are non-zero complex numbers c_1, \ldots, c_{2g} such that

$$res_{P=e_m}\left(\frac{\partial}{\partial \tau}\,\Omega_2(P,\,\tau)\,\bigg|_{\tau=0}\right) = c_m \frac{\partial}{\partial \tau}\,e_m\,\bigg|_{\tau=0}, \qquad m=1,\,\ldots,\,2g. \tag{15}$$

Due to this lemma it is enough to show that

$$\frac{\partial}{\partial \tau} \, \Omega_2(P, \, \tau) \, \bigg|_{\tau \,= \, 0} \equiv 0.$$

This will prove the theorem. We first prove this lemma, before we continue the proof of the theorem.

Proof. To see that the functions $(\partial/\partial \tau)\Omega_i(P, \tau)|_{\tau=0}$ are single-valued, we have to look at the corresponding *b*-periods:

$$\frac{\partial}{\partial \tau} \int_{b_j} \omega_1 \Big|_{\tau=0} = \frac{\partial}{\partial \tau} \int_{b_j} (b_{11}(\Omega_0 + \Omega_\infty) + b_{12}i(\Omega_\infty - \Omega_0)) \Big|_{\tau=0}$$
$$= \int_{b_j} (\dot{b}_{11}(0)(\Omega_0 + \Omega_\infty) + \dot{b}_{12}(0)i(\Omega_\infty - \Omega_0) + \dot{\Omega}_0 + \dot{\Omega}_\infty)$$
$$= 0.$$

The last identity is true due to equation (12). The same is true for ω_2 and the first statement is proved.

Expand $\omega_i(\tau)$ at $e_m(\tau)$ in the local coordinate $(x - e_m(\tau))^{1/2}$:

$$\omega_i(x,\tau) = \sum_{k=-1}^{\infty} (x - e_m(\tau))^{k/2} x_k^{i,m}(e(\tau)) \, dx.$$

Put P = (x, y), then we get

$$\begin{aligned} \frac{\partial}{\partial \tau} \Omega_i(P,\tau) \Big|_{\tau=0} &= \int_{J(P)}^P \frac{\partial}{\partial \tau} \omega_i(x,\tau) \Big|_{\tau=0} \\ &= 2 \sum_{k=-1}^\infty \left(-(x-e_m(0))^{k/2} \frac{\partial}{\partial \tau} e_m(0) x_k^{i,m}(e(0)) \right. \\ &+ \frac{2}{2+k} (x-e_m)^{1+k/2} \frac{\partial}{\partial \tau} x_k^{i,m}(e(\tau)) \Big|_{\tau=0} \right). \end{aligned}$$

It follows that the functions $(\partial/\partial \tau)\Omega_i(P, \tau)|_{\tau=0}$ have first order poles at the points e_1, \ldots, e_{2g} and the same is true for 0 and ∞ by a similar calculation. Due to the assumption (3) in Theorem 2.1 the claim about the numbers c_m is obvious.

Let's continue the proof of the theorem. Take $P \in X_e$ with $\omega_2(P) \neq 0$. The implicit function theorem yields a curve $P(\tau)$ with

$$\Omega_2(P(\tau), \tau) = \Omega_2(P, 0) \tag{16}$$

and after differentiation

$$\frac{d}{d\tau}\Omega_2(P(\tau),\tau)\bigg|_{\tau=0} = \omega_2(P)\frac{d}{d\tau}P(\tau)\bigg|_{\tau=0} + \frac{\partial}{\partial\tau}\Omega_2(P,\tau)\bigg|_{\tau=0} = 0.$$
(17)

Define a new function

$$\dot{\Omega}_1(P) := \frac{d}{d\tau} \,\Omega_1(P(\tau), \tau) \bigg|_{\tau = 0}.$$
(18)

The function $\dot{\Omega}_1$ is welldefined and by the equation (17) above one gets

$$\dot{\Omega}_{1}(P) = \frac{\partial}{\partial \tau} \Omega_{1}(P, \tau) \bigg|_{\tau=0} - \frac{\partial}{\partial \tau} \Omega_{2}(P, \tau) \bigg|_{\tau=0} \cdot \frac{\omega_{1}(P)}{\omega_{2}(P)}.$$
(19)

It follows that $\dot{\Omega}_1$ is a meromorphic function on X. To finish the proof of the theorem we need the following lemma:

LEMMA 3.2. The functions $(\partial/\partial \tau)\Omega_i(P, \tau)|_{\tau=0}$ have a root of order 2 at α .

Proof. By equation (11) the differentials $(\partial/\partial \tau)\omega_i(P, \tau)|_{\tau=0}$ have a root of order 1 at α . The functions

$$h_i(P) := \int_{\alpha}^{P} \frac{\partial}{\partial \tau} \omega_i(P, \tau) \bigg|_{\tau = 0}$$

have a root or order 2 at α . Now look at

$$\begin{aligned} \frac{\partial}{\partial \tau} \Omega_i(P, \tau) \bigg|_{\tau=0} &= \int_{J(P)}^P \frac{\partial}{\partial \tau} \omega_i(P, \tau) \bigg|_{\tau=0} \\ &= \int_{J(P)}^{J(\alpha)} \frac{\partial}{\partial \tau} \omega_i(P, \tau) \bigg|_{\tau=0} + \int_{J(\alpha)}^{\alpha} \frac{\partial}{\partial \tau} \omega_i(P, \tau) \bigg|_{\tau=0} \\ &+ \int_{\alpha}^P \frac{\partial}{\partial \tau} \omega_i(P, \tau) \bigg|_{\tau=0}. \end{aligned}$$

With equation (12) one gets

$$\int_{J(\alpha)}^{\alpha} \frac{\partial}{\partial \tau} \omega_i(P, \tau) \bigg|_{\tau=0} = 0$$

and this implies

$$\frac{\partial}{\partial \tau} \Omega_i(P, \tau) \bigg|_{\tau=0} = 2h_i(P). \qquad \Box$$

 $\dot{\Omega}_1$ has 2g roots at the branch points e_1, \ldots, e_{2g} and another 4 roots over x = 1. The roots of ω_2 lying outside the set $\{\alpha, J(\alpha)\}$ yield 2g poles of $\dot{\Omega}_1$, together with the simple poles at 0 and ∞ we see that $\dot{\Omega}_1$ has at most 2g + 2 poles. Consequently $\dot{\Omega}_1$ is the zero-function and one gets the following equation:

$$\frac{\partial}{\partial \tau} \Omega_2(P, \tau) \bigg|_{\tau=0} \cdot \omega_1(P) = \frac{\partial}{\partial \tau} \Omega_1(P, \tau) \bigg|_{\tau=0} \cdot \omega_2(P).$$
(20)

There are 2g roots of ω_2 outside the set $\{\alpha, J(\alpha)\}$, which can't coincide with roots of ω_1 due to the assumption (2). These 2g roots of ω_2 must be roots of $(\partial/\partial \tau)\Omega_2(P, \tau)|_{\tau=0}$. Together with the 4 roots lying over x = 1 we conclude that $(\partial/\partial \tau)\Omega_2(P, \tau)|_{\tau=0}$ has at least 2g + 4 roots. But $(\partial/\partial \tau)\Omega_2(P, \tau)|_{\tau=0}$ has at most 2g + 2 poles at the branch points. We get $(\partial/\partial \tau)\Omega_2(P, \tau)|_{\tau=0} \equiv 0$ and by Lemma 3.1 $(d/d\tau)e(\tau)|_{\tau=0} = 0$ follows. This proves the theorem.

4. Induction

Theorem 2.2 will be proved by induction on g. We will see that a good configuration of branch points for genus g yields a good configuration of branch points for genus g + 1. Let's first prepare the induction step.

Take a point $e = (e_1, \ldots, e_g)$ for which the conditions (1), (2), (3) are fulfilled. The corresponding curve X_e and differentials Ω_0^g , Ω_∞^g , $\Omega_0^g + \Omega_\infty^g$ look like

$$X_e: y_0^2 = x \prod_{i=1}^g (x - e_i) \left(x - \frac{1}{\bar{e}_i} \right)$$
$$\Omega_0^g = \frac{c_g \prod_{i=1}^g (x - \beta_i)}{xy_0} dx, \qquad \beta_1 = 1, \quad c_g \in \mathbb{C},$$

$$\Omega_{\infty}^{g} = \frac{\prod\limits_{i=1}^{g} (x - \alpha_{i})}{y_{0}} dx, \qquad \alpha_{1} = 1,$$
$$\Omega_{\infty}^{g} - \Omega_{0}^{g} = \frac{d_{g} \prod\limits_{i=1}^{g+1} (x - \xi_{i})}{xy_{0}} dx, \qquad \xi_{1} = 1, \quad d_{g} \in \mathbb{C}.$$

For $(e_1, \ldots, e_g, a, t) \in U \times S^1 \times (-\varepsilon, \varepsilon), \varepsilon > 0$ we define

$$X_{(e,a,t)}: y^{2} = x(x - ae^{t})(x - ae^{-t}) \prod_{i=1}^{g} (x - e_{i})\left(x - \frac{1}{\bar{e}_{i}}\right),$$

and corresponding normalized differentials

$$\Omega_{0}^{g+1} = \frac{c_{g+1} \prod_{i=1}^{g+1} (x - \beta_{i}^{g+1})}{xy} dx, \qquad c_{g+1} \in \mathbb{C},$$

$$\Omega_{\infty}^{g+1} = \frac{\prod_{i=1}^{g+1} (x - \alpha_{i}^{g+1})}{y} dx,$$

$$\Omega_{\infty}^{g+1} - \Omega_{0}^{g+1} = \frac{d_{g+1} \prod_{i=1}^{g+2} (x - \xi_{i}^{g+1})}{xy} dx, \qquad d_{g+1} \in \mathbb{C}.$$

Due to the compactness of $X_{(e,a,t)}$, the normalization conditions and the residue theorem one has the following equations

$$\alpha_{i}^{g+1}(e, a, 0) = \alpha_{i}, \qquad i = 1, \dots, g,$$

$$\alpha_{g+1}^{g+1}(e, a, 0) = a$$

$$\xi_{i}^{g+1}(e, a, 0) = \xi_{i}, \qquad i = 1, \dots, g+1,$$

$$\xi_{g+2}^{g+1}(e, a, 0) = a$$
(21)

and

$$\Omega_{\infty}^{g+1}(e,a,0) = \Omega_{\infty}^{g}.$$
(22)

Due to the reduction (21) we delete the superscript g + 1 from α_i^{g+1} , ξ_i^{g+1} . Now put

 $\alpha_1 = u_1 + iu_2, \qquad e_i = x_i + iy_i, \qquad i = 1, \ldots, g$

and let's impose the further conditions on X_e

(4) rank
$$\left(\frac{\partial u_r}{\partial x_i \partial y_j}\right) = 2, \qquad r = 1, 2,$$

(5) the real part of the meromorphic function

$$k(x) = 1 + x \frac{\frac{\partial}{\partial x} \frac{\Omega_{\infty}^{s}}{dx}}{\frac{\Omega_{\infty}^{s}}{dx}}$$

doesn't vanish identically on S^1 .

The conditions (4) and (5) are used to prove the following lemma:

LEMMA 4.1. The map $h: U \times S^1 \times (-\varepsilon, \varepsilon) \to \mathbb{C} \times \mathbb{R}$

$$(e, a, t^2) \mapsto (\alpha_1, |\alpha_{g+1}|)$$

has maximal rank in a point P = (e, a, 0), where $Re(k(a)) \neq 0$.

REMARK. This lemma together with the property

$$\left.\frac{\partial}{\partial\tau}\,\xi_{g+2}\right|_{t=0}=0$$

yields the existence of curves $X_{(e,a,t)}$ of genus g + 1, which satisfy the conditions (1), (2), (3). Taking t small enough the conditions (4) and (5) are also fulfilled.

Proof. Due to the reduction (21) and condition (4) we have

$$\operatorname{rank}\left(\frac{\partial u_r}{\partial x_i \partial y_j}\right) = 2, \qquad \left(\frac{\partial |\alpha_{g+1}|}{\partial x_i \partial y_j}\Big|_P\right) = 0, \qquad r = 1, 2; \quad i, j = 1, \ldots, g.$$

It remains to prove that

$$\frac{\partial}{\partial t^2} \left| \alpha_{g+1} \right| \bigg|_P = \operatorname{Re} \left(\frac{\partial}{\partial t^2} \alpha_{g+1} \bar{\alpha}_{g+1} \right) \bigg|_P \neq 0.$$

For this we will deduce an equation for $(\partial/\partial t^2)\alpha_{g+1}|_P$. Differentiation of Ω_{∞}^{g+1} yields

$$\frac{\partial}{\partial t^2} \Omega_{\infty}^{g+1} \bigg|_{P} = \frac{\left(-\sum_{i=1}^{g+1} \frac{\partial}{\partial t^2} \alpha_i \bigg|_{P} \frac{1}{x-\alpha_i}\right) \prod_{i=1}^{g} (x-\alpha_i)}{y_0} dx + \frac{ax \prod_{i=1}^{g} (x-\alpha_i)}{2(x-a)^2 y_0} dx.$$

Since

$$\operatorname{res}_{x=a}\left(\frac{\partial}{\partial t^2}\Omega_{\infty}^{g+1}\Big|_{P}\right) = 0$$

we get the equation

$$\operatorname{res}_{x=a}\left(\frac{\partial}{\partial t^2}\,\alpha_{g+1}\,\Big|_{P}\,\Omega_{\infty}^{g}\right)=\operatorname{res}_{x=a}\left(\frac{ax}{2(x-a)^2}\,\Omega_{\infty}^{g}\right),$$

and

$$\frac{\partial}{\partial t^2} \alpha_{g+1} \bigg|_{P} \cdot \bar{a}_{\cdot} = \frac{1}{2} + \frac{1}{2} x \frac{\frac{\partial}{\partial x} \frac{\Omega_{\infty}^{g}}{dx}}{\frac{\Omega_{\infty}^{g}}{dx}} \bigg|_{x=a}.$$

Since $Re(k(a)) \neq 0$ we have

$$\frac{\partial}{\partial t^2} \left| \alpha_{g+1} \right| \bigg|_P \neq 0,$$

and the lemma is proved.

Finally, we have to prove the existence of curves X_e of genus g = 2 which satisfy the conditions (1) up to (5). For the beginning of the induction results of Bobenko [4] and Ercolani-Knörrer-Trubowitz [5] are used.





Let X_e be the hyperelliptic curve (figure (2))

$$X_e: y^2 = x(x-\mu)\left(x-\frac{1}{\bar{\mu}}\right)(x-\bar{\mu})\left(x-\frac{1}{\bar{\mu}}\right)$$
(23)

with normalized differentials

$$\Omega_0 = \frac{\bar{\alpha}_1 \bar{\alpha}_2 (x - \beta_1) (x - \beta_2)}{xy} dx$$
$$\Omega_\infty = \frac{(x - \alpha_1) (x - \alpha_2)}{y} dx.$$

Let C_1 , C_2 be the elliptic curves

$$C_1: y^2 = (z-2)(z-\lambda)(z-\overline{\lambda}), \qquad \lambda = \mu + \frac{1}{\mu}$$
$$C_2: y^2 = (z+2)(z-\lambda)(z-\overline{\lambda})$$

and

$$\varphi_{v}=\frac{(z-\zeta_{v})}{y}\,dz$$

meromorphic differentials on C_{ν} with vanishing *a*-periods (see figure (3)).



Figure 3

There are maps $\tau_v: X_e \to C_v$ given by

$$(x, y) \mapsto \left(x + \frac{1}{x}, \frac{x + (-1)^{\nu}}{x^2}y\right).$$

The pullback of φ_{ν} with respect to τ_{ν} is given by

$$\tau_1^* \varphi_1 = \frac{(x^2 - \zeta_1 x + 1)(x + 1)}{xy} dx,$$

$$\tau_2^* \varphi_2 = \frac{(x^2 - \zeta_2 x + 1)(x - 1)}{xy} dx.$$

Taking the sum and the difference one gets

$$\begin{aligned} \tau_1^* \varphi_1 + \tau_2^* \varphi_2 &= 2\Omega_{\infty}, \\ \tau_1^* \varphi_1 - \tau_2^* \varphi_2 &= 2\Omega_0. \end{aligned}$$

Introduce new parameters r, θ by the equation

$$\lambda = 2 + r e^{i\theta}.$$

Now, look at the following lemma:

LEMMA 4.2.

(i) There is a unique $\theta = \theta_0 \in (0, \pi/2)$, such that $\xi_1(r, \theta_0) = 2$ holds for arbitrary r,

(ii)
$$\frac{\partial \xi_1}{\partial \theta}(r, \theta_0) = \frac{-r}{2\sin(\theta_0)},$$

(iii)
$$\xi_2(r, \theta) = 2 + r \cos(\theta) + \mathcal{O}(r^2)$$
.

Proof. Let's make the change of variables $\xi = z - 2$ and let's define

$$Z(r,\theta) := \xi_1(r,\theta) - 2.$$

The curve C_1 is given by

$$y^2 = \xi(\xi^2 - 2r\xi\,\cos\theta + r^2)$$

and for the differential φ_1 we have

$$\varphi_1 = \frac{\xi - Z(r, \theta)}{y} d\xi.$$

Following Bobenko [4] one has

$$\int_{a} \frac{\xi \, d\xi}{y} = \sqrt{8r} \, \int_{\theta}^{\pi} \frac{\cos t \, dt}{\sqrt{\cos \theta - \cos t}} \, ,$$

and there is a unique $\theta = \theta_0 \in (0, \pi/2)$ for which

$$\int_{\theta}^{\pi} \frac{\cos t \, dt}{\sqrt{\cos \theta - \cos t}} = 0.$$

Consequently, we have the equation

$$Z(r,\theta)=0 \Leftrightarrow \theta=\theta_0.$$

To prove (ii) we first observe that $Z(r, \theta) = rZ(1, \theta)$. Differentiation of φ_1 yields

$$\frac{\partial}{\partial \theta} \varphi_1(1,\theta) \bigg|_{\theta=\theta_0} = \left(-\frac{\partial Z}{\partial \theta} (1,\theta) \right) \bigg|_{\theta=\theta_0} \cdot \frac{d\xi}{y} - \sin \theta_0 \frac{\xi^3 d\xi}{y^3}$$
(25)

and

$$d\left(\frac{-\xi^{2}\cos\theta_{0}+\xi}{y}\right) = -\sin^{2}\theta_{0}\frac{\xi^{3}d\xi}{y^{3}} - \frac{1}{2}\cos\theta_{0}\frac{\xi\,d\xi}{y} + \frac{1}{2}\frac{d\xi}{y}.$$
 (26)

1

Due to

$$\int_{a} \frac{\partial}{\partial \theta} \varphi_{1}(1, \theta) \bigg|_{\theta = \theta_{0}} = 0$$

equation (25) gives rise to

$$\frac{\partial Z}{\partial \theta}(1,\theta)\Big|_{\theta=\theta_0}\cdot\int_a\frac{d\xi}{y}=-\sin\theta_0\int_a\frac{\xi^3\,d\xi}{y^3}\,.$$

Integration of equation (26) yields

$$-\sin\theta_0 \int_a \frac{\xi^3 d\xi}{y^3} = \frac{1}{2} \frac{\cos\theta_0}{\sin\theta_0} \int_a \frac{\xi d\xi}{y} - \frac{1}{2\sin\theta_0} \int_a \frac{d\xi}{y} \,.$$

The first expression on the right is zero and we get

$$\frac{\partial Z}{\partial \theta}(1,\theta) \bigg|_{\theta=\theta_0} = -\frac{1}{2\sin\theta_0},$$

which proves (ii).

The curve C_2 is given by

$$y^2 = (\xi + 4)(\xi^2 - 2r\xi \cos \theta + r^2)$$

and the differential φ_2 looks like

$$\varphi_2 = \frac{\xi - (\xi_2 - 2)}{y} d\xi.$$

Put

$$Q(r,\,\theta):=\frac{1}{2\pi i}\int_a\frac{d\xi}{y}\,,$$

and we have

$$Q(0, \theta) = res_{\xi=0} \left(\frac{d\xi}{\xi\sqrt{\xi+4}} \right) = \frac{1}{2},$$

$$\frac{\partial}{\partial r} Q(r, \theta) \Big|_{r=0} = res_{\xi=0} \left(\frac{\partial}{\partial r} \frac{d\xi}{y} \Big|_{r=0} \right)$$

$$= res_{\xi=0} \left(\frac{\cos\theta \, d\xi}{\xi^2\sqrt{\xi+4}} \right) = -\frac{1}{16}\cos\theta.$$

Consequently,

$$Q(r,\theta) = \frac{1}{2} - \frac{1}{16} r \cos \theta + \mathcal{O}(r^2).$$
(27)

Similarly we put

 $P(r,\theta):=\frac{1}{2\pi i}\int_a\frac{\xi\,d\xi}{y},$

and this yields

$$P(r,\theta) = \frac{1}{2}r\cos\theta + \mathcal{O}(r^2).$$
⁽²⁸⁾

Since the integral of φ_2 over *a* is identically zero, (iii) follows from the equations (27) and (28).

We use this lemma to prove the final step:

PROPOSITION 4.3. There are curves X_e of genus g = 2 which satisfy the conditions $(1), \ldots, (5)$.

Proof. For $\theta = \theta_0$ the differential φ_1 has a root over z = 2. Put $\zeta_1 = 2$. Then Ω_0 and Ω_{∞} have a common root α over x = 1 and condition (1) is fulfilled.

For condition (2) we have to look at α_2 and β_2 . They satisfy the equations

$$\zeta_2\beta_2=2, \qquad 2\alpha_2=\zeta_2.$$

Suppose $\alpha_2 = \beta_2$ holds, then we have $\zeta_2^2 = 4$, but for ζ_2 we know

 $\zeta_2(r,\theta) = 2 + r \cos \theta + \mathcal{O}(r^2).$

For condition (3) we have to examine the roots of $\Omega_{\infty} - \Omega_0 = \tau_2^* \varphi_2$. Due to the equation above for ξ_2 the roots of the polynomial

$$p(x) = (x^2 - \zeta_2 x + 1)(x - 1)$$

don't lie in the branch points and $\Omega_{\infty} - \Omega_0$ has a root of order 1 at α . For small r the conditions (1), (2), (3) are satisfied.

656

Now look at the condition (4). We want to show that the matrix

$$\left(\frac{\partial u_r}{\partial x_i \partial y_j}\right)$$

with $e_1 = \mu$ and $e_2 = \overline{\mu}$ has rank 2. If we rotate the configuration of branch points around the origin, also α_1 is rotated. Moreover, if we move θ for fixed r, the root α_1 can only move on the real axis. Now look at the equations

$$\alpha_1 + \alpha_2 = \frac{1}{2}(\zeta_1 + \zeta_2),$$

 $\alpha_1 \alpha_2 = \frac{1}{2}(\zeta_2 - \zeta_1) + 1.$

Suppose we have

$$\frac{d\alpha_1}{d\theta}\bigg|_{\theta=\theta_0}=0,$$

then we can conclude

$$\left.\frac{d\zeta_1}{d\theta}\right|_{\theta=\theta_0}=0,$$

but

$$\left.\frac{d\zeta_1}{d\theta}\right|_{\theta=\theta_0}=\frac{-r}{2\sin\theta_0}.$$

So, the assumption was false and we get the desired result.

For condition (5) we take the limit $r \to 0$ and we get k(a) = 1/2 (using the identities $\mu = 1, \alpha_2 = 1$). Thus the proof of the theorem is complete.

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CHRISTIAN JAGGY

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