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On the classification of constant mean curvature tori in \mathbb{R}^3

CHRISTIAN JAGGY

1. Introduction

Let S be a compact oriented surface and $i : S \rightarrow \mathbb{R}^3$ an immersion with constant mean curvature. Hopf [6] investigated such immersions, and for genus $(S) = 0$ he showed that $i : S \rightarrow \mathbb{R}^3$ must be an embedding of a round sphere. Conversely, the genus of the surface S is 0, if i is an embedding. This statement was proved by Alexandrov [1]. Only a few years ago Wente [10] and Kapouleas [7] proved the existence of constant mean curvature immersions for genus $(S) = 1$ and genus $(S) \geq 2$, respectively. In this work we will only look at constant mean curvature immersions with genus $(S) = 1$.

First the relation of hyperelliptic curves and constant mean curvature immersions is sketched. For a rigorous formulation see Bobenko [3].

Let u be a solution of the elliptic-sinh Gordon equation

$$u_{w\bar{w}} + \sinh u = 0 \quad (1)$$

on a simply-connected domain $\Omega \subset \mathbb{C}$. There is an algorithm that associates an immersion $i : \Omega \rightarrow \mathbb{R}^3$ to u with constant mean curvature $\frac{1}{2}$. Conversely, every constant mean curvature immersion yields a solution u of equation (1).

On the other hand we can associate quasi-periodic solutions of equation (1) on \mathbb{R}^2 to hyperelliptic curves

$$X : y^2 = x \prod_{i=1}^{2g} (x - e_i) \quad (2)$$

where the branch points are distinct and satisfy

$$e_{i+g} = \frac{1}{\bar{e}_i}, \quad i = 1, \dots, g. \quad (3)$$

We first have to fix some notation to write down an explicit formula for solutions

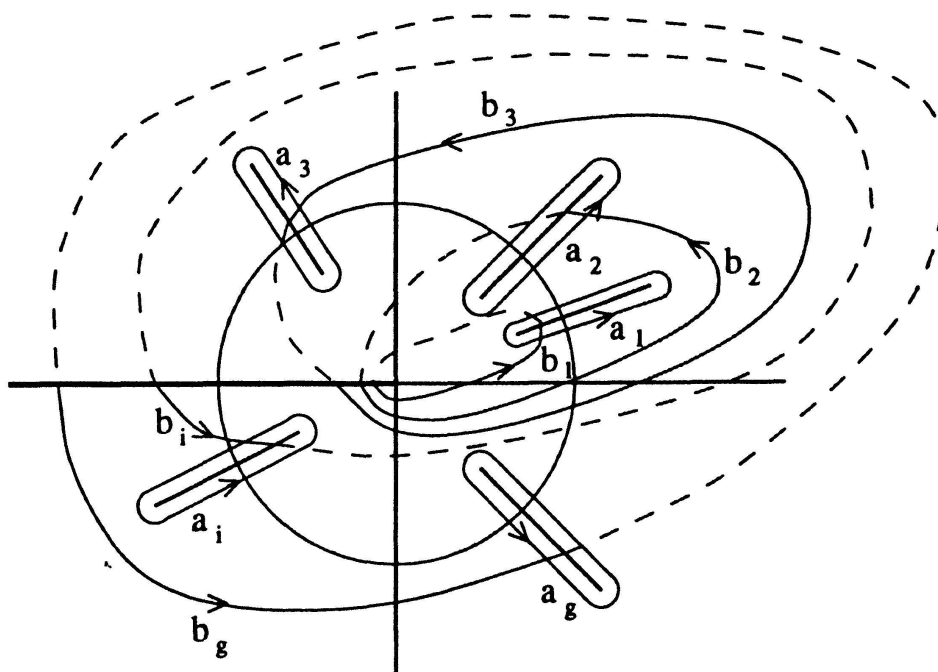


Figure 1

of equation (1). In figure (1) a canonical basis $a_1, \dots, a_g, b_1, \dots, b_g$ of $H_1(X, \mathbb{Z})$ with intersection numbers

$$a_i b_j = \delta_{ij}, \quad a_i a_j = 0, \quad b_i b_j = 0, \quad i, j = 1, \dots, g \quad (4)$$

is introduced. Let Ω_0 and Ω_∞ be meromorphic differentials on X , holomorphic outside 0 and ∞ , respectively, which satisfy the conditions

$$\int_{a_i} \Omega_0 = \int_{a_i} \Omega_\infty = 0, \quad i = 1, \dots, g \quad (5)$$

and

$$\begin{aligned} \Omega_0 &\text{ has a pole of second order at } 0, \\ \Omega_\infty &\text{ has a pole of second order at } \infty. \end{aligned} \quad (6)$$

Define the vectors μ_0, μ_∞ by

$$\begin{aligned} \mu_0 &= \left(\int_{b_1} \Omega_0, \dots, \int_{b_g} \Omega_0 \right) \\ \mu_\infty &= \left(\int_{b_1} \Omega_\infty, \dots, \int_{b_g} \Omega_\infty \right), \end{aligned}$$

and for $\zeta \in \mathbb{C}^g$ put

$$u(\zeta) = 2 \log \frac{\theta\left(\zeta + \left(\frac{1}{2}, \dots, \frac{1}{2}\right)\right)}{\theta(\zeta)}$$

where θ is the Riemann theta function of X for the given homology basis. The function

$$u(\zeta + w\mu_0 + \bar{w}\mu_\infty) \tag{7}$$

is a real quasi-periodic solution of equation (1) for every $\zeta \in \mathbb{R}^g$.

The question arises, whether it is possible to choose X in a way, such that X yields constant mean curvature tori. The answer to this question was given by Bobenko [4] and Pinkall-Sterling [9].

THEOREM 1.1.

(1) *Under the correspondence mentioned above X yields constant mean curvature tori in \mathbb{R}^3 if and only if*

(a) Ω_∞ has a root $p = (x_0, y_0)$ with $|x_0| = 1$;

(b) *Let γ be a path that connects the two points (x_0, y_0) and $(x_0, -y_0)$. Then the span of the vectors*

$$v_0 = \left(\int_\gamma \Omega_0, \int_{b_1} \Omega_0, \dots, \int_{b_g} \Omega_0 \right)$$

$$v_\infty = \left(\int_\gamma \Omega_\infty, \int_{b_1} \Omega_\infty, \dots, \int_{b_g} \Omega_\infty \right)$$

in \mathbb{C}^{g+1} must contain two linearly independent rational vectors. In this case one gets a $(g-2)$ -parameter family of constant mean curvature tori.

(2) *Every constant mean curvature torus arises in such a way.*

It is known that there are no curves satisfying the condition (a) for genus $(X) = 1$. Wente found constant mean curvature tori which are known to correspond to curves with genus $(X) = 2$ or genus $(X) = 3$. In 1991 Ercolani–Knörrer–Trubowitz [5] proved the existence of such curves for even genus (X) . All curves constructed there have the additional property, that the set of branch points is invariant under the map $x \mapsto 1/x$. In this paper the existence of curves X fulfilling the conditions (a) and (b) is proved for genus (X) arbitrary.

2. Preliminaries

The map $\sigma : X \rightarrow X$

$$(x, y) \mapsto \left(\frac{1}{\bar{x}}, \frac{\left(\prod_{i=1}^{2g} e_i \right)^{1/2} \bar{y}}{\bar{x}^{g+1}} \right)$$

is an antiholomorphic involution of X . The sign of $(\prod_{i=1}^{2g} e_i)^{1/2}$ is chosen in such a way, that the points lying over S^1 are fixed points of σ . Then σ_* acts as follows on the cycles:

$$\sigma_*(a_i) = -a_i, \quad i = 1, \dots, g \quad (8)$$

$$\sigma_*(b_i) = b_i + \sum_{j=1}^g \lambda_{ij} a_j, \quad i = 1, \dots, g$$

with $\lambda_{ij} \in \mathbb{Z}$; $i, j = 1, \dots, g$, and

$$\gamma - \sigma_* \gamma = \sum_{j=1}^g \mu_j a_j, \quad (9)$$

with $\mu_j \in \mathbb{Z}$; $j = 1, \dots, g$.

It is possible to choose Ω_0, Ω_∞ in a way, such that

$$\sigma^* \Omega_0 = \bar{\Omega}_\infty$$

holds. It follows that the vectors v_0, v_∞ are complex conjugate. The new vectors

$$v := v_0 + v_\infty$$

$$w := i(v_\infty - v_0)$$

are elements of \mathbb{R}^{g+1} .

Now consider the map $f : \mathbb{C}^g \rightarrow \mathbb{C} \times Gr(2, \mathbb{R}^{g+1})$

$$(e_1, \dots, e_g) \mapsto (\text{root of } \Omega_\infty, \langle v, w \rangle).$$

f is a multivalued function and one should restrict the domain of definition of f to the open subset $U \subset \mathbb{C}^g$, where all the branch points are distinct. $Gr(2, \mathbb{R}^{g+1})$

denotes the Grassmannian of 2-dimensional subspaces of \mathbb{R}^{g+1} . The vectors v and w are linearly independent and $\langle v, w \rangle$ is a welldefined element of $Gr(2, \mathbb{R}^{g+1})$.

It is interesting to look at this map, because if one finds a root $p = (x_0, y_0)$ with $|x_0| = 1$ and if $\langle v, w \rangle$ contains two linearly independent rational vectors, the existence of constant mean curvature tori is guaranteed. In section 3 the following theorem will be proved.

THEOREM 2.1. *Let $e = (e_1, \dots, e_g)$ be in U . Assume that the differentials Ω_0, Ω_∞ on the hyperelliptic curve*

$$X : y^2 = x \prod_{i=1}^{2g} (x - e_i)$$

fulfill the following conditions:

- (1) Ω_0, Ω_∞ have a common root α over $x = 1$,
- (2) Ω_0, Ω_∞ don't have any other common roots,
- (3) $(\Omega_\infty - \Omega_0)(e_m) \neq 0$ for $m = 1, \dots, 2g$, and $\Omega_\infty - \Omega_0$ has a root of order 1 at α .

Then $df(e)$ is invertible.

We denote X_e as the hyperelliptic curve associated to the point $e \in U$. Due to this theorem it follows, that arbitrarily close to e there are points, such that the corresponding curves X_e fulfill conditions (a) and (b). In section 4 we will finally show

THEOREM 2.2. *For every $g \geq 2$ there are curves $X_e, e \in U$, satisfying the conditions (1), (2), (3) above.*

This theorem will be proved by induction on g .

3. Simplification

Proof of Theorem 2.1. Since dimensions are equal it is enough to show that $df(e)$ is injective. The strategy is due to Krichever [8], Bikbaev and Kuksin [2].

Let $e(\tau), \tau \in \mathbb{R}$, be an arbitrary differentiable curve passing through e , such that $f(e(\tau))$ changes only in order τ^2 , in other words

$$\begin{pmatrix} v(\tau) \\ w(\tau) \end{pmatrix} = A(\tau) \begin{pmatrix} v(0) \\ w(0) \end{pmatrix} + \mathcal{O}(\tau^2), \quad \text{with } A(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (10)$$

$$\alpha(\tau) = 1 + \mathcal{O}(\tau^2). \quad (11)$$

We want to conclude that

$$\left. \frac{d}{d\tau} e(\tau) \right|_{\tau=0} = 0$$

holds. This implies that $df(e)$ is injective. Put $B(\tau) := A(\tau)^{-1}$, clearly

$$B(\tau) \begin{pmatrix} v(\tau) \\ w(\tau) \end{pmatrix} = \begin{pmatrix} v(0) \\ w(0) \end{pmatrix} + \mathcal{O}(\tau^2)$$

and after differentiation

$$\dot{B}(0) \begin{pmatrix} v(0) \\ w(0) \end{pmatrix} + \begin{pmatrix} \dot{v}(0) \\ \dot{w}(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (12)$$

These are $2g + 2$ equations, $2g$ of them describe relations among period integrals. Define differentials ω_1, ω_2 by

$$\begin{pmatrix} \omega_1(\tau) \\ \omega_2(\tau) \end{pmatrix} := B(\tau) \begin{pmatrix} \Omega_0(\tau) + \Omega_\infty(\tau) \\ i(\Omega_\infty(\tau) - \Omega_0(\tau)) \end{pmatrix}. \quad (13)$$

By integration of $\omega_1(\tau), \omega_2(\tau)$ one get's multivalued meromorphic functions on $X_{e(\tau)}$:

$$\Omega_i(P, \tau) := \int_{J(P)}^P \omega_i(\tau), \quad i = 1, 2 \quad (14)$$

where J denotes the hyperelliptic involution.

LEMMA 3.1. *The functions*

$$\left. \frac{\partial}{\partial \tau} \Omega_i(P, \tau) \right|_{\tau=0}$$

are single-valued meromorphic functions on X_e . At the points $e_1, \dots, e_{2g}, 0, \infty$ they have first order poles. Furthermore there are non-zero complex numbers c_1, \dots, c_{2g} such that

$$\text{res}_{P=e_m} \left(\left. \frac{\partial}{\partial \tau} \Omega_2(P, \tau) \right|_{\tau=0} \right) = c_m \left. \frac{\partial}{\partial \tau} e_m \right|_{\tau=0}, \quad m = 1, \dots, 2g. \quad (15)$$

Due to this lemma it is enough to show that

$$\left. \frac{\partial}{\partial \tau} \Omega_2(P, \tau) \right|_{\tau=0} \equiv 0.$$

This will prove the theorem. We first prove this lemma, before we continue the proof of the theorem.

Proof. To see that the functions $(\partial/\partial\tau)\Omega_i(P, \tau) \big|_{\tau=0}$ are single-valued, we have to look at the corresponding b -periods:

$$\begin{aligned} \left. \frac{\partial}{\partial \tau} \int_{b_j} \omega_1 \right|_{\tau=0} &= \left. \frac{\partial}{\partial \tau} \int_{b_j} (b_{11}(\Omega_0 + \Omega_\infty) + b_{12}i(\Omega_\infty - \Omega_0)) \right|_{\tau=0} \\ &= \int_{b_j} (\dot{b}_{11}(0)(\Omega_0 + \Omega_\infty) + \dot{b}_{12}(0)i(\Omega_\infty - \Omega_0) + \dot{\Omega}_0 + \dot{\Omega}_\infty) \\ &= 0. \end{aligned}$$

The last identity is true due to equation (12). The same is true for ω_2 and the first statement is proved.

Expand $\omega_i(\tau)$ at $e_m(\tau)$ in the local coordinate $(x - e_m(\tau))^{1/2}$:

$$\omega_i(x, \tau) = \sum_{k=-1}^{\infty} (x - e_m(\tau))^{k/2} x_k^{i,m}(e(\tau)) dx.$$

Put $P = (x, y)$, then we get

$$\begin{aligned} \left. \frac{\partial}{\partial \tau} \Omega_i(P, \tau) \right|_{\tau=0} &= \int_{J(P)} \left. \frac{\partial}{\partial \tau} \omega_i(x, \tau) \right|_{\tau=0} \\ &= 2 \sum_{k=-1}^{\infty} \left(-(x - e_m(0))^{k/2} \frac{\partial}{\partial \tau} e_m(0) x_k^{i,m}(e(0)) \right. \\ &\quad \left. + \frac{2}{2+k} (x - e_m)^{1+k/2} \frac{\partial}{\partial \tau} x_k^{i,m}(e(\tau)) \right) \Big|_{\tau=0}. \end{aligned}$$

It follows that the functions $(\partial/\partial\tau)\Omega_i(P, \tau) \big|_{\tau=0}$ have first order poles at the points e_1, \dots, e_{2g} and the same is true for 0 and ∞ by a similar calculation. Due to the assumption (3) in Theorem 2.1 the claim about the numbers c_m is obvious. \square

Let's continue the proof of the theorem. Take $P \in X_e$ with $\omega_2(P) \neq 0$. The implicit function theorem yields a curve $P(\tau)$ with

$$\Omega_2(P(\tau), \tau) = \Omega_2(P, 0) \quad (16)$$

and after differentiation

$$\left. \frac{d}{d\tau} \Omega_2(P(\tau), \tau) \right|_{\tau=0} = \omega_2(P) \left. \frac{d}{d\tau} P(\tau) \right|_{\tau=0} + \left. \frac{\partial}{\partial \tau} \Omega_2(P, \tau) \right|_{\tau=0} = 0. \quad (17)$$

Define a new function

$$\dot{\Omega}_1(P) := \left. \frac{d}{d\tau} \Omega_1(P(\tau), \tau) \right|_{\tau=0}. \quad (18)$$

The function $\dot{\Omega}_1$ is welldefined and by the equation (17) above one gets

$$\dot{\Omega}_1(P) = \left. \frac{\partial}{\partial \tau} \Omega_1(P, \tau) \right|_{\tau=0} - \left. \frac{\partial}{\partial \tau} \Omega_2(P, \tau) \right|_{\tau=0} \cdot \frac{\omega_1(P)}{\omega_2(P)}. \quad (19)$$

It follows that $\dot{\Omega}_1$ is a meromorphic function on X . To finish the proof of the theorem we need the following lemma:

LEMMA 3.2. *The functions $(\partial/\partial\tau)\Omega_i(P, \tau) \big|_{\tau=0}$ have a root of order 2 at α .*

Proof. By equation (11) the differentials $(\partial/\partial\tau)\omega_i(P, \tau) \big|_{\tau=0}$ have a root of order 1 at α . The functions

$$h_i(P) := \int_{\alpha}^P \frac{\partial}{\partial \tau} \omega_i(P, \tau) \bigg|_{\tau=0}$$

have a root or order 2 at α . Now look at

$$\begin{aligned} \left. \frac{\partial}{\partial \tau} \Omega_i(P, \tau) \right|_{\tau=0} &= \int_{J(P)}^P \frac{\partial}{\partial \tau} \omega_i(P, \tau) \bigg|_{\tau=0} \\ &= \int_{J(P)}^{J(\alpha)} \frac{\partial}{\partial \tau} \omega_i(P, \tau) \bigg|_{\tau=0} + \int_{J(\alpha)}^{\alpha} \frac{\partial}{\partial \tau} \omega_i(P, \tau) \bigg|_{\tau=0} \\ &\quad + \int_{\alpha}^P \frac{\partial}{\partial \tau} \omega_i(P, \tau) \bigg|_{\tau=0}. \end{aligned}$$

With equation (12) one gets

$$\int_{J(\alpha)} \frac{\partial}{\partial \tau} \omega_i(P, \tau) \Big|_{\tau=0} = 0$$

and this implies

$$\frac{\partial}{\partial \tau} \Omega_i(P, \tau) \Big|_{\tau=0} = 2h_i(P). \quad \square$$

$\dot{\Omega}_1$ has $2g$ roots at the branch points e_1, \dots, e_{2g} and another 4 roots over $x = 1$. The roots of ω_2 lying outside the set $\{\alpha, J(\alpha)\}$ yield $2g$ poles of $\dot{\Omega}_1$, together with the simple poles at 0 and ∞ we see that $\dot{\Omega}_1$ has at most $2g + 2$ poles. Consequently $\dot{\Omega}_1$ is the zero-function and one gets the following equation:

$$\frac{\partial}{\partial \tau} \Omega_2(P, \tau) \Big|_{\tau=0} \cdot \omega_1(P) = \frac{\partial}{\partial \tau} \Omega_1(P, \tau) \Big|_{\tau=0} \cdot \omega_2(P). \quad (20)$$

There are $2g$ roots of ω_2 outside the set $\{\alpha, J(\alpha)\}$, which can't coincide with roots of ω_1 due to the assumption (2). These $2g$ roots of ω_2 must be roots of $(\partial/\partial \tau)\Omega_2(P, \tau) \Big|_{\tau=0}$. Together with the 4 roots lying over $x = 1$ we conclude that $(\partial/\partial \tau)\Omega_2(P, \tau) \Big|_{\tau=0}$ has at least $2g + 4$ roots. But $(\partial/\partial \tau)\Omega_2(P, \tau) \Big|_{\tau=0}$ has at most $2g + 2$ poles at the branch points. We get $(\partial/\partial \tau)\Omega_2(P, \tau) \Big|_{\tau=0} \equiv 0$ and by Lemma 3.1 $(d/d\tau)e(\tau) \Big|_{\tau=0} = 0$ follows. This proves the theorem. \square

4. Induction

Theorem 2.2 will be proved by induction on g . We will see that a good configuration of branch points for genus g yields a good configuration of branch points for genus $g + 1$. Let's first prepare the induction step.

Take a point $e = (e_1, \dots, e_g)$ for which the conditions (1), (2), (3) are fulfilled. The corresponding curve X_e and differentials $\Omega_0^e, \Omega_\infty^e, \Omega_0^e + \Omega_\infty^e$ look like

$$X_e : y_0^2 = x \prod_{i=1}^g (x - e_i) \left(x - \frac{1}{\bar{e}_i} \right)$$

$$\Omega_0^e = \frac{c_g \prod_{i=1}^g (x - \beta_i)}{xy_0} dx, \quad \beta_1 = 1, \quad c_g \in \mathbb{C},$$

$$\Omega_{\infty}^g = \frac{\prod_{i=1}^g (x - \alpha_i)}{y_0} dx, \quad \alpha_1 = 1,$$

$$\Omega_{\infty}^g - \Omega_0^g = \frac{d_g \prod_{i=1}^{g+1} (x - \xi_i)}{xy_0} dx, \quad \xi_1 = 1, \quad d_g \in \mathbb{C}.$$

For $(e_1, \dots, e_g, a, t) \in U \times S^1 \times (-\varepsilon, \varepsilon)$, $\varepsilon > 0$ we define

$$X_{(e,a,t)} : y^2 = x(x - ae^t)(x - ae^{-t}) \prod_{i=1}^g (x - e_i) \left(x - \frac{1}{\bar{e}_i} \right),$$

and corresponding normalized differentials

$$\Omega_0^{g+1} = \frac{c_{g+1} \prod_{i=1}^{g+1} (x - \beta_i^{g+1})}{xy} dx, \quad c_{g+1} \in \mathbb{C},$$

$$\Omega_{\infty}^{g+1} = \frac{\prod_{i=1}^{g+1} (x - \alpha_i^{g+1})}{y} dx,$$

$$\Omega_{\infty}^{g+1} - \Omega_0^{g+1} = \frac{d_{g+1} \prod_{i=1}^{g+2} (x - \xi_i^{g+1})}{xy} dx, \quad d_{g+1} \in \mathbb{C}.$$

Due to the compactness of $X_{(e,a,t)}$, the normalization conditions and the residue theorem one has the following equations

$$\begin{aligned} \alpha_i^{g+1}(e, a, 0) &= \alpha_i, & i &= 1, \dots, g, \\ \alpha_{g+1}^{g+1}(e, a, 0) &= a \\ \xi_i^{g+1}(e, a, 0) &= \xi_i, & i &= 1, \dots, g+1, \\ \xi_{g+2}^{g+1}(e, a, 0) &= a \end{aligned} \tag{21}$$

and

$$\Omega_{\infty}^{g+1}(e, a, 0) = \Omega_{\infty}^g. \tag{22}$$

Due to the reduction (21) we delete the superscript $g + 1$ from $\alpha_i^{g+1}, \xi_i^{g+1}$. Now put

$$\alpha_1 = u_1 + iu_2, \quad e_i = x_i + iy_i, \quad i = 1, \dots, g$$

and let's impose the further conditions on X_e

$$(4) \quad \text{rank} \left(\frac{\partial u_r}{\partial x_i \partial y_j} \right) = 2, \quad r = 1, 2,$$

(5) the real part of the meromorphic function

$$k(x) = 1 + x \frac{\frac{\partial}{\partial x} \frac{\Omega_\infty^g}{dx}}{\frac{\Omega_\infty^g}{dx}}$$

doesn't vanish identically on S^1 .

The conditions (4) and (5) are used to prove the following lemma:

LEMMA 4.1. *The map $h : U \times S^1 \times (-\varepsilon, \varepsilon) \rightarrow \mathbb{C} \times \mathbb{R}$*

$$(e, a, t^2) \mapsto (\alpha_1, |\alpha_{g+1}|)$$

has maximal rank in a point $P = (e, a, 0)$, where $\text{Re}(k(a)) \neq 0$.

REMARK. This lemma together with the property

$$\left. \frac{\partial}{\partial \tau} \xi_{g+2} \right|_{t=0} = 0$$

yields the existence of curves $X_{(e,a,t)}$ of genus $g + 1$, which satisfy the conditions (1), (2), (3). Taking t small enough the conditions (4) and (5) are also fulfilled.

Proof. Due to the reduction (21) and condition (4) we have

$$\text{rank} \left(\frac{\partial u_r}{\partial x_i \partial y_j} \right) = 2, \quad \left(\frac{\partial |\alpha_{g+1}|}{\partial x_i \partial y_j} \right) \Big|_P = 0, \quad r = 1, 2; \quad i, j = 1, \dots, g.$$

It remains to prove that

$$\left. \frac{\partial}{\partial t^2} |\alpha_{g+1}| \right|_P = \operatorname{Re} \left(\left. \frac{\partial}{\partial t^2} \alpha_{g+1} \bar{\alpha}_{g+1} \right) \right|_P \neq 0.$$

For this we will deduce an equation for $(\partial/\partial t^2)\alpha_{g+1}|_P$. Differentiation of Ω_∞^{g+1} yields

$$\left. \frac{\partial}{\partial t^2} \Omega_\infty^{g+1} \right|_P = \frac{\left(-\sum_{i=1}^{g+1} \left. \frac{\partial}{\partial t^2} \alpha_i \right|_P \frac{1}{x - \alpha_i} \right) \prod_{i=1}^g (x - \alpha_i)}{y_0} dx + \frac{ax \prod_{i=1}^g (x - \alpha_i)}{2(x - a)^2 y_0} dx.$$

Since

$$\operatorname{res}_{x=a} \left(\left. \frac{\partial}{\partial t^2} \Omega_\infty^{g+1} \right|_P \right) = 0$$

we get the equation

$$\operatorname{res}_{x=a} \left(\left. \frac{\partial}{\partial t^2} \alpha_{g+1} \right|_P \Omega_\infty^g \right) = \operatorname{res}_{x=a} \left(\frac{ax}{2(x-a)^2} \Omega_\infty^g \right),$$

and

$$\left. \frac{\partial}{\partial t^2} \alpha_{g+1} \right|_P \cdot \bar{a} = \frac{1}{2} + \frac{1}{2} x \frac{\frac{\partial}{\partial x} \Omega_\infty^g}{\Omega_\infty^g} \bigg|_{x=a}.$$

Since $\operatorname{Re}(k(a)) \neq 0$ we have

$$\left. \frac{\partial}{\partial t^2} |\alpha_{g+1}| \right|_P \neq 0,$$

and the lemma is proved. \square

Finally, we have to prove the existence of curves X_e of genus $g = 2$ which satisfy the conditions (1) up to (5). For the beginning of the induction results of Bobenko [4] and Ercolani–Knörrer–Trubowitz [5] are used.

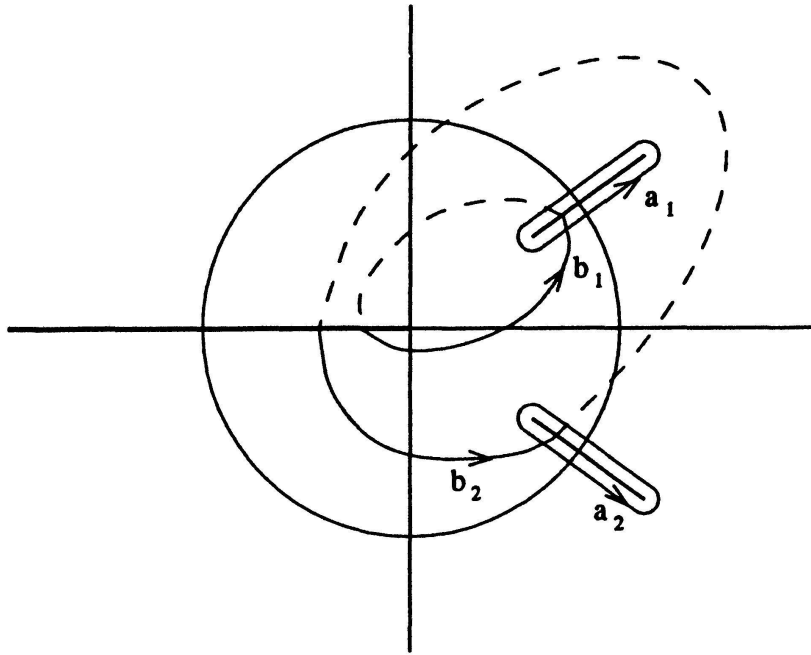


Figure 2

Let X_e be the hyperelliptic curve (figure (2))

$$X_e : y^2 = x(x - \mu) \left(x - \frac{1}{\bar{\mu}} \right) (x - \bar{\mu}) \left(x - \frac{1}{\mu} \right) \quad (23)$$

with normalized differentials

$$\Omega_0 = \frac{\bar{\alpha}_1 \bar{\alpha}_2 (x - \beta_1) (x - \beta_2)}{xy} dx,$$

$$\Omega_\infty = \frac{(x - \alpha_1) (x - \alpha_2)}{y} dx.$$

Let C_1, C_2 be the elliptic curves

$$C_1 : y^2 = (z - 2)(z - \lambda)(z - \bar{\lambda}), \quad \lambda = \mu + \frac{1}{\mu}$$

$$C_2 : y^2 = (z + 2)(z - \lambda)(z - \bar{\lambda})$$

and

$$\varphi_v = \frac{(z - \zeta_v)}{y} dz$$

meromorphic differentials on C_v with vanishing a -periods (see figure (3)).

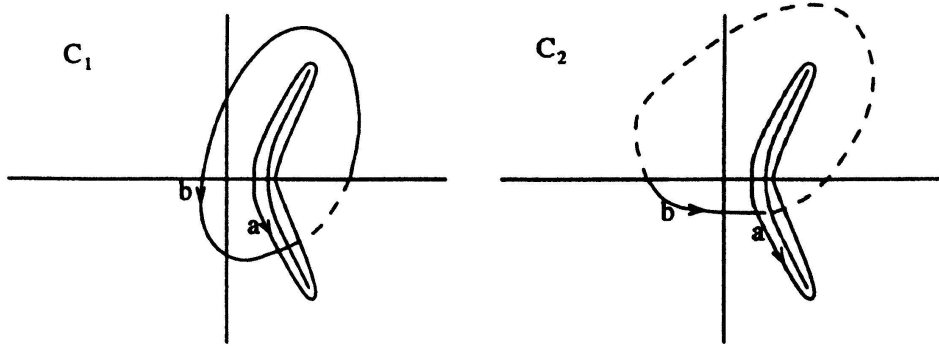


Figure 3

There are maps $\tau_v : X_e \rightarrow C_v$ given by

$$(x, y) \mapsto \left(x + \frac{1}{x}, \frac{x + (-1)^v}{x^2} y \right).$$

The pullback of φ_v with respect to τ_v is given by

$$\tau_1^* \varphi_1 = \frac{(x^2 - \zeta_1 x + 1)(x + 1)}{xy} dx,$$

$$\tau_2^* \varphi_2 = \frac{(x^2 - \zeta_2 x + 1)(x - 1)}{xy} dx.$$

Taking the sum and the difference one gets

$$\tau_1^* \varphi_1 + \tau_2^* \varphi_2 = 2\Omega_\infty,$$

$$\tau_1^* \varphi_1 - \tau_2^* \varphi_2 = 2\Omega_0.$$

Introduce new parameters r, θ by the equation

$$\lambda = 2 + re^{i\theta}. \quad (24)$$

Now, look at the following lemma:

LEMMA 4.2.

(i) *There is a unique $\theta = \theta_0 \in (0, \pi/2)$, such that $\xi_1(r, \theta_0) = 2$ holds for arbitrary r ,*

$$(ii) \quad \frac{\partial \xi_1}{\partial \theta}(r, \theta_0) = \frac{-r}{2 \sin(\theta_0)},$$

$$(iii) \quad \xi_2(r, \theta) = 2 + r \cos(\theta) + \mathcal{O}(r^2).$$

Proof. Let's make the change of variables $\xi = z - 2$ and let's define

$$Z(r, \theta) := \xi_1(r, \theta) - 2.$$

The curve C_1 is given by

$$y^2 = \xi(\xi^2 - 2r\xi \cos \theta + r^2)$$

and for the differential φ_1 we have

$$\varphi_1 = \frac{\xi - Z(r, \theta)}{y} d\xi.$$

Following Bobenko [4] one has

$$\int_a^\pi \frac{\xi d\xi}{y} = \sqrt{8r} \int_\theta^\pi \frac{\cos t dt}{\sqrt{\cos \theta - \cos t}},$$

and there is a unique $\theta = \theta_0 \in (0, \pi/2)$ for which

$$\int_\theta^\pi \frac{\cos t dt}{\sqrt{\cos \theta - \cos t}} = 0.$$

Consequently, we have the equation

$$Z(r, \theta) = 0 \Leftrightarrow \theta = \theta_0.$$

To prove (ii) we first observe that $Z(r, \theta) = rZ(1, \theta)$. Differentiation of φ_1 yields

$$\left. \frac{\partial}{\partial \theta} \varphi_1(1, \theta) \right|_{\theta=\theta_0} = \left(-\frac{\partial Z}{\partial \theta}(1, \theta) \right) \Big|_{\theta=\theta_0} \cdot \frac{d\xi}{y} - \sin \theta_0 \frac{\xi^3 d\xi}{y^3} \quad (25)$$

and

$$d\left(\frac{-\xi^2 \cos \theta_0 + \xi}{y} \right) = -\sin^2 \theta_0 \frac{\xi^3 d\xi}{y^3} - \frac{1}{2} \cos \theta_0 \frac{\xi d\xi}{y} + \frac{1}{2} \frac{d\xi}{y}. \quad (26)$$

Due to

$$\int_a^\pi \frac{\partial}{\partial \theta} \varphi_1(1, \theta) \Big|_{\theta=\theta_0} = 0$$

equation (25) gives rise to

$$\frac{\partial Z}{\partial \theta}(1, \theta) \Big|_{\theta=\theta_0} \cdot \int_a \frac{d\xi}{y} = -\sin \theta_0 \int_a \frac{\xi^3 d\xi}{y^3}.$$

Integration of equation (26) yields

$$-\sin \theta_0 \int_a \frac{\xi^3 d\xi}{y^3} = \frac{1}{2} \frac{\cos \theta_0}{\sin \theta_0} \int_a \frac{\xi d\xi}{y} - \frac{1}{2 \sin \theta_0} \int_a \frac{d\xi}{y}.$$

The first expression on the right is zero and we get

$$\frac{\partial Z}{\partial \theta}(1, \theta) \Big|_{\theta=\theta_0} = -\frac{1}{2 \sin \theta_0},$$

which proves (ii).

The curve C_2 is given by

$$y^2 = (\xi + 4)(\xi^2 - 2r\xi \cos \theta + r^2)$$

and the differential φ_2 looks like

$$\varphi_2 = \frac{\xi - (\xi_2 - 2)}{y} d\xi.$$

Put

$$Q(r, \theta) := \frac{1}{2\pi i} \int_a \frac{d\xi}{y},$$

and we have

$$\begin{aligned} Q(0, \theta) &= \operatorname{res}_{\xi=0} \left(\frac{d\xi}{\xi \sqrt{\xi+4}} \right) = \frac{1}{2}, \\ \frac{\partial}{\partial r} Q(r, \theta) \Big|_{r=0} &= \operatorname{res}_{\xi=0} \left(\frac{\partial}{\partial r} \frac{d\xi}{y} \Big|_{r=0} \right) \\ &= \operatorname{res}_{\xi=0} \left(\frac{\cos \theta d\xi}{\xi^2 \sqrt{\xi+4}} \right) = -\frac{1}{16} \cos \theta. \end{aligned}$$

Consequently,

$$Q(r, \theta) = \frac{1}{2} - \frac{1}{16} r \cos \theta + \mathcal{O}(r^2). \quad (27)$$

Similarly we put

$$P(r, \theta) := \frac{1}{2\pi i} \int_a \frac{\xi d\xi}{y},$$

and this yields

$$P(r, \theta) = \frac{1}{2} r \cos \theta + \mathcal{O}(r^2). \quad (28)$$

Since the integral of φ_2 over a is identically zero, (iii) follows from the equations (27) and (28). \square

We use this lemma to prove the final step:

PROPOSITION 4.3. *There are curves X_e of genus $g = 2$ which satisfy the conditions (1), \dots , (5).*

Proof. For $\theta = \theta_0$ the differential φ_1 has a root over $z = 2$. Put $\zeta_1 = 2$. Then Ω_0 and Ω_∞ have a common root α over $x = 1$ and condition (1) is fulfilled.

For condition (2) we have to look at α_2 and β_2 . They satisfy the equations

$$\zeta_2 \beta_2 = 2, \quad 2\alpha_2 = \zeta_2.$$

Suppose $\alpha_2 = \beta_2$ holds, then we have $\zeta_2^2 = 4$, but for ζ_2 we know

$$\zeta_2(r, \theta) = 2 + r \cos \theta + \mathcal{O}(r^2).$$

For condition (3) we have to examine the roots of $\Omega_\infty - \Omega_0 = \tau_2^* \varphi_2$. Due to the equation above for ζ_2 the roots of the polynomial

$$p(x) = (x^2 - \zeta_2 x + 1)(x - 1)$$

don't lie in the branch points and $\Omega_\infty - \Omega_0$ has a root of order 1 at α . For small r the conditions (1), (2), (3) are satisfied.

Now look at the condition (4). We want to show that the matrix

$$\left(\frac{\partial u_r}{\partial x_i \partial y_j} \right)$$

with $e_1 = \mu$ and $e_2 = \bar{\mu}$ has rank 2. If we rotate the configuration of branch points around the origin, also α_1 is rotated. Moreover, if we move θ for fixed r , the root α_1 can only move on the real axis. Now look at the equations

$$\alpha_1 + \alpha_2 = \frac{1}{2} (\zeta_1 + \zeta_2),$$

$$\alpha_1 \alpha_2 = \frac{1}{2} (\zeta_2 - \zeta_1) + 1.$$

Suppose we have

$$\left. \frac{d\alpha_1}{d\theta} \right|_{\theta=\theta_0} = 0,$$

then we can conclude

$$\left. \frac{d\zeta_1}{d\theta} \right|_{\theta=\theta_0} = 0,$$

but

$$\left. \frac{d\zeta_1}{d\theta} \right|_{\theta=\theta_0} = \frac{-r}{2 \sin \theta_0}.$$

So, the assumption was false and we get the desired result.

For condition (5) we take the limit $r \rightarrow 0$ and we get $k(a) = 1/2$ (using the identities $\mu = 1, \alpha_2 = 1$). Thus the proof of the theorem is complete. \square

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