Minimal singularities for representations of Dynkin quivers.

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Minimal singularities for representations of Dynkin quivers

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Abstract. We develop some reduction techniques for the study of singularities in orbit closures of finite dimensional modules. This enables us to classify all singularities occurring in minimal degenerations of representations of Dynkin quivers. They are all smoothly equivalent to the singularity at the zero-matrix inside the $p \times q$-matrices of rank at most one.

1. Introduction

Given a finitely generated associative algebra over some algebraically closed field, it is an interesting task to study geometric properties of the associated varieties of d-dimensional modules endowed with the natural Gl_d-action. For instance, one would like to know which modules belong to the closure of a fixed orbit and which singularities occur. But even for representation-finite algebras both problems are still open.

However, in characteristic zero the geometric structure of the modules over the truncated polynomial algebra $k[X]/X^n$ and over the path algebra of an equi-oriented Dynkin quiver of type A_n is quite completely analyzed by H. Kraft and C. Procesi (s. [13, 14]) and by S. Abeasis, A. del Fra and H. Kraft (s. [2]) in three nice articles which stimulated and influenced the present paper very much. Later on their methods and results were generalized to representations of an oriented cycle by G. Kempken on one side (s. [11]) and to positive characteristic by S. Donkin on the other side (s. [8]). Their main results are the normality of the orbit closures and – depending on this – the precise description of the singularities occurring in minimal degenerations.

Here we extend by different methods in a characteristic free manner the second result to all path algebras over Dynkin quivers of type A_n, D_n, E_6, E_7 or E_8 with an arbitrary orientation. Remember that P. Gabriel has shown in [9], that these are exactly the connected quivers having only a finite number of indecomposable representations up to isomorphism. To study the minimal singularities, we develop several general reduction techniques some of which have been obtained in special cases by the authors mentioned before. For example, we show that the singularity of a degeneration is not influenced by cancellation of a common direct summand provided the codimensions of the orbit closures remain the same.

Now we describe our results in more detail thereby fixing some notations and conventions. We work always over an algebraically closed field k of arbitrary characteristic and we consider k-varieties, i.e. reduced separated schemes of finite type over k. A point of such a variety means a closed point. The only topology we are dealing with is the Zariski topology. We denote the closure of a set X by $\bar{X}$.

If A is a finite dimensional associative k-algebra with basis $a_1 = 1, \ldots, a_\alpha$, we have the corresponding structure constants defined by

$$a_i a_j = \sum a_{ijk} a_k.$$

The affine variety Mod_A^d of d-dimensional unital left A-modules consists in the α-tuples

$$m = (m_1, \ldots, m_\alpha)$$

of $d \times d$-matrices with coefficients in k such that m_1 is the identity and

$$m_i m_j = \sum a_{ijk} m_k$$

for all indices i and j. The general linear group $Gl_d(k)$ acts on Mod_A^d by conjugation, and the orbits correspond to the isomorphism classes of d-dimensional modules. We denote by $O(m)$ the orbit of a point m in Mod_A^d and by M the A-module on k^d given by m. By abuse of notation we also write M for the isomorphism class of M. Thus N is a degeneration of M if $O(n)$ belongs to the closure of $O(m)$, and we denote this fact by $M \leq_{deg} N$ and *not* by $N \leq_{deg} M$ as one might expect. It is not clear how to characterize the partial order $\leq_{deg}$ on the set of isomorphism classes of d-dimensional modules in terms of representation theory.

However, there are two other partial orders $\leq_{ext}$ and $\leq$ on the isomorphism classes which are defined in terms of representation theory as follows (s. [1, 17, 7]):

- $M \leq_{ext} N :\Leftrightarrow$ there are modules M_i, U_i, V_i and exact sequences $0 \to U_i \to M_i \to V_i \to 0$ such that $M = M_1$, $M_{i+1} = U_i \oplus V_i$ and $N = M_{n+1}$ are true for some natural number n.
- $M \leq N :\Leftrightarrow [M, X] \leq [N, X]$ holds for all modules X.

Here and later on we abbreviate $dim_k Hom_A(M, X)$ by $[M, X]$ and $dim_k Ext_A^i(M, X)$ by $[M, X]^i$. Note that $\leq$ is a partial order on the isomorphism classes by a result of M. Auslander (s. [4, 7]). Furthermore, $M \leq N$ is also equivalent to the inequalities $[X, M] \leq [X, N]$ for all modules X.

It is easy to see that

$$M \leq_{ext} N \Rightarrow M \leq_{deg} N \Rightarrow M \leq N$$

holds for all modules (s. [7]). Unfortunately, the reverse implications are not true in general, and it is interesting to find out when they are. For preprojective modules, i.e. modules all whose indecomposable direct summands live on preprojective components, the above three partial orders all coincide by [7]. We do not want to recall here all the basic notions of representation theory used in the foregoing statement (see e.g. [10, 18]), but we only stress the point that *all* modules over path algebras of Dynkin quivers are preprojective. In the sequel, we call two modules *disjoint* provided they have no direct summand in common.

Two pointed varieties (X, x) and (Y, y) are smoothly equivalent if there are smooth morphisms $\lambda : (Z, z) \to (X, x)$ and $\varrho : (Z, z) \to (Y, y)$ of pointed varieties. A smooth morphism is called *very smooth* provided it does not involve étale morphisms, i.e. it is the composition of an open immersion and a vector bundle projection (see [3] for a good propaganda of smooth morphisms etc.). If the morphisms in the above definition are both very smooth the two pointed varieties are very smoothly equivalent. This is an equivalence relation because very smooth morphisms are obviously stable under composition and base change. We are mainly interested in the case where X is the closure $\overline{O(m)}$ of an orbit of a module M and x is a minimal degeneration N of M. Such a pointed variety $(\overline{O(m)}, n)$ is called a minimal singularity.

For instance, let A be the path algebra of the quiver $1 \to 2$. This algebra has only two one-dimensional indecomposable modules S and T and one other two-dimensional indecomposable P. Given any natural numbers p and q, the module $P \oplus S^{p-1} \oplus T^{q-1}$ has $S^p \oplus T^q$ as the only proper degeneration. The corresponding minimal singularity is very smoothly equivalent to the pointed variety $(D(p, q), 0)$, where $D(p, q)$ is the set of $p \times q$-matrices with rank ≤ 1. These determinantal singularities are reasonably well-understood (s. [2]).

Our main result asserts that all minimal singularities occurring in representations of Dynkin quivers are very smoothly equivalent to some $(D(p, q), 0)$. It is somewhat surprising and disappointing that the complexity of the quiver is not reflected by the minimal singularities. Therefore we believe that the methods used to derive the foregoing result are more interesting than the result itself, and we describe these methods and some other related topics in some detail.

In section 2 we compare the pointed varieties $(\overline{O(m)}, m')$ and $(\overline{O(q)}, q')$ when Q resp. Q' are quotients of M resp. M' by the same module U. Under some – rather strong and technical – conditions both pointed varieties are very smoothly equivalent. Nevertheless the theorem obtained is strong enough to imply the results of H. Kraft and C. Procesi on minimal singularities of matrices (s. [14]).

Section 3 contains the handy cancellation result of common direct summands which was mentioned before. In section 4 we show that a tilting functor behaves very well with respect to geometric properties of torsion modules resp. torsion free modules. The next paragraph asserts that one can shrink certain arrows in the Gabriel quiver of A. We also show that "to understand the geometry of the representations of the double-loop means to understand the geometry of the representations of any finitely generated algebra". This finishes the general part of this note.

In section 6, we study first the set-theoretic structure of minimal degenerations provided the partial orders $\leq_{ext}$ and $\leq$ coincide. Then we show that the geometric structure of certain minimal singularities depends on an irreducible cone with an isolated singularity at its vertex and on two natural numbers p and q. In case the cone is a straight line we find the pointed varieties $(D(p, q), 0)$ introduced before. This case occurs for all minimal singularities of representations of Dynkin quivers as follows from the fact that the codimensions of the orbits in a minimal disjoint degeneration differ by one only. This result has been found by U. Markolf via computer. We include a "theoretical" proof in section 7. It is the only point in the whole article where some sort of classification is used. There is some evidence that the codimension one result holds for the much more general class of modules over representation directed algebras. Unfortunately, we can prove this so far only in some generic situations which include degenerations of indecomposables.

Finally, I want to thank H. Kraft for pointing out some inaccuracies in the first version of this article.

2. Cancellation of submodules

2.1. *The general set-up*

The definitions that follow and slight variations thereof are central for the whole paper and should be read carefully. This section refines chapter 2 of my previous paper [7].

Let r, t and $s = r + t$ be three natural numbers. We want to have a geometric way to produce t-dimensional quotients of s-dimensional modules by r-dimensional submodules. So let $\mathscr{U}$ be a subvariety of Mod_A^r and $\mathscr{M}$ be a subvariety of Mod_A^s. Then we introduce the variety $\mathscr{V}$ consisting of triples $(u, m, g = (g_1, g_2))$ with $u \in \mathscr{U}$, $m \in \mathscr{M}$, $g_1 \in k^{s \times r}$, $g_2 \in k^{s \times t}$ such that $g_1 u = m g_1$, i.e. $g_1 u_i = m_i g_1$ for all i. In the sequel, we will often use this abbreviated notation. One can think of a point $(u, m, g = (g_1, g_2))$ in $\mathscr{V}$ with invertible g as a module M together with a basis, namely the columns of g, such that the first r base vectors generate the submodule U. Clearly, the fibre of the projection $p : \mathscr{V} \to \mathscr{U} \times \mathscr{M}$ over (u, m) is $Hom_A(U, M) \times k^{s \times t}$. The defining conditions of $\mathscr{V}$ can be rewritten in matrix form as a system of homogeneous linear equations where the entries of g_1 are the unknowns and the coefficients depend linearly on the coefficients of u and m. The set of solutions is $Hom_A(U, M)$.

On the open subvariety $\mathscr{V}'$, where g is invertible, we can define the cokernel morphism

$$c : \mathscr{V}' \to Mod_A^t$$

by $c(u, m, g) = v$, where $g^{-1}mg$ has the triangular shape

$$\begin{bmatrix} u & z \\ 0 & v \end{bmatrix}$$

because of $g_1 u = m g_1$.

Of course, $c(\mathscr{V}')$ – which can be empty – is exactly the constructible set of all quotients of some M in $\mathscr{M}$ by some U in $\mathscr{U}$. But without any further assumptions neither the projection $p : \mathscr{V} \to \mathscr{U} \times \mathscr{M}$ nor c have good geometric properties. So we assume in addition that the number $[U, M]$ is independent of $U \in \mathscr{U}$ and $M \in \mathscr{M}$. Then p is a vector bundle projection, whence open. Therefore, $\mathscr{V}$, $\mathscr{V}'$ and $c(\mathscr{V}')$ are all irreducible provided $\mathscr{U}$ and $\mathscr{M}$ are so. If $c(\mathscr{V}')$ contains a dense orbit $O(q)$, we call Q the *generic quotient*. In the special case where $\mathscr{U}$ and $\mathscr{M}$ are (contained in) the orbits of u and m we speak about the *generic quotient* of M by U.

On the other hand, to bring c under control we look at extensions. So we start out with two irreducible subvarieties $\mathscr{U}$ of Mod_A^r and $\mathscr{Q}$ of Mod_A^t, such that the dimension of the space $Z(v, u)$ of 1-cocycles is independent of v in $\mathscr{Q}$ and u in $\mathscr{U}$. Recall that $Z(v, u)$ consists of the tuples $z = (0, z_2, \ldots, z_\alpha)$ in $k^{r \times t}$ such that

$$\sum a_{ijk} z_k = u_i z_j + z_i v_j$$

holds for all i and j. Thus $Z(v, u)$ is nothing but the set of solutions of a system of homogeneous linear equations whose coefficients depend polynomially on the entries of u and v. Therefore, the map $(u, v) \mapsto dim\, Z(u, v)$ is upper semi-continuous. Let $B(v, u)$ be the subspace of coboundaries which is the image of the linear map from $k^{r \times t}$ to $Z(v, u)$ sending h to the tuple with ith entry $hv_i - u_i h$. Because of

$$Ext^1(V, U) = Z(V, U)/B(V, U)$$

we have

$$dim\, Z(V, U) = [V, U]^1 + dim\, B(V, U) = [V, U]^1 - [V, U] + rt.$$

Thus $dim\, Z(V, U)$ is constant if and only if $[V, U]^1 - [V, U]$ is so. In that case we have another vector bundle

$$p' : \mathscr{Z} \to \mathscr{U} \times \mathscr{Q}$$

with irreducible total space

$$\mathscr{Z} = \left\{ \begin{bmatrix} u & z \\ 0 & v \end{bmatrix} \middle| u \in \mathscr{U}, v \in \mathscr{Q}, z \in Z(v, u) \right\}.$$

Clearly, the image of the conjugation

$$Gl_s \times \mathscr{Z} \to Mod_A^s$$

is the irreducible constructible set of all extensions of some V in $\mathscr{Q}$ by some U in $\mathscr{U}$. If this set contains a dense orbit we speak about the *generic extension.*

Now, in the situation of our next theorem both points of view fit together well.

THEOREM. *Let U, M, M', Q and Q' be finite dimensional modules satisfying the following conditions:*

(a) $[U, M] = [U, M']$.

(b) $[Q, U]^1 - [Q, U] = [Q', U]^1 - [Q', U]$.

(c) *Q is the generic quotient of M by U, and M is the generic extension of Q by U.*

(d) *Q' is a quotient of M' by U.*

Then M degenerates to M' if and only if Q degenerates to Q'. In that case the pointed varieties $(\overline{O(m)}, m')$ and $(\overline{O(q)}, q')$ are very smoothly equivalent.

2.2. *An application to nilpotent matrices*

Before we prove the theorem, we derive from it the results of H. Kraft and C. Procesi on minimal singularities of conjugacy classes of nilpotent matrices (s. [14]). They represent such a conjugacy class by the Young diagram of the corresponding partition, and they introduce two reductions, namely "erasing a common first column" and "erasing a common first row", which do not change the type of the singularity. Then it follows easily that any minimal singularity is smooth equivalent to the well-understood subregular singularity inside the set of nilpotent matrices of some smaller size (s. [19]). or to the singularity at 0 inside the set of all nilpotent matrices of rank at most one. Thus we only have to prove that both reductions follow from the theorem.

Now a nilpotent conjugacy class corresponds to a module over some truncated polynomial algebra $A = k[X]/X^r$ whose indecomposable modules V_i are given by $k[X]/X^i$ with $i \leq r$.

"Erasing a common first column" means that M and M' have the same socle U so that assumption (a) holds. It does not harm to suppose that M is a faithful module. Then $Q = M/U$ and $Q' = M'/U$ are annihilated by X^{r-1} so that they contain no copy of V_r as a summand. But for $i \leq r-1$ we have obviously $[V_i, V_1]^1 = [V_i, V_1] = 1$ so that assumption (b) is true. It remains to be seen that M is the generic extension of Q by U. Since A is representation finite, there is a generic extension E. Of course, U belongs to the socle of E which in turn cannot be strictly larger than the socle of M because E degenerates to M. Since any A-module T is determined by its socle S and by T/S we obtain $E \simeq M$.

"Erasing a common first row" is even easier. Again we can suppose that M is faithful. Then we have to divide by a common projective-injective direct summand $U \simeq V_r$. Setting $M = U \oplus Q$ and $M' = U \oplus Q'$, we see that (a) holds because U is projective, (b) and (c) hold because U is injective, and finally (d) holds by definition.

2.3. *The proof of theorem 1*

In the sequel, we use the notations and the remarks of 2.1. So we set $r = dim\ U$, $t = dim\ Q$ and $\mathscr{U} = \{u\}$. By the semi-continuity of the map $L \mapsto [U, L]$, the set

$$\mathscr{M} = \{l \in \overline{O(m)} \mid [U, L] = [U, M]\}$$

is an open irreducible subset of $\overline{O(m)}$. Thus the vector bundle $\mathscr{V}$ contains $p^{-1}(O(m))$ as an open dense subset. By assumption (c), the set $\mathscr{V}'$ is not empty and we have

$$c(\mathscr{V}') = c(\overline{\mathscr{V}' \cap p^{-1}(O(m))}) \subseteq \overline{c(\mathscr{V}' \cap p^{-1}(O(m)))} = \overline{O(q)}.$$

If M degenerates to M', we use properties (a) and (d) to find a point (m', g) in $\mathscr{V}'$ which is mapped to q' under c. Thus Q degenerates to Q'.

On the other hand, by the semi-continuity of the map $V \mapsto dim\, Z(V, U)$, the set

$$\mathscr{Q} = \{v \in \overline{O(q)} \mid [V, U]^1 - [V, U] = [Q, U]^1 - [Q, U]\}$$

is an open subset of $\overline{O(q)}$. We look at the vector bundle

$$p' : \mathscr{X} \to \mathscr{Q}.$$

Clearly, $p'^{-1}(O(q))$ is an open dense subset on $\mathscr{X}$. By assumption (c), $Gl_s \cdot p'^{-1}(O(q))$ is contained in $O(m)$, whence the same is true for $Gl_s \cdot \mathscr{X}$, which is the set of all extensions of modules in $\mathscr{Q}$ by U.

If Q degenerates to Q', then q' belongs to $\mathscr{Q}$ by (b). Using (d), we infer that m' beongs to $\mathscr{X}$. Therefore M degenerates to M'.

Now, we connect both constructions in the next commutative diagram. All morphisms are open immersions, isomorphisms or bundle projections.

$$\begin{tikzcd}
\mathscr{X} \times k^{s\times s} \arrow[d] & (\mathscr{M} \cap \mathscr{X}) \times Gl_s \arrow[l, hook', "i"'] & c^{-1}(\mathscr{Q}) \arrow[l, "\alpha"', "\sim"] \arrow[r, hook] \arrow[dddll, "\lambda"'] \arrow[dddrr, "\varrho"] & \mathscr{V}' \arrow[r, hook] & \mathscr{V} \arrow[dd] \\
\mathscr{X} \arrow[d] & & & & \\
\mathscr{Q} \arrow[d, hook] & & & & \mathscr{M} \arrow[d, hook] \\
\overline{O(q)} & & & & \overline{O(m)}
\end{tikzcd}$$

Here the isomorphism α maps (l, g) to $(g^{-1}lg, g)$. Since $\mathscr{M}$ is open in $\overline{O(m)}$, i is an open immersion.

If M degenerates to M', we choose an embedding g_1 of U into M' with cokernel Q', and we extend g_1 to an invertible matrix $g = (g_1, g_2)$. Then (m', g) belongs to $c^{-1}(\mathscr{Q})$ and we have $\lambda(m', g) = m'$ and $\varrho(m', g) = q'$. Thus the two pointed varieties are very smoothly equivalent.

2.4 *A lemma on vector bundles*

As the alert reader will have observed, our main working tool are various vector bundles. For the convenience of the reader we state a general lemma that produces all the bundles we are dealing with in this article. The straightforward proof uses only basic properties of determinants.

LEMMA 1. *Let $f: X \to k^{m \times n}$ be a morphism. Then we have*:

(a) *For any r, the set $X(r)$ of points x where $f(x)$ has rank r is a locally closed subvariety of X. Moreover, the closed subset $\{(x, v) \mid f(x)v = 0\}$ is a subbundle of rank $n - r$ of the trivial bundle $X(r) \times k^n$.*

(b) *Similarly, the set $\{(x, w) \mid w$ belongs to the image of $f(x)\}$ is a closed subbundle of rank r of the trivial bundle $X(r) \times k^m$.*

3. Cancellation of direct summands

In this section, we adapt the two basic constructions of 2.1 to split-monomorphisms and split-extensions. So let U and Q be modules of dimensions r and t. By semi-continuity, the set

$$\mathscr{Q} = \{v \in \overline{O(q)} \mid [V, U] = [Q, U], [U, V] = [U, Q]\}$$

is an open irreducible subset of $\overline{O(q)}$. We denote by $\mathscr{M}$ the union of all orbits $O(u \oplus v)$ with v in $\mathscr{Q}$, and by M the module $U \oplus Q$.

THEOREM 2. *Using the notation above we have*:

(a) *$\mathscr{M}$ is open in $\overline{O(m)}$.*

(b) *Let Q' be a degeneration of Q. If the codimension of $O(q')$ in $\overline{O(q)}$ is the same as the codimension of $O(q' \oplus u)$ in $\overline{O(q \oplus u)}$, then the two pointed varieties $(\overline{O(q \oplus u)}, q' \oplus u)$ and $(\overline{O(q)}, q')$ are very smoothly equivalent. Furthermore, the map $L \mapsto L \oplus U$ induces an isomorphism between the partially ordered sets*

$$\langle Q, Q' \rangle = \{L \mid Q \leq_{deg} L \leq_{deg} Q'\}$$

and $\langle Q \oplus U, Q' \oplus U \rangle$.

Proof. By semi-continuity, the set

$$\mathscr{M}' = \{l \in \overline{O(m)} \mid [U, L] = [U, M] \quad \text{and} \quad [L, U] = [M, U]\}$$

is an open irreducible subset of $\overline{O(m)}$. Let us look at the vector bundle

$$p: \mathscr{X}' \to M' \quad \text{with } \mathscr{X}' = \{(l, g, h) \mid l \in \mathscr{M}', g = (g_1, g_2) \in k^{s \times s},$$
$$g_1 u = l g_1, h \in k^{r \times s}, hl = uh\}.$$

Clearly, $\mathscr{X}'$ contains the open set $\mathscr{X}$ where hg_1 and g are invertible, and $\mathscr{M}$ is contained in the open set $p(\mathscr{X})$. We claim that equality holds.

Indeed, for any point (l, g, h) in $\mathscr{X}$ we have

$$g^{-1}lg = \begin{bmatrix} u & z \\ 0 & v \end{bmatrix},$$

whence

$$hg\begin{bmatrix} u & z \\ 0 & v \end{bmatrix} = hlg = uhg.$$

Decomposed in appropriate blocs this means

$$uhg_1 = hg_1 u \quad \text{and} \quad uhg_2 = hg_1 z + hg_2 v.$$

Thus we infer $z \in B(v, u)$, so that L is isomorphic to $U \oplus V$, where V belongs to $\mathscr{Q}$ by corollary 2.5 in [7].

Next, we define N to be the subgroup of $G := Gl_s$ consisting of upper triangular bloc-matrices $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$ with a fixing u. Then we have an isomorphism α between $\mathscr{X}$ and $\mathscr{Q} \times G \times N$ sending

$$(l, g, h) \text{ to } \left(v, g, \begin{bmatrix} hg_1 & hg_2 \\ 0 & 1 \end{bmatrix}\right).$$

Here v is constructed as before, and one finds the inverse of α easily from the calculations above. Then we have two very smooth morphisms

$$\lambda : \mathscr{X} \to \overline{O(m)} \quad \text{and} \quad \varrho : \mathscr{X} \to \overline{O(q)}$$

which are given as compositions

$$\mathscr{X} \longrightarrow \mathscr{M} \subseteq \overline{O(m)} \quad \text{and} \quad \mathscr{X} \simeq \mathscr{Q} \times G \times N \longrightarrow \mathscr{Q} \subseteq \overline{O(q)}.$$

Since the isotropy group of a module x is isomorphic to the automorphism group of X, one has for any degeneration Q' of Q the codimension formula

$$codim_{\overline{O(q \oplus u)}} O(q' \oplus u) - codim_{\overline{O(q)}} O(q') = [U, Q'] - [U, Q] + [Q', U] - [Q, U].$$

Thus the codimensions coincide if and only if Q' lies in $\mathscr{Q}$. It is clear then how to find a point in $\mathscr{X}$ mapped onto $q' \oplus u$ by λ and onto q' by ϱ. Finally, the isomorphism between the two partially ordered sets follows from part (a) and the cancellation of degenerations (s. 25 in [7]).

4. The invariance of geometric properties under tilting functors

4.1 *Some known facts from tilting theory*

First of all, we recall some results of "old-fashioned" tilting-theory as described in [18]. An A-module T is called a tilting module provided one has:

(a) $Ext^1(T, T) = 0$.
(b) $Ext^2(T, \ ) = 0$.
(c) T has as many nonisomorphic indecomposable direct summands as A has.

Then T induces a torsion theory on the category $mod\,A$ of finite dimensional A-modules whose torsion part $\mathscr{T}$ consists of the homomorphic images of powers of T. These modules are also characterized by the vanishing of $Ext^1(T, \ )$. Therefore, the set $\mathscr{T}(\underline{d})$ consisting in the torsion modules with dimension vector $\underline{d}$ is an open subset of the connected variety $Mod_A^{\underline{d}}$ of all modules of dimension vector $\underline{d}$. Recall that the dimension vector of a module counts the composition factors (with multiplicities).

One remarkable fact about tilting is that the functors $F = Hom_A({}_A T_B, \ )$ and ${}_A T_b \otimes$ induce inverse equivalences between $\mathscr{T}$ and the full subcategory $\mathscr{Y}$ of $modB$ where $Tor_1^B({}_A T_B, \ )$ vanishes. Here B is the opposite algebra of $End_A T$. Furthermore, the Grothendieck groups of $modA$ and $modB$ are isomorphic under the map which sends the class of X to the difference of the classes of FX and $Ext^1(T, X)$. In particular, for X in $\mathscr{T}(\underline{d})$ all modules FX have the same dimension vector $\underline{e}$. We denote by d or e the total dimension of a module with dimension vector $\underline{d}$ or $\underline{e}$.

4.2 *The main result*

In the next statement we keep the notation introduced in 4.1.

THEOREM 3. *Let T be a tilting module, and let $\underline{d}$ and $\underline{e}$ be two dimension vectors related as in* 4.1. *Then there is a $Gl_d \times Gl_e$-variety $\mathscr{X}$ and two morphisms $\lambda : \mathscr{X} \to \mathscr{T}(\underline{d})$ and $\varrho : \mathscr{X} \to \mathscr{Y}(\underline{e})$ such that the following holds:*

(a) *λ is a Gl_d-equivariant principal Gl_e-bundle.*
(b) *Up to the twist $h \in Gl_e \mapsto (h^{-1})^T$, ϱ is a Gl_e-equivariant principal Gl_d-bundle.*
(c) *For any m in $\mathscr{T}(\underline{d})$, the inverse image $\lambda^{-1}(O(m))$ is mapped under ϱ onto the orbit corresponding to FM.*

Before we prove the theorem, let us state an immediate consequence and some remarks.

COROLLARY 1. *The map $X \mapsto \varrho(\lambda^{-1}(X))$ induces a bijection between Gl_d-stable subsets of $\mathscr{T}(\underline{d})$ and Gl_e-stable subsets of $\mathscr{Y}(\underline{e})$. this correspondence preserves and reflects closures, inclusions, codimensions and types of singularities occurring in orbit closures.*

The most famous examples of tilting functors are Morita-equivalences and reflection functors. In the first case, the theorem above restates the "reduced" part of my previous article [6]. I do not know how to generalize the present result to the scheme-theoretic setting. In the second case, theorem 3 sharpens considerably a result that H. Kraft and C. Riedtmann have obtained in [15] using Grassmannians.

4.3. *The proof of the theorem*

Let t be the dimension of T, which is given by an α-tuple $(t_1, \ldots, t_\alpha)$ of $t \times t$-matrices. We consider the set

$$\mathscr{V} = \{(m, \xi_1, \ldots, \xi_e) \mid m \in \mathscr{T}(\underline{d}), \xi_i t_j = m_j \xi_i\}.$$

Recall that $e = [T, M]$ holds for all M in $\mathscr{T}(\underline{d})$. Then $Gl_d \times GL_e$ acts on $\mathscr{V}$ by

$$(g, h)(m, \xi_1, \ldots, \xi_e) = (gmg^{-1}, \sum h_{1j} g\xi_j, \ldots, \sum h_{ej} g\xi_j).$$

The wanted variety $\mathscr{X}$ is nothing but the $Gl_d \times Gl_e$-stable open subset consisting of those points where the ξ_i's are linearly independent. Defining

$$\lambda : \mathscr{X} \to \mathscr{Y}(\underline{d})$$

as the restriction of the bundle projection $\mathscr{V} \to \mathscr{T}(\underline{d})$, we see that part (a) holds.

Next, we fix a sequence $b_1 = E_t, b_2, \ldots, b_\beta$ of $t \times t$-matrices forming a basis of $End_A T = B^{op}$. Then we get a morphism

$$\phi : \mathscr{X} \to \mathscr{Y}(\underline{e})$$

by $\phi((m, \xi_1, \ldots, \xi_e)) = (n_1, \ldots, n_\beta)$ where the $e \times e$-matrices n_j are obtained from the equation

$$\xi_i b_j = \sum v_{ijk} \xi_k$$

by setting

$$(n_j)_{ki} = v_{ijk}.$$

We always denote by X_{ki} the coefficient of the matrix X sitting in row k and column i. It is clear that the orbit of $\phi((m, \xi_1, \ldots, \xi_e))$ corresponds to the B-module $FM = Hom_A({}_AT_B, M)$, and one verifies easily

$$\phi((g, h)(m, \xi_1, \ldots, \xi_e)) = (h^{-1})^T \phi((m, \xi_1, \ldots, \xi_e))h^T.$$

Now define a sequence $\eta_1, \ldots, \eta_d$ of $e \times t$-matrices by the equations $(\eta_p)_{ik} = (\xi_i)_{pk}$. The point is that the set of matrix equations used to define ϕ is equivalent to the set of matrix equations

$$\eta_p b_j = n_j^T \eta_p.$$

This means exactly that $\eta_1, \ldots, \eta_d$ all belong to

$$Hom_B({}_AT_B, DHom_A({}_AT_B, M)) \simeq D({}_AT_B \otimes_B Hom_A({}_AT_B, M)) \simeq DM,$$

where D is the usual duality functor $Hom_k(\ ,k)$. Thus we are led to introduce the vector bundle

$$\mathscr{W} = \{(n, \eta_1, \ldots, \eta_d) \mid n \in \mathscr{Y}(\underline{e}), \eta_p b_j = (n_j)^T \eta_p\}$$

endowed with the $Gl_d \times Gl_e$-action

$$(g, h)(n, \eta_1, \ldots, \eta_d) = ((h^{-1})^T n h^T, \sum g_{1j} h\eta_j, \ldots, \sum g_{dj} h\eta_j).$$

Up to the twist $h \mapsto (h^{-1})^T$, the restriction ϱ' of the projection onto $\mathscr{Y}(\underline{e})$ turns the open subset $\mathscr{W}'$, where $\eta_1, \ldots, \eta_d$ are linearly independent, into a Gl_e-equivariant principal Gl_d-bundle. It only remains to relate $\mathscr{X}$ and $\mathscr{W}'$ by an equivariant isomorphism.

Using the definitions made before, we have at least an equivariant morphism

$$f : \mathscr{X} \to \mathscr{W}$$

that sends $(m, \xi_1, \ldots, \xi_e)$ to $(\phi((m, \xi_1, \ldots, \xi_e)), \eta_1, \ldots, \eta_d)$. Since M is generated by a power of T, and since $\xi_1, \ldots, \xi_e$ are a basis of $Hom_A(T, M)$, the map from T^e to M with components $\xi_1, \ldots, \xi_e$ is surjective. Therefore, $m = (m_1, \ldots, m_\alpha)$ is the unique solution of the system of equations $\xi_i t_j = m_j \xi_i$. This shows that f is injective, and that the only point in the fibre $f^{-1}(w)$ of $w = f(z)$ depends regularly on w.

Now, $\lambda^{-1}(O(m))$ is a $Gl_d \times GL_e$-orbit of dimension $e^2 + d^2 - [M, M]$. The injection f maps it to an orbit of the same dimension contained in the irreducible set $p^{-1}(O(n))$, where $O(n)$ corresponds to FM. Because F induces an equivalence between $\mathscr{T}$ and $\mathscr{Y}$, the later inverse image has also dimension $e^2 + d^2 - [M, M]$, and $p^{-1}(O(n)) \cap \mathscr{W}'$ is the only orbit of that dimension. We infer that f maps $\mathscr{X}$ to $\mathscr{W}'$. The surjectivity follows again from the equivalence of $\mathscr{T}$ and $\mathscr{Y}$.

5. Reductions of the underlying Gabriel quiver

5.1 *Replacing two arrows by one*

Given any finite dimensional module M over a finitely generated algebra A, the annihilator I is of finite codimension in A. Of course, I annihilates all degenerations of M so that the study of the orbit closure can always be done over a finite dimensional algebra. Moreover, as explained in [6], the singularities in $\overline{O(m)}$ are very smoothly equivalent to those in $\overline{O(r)}$. Here R is the representation of the Gabriel quiver of A/I which corresponds to M.

This point of view is a bit more complicated notationally, but it provides more geometric intuition which was the basis for most of the proofs in the articles cited in the introduction. In this paper we usually stress the categorical aspects of the modules involved like homomorphism spaces and extension groups. But now, we generalize two reduction results of [2] and [11], which are best formulated in the language of quivers. Given an arrow α and a representation M of a quiver, we denote by $M(\alpha)$ the linear map corresponding to α in M.

Let Q be a quiver containing two arrows $\alpha : x \to y$ and $\beta : y \to z$ such that α and β are the only arrows starting or ending at y. Then we delete y and connect x and y by an arrow γ to obtain a new quiver Q'. All the other points and arrows are not touched upon. The obvious contraction functor $M \mapsto M'$ between the categories of representations induces a morphism f between the varieties of representations.

PROPOSITION 1. *We keep the above notations and assumptions. Let M be a representation of Q with $M(\alpha)$ injective and $M(\beta)$ surjective. Furthermore, let N be a degeneration of M such that $N(\alpha)$ is injective or $N(\beta)$ is surjective. Then M' degenerates to N' and the pointed varieties $(\overline{O(m)}, n)$ and $(\overline{O(m')}, n')$ are very smoothly euqivalent.*

Proof. Let d and d' be the dimension vectors of M and M'. Then the corresponding varieties R and R' of representations have the shapes

$$R = X \times k^{d(y) \times d(x)} \times k^{d(z) \times d(y)}$$

and

$$R' = X \times k^{d(z) \times d(x)}.$$

The morphism f sends (x, A, B) to (x, BA). It is equivariant with respect to $G = \Pi_{p \in Q_0} Gl_{d(p)}$ and $G' = \Pi_{p \in Q_0'} Gl_{d(p)}$. Therefore, M' degenerates to N'.

Up to duality, we can assume that $N(\alpha)$ is injective, hence $d(y) \geq d(x)$. We consider the open subset S of $k^{d(y) \times d(x)}$ of matrices of rank $d(x)$. On $S \times k^{d(z)}$ the multiplication can be factorized as a bundle projection $(A, B) \mapsto (A, BA)$ and the composition of an open immersion and a bundle projection. Thus the restriction g of f to

$$R^0 := X \times S \times k^{d(z) \times (d(y)}$$

is a very smooth morphism. By base change, the same is true for the induced morphism

$$g^{-1}(\overline{O(m')}) \longrightarrow \overline{O(m')}.$$

We claim that $g^{-1}(\overline{O(m')})$ equals $\overline{O(m)} \cap R^0$. Then it is clear that the pointed varieties $(\overline{O(m)}, n)$, $(\overline{O(m)} \cap R^0, n)$ and $(\overline{O(m')}, n')$ are very smoothly equivalent.

To prove the claim we note that an arbitrary continuous map $h : Y \to Z$ is open if and only if $h^{-1}(\bar{X}) = \overline{h^{-1}(X)}$ holds for all subsets X of Z. By the openness and equivariance of g we get

$$g^{-1}(\overline{O(m')}) = \overline{g^{-1}(O(m'))} = \overline{Gg^{-1}(m')}.$$

So let m be (x, A_0, B_0) and take any (x, A, B) in $g^{-1}(m')$. Then we have

$$B_0 A_0 = BA, \; ker A = ker A_0 = 0 \quad \text{and} \quad im B \subseteq im B_0 = k^{d(z)}.$$

It follows easily that (A, B) belongs to the closure of the $Gl_{d(y)}$-orbit of (A_0, B_0) (see e.g. [12]). Thus (x, A, B) belongs to $\overline{O(m) \cap R^0}$. The inclusion $\overline{O(m) \cap R^0} \subseteq g^{-1}(\overline{O(m')})$ is obvious.

5.2 *Replacing one arrow by none*

The next reduction allows to shrink arrows different from loops which are represented by bijections. For later use we deduce this from a more general reduction which is a special instance of associated fibre-bundles (s. [20]). For the convenience of the reader we include an elementary proof.

To simplify the notations we work for the moment again with algebras rather than quivers, but an analogous reduction applies to quivers. So let B be a finitely generated not necessarily unital subalgebra of the given finitely generated algebra A. Then the generators of B are linear combinations P_j of products of the generators $a_1, \dots, a_q$ of A. Given an A-module structure $m = (m_1, \dots, m_q)$ in Mod_A^d, we obtain a B-module structure $p(m)$ in Mod_B^d by subsituting the m_i's for the a_i's in the P_j's. Clearly, $p : Mod_A^d \to Mod_B^d$ is equivariant.

PROPOSITION 2. *Under the above notations and assumptions, let n be a point in Mod_B^d with isotropy group $H \subseteq G := Gl_d$ and fibre $F := p^{-1}(n)$. If m, m' are points in F with $m' \in \overline{Gm}$, the pointed varieties $(\overline{O(m)}, m')$ and $(\overline{Hm}, m')$ are very smoothly equivalent.*

Proof. We set $X = p^{-1}(O(n))$ and we look at the vector bundle

$$\mathscr{V} = \{(x, g) \in X \times k^{d \times d} \mid gp(x) = ng\} \longrightarrow X.$$

The open subset where g is invertible is isomorphic to $G \times F$ under the map α defined by $(x, g) \mapsto (g, gxg^{-1})$. The fibres of the conjugation

$$\lambda : G \times F \to X$$

are the H-orbits for the action

$$h(g, f) = (gh^{-1}, hgh^{-1}),$$

and λ is the composition of α^{-1}, an open immersion and the above bundle projection. Therefore, λ and the projection

$$\varrho : G \times F \to F$$

are very smooth morphisms which are $G \times H$-equivariant with respect to the obvious actions. Using the openness and equivariance of λ and ϱ, we find

$$\lambda^{-1}(\overline{Gm}) = \overline{\lambda^{-1}(Gm)} = \overline{G\lambda^{-1}(m)} = \overline{(G \times H)m} = G \times \overline{Hm} = \varrho^{-1}(\overline{Hm}).$$

The proposition follows.

The proof shows that λ identifies X with the associated fibre bundle $G \times {}^H F$, i.e. with the quotient of $G \times F$ by H under the action $h(g, x) = (gh^{-1}, hxh^{-1})$.

Now we apply the proposition to shrink bijective arrows. So let Q be a quiver with a non-loop $\alpha : x \to y$. Choose B as the three-dimensional subalgebra of the path algebra A defined by α. The open subvariety X of all representations of A of dimension vector $\underline{d}$, where α is bijective, is the inverse image of the orbit of the identity matrix. The fibre is the variety R' of all representations with dimension vector $\underline{d}'$ of the shrinked quiver Q', and H is isomorphic to $\Pi_{p \in Q'_0} Gl_{\underline{d}(p)}$. Thus the singularities are not affected by shrinking bijective arrows which are not loops.

5.3 *The geometric wildness of the double-loop*

It is well-known that for each finitely generated algebra C there is a full exact embedding of the category of all finite dimensional C-modules to the corresponding category of modules over the path algebra $A = k\langle X, Y\rangle$ of the double-loop. Using the last proposition, we derive a similar statement on the geometric level.

First, if C is generated by n elements there is an obvious Gl_d-equivariant closed embedding $Mod_C^d \subseteq Mod_D^d$, where D is the free algebra with n generators. Now in A we look at the subalgebra B generated by X, X^pY and YX. Let $n_0 \in Mod_B^{d(p+1)}$ be the module structure where X, X^pY and YX are given by

$$\begin{bmatrix} 0 & 0 & \cdot & \cdot & 0 \\ E & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & E & 0 & 0 \\ \cdot & \cdot & 0 & E & 0 \end{bmatrix}, \quad \begin{bmatrix} E & 0 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad 0.$$

Here E denotes the $d \times d$-identity matrix. Then the isotropy group of n_0 in $Gl_{d(p+1)}$ consists of a diagonal embedded copy of Gl_d, and the fibre F is given by all the matrices Y of the shape

$$\begin{bmatrix} M_1 & 0 & 0 & \cdot & \cdot & 0 \\ M_2 & 0 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ M_p & 0 & 0 & \cdot & \cdot & \\ E & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus the Gl_d-variety F is isomorphic to the Gl_d-variety Mod_D^d. Therefore, to understand the geometry of all finite dimensional modules over all finitely generated algebras is not more difficult than to understand the geometry of pairs of square-matrices.

In a similar vein, one can see that any degeneration of algebras is induced in the sense of [7], 2.2 by a deformation of representations of the double loop.

6. Minimal singularities induced by extensions

6.1 *The set-theoretic structure of certain minimal degenerations*

We want to analyze the minimal degenerations provided the partial orders $\leq_{ext}$ and $\leq$ are equivalent, and we start with the following observation:

LEMMA 2. *Let $E: 0 \to U \to M \to V \to 0$ be an exact sequence with indecomposable end term V such that $M \leq_{deg} U \oplus V$ is a minimal degeneration. Then the radical J of $End_A V$ annihilates E.*

Proof. By the nilpotency of J there is a natural number i such that J^{i+1} annihilates E, but J^i does not. We choose an element x in J^i such that $E' = E \cdot x: 0 \to U \to X \to V \to 0$ does not split. Then we have $M \leq_{deg} X \leq_{deg} U \oplus V$ by lemma 1.1 in [7], whence $M \simeq X$ by minimality. Now, E and E' induce exact sequences of $End_A V$-modules

$$0 \longrightarrow Hom(V, U) \longrightarrow Hom(V, M) \longrightarrow Hom(V, V) \longrightarrow Ext^1(V, U)$$

and

$$0 \longrightarrow Hom(V, U) \longrightarrow Hom(V, X) \longrightarrow Hom(V, V) \longrightarrow Ext^1(V, U).$$

By construction, J is the kernel of the last morphism. Counting lengths of $End_A V$-modules and using the fact that $End_A V$ is local, we conclude that J annihilates E.

Part of our next results is already contained in [7, 16], but we include all arguments for the convenience of the reader.

THEOREM 4. *Let C be a full subcategory of some module category which is closed under isomorphisms, extensions and direct summands. Assume that the partial orders $\leq_{ext}$ and $\leq$ are equivalent on C, and consider two objects M and N in C. Then N is a minimal degeneration of M if and only if there is an exact sequence $E: 0 \to U \to M' \to V \to 0$ with the following properties:*

(a) *U and V are indecomposables with $M = M' \oplus U^{p-1} \oplus V^{q-1} \oplus X$ and $N = U^p \oplus V^q \oplus X$. Here $U \oplus V$ and $M' \oplus X$ are disjoint.*

(b) *$U \oplus V$ is a minimal degeneration of M'.*

(c) *Any common indecomposable direct summand $W \not\simeq V$ of M and N satisfies $[W, N] = [W, M]$.*

(d) *Dually, any common indecomposable direct summand $W \not\simeq U$ of M and N satisfies $[N, W] = [M, W]$.*

Here, U, V, M', p and q are uniquely determined by M and N. Furthermore, we have

$$codim_{\overline{O(m)}} O(n) = codim_{\overline{O(m')}} O(u \oplus v) + \varepsilon(p + q - 2),$$

where ε is 1 for $V \not\simeq U$ and 2 for $V \simeq U$.

Proof. "$\Rightarrow$" We split off the greatest common direct summand (M, N) and we write $M = (M, N) \oplus M'$ and $N = (M, N) \oplus N'$. By the equivalence of the partial orders, N' is a minimal degeneration of M' which is given by an exact sequence E as above with $N' = U \oplus V$. If U is not indecomposable, there is a retraction r onto an indecomposable direct summand U' with kernel K and section s. We consider the pushout of E by r and obtain the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K \arrow[r, "id_K"] \arrow[d] & K \arrow[d] & & \\
0 \arrow[r] & U \arrow[r, "\alpha"] \arrow[d, shift right, "r"'] & M' \arrow[r] \arrow[d, "\beta"] & V \arrow[r] \arrow[d, "id_V"] & 0 \\
0 \arrow[r] & U' \arrow[r, "\gamma"] \arrow[u, shift right, "s"'] \arrow[d] & M'' \arrow[r] \arrow[d] & V \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

This diagram shows

$$M' \leq M'' \oplus K \leq N' = U' \oplus V \oplus K.$$

The last inequality is strict, because otherwise γ has a retraction ϱ, and we get

$$\varrho\beta\alpha s = \varrho\gamma r s = id_{U'},$$

whence U' is a direct summand of M' and N'. Thus we infer $M' = M'' \oplus K$, and K is a common direct summand of M' and N'. This contradiction shows that U is indecomposable. Similarly, V has to be indecomposable.

Up to duality, it only remains to derive property (c). The next argument is due to U. Markolf. So let us assume $[W, N] > [W, M]$ for some common direct summand W of M and N different from V. Then the last map in the exact sequence

$$0 \longrightarrow Hom(W, U) \longrightarrow Hom(W, M') \longrightarrow Hom(W, V) \longrightarrow Ext^1(W, U) \longrightarrow Ext^1(W, M')$$

is not injective. Therefore, we find a non-split exact sequence in $Ext^1(W, U)$ whose pushout under $U \to M'$ splits. Thus we get the diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & U \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M' \arrow[r] \arrow[d] & M' \oplus W \arrow[r] \arrow[d] & W \arrow[r] & 0 \\
 & V \arrow[r] \arrow[d] & V \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Because V does not occur in $M' \oplus W$, we have

$$M' \oplus W < V \oplus Y < V \oplus U \oplus W.$$

Since M' and $U \oplus V$ and disjoint, there is a module Z with $M = M' \oplus W \oplus Z$ and

$N = N' \oplus W \oplus Z$. We arrive at the contradiction

$$M < V \oplus Y \oplus Z < N.$$

"$\Leftarrow$" Only the minimality has to be shown. By theorem 2 and by the assumptions (c) and (d), we can forget about X. So suppose that $M < N$ is not minimal. Then there is a neighbor L of N such that

$$M = M' \oplus U^{p-1} \oplus V^{q-1} < L < N = U^p \oplus V^q.$$

Using the part already proved we find indecomposables U', V' and a non-split exact sequence

$$0 \longrightarrow U' \longrightarrow L' \longrightarrow V' \longrightarrow 0$$

such that

$$L = L' \oplus Y < U' \oplus V' \oplus Y = U^p \oplus V^q.$$

If U is isomorphic to V, one sees the M, L and N have U^{p+q-2} as a common direct summand. Cancelling it leads to the contradiction

$$M' < L' < U^2.$$

If U and V are not isomorphic we divide again by the greatest common direct summand of M, L and N. Up to symmetry in p and q, we arrive in one of the following situations:

- $p = q = 1$, in which case we have an obvious contradiction.
- $p = 1$, $q = 2$ and $U' \simeq V' \simeq V$.

Then property (d) says $[N, V] = [M, V]$, whence also $[N, V] = [L, V]$. By a dimension argument the sequence

$$0 \longrightarrow Hom(V, V) \longrightarrow Hom(L', V) \longrightarrow Hom(V, V) \longrightarrow 0$$

is exact, so that the original sequence

$$0 \longrightarrow U' \longrightarrow L' \longrightarrow V' \longrightarrow 0$$

splits. This contradiction ends the proof of the other implication.

Finally, we look at the codimensions. By lemma 2 and its dual we have

$$[V, U \oplus V] - [V, M'] = [U \oplus V, U] - [M', U] = 1.$$

Using (c) and (d), we obtain the wanted formula by a straightforward computation.

6.2 *A transversal slice*

We describe now under some conditions a transversal slice to the orbit of $u \oplus v$ in the closure $\overline{O(m)}$ of an extension m of V by U (see page 60ff. in [19] for the definition and basic properties of transversal slices).

So let us fix two module structures u and v of dimensions r and t. Inside the space $Z(v, u)$ of cocycles we choose a supplement H of the coboundaries. Given any extension M of V by U, we consider the subvariety $\mathscr{C}$ of $\overline{O(m)}$ consisting of all extensions

$$\begin{bmatrix} u & z \\ 0 & v \end{bmatrix}$$

with z in H.

THEOREM 5. *Let $N = U \oplus V$ be a degeneration of some module M such that $[U, M] = [U, N]$ and $[M, V] = [N, V]$ hold. Then we have*:

(a) *V is the generic quotient of M by U. In particular, M belongs to the set $\mathscr{E}$ of all extensoins of V by U. The intersection $\mathscr{E} \cap \overline{O(m)}$ is open in $\overline{O(m)}$.*

(b) *The variety $\mathscr{C}$ defined above is an irreducible cone whose dimension is given by the codimension of $O(u \oplus v)$ in $\overline{O(m)}$.*

(c) *The singularities of $\overline{O(m)}$ at $u \oplus v$ and of $\mathscr{C}$ at its vertex $\begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix}$ are very smoothly equivalent.*

(d) *$\mathscr{C}$ is a transversal slice in $\overline{O(m)}$ to the orbit $O(u \oplus v)$ at $u \oplus v$.*

The assumptions are obviously satisfied, provided M is an extension of V and U and these two modules have no proper self-extensions. This case occurs for powers of indecomposable preprojectives, and this is the main application of theorem 5 that we have in mind. But there are also other situations where the theorem is useful, e.g. the following structural result on codimension one degenerations.

PROPOSITION 3. *Let $M \leq_{deg} N$ be a degeneration.*

(a) *If $N = N' \oplus U$ with $[U, M] = [U, N]$ and $[M, U] = [N, U]$ then U is a direct summand of M.*

(b) *Set $M = M' \oplus X$ and $N = N' \oplus X$ with disjoint modules M' and N'. If the codimension of the degeneration is one, then M' degenerates to N' and either N' is indecomposable or else the direct sum of two non-isomorphic indecomposables. In this case M' is an extension of these indecomposables and $\overline{O(m)}$ is smooth at n.*

Proof (*of the proposition*): (a) The set $\mathcal{M} = \{l \in \overline{O(m)} \mid [U, L] = [U, M]$ and $[L, U] = [M, U]\}$ is open in $\overline{O(m)}$ Let us look at the vector bundle

$$p : \mathscr{X} = \{(l, g, h) \mid l \in \mathcal{M}, g \in k^{s\times r}, gu = lg, h \in k^{r\times s}, hl = uh\} \rightarrow \mathcal{M}.$$

Since N belongs to $\mathcal{M}$, there is a non-empty open subset of $\mathscr{X}$, where hg is invertible. The projection of this set is open and hits $O(m)$.

(b) From $1 = [N', N'] - [M', M'] + [N', X] - [M', X] + [X, N'] - [X, M']$ we infer $1 = [N', N'] - [M', M']$ and $[N', X] - [M', X] = [X, N'] - [X, M'] = 0$ using lemma 1.2 in [7]. By corollary 2.5 in [7], M' still degenerates to N', and we decompose $N' = \bigoplus U_i^{n_i}$ into indecomposables. Of course we have

$$\begin{aligned} 1 &= ([N', N'] - [N', M']) + ([N', M'] - [M', M']) \geq [N', N'] - [N', M'] \\ &= \sum n_i([U_i, N'] - [U_i, M']) \end{aligned}$$

and similarly

$$1 \geq [N', N'] - [M', N'] = \sum n_i([N', U_i] - [M', U_i]),$$

where all summands are non-negative integers. If there would be an index i with $[N', U_i] - [M', U_i] = [U_i, N'] - [U_i, M'] = 0$, then U_i would be a common direct summand by part (a). It follows that N' is indecomposable or the direct sum of two non-isomorphic indecomposables U and V such that $[U, M'] = [U, N']$ and $[M', V] = [N', V]$ hold. Theorem 5 shows that M' is an extension of V by U, and that there is no singularity at $u \oplus v$ in $\overline{O(m')}$, since the cone is now a straight line. Finally, $\overline{O(m)}$ is smooth at n because of theorem 2.

The proposition implies that the orbit of the middle term of an almost split sequence is smooth at the direct sum of the end-terms provided these are not isomorphic to each other. For then the codimension is one, and an end-term never occurs in the middle. The only almost split sequence of the algebra $k[X]/X^2$ shows that the closure of the middle term can be singular at the sum of the end-terms.

I have no idea what happens in the first case of the proposition. In the very few examples that I looked at there was no singularity.

If one drops one of the assumptions of the theorem, then $\mathscr{C}$ might no longer be a transversal slice. This occurs for representations of a double-arrow. I do not know the exact condition for $\mathscr{C}$ to be a transversal slice.

6.3 *The proof of theorem* 5

We consider the set

$$\mathscr{M} = \{l \in \overline{O(m)} \mid [U, L] - [U, M] = [L, V] - [M, V] = 0\},$$

which is open in $\overline{O(m)}$ and contains $N := U \oplus V$ by assumption. Let

$$p : \mathscr{V} = \{(l, g = (g_1, g_2)) \mid l \in \mathscr{M}, g_1 \in k^{s \times r}, g_2 \in k^{s \times t}, g_1 u = l g_1\} \to \mathscr{M}$$

be the familiar vector bundle of section 2.1. On the open set, where g is invertible, we have the cokernel morphism c to Mod_A^t with $c(l, g) = w$. By semi-continuity, the set of all w's with $[U \oplus W, V] \leq [M, V]$ is open in Mod_A^t, and so is its inverse image $\mathscr{V}'$ under c. In fact we have $[U \oplus W, V] = [M, V]$ for all w in $c(\mathscr{V}')$ because all corresponding $U \oplus W$ are degenerations of M.

Let us introduce another vector bundle

$$q : \mathscr{W} = \{(l, g, h) \mid (l, g) \in \mathscr{V}', h \in k^{t \times t}, vh = hc(l, g)\} \to \mathscr{V}'.$$

Inside $\mathscr{W}$, there is the open set $\mathscr{W}'$ where h is invertible. Note that $\mathscr{W}'$ is not empty, because $U \oplus V$ belongs to $\mathscr{M}$. Set

$$\mathscr{V}'' = q(\mathscr{W}').$$

Any point (l, g) in the open set $\mathscr{V}''$ gives rise to an exact sequence

$$0 \longrightarrow U \longrightarrow L \longrightarrow V \longrightarrow 0$$

and we have

$$p(\mathscr{V}'') = \mathscr{M} \cap \mathscr{E} = \overline{O(m)} \cap \mathscr{E}.$$

Since the projection is open and $\mathscr{M}$ is open in the irreducible set $\overline{O(m)}$, M is an extension of V by U. Our construction also shows that V is the generic quotient of M by U. Thus we have proved part (a) of the theorem.

To go on we introduce the sets

$$\mathscr{Y} = \left\{ \begin{bmatrix} u & z \\ 0 & w \end{bmatrix} \in Mod_A^s \mid w \in O(v) \right\} \cap \mathscr{M}$$

and

$$\mathscr{X} = \left\{ \begin{bmatrix} u & z \\ 0 & v \end{bmatrix} \in Mod_A^s \right\} \cap \mathscr{M},$$

so that $\mathscr{X}$ is isomorphic to $B(v, u) \times \mathscr{C}$. Then we have the following diagram of varieties and open immersions, bundle projections, isomorphisms or compositions of such morphisms:

$$\begin{array}{llll}
\mathscr{V}'' & \longrightarrow & \mathscr{M} \cap \mathscr{E} \hookrightarrow & \overline{O(m)} \\
\downarrow \simeq & (l, g) & & \\
 & \updownarrow & & \\
\mathscr{Y} \times Gl_s & (g^{-1} l g, g) & & \\
\uparrow & & & \\
\multicolumn{4}{l}{\mathscr{U} = \left\{ (y, g, h) \mid y = \begin{bmatrix} u & z \\ 0 & w \end{bmatrix} \in \mathscr{Y}, g \in GL_s, h \in k^{t \times t}, hw = vh \right\}} \\
\uparrow & & & \\
\mathscr{U}_{deth} & \left(\begin{bmatrix} u & zh \\ 0 & h^{-1} v h \end{bmatrix}, g, h \right) & & \\
\uparrow \simeq & \updownarrow & & \\
\mathscr{X} \times Gl_s \times Gl_t & \left(\begin{bmatrix} u & z \\ 0 & v \end{bmatrix}, g, h \right) & & \\
\downarrow & & & \\
\mathscr{X} & \longrightarrow & & \mathscr{C}
\end{array}$$

Now, $x = \left(\begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix}, 1, 1 \right)$ in $\mathscr{X} \times Gl_s \times Gl_t$ is mapped to $u \oplus v$ in $\overline{O(m)}$ and 0 in $\mathscr{C}$, which proves part (c).

Since $\overline{O(m)}$ is irreducible, all varieties involved in the above diagram are irreducible and so is $\mathscr{C}$. It is a cone because it is stable under conjugation with $\begin{bmatrix} E_r & 0 \\ 0 & aE_t \end{bmatrix}$ for all a in $k - \{0\}$.

To obtain the dimension of $\mathscr{C}$, we calculate the dimension of $\mathscr{X} \times Gl_s \times Gl_t$ in two ways using the above diagram:

$$\begin{aligned} \dim \mathscr{X} \times Gl_s \times Gl_t = \dim \mathscr{U} &= \dim \mathscr{V}'' + [V, V] \\ &= \dim \overline{O(m)} + [U, U \oplus V] + st + [V, V] \end{aligned}$$

and

$$\dim \mathscr{X} \times Gl_s \times Gl_t = s^2 + t^2 + \dim \mathscr{X} = s^2 + t^2 + rt + \dim \mathscr{C} - [V, U].$$

The wanted equality follows.

Finally, it is not hard to see now that $\mathscr{C}$ is a transversal slice. Since we will not use this fact later on, we omit this proof.

6.4 *Reduction to disjoint degenerations*

Suppose we are given two indecomposable modules U', V' and an extension

$$E : 0 \to U' \to M' \to V' \to 0$$

such that $[U', M'] = [U', U' \oplus V']$ and $[M', V'] = [U' \oplus V', V']$ hold true. Then we write

$$Z(v', u') = B(v', u') \oplus H'$$

and we consider the irreducible cone

$$\mathscr{C}' = H' \cap \overline{O(m')}$$

defined in the section 6.2. For any natural numbers p and q we set

$$U = (U')^p, \quad V = (V')^q \quad \text{and} \quad M = M' \oplus (U')^{p-1} \oplus (V')^{q-1}.$$

Then $N = U \oplus V$ is also a degeneration of M satisfying the assumptions of theorem 5. We want to relate the corresponding irreducible cones $\mathscr{C}$ and $\mathscr{C}'$.

Of course, we can identify $Z(v, u)$ with $Z(v', u')^{p \times q}$, $B(v, u)$ with $B(v', u')^{p \times q}$ and H with $(H')^{p \times q}$ by writing the elements of $Z(v, u)$ as matrices

$$\begin{bmatrix} u & 0 & \cdot & 0 & z_{11} & z_{12} & \cdot & z_{1q} \\ 0 & u & \cdot & 0 & z_{21} & z_{22} & \cdot & z_{2q} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & u & z_{p1} & z_{p2} & \cdot & z_{pq} \\ 0 & 0 & \cdot & 0 & v & 0 & \cdot & 0 \\ 0 & 0 & \cdot & 0 & 0 & v & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & v \end{bmatrix}$$

and so on. To describe $\mathscr{C}$ in terms of $\mathscr{C}'$ we introduce for an arbitrary cone $\mathscr{C}'$ with vertex 0 inside some vector space H' the new cone $\mathscr{C}'(p, q)$ as the subset of those $[c_{ij}]$ in $(H')^{p \times q}$ such that the following two conditions are satisfied:

- All c_{ij} belong to $\mathscr{C}'$.
- If $c_{i_0 j_0}$ is not zero, there exists a matrix $[t_{ij}]$ of rank 1 such that $c_{ij} = t_{ij} c_{i_0 j_0}$ holds for all indices.

Obviously, $\mathscr{C}'(p, q)$ is a closed irreducible cone inside $(H')^{p \times q}$, and it is isomorphic to the quotient of $\mathscr{C}' \times D(p, q)$ under the k^*-action $t(c', [t_{ij}]) = (c't^{-1}, t[t_{ij}])$. Therefore, the dimension of $\mathscr{C}'(p, q)$ is $\dim \mathscr{C}' + p + q - 2$. If 0 is an isolated singularity in $\mathscr{C}'$, then it is so in $\mathscr{C}'(p, q)$.

LEMMA 3. *We keep all the notations and assumptions of* 6.4. *Then we have*:

(a) *If $\mathscr{C}'$ is one-dimensional the pointed varieties $(\mathscr{C}'(p, q), 0)$ and $(D(p, q), 0)$ are isomorphic.*

(b) *$\mathscr{C}$ always contains $\mathscr{C}'(p, q)$. Both sets coincide if and only if the radicals of the endomorphism algebras of U' and V' annihilate the given exact sequence E.*

Proof. (a) Since $\mathscr{C}'$ is an irreducible cone, we have $\mathscr{C}' = kx$ for any non-zero element x in $\mathscr{C}'$. Then we get

$$\mathscr{C}'(p, q) = \{[t_{ij}x] \mid [t_{ij}] \in D(p, q)\} \simeq D(p, q).$$

(b) Of course, $\mathscr{C}'(p, q)$ is stable under elementary operations on the rows and the columns. Using these operations, any element in $\mathscr{C}'(p, q)$ is conjugate to an element z with z_{11} in $\mathscr{C}'$ and $z_{ij} = 0$ for all other indices. Such an element belongs to $\mathscr{C}$ for obvious reasons.

On the other hand, $\mathscr{C}$ is also invariant under elementary operations, because these are induced by conjugation of

$$\begin{bmatrix} u & 0 & \cdot & 0 & z_{11} & z_{12} & \cdot & z_{1q} \\ 0 & u & \cdot & 0 & z_{21} & z_{22} & \cdot & z_{2q} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & u & z_{p1} & z_{p2} & \cdot & z_{pq} \\ 0 & 0 & \cdot & 0 & v & 0 & \cdot & 0 \\ 0 & 0 & \cdot & 0 & 0 & v & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & v \end{bmatrix}$$

with appropriate bloc matrices

$$\begin{bmatrix} a_{11} & a_{12} & \cdot & a_{1p} & 0 & 0 & \cdot & 0 \\ a_{21} & a_{22} & \cdot & a_{2p} & 0 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{p1} & a_{p2} & \cdot & a_{pp} & 0 & 0 & \cdot & 0 \\ 0 & 0 & \cdot & 0 & b_{11} & b_{12} & \cdot & b_{1q} \\ 0 & 0 & \cdot & 0 & b_{21} & b_{22} & \cdot & b_{2q} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 0 & b_{q1} & b_{q2} & \cdot & b_{qq} \end{bmatrix}$$

where all a_{ij} and b_{ij} are scalar matrices. We infer that $\mathscr{C}'(p, q)$ is contained in $\mathscr{C}$. Since $\mathscr{C}$ is irreducible and $\mathscr{C}'(p, q)$ is a closed subset of $\mathscr{C}$, both sets coincide if and only if the dimensions are equal. Using theorem 5 (b) and the assumptions $[U', M'] = [U', U' \oplus V']$ and $[M', V'] = [U' \oplus V', V']$ we obtain

$$\begin{aligned} \dim \mathscr{C} = \dim \mathscr{C}' &+ (q-1)([V', U' \oplus V'] - [V', M']) \\ &+ (p-1)([U' \oplus V', U'] - [M', U']). \end{aligned}$$

Since the given extension does not split the two differences in the brackets are strictly positive. Thus the dimensions coincide if and only if both differences are equal to one. This means exactly that the radicals annihilate the extension.

I do not know how to analyze the singularity of $\mathscr{C}'(p, q)$ in general. However, if $\mathscr{C}'$ is the closure of a highest weight vector v in an irreducible representation V of a reductive algebraic group G, then $\mathscr{C}'(p, q)$ is again the closure of a highest weight vector in the irreducible representation $V \otimes k^p \otimes (k^q)^*$ of $G \times Gl_p \times Gl_q$. This observation due to H. Kraft and P. Littelmann allows to apply a general result of E. Vinberg and V. Popov in [21], and to conclude that $\mathscr{C}'(p, q)$ is normal. The situation described before occurs for minimal singularities of matrix pencils as we will show in another paper.

7. Minimal singularities of preprojective modules

7.1 *Statement of the results*

Let $M < N$ be a minimal degeneration of preprojective modules. By the equivalence of $\leq_{ext}$ and $\leq$ (s. [7]) and by theorem 4 we have

$$M = M' \oplus (U')^{p-1} \oplus (V')^{q-1} \oplus X \quad \text{and} \quad N = (U')^p \oplus (V')^q \oplus X$$

for some minimal disjoint degeneration $M' \leq N' = U' \oplus V'$. Here we have

$$[X, M'] - [X, N'] = [M', X] - [N', X] = 0,$$

so that theorem 2, theorem 5 and lemma 3 show that

$$(\overline{O(m)}, n), (\overline{O(m' \oplus (u')^{p-1} \oplus (v')^{q-1})}, (u')^p \oplus (v')^q) \quad \text{and} \quad (\mathscr{C}'(p, q), 0)$$

are very smoothly equivalent. Unfortunately, the geometric structure of the cone $\mathscr{C}'$ is still unknown to me in general, but for representation finite quiver algebras we have the following result:

PROPOSITION 4. *Any minimal disjoint degeneration of representations of Dynkin quivers is of codimension one.*

From the discussion above and this proposition we obtain our main result:

THEOREM 6. *Any minimal singularity $(\overline{O(m)}, n)$ of representations of Dynkin quivers is very smoothly equivalent to the pointed variety $(D(p, q), 0)$ for some natural numbers p and q.*

Proposition 4 has been found by U. Markolf in his Diplomarbeit via computer (s. [16]). To check these computer results I figured out a slightly technical theoretical proof which is also reproduced in [16]. The same proof is given here in the last section, because it is essential for the proof of theorem 6, and because it illustrates by a non-trivial example the combinatorial complexity of the problem we are dealing with.

Another remarkable observation in [16] is the fact that the number of minimal degenerations ending in the direct sum of two indecomposable modules U and V is always $r(r+1)/2$ with $r = [V, U]^1$. Furthermore, it is possible that proposition 4 and therefore also theorem 6 remain valid for the much more general class of all representation-directed algebras. I can prove this so far only in some special cases, e.g. for degenerations of indecomposables, which answers a question of C. Riedtmann. The precise statement is this:

PROPOSITION 5. *Let* $0 \to U \to M \to V \to 0$ *induce a minimal disjoint degeneration of preprojective modules. If* $Hom(X, Y) = 0$ *holds for all non-isomorphic indecomposable direct summands* X *and* Y *of* M, *then the codimension is one.*

However, for minimal disjoint degenerations of preprojective representations of the triple arrow the codimensions are no longer bounded, so that the structure of $\mathscr{C}'$ is not clear.

Another interesting question is whether orbit closures of preprojectives are always normal. Again, I can prove this only in some generic situations where they happen to be complete intersections.

PROPOSITION 6. *Let* M *be a stretched preprojective module* (*s.* [7], 3.3 *for the definition*), *e.g. an indecomposable. Then* $\overline{O(m)}$ *is a complete intersection which is regular in codimension one. In particular,* $\overline{O(m)}$ *is Cohen-Macaulay and normal.*

This result is obvious for quiver algebras, but there exist many more algebras having preprojective modules (s. [10]).

The proof of proposition 6 is essentially contained in [6]. It only remains to verify in addition that each irreducible component of $\overline{O(m)} - O(m)$ contains a smooth point n. To see this one follows the argument of [6] and one finds a module N with $Ext^2(N, N) = 0$.

In the next section we prove proposition 5, and in the last section a sharper version of proposition 4. Thus we conclude the present article with the only proof that involves some sort of classification.

7.2 *The proof of proposition 5*

Because of

$$0 = [V, V]^1 = [V, U] = [M, U] \quad \text{and} \quad [U, U] = 1$$

we can calculate the codimension as follows:

$$[U \oplus V, U \oplus V] - [M, M] = [U \oplus V, U \oplus V] - [M, U \oplus V] + [M, U \oplus V]$$
$$- [M, M] = 1 + [M, U \oplus V] - [M, M].$$

So assume that the codimension is greater than one. Then there is an indecomposable direct summand M_1 of $M = \bigoplus M_i^{n_i}$, such that

$$[M_1, U \oplus V] = [M_1, V] > [M_1, M] \geq n_1.$$

Thus we can choose linearly independent elements

$$f_{11}, \ldots, f_{1n_1+1} \text{ in } Hom(M_1, V).$$

Similarly, we can find linearly independent functions

$$f_{i1}, \ldots, f_{in_i} \text{ in } Hom(M_i, V)$$

for the other indices. We take these homomorphisms as the components of a map

$$h : \bigoplus_{i \geq 2} M_i^{n_i} \to V,$$

and we take $f_{11}, \ldots, f_{1n_1}$ and $f_{12}, \ldots, f_{1n_1+1}$ as the components of two maps g_1 and g_2 from $M_1^{n_1}$ to V. For any pair $(a, b) \in k^2 - \{0\}$ the morphism $f(a, b)$ from $M = \bigoplus M_i^{n_i}$ to V with components $ag_1 + bg_2$ and h has the property that none of its components factors through the others. This follows from $[M_i, M_i] = 1$ for all i and from the assumption $Hom(M_j, M_i) = 0$ for $i \neq j$. Argueing as in theorem 4.1 of [7], we see that $f(a, b)$ induces for all $(a, b) \neq (0, 0)$ an exact sequence

$$0 \longrightarrow U \longrightarrow M \longrightarrow V \longrightarrow 0.$$

This leads to a contradiction.

Namely, suppose for a moment that there is a point x in the Gabriel quiver of A with

$$U(x) = 0 \neq M_1(x).$$

Then we can choose appropriate bases of $M(x)$ and $U(x)$ such that $(g_1, h)(x)$ is represented by the identity and $(g_2, h)(x)$ by a bloc matrix

$$\begin{bmatrix} H & 0 \\ I & E \end{bmatrix}.$$

If λ is an eigenvalue of H then $f(\lambda, -1)$ is not bijective as it should be.

The general case is reduced to the former by an appropriate tilting functor. First of all, we can forget about all indecomposable projectives which are not predecessors of V, and all indecomposable projective injectives which are not successors of U, because the corresponding simples do not occur as composition factors of M. Thus we can assume that all preprojective injectives are successors of U. Then we take in the TrD-orbit of each projective indecomposable P the lowest power TrD^i P, which is a successor of U. Standard arguments show that the direct sum T of all these modules is a tilting module. By construction, all preprojective successors of U are generated by T. Therefore,

$$0 \longrightarrow FU \longrightarrow FM \longrightarrow FV \longrightarrow 0$$

remains exact, and all the properties of homomorphisms between indecomposables occurring in the original exact sequence are preserved by $F = Hom(T,)$. But now, FU is a simple not isomorphic to FM_1, so that our previous arguments apply.

7.3 *The proof of proposition 4*

We will use the following consequence of theorem 4 (compare 4.6 in [7]):

LEMMA 4. *Let $M < N = U \oplus V$ be a minimal disjoint degeneration of preprojective modules. If the codimension is not one, there is an indecomposable direct summand X of M and a minimal disjoint degeneration $X^2 \oplus Y < N'$ of preprojectives.*

Proof. By the first section in the last proof, there is an indecomposable direct summand X of M with $[X, U \oplus V] > [X, M]$. By theorem 4,

$$M \oplus X < U \oplus V \oplus X$$

is no longer minimal. So we can take a minimal degeneration L of M in between. Again by theorem 4, we have

$$M \oplus X = M' \oplus Z < L = U' \oplus V' \oplus Z < U \oplus V \oplus X$$

for some modules M', U', V' and Z and some minimal disjoint degeneration $M' < U' \oplus V'$. If Z would be $Z' \oplus X$, we would get

$$M < M' \oplus Z' < U \oplus V,$$

contradicting the minimality of $M < U \oplus V$. Thus $M' < U' \oplus V'$ is the wanted degeneration.

Of course, proposition 4 follows now from the following sharper result, which is definitely wrong for arbitrary representation-directed algebras:

LEMMA 5. *Let $M < N$ be a minimal disjoint degeneration of modules over the path algebra A of a Dynkin quiver. Then no indecomposable direct summand X occurs twice in M.*

Proof. Suppose not. Then we consider the exact sequence

$$0 \longrightarrow U \longrightarrow M = X^2 \oplus Y \longrightarrow V \longrightarrow 0$$

inducing the minimal degeneration $M < N$. We will show that in that case the quiver is of type E_8, Y has two indecomposable direct summands Y_1 and Y_2 and the position of U, V, X, Y_1 and Y_2 in the Auslander–Reiten quiver is the one shown in figure 1.

Then $X^2 \oplus Y < U \oplus V$ is not a minimal degeneration as can be seen in figure 2, which shows a sequence of five minimal degenerations from the generic extension of V by U through $X^2 \oplus Y$ to $U \oplus V$. The six modules involved are the direct sums of the thick points in the Auslander–Reiten quiver Γ_A, and only X occurs twice. The first three degenerations are induced by almost split sequences, whence obvious. The last two have been verified using the equivalence of $\leq_{\mathrm{deg}}$ and $\leq$. Also proposition 5 shows that $X^2 \oplus Y < U \oplus V$ is not a minimal degeneration.

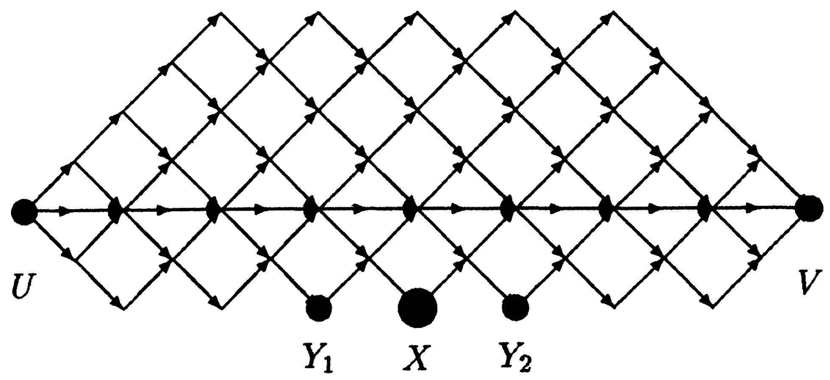

Figure 1

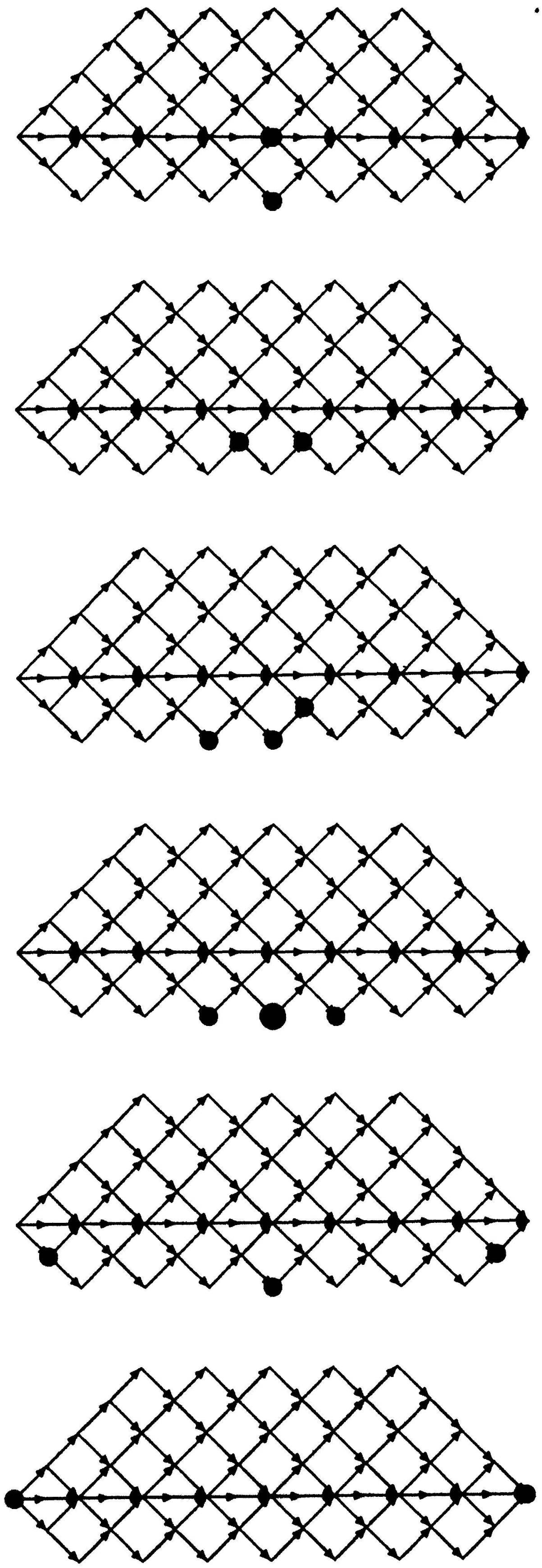

Figure 2

Using reflection functors, we reduce to the situation where U is the only simple projective. Then we infer from

$$4 \leq [X^2 \oplus Y, X^2 \oplus Y] < [U \oplus V, U \oplus V] = 2 + [U, V]$$

and from the fact that the roots of A_n and D_n have only components ≤ 2, that A is of type E_n. By the above inequality we have $[U, V] > 0$, so that we can assume V to be faithful. Next, we look at the full subquiver S of Γ_A consisting of all successors T of X such that $DTrT$ is not a successor. Because V is faithful and $[X, V]$ is not zero, S is an oriented tree of the same type as A with X as the only source. Recall that such subquivers are often called slices. Let Z be the indecomposable at the end of a longest branch of S. Then we have

$$2 \leq [X, X^2 \oplus Y] \leq [X, U \oplus V] \leq [X, V],$$

whence V does not belong to S. Similarly, we have

$$2 \leq [X^2 \oplus Y, Z] \leq [U \oplus V, Z] \leq [U, Z].$$

Now, for any indecomposable U' over a Dynkin quiver of type E_n the functions

$$T \mapsto [U', T]$$

from $(\Gamma_A)_0$ to the natural numbers are well-known and easy to determine (s. [5]). The case $[U', Z] \geq 2$ for a Z sitting at a longest branch occurs only for E_8, only for U' in the DTr-orbit with three neighbors and only for at most one Z once U' is given and vice versa. Since we have fixed U already, we know that X lies on the slice through Γ_A with Z as the only sink. Dual arguments show, that V lies in the TrD-orbit of U, and that its position is uniquely determined by X. We are left with eight possibilities for X. Looking at the non-negative function

$$T \mapsto [U \oplus V, T] - [X^2, T]$$

one verifies that U, V and X have to be in the position of figure 1. Since the difference of the dimension vectors of $U \oplus V$ and X^2 is neither zero nor a root, we infer that Y has at least two indecomposable direct summands. By duality, we can assume that Y_1 belongs to the left half of figure 1. The dashed zeros in the function

$$T \mapsto [U \oplus V, T] - [X^2, T],$$

				1		1		1		1		1		0′		1		1		0		2		0′		1
			1		2		2		2		2		1		1		2		1		2		2		1	
		1		2		3		3		3		2		2		2		2		3		2		3		2
	1		2		3		4		4		3		3		3		2		4		3		3		4	
1	1	2	1	3	2	4	2	5	3	4	1	4	3	4	1	3	2	4	2	4	2	4	2	4	2	4
	1		2		2		3		2		4		2		2		3		2		3		3		2	
		1		1		1′		0		2		2		0′		2		1		1		2		1		1

Figure 3

which is reproduced to a large part in figure 3, leave only the possibility of figure 1 for Y_1. Its multiplicity is one because of the dashed one. The dual argument finishes the proof of the lemma.

As shown in [16] there are 120 isomorphism classes of extensions of V by U in the case of lemma 5. For Dynkin quivers the maximal number of such extensions is 132, whereas this number becomes arbitrarily large for general representation-directed algebras.

REFERENCES

[1] Abeasis S. and del Fra A. Degenerations for the representations of an equioriented quiver of type A_m, J. of Algebra *93* (1985), 376–412.

[2] Abeasis, S., del Fra, A. and Kraft, H. The geometry of representations of A_m, Math. Ann. *256* (1981), 401–418.

[3] Altman, A. and Kleiman, S. Introduction to Grothendieck Duality Theory, Lecture Notes in Math. *146*, Springer Verlag (1970).

[4] Auslander, M. and Reiten, I. Modules determined by their composition factors, Illinois J. Math. *29* (1985), 289–301.

[5] Bongartz, K. Critical simply connected algebras, Man. Math. *46* (1981), 117–136.

[6] Bongartz, K. A geometric version of the Morita equivalence, J. Algebra *139* (1991), 159–179

[7] Bongartz, K. On degenerations and extensions of finite dimensional modules, to appear in Adv. Math., preprint 1–53 (1990).

[8] Donkin, S. The normality of closures of conjugacy classes of matrices, Invent. math. *101* (1990), 717–736.

[9] Gabriel, P. Unzerlegbare Darstellungen I, Man. Math. *6*, 71–103 (1972).

[10] Gabriel, P., Keller, B. and Roiter, A. V. Representations of finite-dimensional algebras, Encycl. Math. Sc. *73* (1992), 1–176.

[11] Kempken, G. Eine Darstellung des Köchers $\tilde{A}_k$, Bonner Math. Schriften *137* (1982), 1–159.

[12] Kraft, H. Geometrische Methoden in der Invariantentheorie, Vieweg Verlag (1984).

[13] Kraft, H. and Procesi, C. Closures of conjugacy classes of matrices are normal, Invent. math. *53* (1979), 227–247.

[14] Kraft, H. and Procesi, C. Minimal singularities in Gl_n, Invent. math. *62*, 503–515 (1981).

[15] Kraft, H. and Riedtmann, C. Geometry of representations of quivers, LMS lecture notes *116* (1985), 109–147.

[16] Markolf, U. Entartungen von Darstellungen zu Dynkin–Köchern, Diplomarbeit Uni Wuppertal, (1990), 1–68.

[17] Riedtmann, C. Degenerations for representations of quivers with relations, Ann. Sci. Ecole Norm. Sup. *4* (1986), 275–301.
[18] Ringel, C. M. Tame algebras and quadratic forms, Lecture Notes in Math. 1099, Springer Verlag (1984).
[19] Slodowy, P. Simple singularities and simple algebraic groups, Lecture Notes in Math. 815, Springer Verlag (1980).
[20] Serre, J. P. Espaces fibrés algébriques, Séminaire Chevalley 1958, 1–36.
[21] Vinberg, E. B. and Popov, V. L. On a class of quasihomogeneous affine varieties, Math. USSR Izvestija *6* (1972), 743–758.

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