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New applications of Luttinger's surgery

YAKOV ELIASHBERG and LEONID POLTEROVICH

§1. Introduction and main results

Recently Karl Luttinger [L] made a remarkable observation that certain surgeries along a Lagrangian 2-torus in the standard symplectic space (\mathbb{C}^2 , ω) do not change the ambient topology. As a consequence he found restrictions on isotopy classes of embeddings $\mathbb{T}^2 \to \mathbb{C}^2$ which can be represented by Lagrangian ones.

In the present paper, we discuss some new applications of this technique to *linking* of Lagrangian 2-tori in \mathbb{C}^2 , to *contact geometry* on the 3-torus as well as to study of *complex structures with pseudo-convex boundary* on $\mathbb{T}^2 \times \mathbb{D}^2$.

1.1. Linking class of totally real tori

A field of lines on a 2-torus is called *homotopically trivial* if it is homotopic to the kernel of a non-singular closed 1-form. All homotopically trivial line fields are homotopic. A 2-torus in \mathbb{C}^2 is called *totally real* if it has no complex tangent lines. From now on we denote by $\ell k(\cdot, \cdot)$ the linking number, and by J the standard complex structure on \mathbb{C}^2 . All (co)homology groups considered below are integer.

Assume that $L \subset \mathbb{C}^2$ is an embedded oriented totally real 2-torus. Take an arbitrary non-singular tangent vector field, say v on L which generates a homotopically trivial field of lines. For a 1-cycle α on L set

 $\sigma(\alpha) = \ell k(\alpha + \varepsilon J v, L),$

where ε is sufficiently small.

One can easily check that σ is a well defined element of $H^1(L)$, in particular σ does not depend on the choice of v. We call σ the linking class of a totally real torus L (see [P1], [P2]). Note that this class is closely related to the Viro quadratic form.

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As it was shown in [P1] for each cohomology class $\sigma \in H^1(L)$ there exists a totally real embedding $L \to \mathbb{C}^2$ whose linking class is equal to σ . However for Lagrangian submanifolds the situation is quite different. Namely, we prove the following result which was conjectured in [P1], [P2].

THEOREM 1.1.A. The linking class of every embedded Lagrangian torus in \mathbb{C}^2 vanishes.

The theorem is proved below in 3.1. As a consequence we obtain the following

COROLLARY 1.1.B. (see [P1]). Let $M \subset \mathbb{C}^2$ be an embedded closed 3-manifold whose characteristic foliation admits an embedded invariant 2-torus L. If L divides M then the restriction of the characteristic foliation to L is homotopically trivial.

Proof. Notice that L is a Lagrangian torus. Let l be the field of Euclidian normal lines to M along L. Then the field Jl is tangent to the characteristic foliation on L. The needed assertion easily follows now from 1.1.A.

1.2. Giroux' theorem

Homotopically trivial fields of lines on \mathbb{T}^2 allow to identify canonically (up to a homotopy) the cotangent bundle $T^*\mathbb{T}^2$ with $\mathbb{T}^2 \times \mathbb{R}^2$ (with this language the zero section is identified with $\mathbb{T}^2 \times \{0\}$).

THEOREM 1.2.A. Consider an embedded Lagrangian torus in $T^*\mathbb{T}^2 = \mathbb{T}^2 \times \mathbb{R}^2$ which does not intersect the zero section and is homologous to it. Then its projection to $\mathbb{R}^2 - \{0\}$ is homotopic to a point.

This result was conjectured by J.-C. Sikorav in [S] who verified it under an additional assumption that the torus is *Lagrangian isotopic* to the zero section. It was proved recently by E. Giroux (see [Gi]) using, in particular, some tools from contact geometry. We give here a different purely symplectic proof (see section 3.2 below).

1.3. Contact geometry of the 3-torus

Consider the 3-torus $\mathbb{T}^2 = S^1(\theta) \times \mathbb{T}^2(x, y)$, where $(\theta, x, y) \pmod{1}$ are angular coordinates. Let $\xi = \text{Ker } \lambda$, where

 $\lambda = \cos 2\pi\theta \, dx + \sin 2\pi\theta \, dy$

be the standard contact structure.

We identify $H_1(\mathbb{T}^3)$ with $\mathbb{Z} \oplus \mathbb{Z}^2$ and the automorphisms group of $H_1(\mathbb{T}^3)$ with GL(3, Z). Recall [La] that isotopy classes of 3-torus diffeomorphisms are defined by their action on homology. Let $\mathscr{D} \subset SL(3, \mathbb{Z})$ be the stabilizer of the subspace $0 \oplus \mathbb{Z}^2$.

THEOREM 1.3.A. An element from $SL(3,\mathbb{Z})$ can be represented by a contactomorphism of the standard contact structure ξ if and only if it belongs to \mathcal{D} .

The proof which is based on 1.2.A is given in Section 3.4 below.

We apply this theorem in order to construct an infinite sequence of pairwise non-isotopic tight contact structures on \mathbb{T}^3 with the same Euler class (see Question 8.6.1 in [E2]). Recall that two contact structures are called *isotopic* if there exists a diffeomorphism isotopic to the identity which takes one to another. An immediate consequence of 1.3.A is the following

COROLLARY 1.3.B. For $f, g \in SL(3, \mathbb{Z})$, contact structures $f_*(\xi)$ and $g_*(\xi)$ are isotopic if and only if $f^{-1} \circ g$ belongs to \mathcal{D} .

A theorem by J. Gray states that two contact structures on a compact manifold which are homotopic through contact structures are isotopic. On the other hand the image of the standard contact structure ξ under an arbitrary diffeomorphism of \mathbb{T}^3 is homotopic to ξ through plane distributions.

Hence, we have, in particular

COROLLARY 1.3.C. There exists a sequence ξ_n , $n \ge 0$, of contact structures on \mathbb{T}^3 such that

- (i) ξ_n is contactomorphic to ξ for every n, and $\xi_0 = \xi$;
- (ii) all ξ_n are homotopic to ξ through two-dimensional distributions;
- (iii) for $m \neq n$ the structures ξ_m and ξ_n are not homotopic through contact structures on \mathbb{T}^3 .

Proof. Take a diffeomorphism f of \mathbb{T}^3 such that $[f^n] \notin \mathcal{D}$ for every $n \in \mathbb{Z} - \{0\}$. It follows from 1.3.B and the previous discussion that the structures $\xi_n = f_*^n(\xi)$, $n = 0, \ldots$, are homotopic through plane distributions but not through contact structures.

REMARK 1.3.D. Giroux in [Gi] used Theorem 1.2.A to construct a tight (see [E2]) contact structure on T^3 which is homotopic (through two-dimensional distributions) but *not isomorphic* to the standard contact structure ξ_0 . His structure

is symplectically fillable (see [E1] for the definition of symplectically and holomorphically fillable structures) while at least some of structures constructed above are holomorphically fillable (see the next section).

1.4. Complex structures on $\mathbb{T}^2 \times \mathbb{D}^2$

A contact structure on an oriented 3-manifold is called *positive* if it is (locally) defined by a 1-form, say λ with $\lambda \wedge d\lambda > 0$. A boundary of a complex surface is called *strictly pseudo-convex* if its field of tangent lines is a positive (with respect to the canonical orientation) contact structure.

It was shown in [E1] that the manifold $S^2 \times \mathbb{D}^2$ does not admit a complex structure with strictly pseudo-convex boundary. In the present section we study the space of such structures on $\mathbb{T}^2 \times \mathbb{D}^2$.

THEOREM 1.4.A. There exists a sequence J_n , $n \ge 0$, of complex structures with strictly pseudo-convex boundary on $\mathbb{T}^2 \times \mathbb{D}^2$ such that

- (i) any two of them are biholomorphically equivalent and homotopic through complex structures;
- (ii) for $m \neq n$ the structures J_m and J_n are not homotopic through complex structures with strictly pseudo-convex boundary.

Proof. We represent $V = \mathbb{T}^2 \times \mathbb{R}^2$ as the quotient space of \mathbb{C}^2 by the imaginary lattice $i\mathbb{Z}^2$. We still denote by J the induced complex structure on V. Let $(x, y) \pmod{1}$ be angular coordinates on \mathbb{T}^2 and $(r, \theta \pmod{1})$ be polar coordinates on \mathbb{R}^2 . Set

 $N = \mathbb{T}^2 \times \mathbb{D}^2 = \{r \le 1\}.$

Denote by $\Sigma = \mathbb{T}^3$ the boundary of N. Obviously, Σ is strictly pseudo-convex with respect to J since its field of tangent complex lines is just the standard contact structure ξ defined in 1.3.

Consider a diffeomorphism $F: V \rightarrow V$,

$$(r, \theta, x, y) \rightarrow (r, \theta + 2x, x, y),$$

and set

 $J_n = DF^n \circ J \circ DF^{-n}.$

We claim that the sequence $\{J_n\}$ has the desired properties. Indeed, since F preserves Σ we conclude that all $J_n |_N$ are pairwise biholomorphically equivalent and with strictly pseudo-convex boundary. Moreover, for $n \neq 0$ the restriction of F to Σ does not belong to the group \mathcal{D} (see 1.3). Therefore for different values of n the fields of J_n -complex tangent lines to Σ are pairwise non-isotopic through contact structures on \mathbb{T}^3 (see 1.3.B) and thus we get (ii).

It remains to check that J_m and J_n are homotopic through complex structures for all m and n. In order to do it we notice that the map $DF: TV \to TV$ is homotopic to the identity through fiberwise linear maps whose restriction to each fiber is an isomorphism (verification of this fact is straightforward and we omit it). Hence the parametric h-principle for immersions of open manifolds (see [H] or [G2, 2.1.2]) implies that F is homotopic to the identity through immersions $V \to V$. Let $F_t, t \in [0; n]$ be such a homotopy with $F_0 = F$ and $F_n = id$. Then

$$J_t(v) = (DF_t^n(v))^{-1} \circ J_n(F_t^n(v)) \circ DF_t^n(v)$$

is the desired homotopy between J_0 and J_n . This completes the proof.

REMARK 1.4.B. It follows easily from a Bennequin-type inequality proved in [E1, 4.1] that all complex structures with strictly pseudo-convex boundary on $\mathbb{T}^2 \times \mathbb{D}^2$ are homotopic one to another through almost complex structures. Moreover, using additional arguments from [G2] one can show that they are homotopic through complex structures.

REMARK 1.4.C. Let \mathscr{J}_{conv} be the space of complex structures with strictly pseudo-convex boundary on $N = \mathbb{T}^2 \times \mathbb{D}^2$. How to describe the connected components of \mathscr{J}_{conv} ? In order to formulate this question in a more precise way define a diffeomorphism $G_{m,n}$ of N by

$$G_{m,n}(r, \theta, x, y) = (r, \theta + mx + ny, x, y),$$

and consider a complex structure

$$J_{m,n} = DG_{m,n} \circ J \circ DG_{-m,-n}$$

which evidently belongs to \mathcal{J}_{conv} . It follows immediately from 1.3.B that for different pairs of integers (m, n) the structures $J_{m,n}$ represent different connected components of \mathcal{J}_{conv} . Is it true that each such a component contains some $J_{m,n}$?

§2. Surgery along Lagrangian tori

2.1. The standard model

Consider cotangent bundle $T^*\mathbb{T}^2$ of the 2-torus \mathbb{T}^2 endowed with the standard symplectic structure ω_0 . Let $(x, y) \pmod{1}$ be angular coordinates on the base, and let $(r, \theta \pmod{1})$ be polar coordinates on fibers. We identify the hypersurface $\Sigma_0 = \{r = 1\}$ with the 3-torus $\mathbb{T}^3(\theta, x, y)$, and set $N_0 = \{r \le 1\}$.

For $m, n \in \mathbb{Z}$ we define the Dehn twist $f_{m,n} : \Sigma_0 \to \Sigma_0$ by

 $(\theta, x, y) \rightarrow (\theta, x + m\theta, y + n\theta).$

Note that $f_{m,n}$ preserves the restriction of ω_0 to $T\Sigma_0$.

2.2. Configurations of marked Lagrangian tori

Let $L_1, \ldots, L_k \subset \mathbb{C}^2$ be a set of embedded disjoint Lagrangian tori. By marking we mean the choice of a basis in $H_1(L_i)$, say α_i, β_i .

Given such a marking, we can identify sufficiently small closed tubular neighbourhood N_j of L_j with N_0 by a conformally symplectic diffeomorphism in such a way that L_j goes to the zero section, and the cycles α_j , β_j correspond to the x- and y-coordinate cycles respectively. We assume that all N_j are disjoint. Set $\Sigma_j = \partial N_j \approx \mathbb{T}^3$, and $K = \mathbb{C}^2 - \bigcup_{j=1}^k (IntN_j)$. Let $f^{(j)}: \Sigma_j \to \Sigma_j$ be some Dehn twists. Denote by V a manifold obtained as the sum

$$K \cup_{f^{(1)}, \Sigma_1} N_1 \cup \cdots \cup_{f^{(k)}, \Sigma_k} N_k.$$

The main observation of Luttinger is the following

PROPOSITION 2.2.A. ([L]). The manifold V associated with an arbitrary configuration L_1, \ldots, L_k of marked Lagrangian tori and an arbitrary sequence $f^{(1)}, \ldots, f^{(k)}$ of Dehn twists is diffeomorphic to \mathbb{C}^2 . In particular, $H_1(V) = 0$.

Proof. Note that V admits a symplectic structure which outside a compact set coincides with the standard one on \mathbb{C}^2 . It follows immediately from well known theorems by M. Gromov and D. McDuff (see [G1], [M]) that V is diffeomorphic to \mathbb{C}^2 , maybe blown up at finite number of points. On the other hand the signature of V vanishes in view of Novikov's additivity theorem (we thank R. Gompf for this argument), and hence the proposition follows.

We need below the following corollary of 2.2.A. Set $\Sigma = \coprod \Sigma_j$, $N = \coprod N_j$. Let $\Phi: H_1(\Sigma) \to H_1(K)$ be a homomorphism induced by the inclusion, and let $\Psi: H_1(\Sigma) \to H_1(N)$ be a homomorphism induced by the composition

 $\Sigma \xrightarrow{\amalg f^{(j)}} \Sigma \longrightarrow N,$

where the last arrow is the inclusion.

COROLLARY 2.2.B. The homomorphism

 $\Phi \oplus (-\Psi) : H_1(\Sigma) \to H_1(K) \oplus H_1(N)$

is an isomorphism.

Proof. Consider the Mayer-Vietoris sequence

 $H_1(\Sigma) \xrightarrow{\Phi \oplus (-\Psi)} H_1(K) \oplus H_1(N) \longrightarrow H_1(V).$

Since $H_1(V) = 0$ due to 2.2.A, we have that $\Phi \oplus (-\Psi)$ is an epimorphism. But $H_1(\Sigma)$ and $H_1(K) \oplus H_1(N)$ are free \mathbb{Z} -modules of the same dimension 3k. Hence $\Phi \oplus (-\Psi)$ is an isomorphism.

For our purposes we have to fix a basis in each space $H_1(\Sigma)$, $H_1(K)$, $H_1(N)$. Let $h_1, a_1, b_1, \ldots, h_k, a_k, b_k$ be a basis in $H_1(\Sigma)$ such that for every *j* the cycles h_j, a_j, b_j correspond to θ -, *x*- and *y*-coordinate cycles on \mathbb{T}^3 respectively. Let $A_1, B_1, \ldots, A_k, B_k$ be a basis in $H_1(N)$, where for every *j* the cycles A_j, B_j correspond to *x*- and *y*-coordinate cycles on \mathbb{T}^2 respectively. Finally, let H_1, \ldots, H_k be the basis in $H_1(K)$ which is defined by relations

$$\ell k(H_i, L_j) = \begin{bmatrix} 1, & i = j \\ 0, & i \neq j \end{bmatrix}$$

(here the orientation of L_i is determined by the marking).

§3. Proof of main theorems

3.1. Proof of 1.1.A

Let $L \subset \mathbb{C}^2$ be an embedded Lagrangian torus, and let σ be its linking class. Choose a marking α , β on L and apply the construction of 2.2 with respect to a Dehn twist $f_{m,n}$. Recall that using homotopically trivial fields of lines one can define the canonical trivialisation of the (co)tangent bundle to a 2-torus. Consider a trivialisation of the normal bundle to L which is obtained from the canonical one of TL by the multiplication by J. It is easy to see that after the identification of a tubular neighbourhood of L with N_0 (see 2.2) this trivialisation coincides with the canonical one of $T^*\mathbb{T}^2$.

In view of this we have that the maps $\Phi: H_1(\Sigma) \to H_1(K)$ and $\Psi: H_1(\Sigma) \to H_1(N)$ act as follows:

$$\Phi(h) = H,$$
 $\Phi(a) = \sigma(\alpha)H,$ $\Phi(b) = \sigma(\beta)H;$
 $\Psi(h) = mA + nB,$ $\Psi(a) = A,$ $\Psi(b) = B.$

(The numeration of the basis elements is omitted since we work with one torus). Hence in the bases (h, a, b) and (H, A, B) the map $\Phi \oplus (-\Psi)$ is given by the matrix

$$\begin{pmatrix} 1 & \sigma(\alpha) & \sigma(\beta) \\ -m & -1 & 0 \\ -n & 0 & -1 \end{pmatrix}.$$

Its determinant equals to $1 - \sigma(\alpha)m - \sigma(\beta)n$. On the other hand 2.2.B implies that this determinant equals to ± 1 for all *m* and *n*. Hence $\sigma(\alpha) = \sigma(\beta) = 0$. This completes the proof.

3.2. Proof of 1.2.A

Let us represent a neighbourhood of the zero section in $T^*\mathbb{T}^2$ as a tubular neighbourhood \mathscr{U} of the standard Lagrangian torus $L_1 = \mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{C}^2$. Let L_2 be an embedded Lagrangian torus in \mathscr{U} which is disjoint from L_1 and homologous to L_1 inside \mathscr{U} . The assertion we have to prove can be reformulated as follows: every cycle $e \in H_1(L_2)$ is unlinked with L_1 :

 $\ell k(e, L_1) = 0.$

Denote by $\tau : \mathcal{U} \to L_1$ the natural projection and by $\tau_* : H_1(L_2) \to H_1(L_1)$ the induced *isomorphism*. We need the following simple topological

LEMMA 3.2.A. For every $e \in H_1(L_2)$ the following equality holds:

 $\ell k(e, L_1) = \ell k(\tau_* e, L_2),$

where we assume that τ preserves orientations of L_1 and L_2 .

The proof is given in 3.3 below.

Let α_2 , β_2 be a marking of L_2 , and let $\alpha_1 = \tau_* \alpha_2$, $\beta_1 = \tau_* \beta_2$ be the "coherent" marking of L_1 . Set $u = \ell k(\alpha_1, L_2) = \ell k(\alpha_2, L_1)$, $v = \ell k(\beta_1, L_2) = \ell k(\beta_2, L_1)$. Choose disjoint tubular neighbourhoods N_1 , N_2 of L_1 , L_2 respectively *inside* \mathcal{U} , and apply the surgery procedure 2.2 associated with Dehn twists $f^{(1)} = f_{m,n}$ and $f^{(2)} = f_{m,n}$ for some integer m, n. Now consider the action of Φ and Ψ in corresponding bases $(h_1, a_1, b_1, h_2, a_2, b_2)$ and $(H_1, A_1, B_1, H_2, A_2, B_2)$. A straightforward computation (which uses also 1.1.A) shows that $\Phi \oplus (-\Psi)$ is given by the matrix

| | <i>h</i> ₁ | a 1 | b 1 | h2 | a 2 | b2 |
|----------------|-----------------------|------------|------------|----|------------|----|
| H ₁ | 1 | .0 | 0 | 0 | u | v |
| A 1 | -m | -1 | 0 | 0 | 0 | 0 |
| B 1 | -n | 0 | -1 | 0 | 0 | 0 |
| H ₂ | 0 | u | υ | 1 | 0 | 0 |
| A2 | 0 | 0 | 0 | -m | -1 | 0 |
| B 2 | 0 | 0 | 0 | -n | 0 | -1 |

whose determinant is equal to $1 - (um + vn)^2$. On the other hand, this determinant equals to ± 1 for each choice of m and n due to 2.2.B. Hence u = v = 0, and the desired assertion follows.

3.3. Proof of 3.2.A

Let $v_1 \in H_1(L_1)$ (respectively, $v_2 \in H_1(L_2)$) be a class Poincare dual to $\ell k(\cdot, L_2)$ (respectively, to $\ell k(\cdot, L_1)$). We have to show that $\tau_* v_2 = v_1$, in other words that 1-cycles representing these classes are homologous inside \mathscr{U} . Let \mathscr{R} be a smooth embedded 3-chain which spans L_1 in \mathbb{C}^2 and has the following properties:

- \mathcal{R} is transversal to $\partial \mathcal{U}$ and to L_2 ;
- $\mathscr{R} \cap \mathscr{U} \approx \mathbb{T}^2 \times [0; 1]$, where $\mathbb{T}^2 \times \{0\} = L_1$ and $\mathbb{T}^2 \times \{1\} \subset \partial \mathscr{U}$.

Let \mathscr{R}' be a small shift of \mathscr{R} along the field of normals, such that $\mathscr{R} \cap \mathscr{R}' = \emptyset$ and \mathscr{R}' intersects $\partial \mathscr{U}$ transversally along a torus L. Note that L and L_2 are homologous inside \mathscr{U} . Let Q be a 3-chain such that $Q \subset \mathscr{U}$ and $\partial Q = L \cup L_2$. We shall assume that Q is an immersed 3-manifold transversal to \mathscr{R} and to L_1 . Finally, set $S = Q \cup (\mathscr{R}' - \mathscr{U})$. Note that S is a 3-chain with the following properties:

- S spans L_2 in \mathbb{C}^2 ;
- S is transversal to \mathcal{R} and to L_1 and intersects \mathcal{R} inside \mathcal{U} .

Set $W = S \cap \mathscr{R}$. Obviously, W is a 2-chain in \mathscr{U} whose boundary components are $S \cap L_1$ and $\mathscr{R} \cap L_2$. Moreover, 1-cycles $S \cap L_1$ on L_1 and $\mathscr{R} \cap L_2$ on L_2 represent classes v_1 and v_2 respectively. Hence $\tau_* v_2 = v_1$, and the proof is complete.

3.4. Proof of 1.3.A

Assume that f is a linear automorphism of \mathbb{T}^3 with $[f] \in \mathcal{D}$. One can easily check that the form $f^*\lambda$ is isotopic to λ through contact forms, and hence f is isotopic to a contactomorphism.

The proof of the inverse assertion is divided into several steps.

(1) We represent \mathbb{T}^3 as the hypersurface $\Sigma_0 = \{r = 1\}$ in $T^*\mathbb{T}^2$ (see 2.1). Then λ is just the restriction of the standard Liouville form

 $r\cos 2\pi\theta \, dx + r\sin 2\pi\theta \, dy$

on $T^*\mathbb{T}^2$. Let $f:\mathbb{T}^3 \to \mathbb{T}^3$ be a contactomorphism, that is $f^*\lambda = \varphi\lambda$ for some non-vanishing function $\varphi(\theta, x, y)$. Since α and $-\alpha$ are isotopic through contact forms, we can assume that φ is *positive*.

(2) We claim that the map $F: \Sigma_0 \to T^* \mathbb{T}^2$, given in coordinates (r, θ, x, y) on $T^* \mathbb{T}^2$ by

$$(\theta, x, y) \mapsto \left(\frac{1}{\varphi(\theta, x, y)}, f(\theta, x, y)\right)$$

is symplectic, that is $F^*\omega_0 = \omega_0 |_{T\Sigma}$. Indeed,

$$F^*\omega_0 = F^* d(r \cdot (\cos 2\pi\theta \, dx + \sin 2\pi\theta \, dy))$$
$$= d\left(\frac{1}{\varphi} \cdot f^*\lambda\right) = d\lambda = \omega_0.$$

(3) Take a Lagrangian torus $L = \{\theta = \text{const}\} \subset \Sigma_0$. Due to the previous step, its image F(L) is a Lagrangian torus in $T^*\mathbb{T}^2$ disjoint from the zero section. Obviously, the projection $F(L) \to \mathbb{R}^2 - \{0\}$ (see 1.2) is homotopic to a point if and only if $[f] \in \mathcal{D}$. The desired assertion follows now from 1.2.A.

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