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## New applications of Luttinger's surgery

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### §1. Introduction and main results

Recently Karl Luttinger [L] made a remarkable observation that certain surgeries along a Lagrangian 2-torus in the standard symplectic space  $(\mathbb{C}^2, \omega)$  do not change the ambient topology. As a consequence he found restrictions on isotopy classes of embeddings  $\mathbb{T}^2 \rightarrow \mathbb{C}^2$  which can be represented by Lagrangian ones.

In the present paper, we discuss some new applications of this technique to *linking* of Lagrangian 2-tori in  $\mathbb{C}^2$ , to *contact geometry* on the 3-torus as well as to study of *complex structures with pseudo-convex boundary* on  $\mathbb{T}^2 \times \mathbb{D}^2$ .

#### 1.1. *Linking class of totally real tori*

A field of lines on a 2-torus is called *homotopically trivial* if it is homotopic to the kernel of a non-singular closed 1-form. All homotopically trivial line fields are homotopic. A 2-torus in  $\mathbb{C}^2$  is called *totally real* if it has no complex tangent lines. From now on we denote by  $\ell k(\cdot, \cdot)$  the linking number, and by  $J$  the standard complex structure on  $\mathbb{C}^2$ . All (co)homology groups considered below are integer.

Assume that  $L \subset \mathbb{C}^2$  is an embedded oriented totally real 2-torus. Take an arbitrary non-singular tangent vector field, say  $v$  on  $L$  which generates a homotopically trivial field of lines. For a 1-cycle  $\alpha$  on  $L$  set

$$\sigma(\alpha) = \ell k(\alpha + \varepsilon Jv, L),$$

where  $\varepsilon$  is sufficiently small.

One can easily check that  $\sigma$  is a well defined element of  $H^1(L)$ , in particular  $\sigma$  does not depend on the choice of  $v$ . We call  $\sigma$  the *linking class* of a totally real torus  $L$  (see [P1], [P2]). Note that this class is closely related to the *Viro quadratic form*.

As it was shown in [P1] for each cohomology class  $\sigma \in H^1(L)$  there exists a totally real embedding  $L \rightarrow \mathbb{C}^2$  whose linking class is equal to  $\sigma$ . However for Lagrangian submanifolds the situation is quite different. Namely, we prove the following result which was conjectured in [P1], [P2].

**THEOREM 1.1.A.** *The linking class of every embedded Lagrangian torus in  $\mathbb{C}^2$  vanishes.*

The theorem is proved below in 3.1.

As a consequence we obtain the following

**COROLLARY 1.1.B.** (see [P1]). *Let  $M \subset \mathbb{C}^2$  be an embedded closed 3-manifold whose characteristic foliation admits an embedded invariant 2-torus  $L$ . If  $L$  divides  $M$  then the restriction of the characteristic foliation to  $L$  is homotopically trivial.*

*Proof.* Notice that  $L$  is a Lagrangian torus. Let  $l$  be the field of Euclidian normal lines to  $M$  along  $L$ . Then the field  $Jl$  is tangent to the characteristic foliation on  $L$ . The needed assertion easily follows now from 1.1.A.  $\square$

## 1.2. *Giroux' theorem*

Homotopically trivial fields of lines on  $\mathbb{T}^2$  allow to identify canonically (up to a homotopy) the cotangent bundle  $T^*\mathbb{T}^2$  with  $\mathbb{T}^2 \times \mathbb{R}^2$  (with this language the zero section is identified with  $\mathbb{T}^2 \times \{0\}$ ).

**THEOREM 1.2.A.** *Consider an embedded Lagrangian torus in  $T^*\mathbb{T}^2 = \mathbb{T}^2 \times \mathbb{R}^2$  which does not intersect the zero section and is homologous to it. Then its projection to  $\mathbb{R}^2 - \{0\}$  is homotopic to a point.*

This result was conjectured by J.-C. Sikorav in [S] who verified it under an additional assumption that the torus is *Lagrangian isotopic* to the zero section. It was proved recently by E. Giroux (see [Gi]) using, in particular, some tools from contact geometry. We give here a different purely symplectic proof (see section 3.2 below).

## 1.3. *Contact geometry of the 3-torus*

Consider the 3-torus  $\mathbb{T}^2 = S^1(\theta) \times \mathbb{T}^2(x, y)$ , where  $(\theta, x, y) \pmod{1}$  are angular coordinates. Let  $\xi = \text{Ker } \lambda$ , where

$$\lambda = \cos 2\pi\theta \, dx + \sin 2\pi\theta \, dy$$

be the standard contact structure.

We identify  $H_1(\mathbb{T}^3)$  with  $\mathbb{Z} \oplus \mathbb{Z}^2$  and the automorphisms group of  $H_1(\mathbb{T}^3)$  with  $\mathrm{GL}(3, \mathbb{Z})$ . Recall [La] that isotopy classes of 3-torus diffeomorphisms are defined by their action on homology. Let  $\mathcal{D} \subset \mathrm{SL}(3, \mathbb{Z})$  be the stabilizer of the subspace  $0 \oplus \mathbb{Z}^2$ .

**THEOREM 1.3.A.** *An element from  $\mathrm{SL}(3, \mathbb{Z})$  can be represented by a contactomorphism of the standard contact structure  $\xi$  if and only if it belongs to  $\mathcal{D}$ .*

The proof which is based on 1.2.A is given in Section 3.4 below.

We apply this theorem in order to construct an infinite sequence of pairwise non-isotopic tight contact structures on  $\mathbb{T}^3$  with the same Euler class (see Question 8.6.1 in [E2]). Recall that two contact structures are called *isotopic* if there exists a diffeomorphism isotopic to the identity which takes one to another. An immediate consequence of 1.3.A is the following

**COROLLARY 1.3.B.** *For  $f, g \in \mathrm{SL}(3, \mathbb{Z})$ , contact structures  $f_*(\xi)$  and  $g_*(\xi)$  are isotopic if and only if  $f^{-1} \circ g$  belongs to  $\mathcal{D}$ .*

A theorem by J. Gray states that two contact structures on a compact manifold which are homotopic through contact structures are isotopic. On the other hand the image of the standard contact structure  $\xi$  under an arbitrary diffeomorphism of  $\mathbb{T}^3$  is homotopic to  $\xi$  through plane distributions.

Hence, we have, in particular

**COROLLARY 1.3.C.** *There exists a sequence  $\xi_n$ ,  $n \geq 0$ , of contact structures on  $\mathbb{T}^3$  such that*

- (i)  $\xi_n$  is contactomorphic to  $\xi$  for every  $n$ , and  $\xi_0 = \xi$ ;
- (ii) all  $\xi_n$  are homotopic to  $\xi$  through two-dimensional distributions;
- (iii) for  $m \neq n$  the structures  $\xi_m$  and  $\xi_n$  are not homotopic through contact structures on  $\mathbb{T}^3$ .

*Proof.* Take a diffeomorphism  $f$  of  $\mathbb{T}^3$  such that  $[f^n] \notin \mathcal{D}$  for every  $n \in \mathbb{Z} - \{0\}$ . It follows from 1.3.B and the previous discussion that the structures  $\xi_n = f_*^n(\xi)$ ,  $n = 0, \dots$ , are homotopic through plane distributions but not through contact structures.  $\square$

**REMARK 1.3.D.** Giroux in [Gi] used Theorem 1.2.A to construct a tight (see [E2]) contact structure on  $T^3$  which is homotopic (through two-dimensional distributions) but *not isomorphic* to the standard contact structure  $\xi_0$ . His structure

is symplectically fillable (see [E1] for the definition of symplectically and holomorphically fillable structures) while at least some of structures constructed above are holomorphically fillable (see the next section).

#### 1.4. Complex structures on $\mathbb{T}^2 \times \mathbb{D}^2$

A contact structure on an oriented 3-manifold is called *positive* if it is (locally) defined by a 1-form, say  $\lambda$  with  $\lambda \wedge d\lambda > 0$ . A boundary of a complex surface is called *strictly pseudo-convex* if its field of tangent lines is a positive (with respect to the canonical orientation) contact structure.

It was shown in [E1] that the manifold  $\mathbb{S}^2 \times \mathbb{D}^2$  does not admit a complex structure with strictly pseudo-convex boundary. In the present section we study the space of such structures on  $\mathbb{T}^2 \times \mathbb{D}^2$ .

**THEOREM 1.4.A.** *There exists a sequence  $J_n$ ,  $n \geq 0$ , of complex structures with strictly pseudo-convex boundary on  $\mathbb{T}^2 \times \mathbb{D}^2$  such that*

- (i) *any two of them are biholomorphically equivalent and homotopic through complex structures;*
- (ii) *for  $m \neq n$  the structures  $J_m$  and  $J_n$  are not homotopic through complex structures with strictly pseudo-convex boundary.*

*Proof.* We represent  $V = \mathbb{T}^2 \times \mathbb{R}^2$  as the quotient space of  $\mathbb{C}^2$  by the imaginary lattice  $i\mathbb{Z}^2$ . We still denote by  $J$  the induced complex structure on  $V$ . Let  $(x, y)(\text{mod } 1)$  be angular coordinates on  $\mathbb{T}^2$  and  $(r, \theta)(\text{mod } 1)$  be polar coordinates on  $\mathbb{R}^2$ . Set

$$N = \mathbb{T}^2 \times \mathbb{D}^2 = \{r \leq 1\}.$$

Denote by  $\Sigma = \mathbb{T}^3$  the boundary of  $N$ . Obviously,  $\Sigma$  is strictly pseudo-convex with respect to  $J$  since its field of tangent complex lines is just the standard contact structure  $\xi$  defined in 1.3.

Consider a diffeomorphism  $F : V \rightarrow V$ ,

$$(r, \theta, x, y) \rightarrow (r, \theta + 2x, x, y),$$

and set

$$J_n = DF^n \circ J \circ DF^{-n}.$$

We claim that the sequence  $\{J_n\}$  has the desired properties. Indeed, since  $F$  preserves  $\Sigma$  we conclude that all  $J_n|_N$  are pairwise biholomorphically equivalent and with strictly pseudo-convex boundary. Moreover, for  $n \neq 0$  the restriction of  $F$  to  $\Sigma$  does not belong to the group  $\mathcal{D}$  (see 1.3). Therefore for different values of  $n$  the fields of  $J_n$ -complex tangent lines to  $\Sigma$  are pairwise non-isotopic through contact structures on  $\mathbb{T}^3$  (see 1.3.B) and thus we get (ii).

It remains to check that  $J_m$  and  $J_n$  are homotopic through complex structures for all  $m$  and  $n$ . In order to do it we notice that the map  $DF: TV \rightarrow TV$  is homotopic to the identity through fiberwise linear maps whose restriction to each fiber is an isomorphism (verification of this fact is straightforward and we omit it). Hence the parametric  $h$ -principle for immersions of open manifolds (see [H] or [G2, 2.1.2]) implies that  $F$  is homotopic to the identity through immersions  $V \rightarrow V$ . Let  $F_t$ ,  $t \in [0; n]$  be such a homotopy with  $F_0 = F$  and  $F_n = id$ . Then

$$J_t(v) = (DF_t^n(v))^{-1} \circ J_n(F_t^n(v)) \circ DF_t^n(v)$$

is the desired homotopy between  $J_0$  and  $J_n$ . This completes the proof.  $\square$

**REMARK 1.4.B.** It follows easily from a Bennequin-type inequality proved in [E1, 4.1] that all complex structures with strictly pseudo-convex boundary on  $\mathbb{T}^2 \times \mathbb{D}^2$  are homotopic one to another through almost complex structures. Moreover, using additional arguments from [G2] one can show that they are homotopic through complex structures.

**REMARK 1.4.C.** Let  $\mathcal{J}_{conv}$  be the space of complex structures with strictly pseudo-convex boundary on  $N = \mathbb{T}^2 \times \mathbb{D}^2$ . *How to describe the connected components of  $\mathcal{J}_{conv}$ ?* In order to formulate this question in a more precise way define a diffeomorphism  $G_{m,n}$  of  $N$  by

$$G_{m,n}(r, \theta, x, y) = (r, \theta + mx + ny, x, y),$$

and consider a complex structure

$$J_{m,n} = DG_{m,n} \circ J \circ DG_{-m,-n}$$

which evidently belongs to  $\mathcal{J}_{conv}$ . It follows immediately from 1.3.B that for different pairs of integers  $(m, n)$  the structures  $J_{m,n}$  represent different connected components of  $\mathcal{J}_{conv}$ . *Is it true that each such a component contains some  $J_{m,n}$ ?*

## §2. Surgery along Lagrangian tori

### 2.1. The standard model

Consider cotangent bundle  $T^*\mathbb{T}^2$  of the 2-torus  $\mathbb{T}^2$  endowed with the standard symplectic structure  $\omega_0$ . Let  $(x, y) \pmod{1}$  be angular coordinates on the base, and let  $(r, \theta \pmod{1})$  be polar coordinates on fibers. We identify the hypersurface  $\Sigma_0 = \{r = 1\}$  with the 3-torus  $\mathbb{T}^3(\theta, x, y)$ , and set  $N_0 = \{r \leq 1\}$ .

For  $m, n \in \mathbb{Z}$  we define the *Dehn twist*  $f_{m,n} : \Sigma_0 \rightarrow \Sigma_0$  by

$$(\theta, x, y) \rightarrow (\theta, x + m\theta, y + n\theta).$$

Note that  $f_{m,n}$  preserves the restriction of  $\omega_0$  to  $T\Sigma_0$ .

### 2.2. Configurations of marked Lagrangian tori

Let  $L_1, \dots, L_k \subset \mathbb{C}^2$  be a set of embedded disjoint Lagrangian tori. By *marking* we mean the choice of a basis in  $H_1(L_j)$ , say  $\alpha_j, \beta_j$ .

Given such a marking, we can identify sufficiently small closed tubular neighbourhood  $N_j$  of  $L_j$  with  $N_0$  by a *conformally symplectic diffeomorphism* in such a way that  $L_j$  goes to the zero section, and the cycles  $\alpha_j, \beta_j$  correspond to the  $x$ - and  $y$ -coordinate cycles respectively. We assume that all  $N_j$  are disjoint. Set  $\Sigma_j = \partial N_j \approx \mathbb{T}^3$ , and  $K = \mathbb{C}^2 - \bigcup_{j=1}^k (\text{Int} N_j)$ . Let  $f^{(j)} : \Sigma_j \rightarrow \Sigma_j$  be some Dehn twists. Denote by  $V$  a manifold obtained as the sum

$$K \cup_{f^{(1)}, \Sigma_1} N_1 \cup \dots \cup_{f^{(k)}, \Sigma_k} N_k.$$

The main observation of Luttinger is the following

**PROPOSITION 2.2.A.** ([L]). *The manifold  $V$  associated with an arbitrary configuration  $L_1, \dots, L_k$  of marked Lagrangian tori and an arbitrary sequence  $f^{(1)}, \dots, f^{(k)}$  of Dehn twists is diffeomorphic to  $\mathbb{C}^2$ . In particular,  $H_1(V) = 0$ .*

*Proof.* Note that  $V$  admits a symplectic structure which outside a compact set coincides with the standard one on  $\mathbb{C}^2$ . It follows immediately from well known theorems by M. Gromov and D. McDuff (see [G1], [M]) that  $V$  is diffeomorphic to  $\mathbb{C}^2$ , maybe blown up at finite number of points. On the other hand the signature of  $V$  vanishes in view of Novikov's additivity theorem (we thank R. Gompf for this argument), and hence the proposition follows.  $\square$

We need below the following corollary of 2.2.A. Set  $\Sigma = \amalg \Sigma_j$ ,  $N = \amalg N_j$ . Let  $\Phi : H_1(\Sigma) \rightarrow H_1(K)$  be a homomorphism induced by the inclusion, and let  $\Psi : H_1(\Sigma) \rightarrow H_1(N)$  be a homomorphism induced by the composition

$$\Sigma \xrightarrow{\amalg f^j} \Sigma \longrightarrow N,$$

where the last arrow is the inclusion.

**COROLLARY 2.2.B.** *The homomorphism*

$$\Phi \oplus (-\Psi) : H_1(\Sigma) \rightarrow H_1(K) \oplus H_1(N)$$

*is an isomorphism.*

*Proof.* Consider the Mayer–Vietoris sequence

$$H_1(\Sigma) \xrightarrow{\Phi \oplus (-\Psi)} H_1(K) \oplus H_1(N) \longrightarrow H_1(V).$$

Since  $H_1(V) = 0$  due to 2.2.A, we have that  $\Phi \oplus (-\Psi)$  is an epimorphism. But  $H_1(\Sigma)$  and  $H_1(K) \oplus H_1(N)$  are free  $\mathbb{Z}$ -modules of the same dimension  $3k$ . Hence  $\Phi \oplus (-\Psi)$  is an isomorphism.  $\square$

For our purposes we have to fix a basis in each space  $H_1(\Sigma)$ ,  $H_1(K)$ ,  $H_1(N)$ . Let  $h_1, a_1, b_1, \dots, h_k, a_k, b_k$  be a basis in  $H_1(\Sigma)$  such that for every  $j$  the cycles  $h_j, a_j, b_j$  correspond to  $\theta$ -,  $x$ - and  $y$ -coordinate cycles on  $\mathbb{T}^3$  respectively. Let  $A_1, B_1, \dots, A_k, B_k$  be a basis in  $H_1(N)$ , where for every  $j$  the cycles  $A_j, B_j$  correspond to  $x$ - and  $y$ -coordinate cycles on  $\mathbb{T}^2$  respectively. Finally, let  $H_1, \dots, H_k$  be the basis in  $H_1(K)$  which is defined by relations

$$\ell k(H_i, L_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

(here the orientation of  $L_j$  is determined by the marking).

### §3. Proof of main theorems

#### 3.1. Proof of 1.1.A

Let  $L \subset \mathbb{C}^2$  be an embedded Lagrangian torus, and let  $\sigma$  be its linking class. Choose a marking  $\alpha, \beta$  on  $L$  and apply the construction of 2.2 with respect to a Dehn twist  $f_{m,n}$ .

Recall that using homotopically trivial fields of lines one can define the canonical trivialisation of the (co)tangent bundle to a 2-torus. Consider a trivialisation of the normal bundle to  $L$  which is obtained from the canonical one of  $TL$  by the multiplication by  $J$ . It is easy to see that after the identification of a tubular neighbourhood of  $L$  with  $N_0$  (see 2.2) this trivialisation coincides with the canonical one of  $T^*\mathbb{T}^2$ .

In view of this we have that the maps  $\Phi : H_1(\Sigma) \rightarrow H_1(K)$  and  $\Psi : H_1(\Sigma) \rightarrow H_1(N)$  act as follows:

$$\Phi(h) = H, \quad \Phi(a) = \sigma(\alpha)H, \quad \Phi(b) = \sigma(\beta)H;$$

$$\Psi(h) = mA + nB, \quad \Psi(a) = A, \quad \Psi(b) = B.$$

(The numeration of the basis elements is omitted since we work with one torus). Hence in the bases  $(h, a, b)$  and  $(H, A, B)$  the map  $\Phi \oplus (-\Psi)$  is given by the matrix

$$\begin{pmatrix} 1 & \sigma(\alpha) & \sigma(\beta) \\ -m & -1 & 0 \\ -n & 0 & -1 \end{pmatrix}.$$

Its determinant equals to  $1 - \sigma(\alpha)m - \sigma(\beta)n$ . On the other hand 2.2.B implies that this determinant equals to  $\pm 1$  for all  $m$  and  $n$ . Hence  $\sigma(\alpha) = \sigma(\beta) = 0$ . This completes the proof.  $\square$

### 3.2. Proof of 1.2.A

Let us represent a neighbourhood of the zero section in  $T^*\mathbb{T}^2$  as a tubular neighbourhood  $\mathcal{U}$  of the standard Lagrangian torus  $L_1 = \mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{C}^2$ . Let  $L_2$  be an embedded Lagrangian torus in  $\mathcal{U}$  which is disjoint from  $L_1$  and homologous to  $L_1$  inside  $\mathcal{U}$ . The assertion we have to prove can be reformulated as follows: every cycle  $e \in H_1(L_2)$  is unlinked with  $L_1$ :

$$\ell k(e, L_1) = 0.$$

Denote by  $\tau : \mathcal{U} \rightarrow L_1$  the natural projection and by  $\tau_* : H_1(L_2) \rightarrow H_1(L_1)$  the induced isomorphism. We need the following simple topological

**LEMMA 3.2.A.** *For every  $e \in H_1(L_2)$  the following equality holds:*

$$\ell k(e, L_1) = \ell k(\tau_* e, L_2),$$

where we assume that  $\tau$  preserves orientations of  $L_1$  and  $L_2$ .

The proof is given in 3.3 below.

Let  $\alpha_2, \beta_2$  be a marking of  $L_2$ , and let  $\alpha_1 = \tau_* \alpha_2, \beta_1 = \tau_* \beta_2$  be the “coherent” marking of  $L_1$ . Set  $u = \ell k(\alpha_1, L_2) = \ell k(\alpha_2, L_1), v = \ell k(\beta_1, L_2) = \ell k(\beta_2, L_1)$ . Choose disjoint tubular neighbourhoods  $N_1, N_2$  of  $L_1, L_2$  respectively *inside*  $\mathcal{U}$ , and apply the surgery procedure 2.2 associated with Dehn twists  $f^{(1)} = f_{m,n}$  and  $f^{(2)} = f_{m,n}$  for some integer  $m, n$ . Now consider the action of  $\Phi$  and  $\Psi$  in corresponding bases  $(h_1, a_1, b_1, h_2, a_2, b_2)$  and  $(H_1, A_1, B_1, H_2, A_2, B_2)$ . A straightforward computation (which uses also 1.1.A) shows that  $\Phi \oplus (-\Psi)$  is given by the matrix

	$h_1$	$a_1$	$b_1$	$h_2$	$a_2$	$b_2$
$H_1$	1	0	0	0	$u$	$v$
$A_1$	$-m$	-1	0	0	0	0
$B_1$	$-n$	0	-1	0	0	0
$H_2$	0	$u$	$v$	1	0	0
$A_2$	0	0	0	$-m$	-1	0
$B_2$	0	0	0	$-n$	0	-1

whose determinant is equal to  $1 - (um + vn)^2$ . On the other hand, this determinant equals to  $\pm 1$  for each choice of  $m$  and  $n$  due to 2.2.B. Hence  $u = v = 0$ , and the desired assertion follows.  $\square$

### 3.3. Proof of 3.2.A

Let  $v_1 \in H_1(L_1)$  (respectively,  $v_2 \in H_1(L_2)$ ) be a class Poincare dual to  $\ell k(\cdot, L_2)$  (respectively, to  $\ell k(\cdot, L_1)$ ). We have to show that  $\tau_* v_2 = v_1$ , in other words that 1-cycles representing these classes are homologous inside  $\mathcal{U}$ . Let  $\mathcal{R}$  be a smooth embedded 3-chain which spans  $L_1$  in  $\mathbb{C}^2$  and has the following properties:

- $\mathcal{R}$  is transversal to  $\partial \mathcal{U}$  and to  $L_2$ ;
- $\mathcal{R} \cap \mathcal{U} \approx \mathbb{T}^2 \times [0; 1]$ , where  $\mathbb{T}^2 \times \{0\} = L_1$  and  $\mathbb{T}^2 \times \{1\} \subset \partial \mathcal{U}$ .

Let  $\mathcal{R}'$  be a small shift of  $\mathcal{R}$  along the field of normals, such that  $\mathcal{R} \cap \mathcal{R}' = \emptyset$  and  $\mathcal{R}'$  intersects  $\partial \mathcal{U}$  transversally along a torus  $L$ . Note that  $L$  and  $L_2$  are

homologous inside  $\mathcal{U}$ . Let  $Q$  be a 3-chain such that  $Q \subset \mathcal{U}$  and  $\partial Q = L \cup L_2$ . We shall assume that  $Q$  is an immersed 3-manifold transversal to  $\mathcal{R}$  and to  $L_1$ . Finally, set  $S = Q \cup (\mathcal{R}' - \mathcal{U})$ . Note that  $S$  is a 3-chain with the following properties:

- $S$  spans  $L_2$  in  $\mathbb{C}^2$ ;
- $S$  is transversal to  $\mathcal{R}$  and to  $L_1$  and intersects  $\mathcal{R}$  inside  $\mathcal{U}$ .

Set  $W = S \cap \mathcal{R}$ . Obviously,  $W$  is a 2-chain in  $\mathcal{U}$  whose boundary components are  $S \cap L_1$  and  $\mathcal{R} \cap L_2$ . Moreover, 1-cycles  $S \cap L_1$  on  $L_1$  and  $\mathcal{R} \cap L_2$  on  $L_2$  represent classes  $v_1$  and  $v_2$  respectively. Hence  $\tau_* v_2 = v_1$ , and the proof is complete.  $\square$

### 3.4. Proof of 1.3.A

Assume that  $f$  is a linear automorphism of  $\mathbb{T}^3$  with  $[f] \in \mathcal{D}$ . One can easily check that the form  $f^*\lambda$  is isotopic to  $\lambda$  through contact forms, and hence  $f$  is isotopic to a contactomorphism.

The proof of the inverse assertion is divided into several steps.

(1) We represent  $\mathbb{T}^3$  as the hypersurface  $\Sigma_0 = \{r = 1\}$  in  $T^*\mathbb{T}^2$  (see 2.1). Then  $\lambda$  is just the restriction of the standard Liouville form

$$r \cos 2\pi\theta \, dx + r \sin 2\pi\theta \, dy$$

on  $T^*\mathbb{T}^2$ . Let  $f: \mathbb{T}^3 \rightarrow \mathbb{T}^3$  be a contactomorphism, that is  $f^*\lambda = \varphi\lambda$  for some non-vanishing function  $\varphi(\theta, x, y)$ . Since  $\alpha$  and  $-\alpha$  are isotopic through contact forms, we can assume that  $\varphi$  is positive.

(2) We claim that the map  $F: \Sigma_0 \rightarrow T^*\mathbb{T}^2$ , given in coordinates  $(r, \theta, x, y)$  on  $T^*\mathbb{T}^2$  by

$$(\theta, x, y) \mapsto \left( \frac{1}{\varphi(\theta, x, y)}, f(\theta, x, y) \right)$$

is symplectic, that is  $F^*\omega_0 = \omega_0|_{T\Sigma}$ . Indeed,

$$F^*\omega_0 = F^* d(r \cdot (\cos 2\pi\theta \, dx + \sin 2\pi\theta \, dy))$$

$$= d\left(\frac{1}{\varphi} \cdot f^*\lambda\right) = d\lambda = \omega_0.$$

(3) Take a Lagrangian torus  $L = \{\theta = \text{const}\} \subset \Sigma_0$ . Due to the previous step, its image  $F(L)$  is a Lagrangian torus in  $T^*\mathbb{T}^2$  disjoint from the zero section. Obviously, the projection  $F(L) \rightarrow \mathbb{R}^2 - \{0\}$  (see 1.2) is homotopic to a point if and only if  $[f] \in \mathcal{D}$ . The desired assertion follows now from 1.2.A.  $\square$

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