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## Manifolds of even dimension with amenable fundamental group

BENO ECKMANN

### 0. Introduction

0.1. If the fundamental group  $G$  of a closed (orientable) 4-manifold  $X$  is *infinite and amenable* then the Euler characteristic  $\chi(X)$  is  $\geq 0$ . This has been proved in a previous paper [E] using the Følner criterion for amenability [F], in a geometrical version. If  $X$  is aspherical, i.e., an Eilenberg-MacLane space  $K(G, 1)$  (whence  $G$  a Poincaré duality group of dimension 4, in short a  $PD^4$ -group) then  $\chi(X) = \chi(G) = 0$  by [E], Corollary 2.3.

The main purpose of the present paper is to examine, conversely, 4-manifolds  $X$  as above *assuming*  $\chi(X) = 0$ . We recall (see [E], Section 0.3) that infinite amenable groups  $G$  have one or two ends, i.e.,  $H^1(G; \mathbb{Z}G) = 0$  or  $\mathbb{Z}$ . It is easily seen that the universal cover  $\tilde{X}$  of  $X$  has integral homology  $H_1(\tilde{X}) = H_4(\tilde{X}) = 0$  and  $H_3(\tilde{X}) \cong H^1(G; \mathbb{Z}G)$ . We will prove (Theorem 3.4):

(A) *If  $\chi(X) = 0$  then  $H_2(\tilde{X}) \cong H^2(G; \mathbb{Z}G)$ , the “second end-group” of  $G$ . From this we get the result*

(B) *If  $H^1(G; \mathbb{Z}G) = H^2(G; \mathbb{Z}G) = 0$  then  $\chi(X) = 0$  implies that  $\tilde{X}$  is contractible, whence  $X = K(G, 1)$  and  $G$  is a  $PD^4$ -group.*

These statements can be expressed in terms of the Hausmann-Weinberger invariant  $q(G)$ , see [H-W], for finitely presented groups  $G$  (Corollaries 2.5 and 3.6):

(C) *If  $G$  is infinite amenable then  $q(G)$  is  $\geq 0$ . If  $H^1(G; \mathbb{Z}G) = H^2(G; \mathbb{Z}G) = 0$  then  $q(G) = 0$  implies that  $G$  is a  $PD^4$ -group.*

In the context of these results it is of interest to look at 2-knot groups  $G$  since for these  $q(G)$  is always  $= 0$ ; see Section 4 below.

0.2. The proofs make use of (reduced and non-reduced)  $l_2$ -cohomology of the infinite cell-complex  $\tilde{X}$  combined with the free cocompact action of  $G$  on  $\tilde{X}$ . The main tool then is a lemma of Cheeger-Gromov [Ch-G], see Section 2.2. We apply it not only to get the results for  $\chi(X) = 0$  but also to give a new proof of the statement  $\chi(X) \geq 0$  above. This is done in the more general context of a closed manifold of even dimension  $n = 2k \geq 4$  which, if  $k > 2$ , is *aspherical* up to the middle dimension  $k$ ; for  $n = 4$  there is no asphericity assumption.

These  $2k$ -manifolds can be used to define a new invariant  $\gamma_k(G)$  for groups  $G$  of type  $F_k$ ,  $k \geq 2$ , generalizing the Hausmann-Weinberger invariant  $q(G)$ . For  $G$  of type  $F_2$  (i.e., finitely presented) one has  $\gamma_2(G) = q(G)$ .

0.3. Section 1 contains various facts concerning  $l_2$ -cohomology of  $\tilde{X}$ , ordinary cohomology of  $\tilde{X}$ , and  $G$ -cohomology of  $\tilde{X}$  for  $G$ -module coefficients such as  $l_2G$  and  $\mathbb{Z}G$ . They go a little beyond the minimum necessary for the following sections in view of later use.

0.4. Section 2 deals with  $\chi(X) \geq 0$  for the  $2k$ -manifolds as above and with  $\gamma_k(G)$ , Section 3 with the vanishing of  $\chi(X)$  and the main results. Section 5 is an appendix on the “partial Euler characteristic” of groups  $G$  fulfilling certain finiteness conditions; the results appear already in [E] but are given new proofs by the  $l_2$ -cohomology methods of the present paper.

0.5. Our results on 4-manifolds should be compared with some of those given by Hillman [H] for the case of “elementary amenable” groups, which constitute a special, but important class of amenable groups. The results of [H] are, however, more general in another sense, namely that  $G$  need only have a non-trivial normal subgroup which is elementary amenable.

0.6. Although this paper deals with amenable groups we want to emphasize that the results above on 4-manifolds and the invariant  $q(G)$  are valid for other types of groups, in particular for all finitely presented groups with vanishing first  $l_2$ -Betti number; see Section 6 below (Addendum).

## 1. Infinite cell-complexes and $l_2$ -cohomology

1.1. For a cell-complex  $X$  with  $\pi_1 X = G$  and a  $G$ -module  $A$  we consider cohomology with local coefficients  $H^i(X; A)$ ; i.e.,  $G$ -cohomology  $H_G^i(\tilde{X}; A)$  of the universal cover, relative to the  $G$ -module  $A$  ( $G$  operates on the cell complex  $\tilde{X}$  and on  $A$ ). A special situation occurs if  $X$  is a *finite* complex and  $G$  an *infinite* group, with regard to the coefficient modules  $\mathbb{Z}G$  and  $l_2G$  (the Hilbert space of linear combinations  $\sum_{x \in G} c_x x$ ,  $c_x \in \mathbb{R}$ , with  $\sum_x c_x^2 < \infty$ );  $G$  operates on  $\mathbb{Z}G$  and on  $l_2G$  by left translations.

Namely, one has for the cochains  $C^i(\tilde{X}; \mathbb{Z}G) = \text{Hom}_G(C_i(\tilde{X}), \mathbb{Z}G)$  and  $C^i(\tilde{X}; l_2G) = \text{Hom}_G(C_i(\tilde{X}), l_2G)$  the isomorphisms

- (1)  $C^i(\tilde{X}; \mathbb{Z}G) \cong C_{\text{fin}}^i(\tilde{X}; \mathbb{Z})$ ,
- (2)  $C^i(\tilde{X}; l_2G) \cong C_{(2)}^i(\tilde{X}; \mathbb{R})$ .

$C_{\text{fin}}^i$  is the group of *finite cochains* of  $\tilde{X}$ , and  $C_{(2)}^i$  the group of  $l_2$ -cochains (functions  $f(\sigma_i)$  of the cells  $\sigma_i$  of  $\tilde{X}$  with  $\sum_{\sigma_i} f(\sigma_i)^2 < \infty$ ). The corresponding cohomology groups are respectively  $H_{\text{comp}}^i(\tilde{X}; \mathbb{Z})$ , cohomology with compact support; and  $H_{(2)}^i(\tilde{X}; \mathbb{R})$ ,  $l_2$ -cohomology of  $\tilde{X}$ .

1.2. For the convenience of the reader we recall the proof of (1) and (2).

We choose a (finite)  $\mathbb{Z}G$ -basis  $\{\tau_i\}$  of the chain group  $C_i(\tilde{X})$  corresponding to the cells of  $X$  (one cell in each  $G$ -orbit). Given  $f \in C^i(\tilde{X}; \mathbb{Z}G) = \text{Hom}_G(C_i(\tilde{X}), \mathbb{Z}G)$  we put  $g(x\tau_i) = m_{x^{-1}} \in \mathbb{Z}$  where  $f(\tau_i) = \sum_x m_x x$ ; clearly  $g$  is a finite cochain in  $\tilde{X}$ . Conversely, given  $g \in C_{\text{fin}}^i(\tilde{X}; \mathbb{Z})$  we put  $f(\tau_i) = \sum_x g(x^{-1}\tau_i)x \in \mathbb{Z}G$ . The correspondence  $f \mapsto g$  yields the isomorphism (1). Note that it is independent of the choice of basis  $\{\tau_i\}$ : Indeed if we replace  $\tau_i$  by  $y\tau_i$ ,  $y \in G$ , then  $g(x\tau_i) = g(xy^{-1}y\tau_i) = m'_{yx^{-1}}$  where  $f(y\tau_i) = \sum_x m_x yx = \sum m'_x x$ , i.e.,  $m'_x = m_{y^{-1}x}$ ; thus  $g(x\tau_i) = m'_{yx^{-1}} = m_{x^{-1}}$  as before.

Similarly, given  $f \in C^i(\tilde{X}; l_2 G)$  we put  $g(x\tau_i) = c_{x^{-1}}$  where  $f(\tau_i) = \sum_x c_x x$  with  $\sum_x c_x^2 < \infty$ . Then

$$\sum_{\text{all } \sigma} g(\sigma)^2 = \sum_{\tau_i} \sum_x g(x\tau_i)^2 < \infty,$$

so  $g$  is an  $l_2$ -cochain. This yields the isomorphism (2). We summarize:

**PROPOSITION 1.1.** *For a finite cell complex  $X$  (with infinite fundamental group  $G$ ) the cohomology groups with local coefficients  $H^i(X; \mathbb{Z}G)$  and  $H^i(X; l_2 G)$  are isomorphic respectively to  $H_{\text{comp}}^i(\tilde{X}; \mathbb{Z})$  and  $H_{(2)}^i(\tilde{X}; \mathbb{R})$  of the universal cover  $\tilde{X}$  of  $X$ .*

*Remark.* Everything above holds if instead of  $\tilde{X}$  we take any free cocompact  $G$ -space (=cell complex)  $Y$  with  $Y/G = X$ ;  $G$  is a factor group of  $\pi_1 X$ . The isomorphisms are of interest only if  $G$  is infinite.

1.3. We will also consider *reduced*  $l_2$ -cohomology of  $\tilde{X}$ , denoted by  $\bar{H}^i(\tilde{X})$ . It differs from  $H_{(2)}^i(X; \mathbb{R})$  by  $\delta C_{(2)}^{i-1}(\tilde{X}; \mathbb{R})$  being replaced by its  $l_2$ -closure  $\overline{\delta C_{(2)}^{i-1}}$ . It imbeds equivariantly and isometrically in  $Z^i$ , the kernel of  $\delta : C_{(2)}^i \rightarrow C_{(2)}^{i+1}$ , and its von Neumann dimension relative to  $G$  is denoted by  $\beta_i(\tilde{X} \text{ rel. } G)$ , cf. [Ch-G].

There is an obvious map  $\Phi$  of  $H_{(2)}^i(\tilde{X}; \mathbb{R})$ , i.e. the  $G$ -cohomology group  $H_G^i(\tilde{X}; l_2 G)$  based on  $G$ -homomorphisms  $C_i(\tilde{X}) \rightarrow l_2 G$ , into the *ordinary* cohomology group  $H^i(\tilde{X}; l_2 G)$  disregarding the  $G$ -action on  $\tilde{X}$  and  $l_2 G$ . Under that map  $\Phi$  the closure of  $\delta C^{i-1}(\tilde{X}; l_2 G)$  goes to 0. Indeed, the  $l_2$ -limit  $f$  of a sequence of

$i$ -coboundaries is  $=0$  on the  $i$ -cycles; it thus defines  $\varphi : \partial C_i(\tilde{X}) \rightarrow l_2 G$  which can be extended to all of  $C_{i-1}$  (since  $l_2 G$  is divisible, i.e.  $\mathbb{Z}$ -injective), and  $\delta\varphi = f$ .

**PROPOSITION 1.2.** *The natural map  $H_G^i(\tilde{X}; l_2 G) \rightarrow H^i(\tilde{X}; l_2 G)$  factors through the reduced  $l_2$ -cohomology group  $\bar{H}^i(\tilde{X})$ .*

Of course  $H^i(\tilde{X}; l_2 G)$  can be regarded as a  $\mathbb{Z}G$ -module through the action of  $G$  on  $\tilde{X}$  and on  $l_2 G$ . The image of  $\Phi$  lies in the invariant part  $H^i(\tilde{X}; l_2 G)^G$ .

1.4. The map  $\Phi : H_G^n(\tilde{X}; l_2 G) \rightarrow H^n(\tilde{X}; l_2 G)^G$  occurs in a well-known exact sequence, available if  $\tilde{X}$  is  $(n-1)$ -connected, i.e., if  $\pi_i(X) = 0$  for  $1 < i < n$  (deduced from the spectral sequence of the covering  $\tilde{X} \rightarrow X$ ):

$$0 \rightarrow H^n(G; l_2 G) \rightarrow H_G^n(\tilde{X}; l_2 G) \xrightarrow{\Phi} H^n(\tilde{X}; l_2 G)^G \rightarrow H^{n+1}(G; l_2 G) \rightarrow H_G^{n+1}(\tilde{X}; l_2 G).$$

There is, of course, an analogous exact sequence for  $\mathbb{Z}G$ -coefficients. The coefficient map  $\mathbb{Z}G \rightarrow l_2 G$  by inclusion yields, in combination with Proposition 1.1, the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^n(G; \mathbb{Z}G) & \longrightarrow & H_{\text{comp}}^n(\tilde{X}; \mathbb{Z}) & \xrightarrow{\Phi'} & H^n(\tilde{X}; \mathbb{Z}G)^G \longrightarrow H^{n+1}(G; \mathbb{Z}G) \\ & & \downarrow & & \downarrow & & \downarrow \Omega \\ 0 & \longrightarrow & H^n(G; l_2 G) & \longrightarrow & H_{(2)}^n(\tilde{X}; \mathbb{R}) & \xrightarrow{\Phi} & H^n(\tilde{X}; l_2 G)^G \longrightarrow H^{n+1}(G; l_2 G) \end{array} \quad (4)$$

1.5. There is a further natural map  $\Psi : H_{(2)}^i(\tilde{X}; \mathbb{R}) \rightarrow H^i(\tilde{X}; \mathbb{R})$ ; it clearly factors through  $\bar{H}^i(\tilde{X})$  since the limit of a sequence of  $l_2$ -coboundaries is an ordinary coboundary.

1.6. There is an  $l_2$ -homology analogue of the above statements for  $l_2$ -cohomology; we leave it to the reader. We just remark that it is based on the boundary operator  $\partial : C_{(2)}^i \rightarrow C_{(2)}^{i-1}$  instead of the coboundary  $\delta : C_{(2)}^i \rightarrow C_{(2)}^{i+1}$ ; and that the reduced homology groups  $\bar{H}_i(\tilde{X})$  are isometrically isomorphic to the  $\bar{H}^i(\tilde{X})$  – indeed, they are both isomorphic to the intersection  $Z^i(\tilde{X}) \cap Z_i(\tilde{X})$  in  $C_{(2)}^i$ , where  $Z^i$  denotes the cocycle subspace,  $Z_i$  the cycle subspace of  $C_{(2)}^i$ , and  $Z^i(\tilde{X}) \cap Z_i(\tilde{X})$  is (a) the orthogonal complement of  $\delta C_{(2)}^{i-1}$  in  $Z^i$ , (b) the orthogonal complement of  $\partial C_{(2)}^{i+1}$  in  $Z_i$  (Hodge-de Rham decomposition of  $C_{(2)}^i$ ). We further remark that this yields a simple proof of  $l_2$ -Poincaré duality for a closed  $n$ -manifold  $X$  by using (2) and ordinary Poincaré duality of  $X$ ; one gets  $\bar{H}^i(\tilde{X}) \cong \bar{H}_{n-i}(\tilde{X}) \cong \bar{H}^{n-i}(\tilde{X})$  as Hilbert  $G$ -modules.

## 2. Closed manifolds of dimension $n = 2k$ and an invariant for groups of type $F_k$

2.1. We take for  $X$  a closed orientable (differentiable)  $n$ -manifold,  $n = 2k \geq 4$  which if  $k > 2$  is  $(k - 1)$ -aspherical; i.e., with  $\pi_i(X) = 0$  for  $1 < i < k$ . We assume again  $G = \pi_1(X)$  infinite.

We note that  $H_i(\tilde{X}) = 0$  for  $1 \leq i < k$ , and that  $H_{2k}(\tilde{X}) = 0$  since  $G$  is infinite (if in ordinary homology coefficients are not indicated they are meant to be  $\mathbb{Z}$ ).

**PROPOSITION 2.1.** *For  $k < i \leq 2k$  one has  $H_i(\tilde{X}) \cong H^{2k-i}(G; \mathbb{Z}G)$ .*

*Proof.*  $H_i(\tilde{X}) \cong H_{\text{comp}}^{2k-i}(\tilde{X}) \cong H^{2k-i}(X; \mathbb{Z}G)$  by Poincaré duality. But since  $X$  is  $(k - 1)$ -aspherical  $H^i(X; \mathbb{Z}G) \cong H^i(G; \mathbb{Z}G)$  for  $0 \leq i < k$ . If  $n = 2k = 4$ , there are no asphericity assumptions, and one simply has  $H_3(\tilde{X}) \cong H^1(X; \mathbb{Z}G) \cong H^1(G; \mathbb{Z}G)$ .

If the “end-groups”  $H^i(G; \mathbb{Z}G)$  are 0 for  $0 \leq i < k$  then  $H_k(\tilde{X})$  is the only homology group of  $\tilde{X}$  which is possibly non-zero. If moreover  $H_k(\tilde{X}) = 0$  then  $\tilde{X}$  is contractible,  $X$  is a  $K(G, 1)$ , and  $G$  is a  $PD^{2k}$ -group.

2.2. We now consider the Euler characteristic  $\chi(X) = \sum_{i=0}^n (-1)^i \alpha_i = \sum_{i=0}^n (-1)^i \beta_i(X)$ ;  $\alpha_i$  is the number of  $i$ -cells of a cell-decomposition of  $X$ , and  $\beta_i(X) = \dim_{\mathbb{Q}} H_i(X; \mathbb{Q})$  the  $i$ -th Betti number. We recall ([Ch-G] and [E]) that  $\chi(X)$  can also be expressed by the reduced Betti numbers  $\bar{\beta}_i(\tilde{X} \text{ rel. } G)$  as

$$\chi(X) = \sum_{i=0}^n (-1)^i \bar{\beta}_i(\tilde{X} \text{ rel. } G).$$

$\bar{\beta}_i(\tilde{X} \text{ rel. } G)$  is the von Neumann dimension of  $\bar{H}^i(\tilde{X})$  considered as a Hilbert  $G$ -module.

A lemma of Cheeger-Gromov [Ch-G] tells that if  $G$  is amenable then the natural map  $\bar{H}^i(\tilde{X}) \rightarrow H^i(\tilde{X}; \mathbb{R})$  is *injective*. From our assumptions it follows that  $H^i(\tilde{X}; \mathbb{R}) = 0$  for  $0 < i < k$  whence  $\bar{H}^i(\tilde{X}) = 0$  and  $\bar{\beta}_i(\tilde{X} \text{ rel. } G) = 0$  for  $0 \leq i < k$  ( $\bar{\beta}_0 = 0$  since  $G$  is infinite). By Poincaré duality for the  $\bar{\beta}_i$  (cf. 1.6, or [L-L], Proposition 4.2) it follows that  $\bar{\beta}_i(\tilde{X} \text{ rel. } G) = 0$  for  $k < i \leq 2k$ . The Euler characteristic can thus be expressed by  $\bar{\beta}_k$  alone:

**THEOREM 2.2.** *Let  $X$  be a closed orientable  $n$ -manifold,  $n = 2k$ , which for  $k > 2$  is  $(k - 1)$ -aspherical, and with infinite amenable fundamental group  $G$ . Then*

$$\chi(X) = (-1)^k \bar{\beta}_k(\tilde{X} \text{ rel. } G).$$

**COROLLARY 2.3.** *For  $X$  as in Theorem 2.2 one has*

$$(-1)^k \chi(X) \geq 0.$$

This is due to the fact that  $\bar{\beta}_k$  is a non-negative real number.

In the case  $n = 4$  there are no asphericity assumptions and we get the result proved by a different method (“Følner sequence”) in [E]:

**THEOREM 2.4.** *Let  $X$  be a closed orientable 4-manifold with infinite amenable fundamental group  $G$ . Then  $\chi(X)$  is  $\geq 0$ .*

Or in terms of the Hausmann-Weinberger invariant  $q(G)$ :

**COROLLARY 2.5.** *For a finitely presented infinite amenable group  $G$  the invariant  $q(G)$  is  $\geq 0$ .*

2.3. For manifolds  $X$  as considered in 2.1 the fundamental group  $G = \pi_1(X)$  is of type  $F_k$  (finitely presented and of type  $FP_k$ ). Indeed, the (finite)  $k$ -skeleton of a cell-decomposition of  $X$  can be extended to a  $K(G, 1)$  by attaching cells of dimensions  $> k$ .

Conversely there exists for any group  $G$  of type  $F_k$ ,  $k \geq 2$ , a closed orientable  $2k$ -manifold with  $\pi_1(X) = G$  and  $\pi_i(X) = 0$  for  $1 < i < k$ . To find  $X$  one starts with any closed orientable differentiable  $2k$ -manifold  $M$  with  $\pi_1(M) = G$ . For  $k > 2$ , type  $FP_k$  of  $G$  guarantees that  $\pi_2(M) = H_2(\tilde{M})$  is finitely generated as a  $\mathbb{Z}G$ -module. Thus  $\pi_2(M)$  can be annihilated by a finite number of surgeries in  $M$  (see [M]), and there results a closed manifold  $M'$  with  $\pi_1(M') = G$ ,  $\pi_2(M') = 0$ . If  $k > 3$  then  $\pi_3(M')$  is finitely generated over  $\mathbb{Z}G$ , and the procedure can be repeated until one has a manifold  $X$  as required.

Now we define for a group  $G$  of type  $F_k$ ,  $k \geq 2$ , the invariant  $\gamma_k(G)$  to be the minimum of  $(-1)^k \chi(X)$  for all  $2k$ -manifolds as above with  $\pi_1(X) = G$ ,  $\pi_i(X) = 0$  for  $1 < i < k$ . The minimum exists since

$$\begin{aligned} (-1)^k \chi(X) &= \beta_k(X) + 2 \sum_0^{k-1} (-1)^{i+k} \beta_i(X) \\ &= \beta_k(X) + 2 \sum_0^{k-1} (-1)^{i+k} \beta_i(G) \end{aligned}$$

and  $\beta_k(X) \geq \beta_k(G)$ .

Clearly  $\gamma_2(G) = q(G)$ .

**COROLLARY 2.6.** *For an infinite amenable group  $G$  of type  $F_k$ ,  $k \geq 2$ , the invariant  $\gamma_k(G)$  is  $\geq 0$ .*

### 3. The vanishing of $\chi(X)$

3.1. We return to a closed orientable manifold  $X$  of even dimension  $n = 2k$  as in Section 2, aspherical up to the middle dimension  $k$  (if  $k > 2$ ) and with infinite amenable fundamental group.

If  $\chi(X) = 0$  then by Theorem 2.2  $\bar{\beta}_k(\tilde{X} \text{ rel. } G) = 0$ , whence  $\bar{H}^k(\tilde{X}) = 0$ . We will show that this implies, in addition to Proposition 2.1,  $H_k(\tilde{X}) \cong H^k(G; \mathbb{Z}G)$ .

Since  $\tilde{X}$  is  $(k-1)$ -connected we can use (part of) diagram (4) with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^k(G; \mathbb{Z}G) & \longrightarrow & H_{\text{comp}}^k(\tilde{X}; \mathbb{Z}) & \xrightarrow{\Phi'} & H^k(\tilde{X}; \mathbb{Z}G)^G \\ & & \downarrow & & \downarrow & & \downarrow \Omega \\ 0 & \longrightarrow & H^k(G; l_2 G) & \longrightarrow & H_{(2)}^k(\tilde{X}; \mathbb{R}) & \xrightarrow{\Phi} & H^k(\tilde{X}; l_2 G)^G \end{array}$$

Since  $\Phi$  factors through  $\bar{H}^k(\tilde{X})$  (see Proposition 1.2) which is 0 if  $\chi(X) = 0$  the map

$$H_{\text{comp}}^k(\tilde{X}; \mathbb{Z}) \xrightarrow{\Phi'} H^k(\tilde{X}; \mathbb{Z}G)^G \xrightarrow{\Omega} H^k(\tilde{X}; l_2 G)$$

is  $=0$ . The coefficient map  $\Omega$  is injective since  $H^{k-1}(\tilde{X}; -) = 0$ . Thus  $\Phi' = 0$  and  $H^k(G; \mathbb{Z}G) \cong H_{\text{comp}}^k(\tilde{X}; \mathbb{Z}) \cong H_k(\tilde{X})$ .

**THEOREM 3.1.** *Let  $X$  be a compact orientable  $n$ -manifold,  $n = 2k$ , which for  $k > 2$  is  $(k-1)$ -aspherical, and with infinite amenable fundamental group  $G$ . If  $\chi(X) = 0$  then*

$$H_k(\tilde{X}) \cong H^k(G; \mathbb{Z}G).$$

We recall that  $H_i(\tilde{X}) = 0$  for  $0 < i < k$ , and that  $H_i(\tilde{X}) \cong H^{2k-i}(G; \mathbb{Z}G)$  for  $k < i < 2k$  (by Proposition 2.1); whence

**COROLLARY 3.2.** *Let  $X$  be as in Theorem 3.1. If  $\chi(X) = 0$  and  $H^i(G; \mathbb{Z}G) = 0$  for  $0 \leq i \leq k$  then  $\tilde{X}$  is contractible,  $X$  a  $K(G, 1)$ , and  $G$  is a  $PD^{2k}$ -group.*

In terms of the invariant  $\gamma_k(G)$  defined in 2.3:

**COROLLARY 3.3.** *If  $G$  is an infinite amenable group of type  $F_k$ ,  $k \geq 2$ , with  $H^i(G; \mathbb{Z}G) = 0$  for  $0 \leq i \leq k$ , then  $\gamma_k(G) = 0$  implies that  $G$  is a  $PD^{2k}$ -group.*

3.2. Again  $n = 2k = 4$  does not require any asphericity assumptions:

**THEOREM 3.4.** *Let  $X$  be a closed orientable 4-manifold with infinite amenable fundamental group  $G$ . If  $\chi(X) = 0$  then  $H_2(\tilde{X}) \cong H^2(G; \mathbb{Z}G)$ .*

**COROLLARY 3.5.** *If for  $X$  as in Theorem 3.3,  $H^1(G; \mathbb{Z}G) = H^2(G; \mathbb{Z}G) = 0$  and  $\chi(X) = 0$  then  $X$  is a  $PD^4$ -group.*

We recall that  $H^1(G; \mathbb{Z}G)$  must be 0 or  $\mathbb{Z}$ ; it is  $=\mathbb{Z}$  if and only if  $G$  is virtually infinite cyclic; whence

**COROLLARY 3.6.** *If  $G$  is a finitely presented infinite amenable group, not virtually infinite cyclic, with  $H^2(G; \mathbb{Z}G) = 0$ , then  $q(G) = 0$  implies that  $G$  is  $PD^4$ -group.*

#### 4. Amenable 2-knot groups

4.1. A 2-knot, or a knot in dimension 4, is a differentiable embedding  $f: S^2 \rightarrow S^4$  of the 2-sphere into the 4-sphere. The group  $G$  is called a 2-knot group if there is a 2-knot such that the fundamental group  $\pi_1(S^4 - f(S^2))$  of the complement is  $\cong G$ . For such a group one has  $H_1(G) = \mathbb{Z}$  and  $H_2(G) = 0$  (cf. Kervaire [K]).

Let  $C$  be the closed complement of  $f(S^2)$  in  $S^4$ , obtained by removing an open tubular neighborhood of  $f(S^2)$ . Clearly  $\pi_1 C = G$ , and  $\partial C$  is homeomorphic to  $f(S^2) \times S^1$ . Attaching a handle  $V^3 \times S^1$  to  $\partial C$  ("surgery along  $f(S^2)$ ") yields a closed 4-manifolds  $X$ , with  $\pi_1 X = G$ ,  $H_1 X = H_1 G = \mathbb{Z}$ , and  $H_2 X = 0$ . The invariant  $q(G)$  is  $\geq 2 - 2\beta_1(G) + \beta_2(G) = 0$ , and  $q(G) \leq \chi(X) = 0$ .

Thus one has quite generally  $q(G) = 0$  for all 2-knot groups.

4.2. If the 2-knot group  $G$  is amenable then Theorem 3.3 can be applied, whence

**THEOREM 4.1.** *Let  $G$  be an amenable 2-knot group, not virtually  $\mathbb{Z}$ , and  $X$  the closed 4-manifold obtained by surgery from a 2-knot with fundamental group  $G$ . Then  $H^2(G; \mathbb{Z}G) = H_2(\tilde{X})$ .*

**COROLLARY 4.2.** *If  $G$  is an amenable 2-knot group with  $H^1(G; \mathbb{Z}G) = H^2(G; \mathbb{Z}G) = 0$  then  $\tilde{X}$  is contractible, and  $G$  is a  $PD^4$ -group.*

4.3. *Remark.* Since  $H_1(G) = \mathbb{Z}$  for a 2-knot group (actually for any knot group) one can write  $G$  as an HNN extension over a finitely generated group; if  $G$  is amenable the HNN extension must be ascending, i.e.  $G = H_{*H,p}$  (cf. [E], p. 389). Here  $H$  also being amenable is either finite or has one or two ends.

If  $H$  is finite then  $G$  is virtually infinite cyclic, i.e.  $G$  has two ends. If  $H$  has one end, and if we assume that  $H$  is almost finitely presented, then  $H^1(G; \mathbb{Z}G) = H^2(G; \mathbb{Z}G) = 0$  by [B-G], thus  $G$  is a  $PD^4$ -group. If  $H$  has 2 ends it must be infinite cyclic  $= \langle a \rangle$ ; this yields  $G = \langle a, p \mid pap^{-1} = a^k \rangle$  where  $H_1(G) = \mathbb{Z}$  forces  $k$  to be  $= 2$ .

4.4. *Remark.* All 2-knot groups with 2 ends are determined by Hillman in [H2], Chapter 4. All *elementary amenable* 2-knot groups which are  $PD^4$ -groups are virtually solvable (cf. [H-L]) and thus torsion-free virtually polycyclic; all such 2-knot groups have been determined in [H2], Chapter 6.

## 5. Partial Euler characteristic of groups

5.1. In this appendix we use the method of  $l_2$ -cohomology to prove results concerning the “partial Euler characteristic” of an amenable group  $G$  which were already established earlier [E], partly by an entirely different method.

We assume that  $G$  is of type  $F_m$ ; i.e.,  $G$  admits a  $K(G, 1)$  which has a finite  $m$ -skeleton ( $G$  is of type  $FP_m$  and finitely presented if  $m \geq 2$ ). We denote by  $X$  the  $m$ -skeleton of  $K(G, 1)$  and consider its Euler characteristic  $\chi(X)$ . The minimum value of  $(-1)^m \chi(X)$  for all such  $K(G, 1)$  is written  $q_m(G)$ . The minimum exists since  $\beta_i(X) = \beta_i(G)$  for  $i < m$  and  $\beta_m(X) \geq \beta_m(G)$ .

5.2. Since  $H_i(\tilde{X}) = 0$  for  $0 < i < m$  the Cheeger-Gromov lemma yields, for amenable  $G$ ,  $\bar{H}^i(\tilde{X}) = 0$  for  $0 \leq i < m$ , whence  $\bar{\beta}_i(\tilde{X} \text{ rel. } G) = 0$  for these  $i$ . Thus

$$\chi(X) = (-1)^m \bar{\beta}_m(\tilde{X} \text{ rel. } G).$$

**THEOREM 5.1.** *For an infinite amenable group  $G$  of type  $F_m$  the group invariant  $q_m(G)$  is  $\geq 0$ .*

We recall that this yields explicit results of the following type: If  $G$  is a finitely presented infinite amenable group then the defect  $d(G)$  is  $\leq 1$ , cf. [E].

5.3. The vanishing of  $q_m(G)$  is of special interest. It means that there is a certain  $K(G, 1)$  – with finite  $m$ -skeleton  $X$  – such that  $\chi(X) = 0$ .

From 5.2 it follows that this implies  $\bar{\beta}_m(\tilde{X} \text{ rel. } G) = 0$ , whence  $\bar{H}^m(\tilde{X}) = 0$ . The map  $\Psi : H_{(2)}^m(\tilde{X}; \mathbb{R}) \rightarrow H^m(\tilde{X}; \mathbb{R})$ , see (5) in 1.5, factors through  $\bar{H}^m(\tilde{X})$  and is therefore  $= 0$ .

We now consider an arbitrary *finite* subcomplex  $S$  of  $\tilde{X}$ . The restriction of  $\tilde{X}$  to  $S$  yields the commutative diagram

$$\begin{array}{ccc} H_{(2)}^m(\tilde{X}; \mathbb{R}) & \xrightarrow{\Psi=0} & H^m(\tilde{X}; \mathbb{R}) \\ \downarrow & & \downarrow \\ H_{(2)}^m(S; \mathbb{R}) & \xrightarrow{=} & H^m(S; \mathbb{R}) \end{array}$$

The vertical maps are surjective due to exactness of the relative sequence of  $\tilde{X}$  modulo  $S$ , and to the fact that there are no  $(m+1)$ -cells.

Thus  $H^m(S; \mathbb{R}) = \text{Hom}(H_m(S), \mathbb{R}) = 0$ . As  $H_m(S)$  is  $\mathbb{Z}$ -free, it must be 0. Thus  $H_m(\tilde{X}) = 0$ , and  $\tilde{X}$  is contractible; i.e., we can take  $X = K(G, 1)$ .

**THEOREM 5.2.** *If for an infinite amenable group  $G$  of type  $F_m$  the group invariant  $q_m(G) = 0$  then  $G$  admits a finite  $K(G, 1)$ -complex of dimension  $\leq m$ ; in particular the cohomology dimension  $cdG$  is  $\leq m$ .*

5.4. We finally remark that results such as Theorems 2.2 and 5.1 hold in the more general setting of [E], Section 5: namely for a group  $G$  of the appropriate type which need not be amenable, but is an *extension*  $G/N = A$  of an infinite amenable group  $A$  by a normal subgroup  $N$  with  $\beta_i(N)$  *finite* for the respective  $i$ . These results can be established by the  $l_2$ -cohomology methods of the present paper. One takes, instead of  $\tilde{X}$ , the covering space  $Y$  corresponding to the subgroup  $N$  of  $G$ , which is a free cocompact  $A$ -space. Since  $H^i(Y; \mathbb{R}) = H^i(N; \mathbb{R})$  has finite  $\mathbb{R}$ -dimension and  $\bar{H}^i(Y) \rightarrow H^i(Y; \mathbb{R})$  is injective,  $\bar{H}^i(Y)$  must be 0 ( $\bar{H}^i(Y)$  is an invariant subspace of  $C_{(2)}^i(Y; \mathbb{R})$  and cannot be of finite  $\mathbb{R}$ -dimension unless it is 0). Thus  $\bar{\beta}_i(Y \text{ rel. } A) = 0$  and the arguments are as before. – These remarks, of course, do not apply to the “converse” statements concerning the vanishing of the Euler characteristic.

## 6. Addendum\*) on groups with vanishing first $l_2$ -Betti number

6.1. For any finite complex  $X$  with fundamental group  $G$ , i.e., for any finitely presented group,  $\bar{\beta}_1(\tilde{X} \text{ rel. } G)$  depends on  $G$  only; it can be written  $\bar{\beta}_1(G)$ . If  $X$  is a closed orientable 4-manifold with  $\pi_1(X) = G$ , and if  $\bar{\beta}_1(G) = 0$ , then

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\*)January 1994

$\chi(X) = \bar{\beta}_2(X \text{ rel. } G)$ . Thus all arguments of Sections 2 and 3 concerning 4-manifolds can be carried through. Moreover, via the  $l_2$ -signature theorem, one can obtain statements concerning the signature of  $X$ . We plan to return to these aspects in a separate paper.

6.2. Here we only note as an immediate consequence of Proposition 1.1 that finitely presented groups  $G$  with the Kazhdan ( $T$ ) property have  $\bar{\beta}_1(G) = 0$ . Indeed, ( $T$ ) implies  $H^1(G; l_2 G) = 0$ ; but  $H^1(G; l_2 G) = H^1(X; l_2 G) = H^1_{(2)}(\tilde{X})$ , and since  $H^1_{(2)}(\tilde{X})$  maps onto  $\bar{H}^1(\tilde{X})$  it follows that  $\bar{\beta}_1(\tilde{X} \text{ rel. } G) = \bar{\beta}_1(G) = 0$ .

## REFERENCES

- [Ch-G] J. CHEEGER and M. GROMOV,  *$L_2$ -cohomology and group cohomology*, Topology 25 (1986), 189–215.
- [B-G] K. BROWN and R. GEOGHEGAN, *Cohomology with free coefficients of the fundamental group of a graph of groups*, Comment. Math. Helvetici 60 (1985), 31–45.
- [E] B. ECKMANN, *Amenable groups and Euler characteristic*, Comment. Math. Helvetici 67 (1992), 383–393.
- [F] E. FØLNER, *On groups with full Banach mean value*, Math. Scand. 3 (1955), 336–354.
- [G] K. W. GRUENBERG, *Partial Euler characteristics of finite groups and the decomposition of lattices*, Proc. London Math. Soc. (3), 48 (1984), 91–107.
- [H-W] J.-CL. HAUSMANN and S. WEINBERGER, *Caractéristique d'Euler et groupes fondamentaux des variétés de dimension 4*, Comment. Math. Helvetici 60 (1985), 139–144.
- [H-L] J. A. HILLMAN and PETER LINNELL, *Elementary amenable groups of finite Hirsch length are locally-finite by virtually-solvable*, J. Austral. Math. Soc. 52 (1992), 237–241.
- [H] J. A. Hillman, *Elementary amenable groups and 4-manifolds with Euler characteristic 0*, J. Austral. Math. Soc. Ser. A 50 (1991), 160–170.
- [H2] J. A. Hillman, *2-knots and their groups*, Cambridge University Press (1989).
- [K] M. KERVAIRE, *Les noeuds de dimensions supérieures*, Bull. Soc. Math. de France 93 (1965), 225–271.
- [L-L] J. LOTT and W. LÜCK,  *$l^2$ -topological invariants of 3-manifolds*, Preprint 1992.
- [M] J. MILNOR, *A procedure for killing homotopy groups of differentiable manifolds*, Proc. Symposia in Pure Mathematics. AMS, vol. III (1961), 39–55.

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