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Manifolds of even dimension with amenable fundamental group

BENO ECKMANN

0. Introduction

0.1. If the fundamental group G of a closed (orientable) 4-manifold X is *infinite and amenable* then the Euler characteristic $\chi(X)$ is ≥ 0 . This has been proved in a previous paper [E] using the Følner criterion for amenability [F], in a geometrical version. If X is aspherical, i.e., an Eilenberg-MacLane space $K(G, 1)$ (whence G a Poincaré duality group of dimension 4, in short a PD^4 -group) then $\chi(X) = \chi(G) = 0$ by [E], Corollary 2.3.

The main purpose of the present paper is to examine, conversely, 4-manifolds X as above *assuming* $\chi(X) = 0$. We recall (see [E], Section 0.3) that infinite amenable groups G have one or two ends, i.e., $H^1(G; \mathbb{Z}G) = 0$ or \mathbb{Z} . It is easily seen that the universal cover \tilde{X} of X has integral homology $H_1(\tilde{X}) = H_4(\tilde{X}) = 0$ and $H_3(\tilde{X}) \cong H^1(G; \mathbb{Z}G)$. We will prove (Theorem 3.4):

(A) If $\chi(X) = 0$ then $H_2(\tilde{X}) \cong H^2(G; \mathbb{Z}G)$, the “second end-group” of G . From this we get the result

(B) If $H^1(G; \mathbb{Z}G) = H^2(G; \mathbb{Z}G) = 0$ then $\chi(X) = 0$ implies that \tilde{X} is contractible, whence $X = K(G, 1)$ and G is a PD^4 -group.

These statements can be expressed in terms of the Hausmann-Weinberger invariant $q(G)$, see [H-W], for finitely presented groups G (Corollaries 2.5 and 3.6):

(C) If G is infinite amenable then $q(G)$ is ≥ 0 . If $H^1(G; \mathbb{Z}G) = H^2(G; \mathbb{Z}G) = 0$ then $q(G) = 0$ implies that G is a PD^4 -group.

In the context of these results it is of interest to look at 2-knot groups G since for these $q(G)$ is always $= 0$; see Section 4 below.

0.2. The proofs make use of (reduced and non-reduced) l_2 -cohomology of the infinite cell-complex \tilde{X} combined with the free cocompact action of G on \tilde{X} . The main tool then is a lemma of Cheeger-Gromov [Ch-G], see Section 2.2. We apply it not only to get the results for $\chi(X) = 0$ but also to give a new proof of the statement $\chi(X) \geq 0$ above. This is done in the more general context of a closed manifold of even dimension $n = 2k \geq 4$ which, if $k > 2$, is *aspherical* up to the middle dimension k ; for $n = 4$ there is no asphericity assumption.

These $2k$ -manifolds can be used to define a new invariant $\gamma_k(G)$ for groups G of type F_k , $k \geq 2$, generalizing the Hausmann-Weinberger invariant $q(G)$. For G of type F_2 (i.e., finitely presented) one has $\gamma_2(G) = q(G)$.

0.3. Section 1 contains various facts concerning l_2 -cohomology of \tilde{X} , ordinary cohomology of \tilde{X} , and G -cohomology of \tilde{X} for G -module coefficients such as l_2G and $\mathbb{Z}G$. They go a little beyond the minimum necessary for the following sections in view of later use.

0.4. Section 2 deals with $\chi(X) \geq 0$ for the $2k$ -manifolds as above and with $\gamma_k(G)$, Section 3 with the vanishing of $\chi(X)$ and the main results. Section 5 is an appendix on the “partial Euler characteristic” of groups G fulfilling certain finiteness conditions; the results appear already in [E] but are given new proofs by the l_2 -cohomology methods of the present paper.

0.5. Our results on 4-manifolds should be compared with some of those given by Hillman [H] for the case of “elementary amenable” groups, which constitute a special, but important class of amenable groups. The results of [H] are, however, more general in another sense, namely that G need only have a non-trivial normal subgroup which is elementary amenable.

0.6. Although this paper deals with amenable groups we want to emphasize that the results above on 4-manifolds and the invariant $q(G)$ are valid for other types of groups, in particular for all finitely presented groups with vanishing first l_2 -Betti number; see Section 6 below (Addendum).

1. Infinite cell-complexes and l_2 -cohomology

1.1. For a cell-complex X with $\pi_1 X = G$ and a G -module A we consider cohomology with local coefficients $H^i(X; A)$; i.e., G -cohomology $H_G^i(\tilde{X}; A)$ of the universal cover, relative to the G -module A (G operates on the cell complex \tilde{X} and on A). A special situation occurs if X is a *finite* complex and G an *infinite* group, with regard to the coefficient modules $\mathbb{Z}G$ and l_2G (the Hilbert space of linear combinations $\sum_{x \in G} c_x x$, $c_x \in \mathbb{R}$, with $\sum_x c_x^2 < \infty$); G operates on $\mathbb{Z}G$ and on l_2G by left translations.

Namely, one has for the cochains $C^i(\tilde{X}; \mathbb{Z}G) = \text{Hom}_G(C_i(\tilde{X}), \mathbb{Z}G)$ and $C^i(\tilde{X}; l_2G) = \text{Hom}_G(C_i(\tilde{X}), l_2G)$ the isomorphisms

- (1) $C^i(\tilde{X}; \mathbb{Z}G) \cong C_{\text{fin}}^i(\tilde{X}; \mathbb{Z})$,
- (2) $C^i(\tilde{X}; l_2G) \cong C_{(2)}^i(\tilde{X}; \mathbb{R})$.

C_{fin}^i is the group of *finite cochains* of \tilde{X} , and $C_{(2)}^i$ the group of l_2 -cochains (functions $f(\sigma_i)$ of the cells σ_i of \tilde{X} with $\sum_{\sigma_i} f(\sigma_i)^2 < \infty$). The corresponding cohomology groups are respectively $H_{\text{comp}}^i(\tilde{X}; \mathbb{Z})$, cohomology with compact support; and $H_{(2)}^i(\tilde{X}; \mathbb{R})$, l_2 -cohomology of \tilde{X} .

1.2. For the convenience of the reader we recall the proof of (1) and (2).

We choose a (finite) $\mathbb{Z}G$ -basis $\{\tau_i\}$ of the chain group $C_i(\tilde{X})$ corresponding to the cells of X (one cell in each G -orbit). Given $f \in C^i(\tilde{X}; \mathbb{Z}G) = \text{Hom}_G(C_i(\tilde{X}), \mathbb{Z}G)$ we put $g(x\tau_i) = m_{x^{-1}} \in \mathbb{Z}$ where $f(\tau_i) = \sum_x m_x x$; clearly g is a finite cochain in \tilde{X} . Conversely, given $g \in C_{\text{fin}}^i(\tilde{X}; \mathbb{Z})$ we put $f(\tau_i) = \sum_x g(x^{-1}\tau_i)x \in \mathbb{Z}G$. The correspondence $f \mapsto g$ yields the isomorphism (1). Note that it is independent of the choice of basis $\{\tau_i\}$: Indeed if we replace τ_i by $y\tau_i$, $y \in G$, then $g(x\tau_i) = g(xy^{-1}y\tau_i) = m'_{yx^{-1}}$ where $f(y\tau_i) = \sum_x m_x yx = \sum m'_x x$, i.e., $m'_x = m_{y^{-1}x}$; thus $g(x\tau_i) = m'_{yx^{-1}} = m_{x^{-1}}$ as before.

Similarly, given $f \in C^i(\tilde{X}; l_2 G)$ we put $g(x\tau_i) = c_{x^{-1}}$ where $f(\tau_i) = \sum_x c_x x$ with $\sum_x c_x^2 < \infty$. Then

$$\sum_{\text{all } \sigma} g(\sigma)^2 = \sum_{\tau_i} \sum_x g(x\tau_i)^2 < \infty,$$

so g is an l_2 -cochain. This yields the isomorphism (2). We summarize:

PROPOSITION 1.1. *For a finite cell complex X (with infinite fundamental group G) the cohomology groups with local coefficients $H^i(X; \mathbb{Z}G)$ and $H^i(X; l_2 G)$ are isomorphic respectively to $H_{\text{comp}}^i(\tilde{X}; \mathbb{Z})$ and $H_{(2)}^i(\tilde{X}; \mathbb{R})$ of the universal cover \tilde{X} of X .*

Remark. Everything above holds if instead of \tilde{X} we take any free cocompact G -space (=cell complex) Y with $Y/G = X$; G is a factor group of $\pi_1 X$. The isomorphisms are of interest only if G is infinite.

1.3. We will also consider *reduced* l_2 -cohomology of \tilde{X} , denoted by $\bar{H}^i(\tilde{X})$. It differs from $H_{(2)}^i(X; \mathbb{R})$ by $\delta C_{(2)}^{i-1}(\tilde{X}; \mathbb{R})$ being replaced by its l_2 -closure $\overline{\delta C_{(2)}^{i-1}}$. It imbeds equivariantly and isometrically in Z^i , the kernel of $\delta : C_{(2)}^i \rightarrow C_{(2)}^{i+1}$, and its von Neumann dimension relative to G is denoted by $\beta_i(\tilde{X} \text{ rel. } G)$, cf. [Ch-G].

There is an obvious map Φ of $H_{(2)}^i(\tilde{X}; \mathbb{R})$, i.e. the G -cohomology group $H_G^i(\tilde{X}; l_2 G)$ based on G -homomorphisms $C_i(\tilde{X}) \rightarrow l_2 G$, into the *ordinary* cohomology group $H^i(\tilde{X}; l_2 G)$ disregarding the G -action on \tilde{X} and $l_2 G$. Under that map Φ the closure of $\delta C^{i-1}(\tilde{X}; l_2 G)$ goes to 0. Indeed, the l_2 -limit f of a sequence of

i -coboundaries is $=0$ on the i -cycles; it thus defines $\varphi : \partial C_i(\tilde{X}) \rightarrow l_2 G$ which can be extended to all of C_{i-1} (since $l_2 G$ is divisible, i.e. \mathbb{Z} -injective), and $\delta\varphi = f$.

PROPOSITION 1.2. *The natural map $H_G^i(\tilde{X}; l_2 G) \rightarrow H^i(\tilde{X}; l_2 G)$ factors through the reduced l_2 -cohomology group $\bar{H}^i(\tilde{X})$.*

Of course $H^i(\tilde{X}; l_2 G)$ can be regarded as a $\mathbb{Z}G$ -module through the action of G on \tilde{X} and on $l_2 G$. The image of Φ lies in the invariant part $H^i(\tilde{X}; l_2 G)^G$.

1.4. The map $\Phi : H_G^n(\tilde{X}; l_2 G) \rightarrow H^n(\tilde{X}; l_2 G)^G$ occurs in a well-known exact sequence, available if \tilde{X} is $(n-1)$ -connected, i.e., if $\pi_i(X) = 0$ for $1 < i < n$ (deduced from the spectral sequence of the covering $\tilde{X} \rightarrow X$):

$$0 \rightarrow H^n(G; l_2 G) \rightarrow H_G^n(\tilde{X}; l_2 G) \xrightarrow{\Phi} H^n(\tilde{X}; l_2 G)^G \rightarrow H^{n+1}(G; l_2 G) \rightarrow H_G^{n+1}(\tilde{X}; l_2 G).$$

There is, of course, an analogous exact sequence for $\mathbb{Z}G$ -coefficients. The coefficient map $\mathbb{Z}G \rightarrow l_2 G$ by inclusion yields, in combination with Proposition 1.1, the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^n(G; \mathbb{Z}G) & \longrightarrow & H_{\text{comp}}^n(\tilde{X}; \mathbb{Z}) & \xrightarrow{\Phi'} & H^n(\tilde{X}; \mathbb{Z}G)^G \longrightarrow H^{n+1}(G; \mathbb{Z}G) \\ & & \downarrow & & \downarrow & & \downarrow \Omega \\ 0 & \longrightarrow & H^n(G; l_2 G) & \longrightarrow & H_{(2)}^n(\tilde{X}; \mathbb{R}) & \xrightarrow{\Phi} & H^n(\tilde{X}; l_2 G)^G \longrightarrow H^{n+1}(G; l_2 G) \end{array} \quad (4)$$

1.5. There is a further natural map $\Psi : H_{(2)}^i(\tilde{X}; \mathbb{R}) \rightarrow H^i(\tilde{X}; \mathbb{R})$; it clearly factors through $\bar{H}^i(\tilde{X})$ since the limit of a sequence of l_2 -coboundaries is an ordinary coboundary.

1.6. There is an l_2 -homology analogue of the above statements for l_2 -cohomology; we leave it to the reader. We just remark that it is based on the boundary operator $\partial : C_{(2)}^i \rightarrow C_{(2)}^{i-1}$ instead of the coboundary $\delta : C_{(2)}^i \rightarrow C_{(2)}^{i+1}$; and that the reduced homology groups $\bar{H}_i(\tilde{X})$ are isometrically isomorphic to the $\bar{H}^i(\tilde{X})$ – indeed, they are both isomorphic to the intersection $Z^i(\tilde{X}) \cap Z_i(\tilde{X})$ in $C_{(2)}^i$, where Z^i denotes the cocycle subspace, Z_i the cycle subspace of $C_{(2)}^i$, and $Z^i(\tilde{X}) \cap Z_i(\tilde{X})$ is (a) the orthogonal complement of $\delta C_{(2)}^{i-1}$ in Z^i , (b) the orthogonal complement of $\partial C_{(2)}^{i+1}$ in Z_i (Hodge-de Rham decomposition of $C_{(2)}^i$). We further remark that this yields a simple proof of l_2 -Poincaré duality for a closed n -manifold X by using (2) and ordinary Poincaré duality of X ; one gets $\bar{H}^i(\tilde{X}) \cong \bar{H}_{n-i}(\tilde{X}) \cong \bar{H}^{n-i}(\tilde{X})$ as Hilbert G -modules.

2. Closed manifolds of dimension $n = 2k$ and an invariant for groups of type F_k

2.1. We take for X a closed orientable (differentiable) n -manifold, $n = 2k \geq 4$ which if $k > 2$ is $(k - 1)$ -aspherical; i.e., with $\pi_i(X) = 0$ for $1 < i < k$. We assume again $G = \pi_1(X)$ infinite.

We note that $H_i(\tilde{X}) = 0$ for $1 \leq i < k$, and that $H_{2k}(\tilde{X}) = 0$ since G is infinite (if in ordinary homology coefficients are not indicated they are meant to be \mathbb{Z}).

PROPOSITION 2.1. *For $k < i \leq 2k$ one has $H_i(\tilde{X}) \cong H^{2k-i}(G; \mathbb{Z}G)$.*

Proof. $H_i(\tilde{X}) \cong H_{\text{comp}}^{2k-i}(\tilde{X}) \cong H^{2k-i}(X; \mathbb{Z}G)$ by Poincaré duality. But since X is $(k - 1)$ -aspherical $H^i(X; \mathbb{Z}G) \cong H^i(G; \mathbb{Z}G)$ for $0 \leq i < k$. If $n = 2k = 4$, there are no asphericity assumptions, and one simply has $H_3(\tilde{X}) \cong H^1(X; \mathbb{Z}G) \cong H^1(G; \mathbb{Z}G)$.

If the “end-groups” $H^i(G; \mathbb{Z}G)$ are 0 for $0 \leq i < k$ then $H_k(\tilde{X})$ is the only homology group of \tilde{X} which is possibly non-zero. If moreover $H_k(\tilde{X}) = 0$ then \tilde{X} is contractible, X is a $K(G, 1)$, and G is a PD^{2k} -group.

2.2. We now consider the Euler characteristic $\chi(X) = \sum_{i=0}^n (-1)^i \alpha_i = \sum_{i=0}^n (-1)^i \beta_i(X)$; α_i is the number of i -cells of a cell-decomposition of X , and $\beta_i(X) = \dim_{\mathbb{Q}} H_i(X; \mathbb{Q})$ the i -th Betti number. We recall ([Ch-G] and [E]) that $\chi(X)$ can also be expressed by the reduced Betti numbers $\bar{\beta}_i(\tilde{X} \text{ rel. } G)$ as

$$\chi(X) = \sum_{i=0}^n (-1)^i \bar{\beta}_i(\tilde{X} \text{ rel. } G).$$

$\bar{\beta}_i(\tilde{X} \text{ rel. } G)$ is the von Neumann dimension of $\bar{H}^i(\tilde{X})$ considered as a Hilbert G -module.

A lemma of Cheeger-Gromov [Ch-G] tells that if G is amenable then the natural map $\bar{H}^i(\tilde{X}) \rightarrow H^i(\tilde{X}; \mathbb{R})$ is *injective*. From our assumptions it follows that $H^i(\tilde{X}; \mathbb{R}) = 0$ for $0 < i < k$ whence $\bar{H}^i(\tilde{X}) = 0$ and $\bar{\beta}_i(\tilde{X} \text{ rel. } G) = 0$ for $0 \leq i < k$ ($\bar{\beta}_0 = 0$ since G is infinite). By Poincaré duality for the $\bar{\beta}_i$ (cf. 1.6, or [L-L], Proposition 4.2) it follows that $\bar{\beta}_i(\tilde{X} \text{ rel. } G) = 0$ for $k < i \leq 2k$. The Euler characteristic can thus be expressed by $\bar{\beta}_k$ alone:

THEOREM 2.2. *Let X be a closed orientable n -manifold, $n = 2k$, which for $k > 2$ is $(k - 1)$ -aspherical, and with infinite amenable fundamental group G . Then*

$$\chi(X) = (-1)^k \bar{\beta}_k(\tilde{X} \text{ rel. } G).$$

COROLLARY 2.3. *For X as in Theorem 2.2 one has*

$$(-1)^k \chi(X) \geq 0.$$

This is due to the fact that $\bar{\beta}_k$ is a non-negative real number.

In the case $n = 4$ there are no asphericity assumptions and we get the result proved by a different method (“Følner sequence”) in [E]:

THEOREM 2.4. *Let X be a closed orientable 4-manifold with infinite amenable fundamental group G . Then $\chi(X)$ is ≥ 0 .*

Or in terms of the Hausmann-Weinberger invariant $q(G)$:

COROLLARY 2.5. *For a finitely presented infinite amenable group G the invariant $q(G)$ is ≥ 0 .*

2.3. For manifolds X as considered in 2.1 the fundamental group $G = \pi_1(X)$ is of type F_k (finitely presented and of type FP_k). Indeed, the (finite) k -skeleton of a cell-decomposition of X can be extended to a $K(G, 1)$ by attaching cells of dimensions $> k$.

Conversely there exists for any group G of type F_k , $k \geq 2$, a closed orientable $2k$ -manifold with $\pi_1(X) = G$ and $\pi_i(X) = 0$ for $1 < i < k$. To find X one starts with any closed orientable differentiable $2k$ -manifold M with $\pi_1(M) = G$. For $k > 2$, type FP_k of G guarantees that $\pi_2(M) = H_2(\tilde{M})$ is finitely generated as a $\mathbb{Z}G$ -module. Thus $\pi_2(M)$ can be annihilated by a finite number of surgeries in M (see [M]), and there results a closed manifold M' with $\pi_1(M') = G$, $\pi_2(M') = 0$. If $k > 3$ then $\pi_3(M')$ is finitely generated over $\mathbb{Z}G$, and the procedure can be repeated until one has a manifold X as required.

Now we define for a group G of type F_k , $k \geq 2$, the invariant $\gamma_k(G)$ to be the minimum of $(-1)^k \chi(X)$ for all $2k$ -manifolds as above with $\pi_1(X) = G$, $\pi_i(X) = 0$ for $1 < i < k$. The minimum exists since

$$\begin{aligned} (-1)^k \chi(X) &= \beta_k(X) + 2 \sum_0^{k-1} (-1)^{i+k} \beta_i(X) \\ &= \beta_k(X) + 2 \sum_0^{k-1} (-1)^{i+k} \beta_i(G) \end{aligned}$$

and $\beta_k(X) \geq \beta_k(G)$.

Clearly $\gamma_2(G) = q(G)$.

COROLLARY 2.6. *For an infinite amenable group G of type F_k , $k \geq 2$, the invariant $\gamma_k(G)$ is ≥ 0 .*

3. The vanishing of $\chi(X)$

3.1. We return to a closed orientable manifold X of even dimension $n = 2k$ as in Section 2, aspherical up to the middle dimension k (if $k > 2$) and with infinite amenable fundamental group.

If $\chi(X) = 0$ then by Theorem 2.2 $\bar{\beta}_k(\tilde{X} \text{ rel. } G) = 0$, whence $\bar{H}^k(\tilde{X}) = 0$. We will show that this implies, in addition to Proposition 2.1, $H_k(\tilde{X}) \cong H^k(G; \mathbb{Z}G)$.

Since \tilde{X} is $(k-1)$ -connected we can use (part of) diagram (4) with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^k(G; \mathbb{Z}G) & \longrightarrow & H_{\text{comp}}^k(\tilde{X}; \mathbb{Z}) & \xrightarrow{\Phi'} & H^k(\tilde{X}; \mathbb{Z}G)^G \\ & & \downarrow & & \downarrow & & \downarrow \Omega \\ 0 & \longrightarrow & H^k(G; l_2 G) & \longrightarrow & H_{(2)}^k(\tilde{X}; \mathbb{R}) & \xrightarrow{\Phi} & H^k(\tilde{X}; l_2 G)^G \end{array}$$

Since Φ factors through $\bar{H}^k(\tilde{X})$ (see Proposition 1.2) which is 0 if $\chi(X) = 0$ the map

$$H_{\text{comp}}^k(\tilde{X}; \mathbb{Z}) \xrightarrow{\Phi'} H^k(\tilde{X}; \mathbb{Z}G)^G \xrightarrow{\Omega} H^k(\tilde{X}; l_2 G)$$

is 0. The coefficient map Ω is injective since $H^{k-1}(\tilde{X}; -) = 0$. Thus $\Phi' = 0$ and $H^k(G; \mathbb{Z}G) \cong H_{\text{comp}}^k(\tilde{X}; \mathbb{Z}) \cong H_k(\tilde{X})$.

THEOREM 3.1. *Let X be a compact orientable n -manifold, $n = 2k$, which for $k > 2$ is $(k-1)$ -aspherical, and with infinite amenable fundamental group G . If $\chi(X) = 0$ then*

$$H_k(\tilde{X}) \cong H^k(G; \mathbb{Z}G).$$

We recall that $H_i(\tilde{X}) = 0$ for $0 < i < k$, and that $H_i(\tilde{X}) \cong H^{2k-i}(G; \mathbb{Z}G)$ for $k < i < 2k$ (by Proposition 2.1); whence

COROLLARY 3.2. *Let X be as in Theorem 3.1. If $\chi(X) = 0$ and $H^i(G; \mathbb{Z}G) = 0$ for $0 \leq i \leq k$ then \tilde{X} is contractible, X a $K(G, 1)$, and G is a PD^{2k} -group.*

In terms of the invariant $\gamma_k(G)$ defined in 2.3:

COROLLARY 3.3. *If G is an infinite amenable group of type F_k , $k \geq 2$, with $H^i(G; \mathbb{Z}G) = 0$ for $0 \leq i \leq k$, then $\gamma_k(G) = 0$ implies that G is a PD^{2k} -group.*

3.2. Again $n = 2k = 4$ does not require any asphericity assumptions:

THEOREM 3.4. *Let X be a closed orientable 4-manifold with infinite amenable fundamental group G . If $\chi(X) = 0$ then $H_2(\tilde{X}) \cong H^2(G; \mathbb{Z}G)$.*

COROLLARY 3.5. *If for X as in Theorem 3.3, $H^1(G; \mathbb{Z}G) = H^2(G; \mathbb{Z}G) = 0$ and $\chi(X) = 0$ then X is a PD^4 -group.*

We recall that $H^1(G; \mathbb{Z}G)$ must be 0 or \mathbb{Z} ; it is $=\mathbb{Z}$ if and only if G is virtually infinite cyclic; whence

COROLLARY 3.6. *If G is a finitely presented infinite amenable group, not virtually infinite cyclic, with $H^2(G; \mathbb{Z}G) = 0$, then $q(G) = 0$ implies that G is PD^4 -group.*

4. Amenable 2-knot groups

4.1. A 2-knot, or a knot in dimension 4, is a differentiable embedding $f: S^2 \rightarrow S^4$ of the 2-sphere into the 4-sphere. The group G is called a 2-knot group if there is a 2-knot such that the fundamental group $\pi_1(S^4 - f(S^2))$ of the complement is $\cong G$. For such a group one has $H_1(G) = \mathbb{Z}$ and $H_2(G) = 0$ (cf. Kervaire [K]).

Let C be the closed complement of $f(S^2)$ in S^4 , obtained by removing an open tubular neighborhood of $f(S^2)$. Clearly $\pi_1 C = G$, and ∂C is homeomorphic to $f(S^2) \times S^1$. Attaching a handle $V^3 \times S^1$ to ∂C ("surgery along $f(S^2)$ ") yields a closed 4-manifolds X , with $\pi_1 X = G$, $H_1 X = H_1 G = \mathbb{Z}$, and $H_2 X = 0$. The invariant $q(G)$ is $\geq 2 - 2\beta_1(G) + \beta_2(G) = 0$, and $q(G) \leq \chi(X) = 0$.

Thus one has quite generally $q(G) = 0$ for all 2-knot groups.

4.2. If the 2-knot group G is amenable then Theorem 3.3 can be applied, whence

THEOREM 4.1. *Let G be an amenable 2-knot group, not virtually \mathbb{Z} , and X the closed 4-manifold obtained by surgery from a 2-knot with fundamental group G . Then $H^2(G; \mathbb{Z}G) = H_2(\tilde{X})$.*

COROLLARY 4.2. *If G is an amenable 2-knot group with $H^1(G; \mathbb{Z}G) = H^2(G; \mathbb{Z}G) = 0$ then \tilde{X} is contractible, and G is a PD^4 -group.*

4.3. *Remark.* Since $H_1(G) = \mathbb{Z}$ for a 2-knot group (actually for any knot group) one can write G as an HNN extension over a finitely generated group; if G is amenable the HNN extension must be ascending, i.e. $G = H_{*H,p}$ (cf. [E], p. 389). Here H also being amenable is either finite or has one or two ends.

If H is finite then G is virtually infinite cyclic, i.e. G has two ends. If H has one end, and if we assume that H is almost finitely presented, then $H^1(G; \mathbb{Z}G) = H^2(G; \mathbb{Z}G) = 0$ by [B-G], thus G is a PD^4 -group. If H has 2 ends it must be infinite cyclic $= \langle a \rangle$; this yields $G = \langle a, p \mid pap^{-1} = a^k \rangle$ where $H_1(G) = \mathbb{Z}$ forces k to be $= 2$.

4.4. *Remark.* All 2-knot groups with 2 ends are determined by Hillman in [H2], Chapter 4. All *elementary amenable* 2-knot groups which are PD^4 -groups are virtually solvable (cf. [H-L]) and thus torsion-free virtually polycyclic; all such 2-knot groups have been determined in [H2], Chapter 6.

5. Partial Euler characteristic of groups

5.1. In this appendix we use the method of l_2 -cohomology to prove results concerning the “partial Euler characteristic” of an amenable group G which were already established earlier [E], partly by an entirely different method.

We assume that G is of type F_m ; i.e., G admits a $K(G, 1)$ which has a finite m -skeleton (G is of type FP_m and finitely presented if $m \geq 2$). We denote by X the m -skeleton of $K(G, 1)$ and consider its Euler characteristic $\chi(X)$. The minimum value of $(-1)^m \chi(X)$ for all such $K(G, 1)$ is written $q_m(G)$. The minimum exists since $\beta_i(X) = \beta_i(G)$ for $i < m$ and $\beta_m(X) \geq \beta_m(G)$.

5.2. Since $H_i(\tilde{X}) = 0$ for $0 < i < m$ the Cheeger-Gromov lemma yields, for amenable G , $\bar{H}^i(\tilde{X}) = 0$ for $0 \leq i < m$, whence $\bar{\beta}_i(\tilde{X} \text{ rel. } G) = 0$ for these i . Thus

$$\chi(X) = (-1)^m \bar{\beta}_m(\tilde{X} \text{ rel. } G).$$

THEOREM 5.1. *For an infinite amenable group G of type F_m the group invariant $q_m(G)$ is ≥ 0 .*

We recall that this yields explicit results of the following type: If G is a finitely presented infinite amenable group then the defect $d(G)$ is ≤ 1 , cf. [E].

5.3. The vanishing of $q_m(G)$ is of special interest. It means that there is a certain $K(G, 1)$ – with finite m -skeleton X – such that $\chi(X) = 0$.

From 5.2 it follows that this implies $\bar{\beta}_m(\tilde{X} \text{ rel. } G) = 0$, whence $\bar{H}^m(\tilde{X}) = 0$. The map $\Psi : H_{(2)}^m(\tilde{X}; \mathbb{R}) \rightarrow H^m(\tilde{X}; \mathbb{R})$, see (5) in 1.5, factors through $\bar{H}^m(\tilde{X})$ and is therefore $= 0$.

We now consider an arbitrary *finite* subcomplex S of \tilde{X} . The restriction of \tilde{X} to S yields the commutative diagram

$$\begin{array}{ccc} H_{(2)}^m(\tilde{X}; \mathbb{R}) & \xrightarrow{\Psi=0} & H^m(\tilde{X}; \mathbb{R}) \\ \downarrow & & \downarrow \\ H_{(2)}^m(S; \mathbb{R}) & \xrightarrow{=} & H^m(S; \mathbb{R}) \end{array}$$

The vertical maps are surjective due to exactness of the relative sequence of \tilde{X} modulo S , and to the fact that there are no $(m+1)$ -cells.

Thus $H^m(S; \mathbb{R}) = \text{Hom}(H_m(S), \mathbb{R}) = 0$. As $H_m(S)$ is \mathbb{Z} -free, it must be 0. Thus $H_m(\tilde{X}) = 0$, and \tilde{X} is contractible; i.e., we can take $X = K(G, 1)$.

THEOREM 5.2. *If for an infinite amenable group G of type F_m the group invariant $q_m(G) = 0$ then G admits a finite $K(G, 1)$ -complex of dimension $\leq m$; in particular the cohomology dimension cdG is $\leq m$.*

5.4. We finally remark that results such as Theorems 2.2 and 5.1 hold in the more general setting of [E], Section 5: namely for a group G of the appropriate type which need not be amenable, but is an *extension* $G/N = A$ of an infinite amenable group A by a normal subgroup N with $\beta_i(N)$ *finite* for the respective i . These results can be established by the l_2 -cohomology methods of the present paper. One takes, instead of \tilde{X} , the covering space Y corresponding to the subgroup N of G , which is a free cocompact A -space. Since $H^i(Y; \mathbb{R}) = H^i(N; \mathbb{R})$ has finite \mathbb{R} -dimension and $\bar{H}^i(Y) \rightarrow H^i(Y; \mathbb{R})$ is injective, $\bar{H}^i(Y)$ must be 0 ($\bar{H}^i(Y)$ is an invariant subspace of $C_{(2)}^i(Y; \mathbb{R})$ and cannot be of finite \mathbb{R} -dimension unless it is 0). Thus $\bar{\beta}_i(Y \text{ rel. } A) = 0$ and the arguments are as before. – These remarks, of course, do not apply to the “converse” statements concerning the vanishing of the Euler characteristic.

6. Addendum*) on groups with vanishing first l_2 -Betti number

6.1. For any finite complex X with fundamental group G , i.e., for any finitely presented group, $\bar{\beta}_1(\tilde{X} \text{ rel. } G)$ depends on G only; it can be written $\bar{\beta}_1(G)$. If X is a closed orientable 4-manifold with $\pi_1(X) = G$, and if $\bar{\beta}_1(G) = 0$, then

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$\chi(X) = \bar{\beta}_2(X \text{ rel. } G)$. Thus all arguments of Sections 2 and 3 concerning 4-manifolds can be carried through. Moreover, via the l_2 -signature theorem, one can obtain statements concerning the signature of X . We plan to return to these aspects in a separate paper.

6.2. Here we only note as an immediate consequence of Proposition 1.1 that finitely presented groups G with the Kazhdan (T) property have $\bar{\beta}_1(G) = 0$. Indeed, (T) implies $H^1(G; l_2 G) = 0$; but $H^1(G; l_2 G) = H^1(X; l_2 G) = H^1_{(2)}(\tilde{X})$, and since $H^1_{(2)}(\tilde{X})$ maps onto $\bar{H}^1(\tilde{X})$ it follows that $\bar{\beta}_1(\tilde{X} \text{ rel. } G) = \bar{\beta}_1(G) = 0$.

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