

Zeitschrift:	Commentarii Mathematici Helvetici
Herausgeber:	Schweizerische Mathematische Gesellschaft
Band:	69 (1994)
Artikel:	Manifolds of even dimension with amenable fundamental group.
Autor:	Eckmann, Beno
DOI:	https://doi.org/10.5169/seals-52273

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 05.02.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Manifolds of even dimension with amenable fundamental group

BENO ECKMANN

0. Introduction

0.1. If the fundamental group G of a closed (orientable) 4-manifold X is *infinite and amenable* then the Euler characteristic $\chi(X)$ is ≥ 0 . This has been proved in a previous paper [E] using the Følner criterion for amenability [F], in a geometrical version. If X is aspherical, i.e., an Eilenberg-MacLane space $K(G, 1)$ (whence G a Poincaré duality group of dimension 4, in short a PD^4 -group) then $\chi(X) = \chi(G) = 0$ by [E], Corollary 2.3.

The main purpose of the present paper is to examine, conversely, 4-manifolds X as above *assuming* $\chi(X) = 0$. We recall (see [E], Section 0.3) that infinite amenable groups G have one or two ends, i.e., $H^1(G; \mathbb{Z}G) = 0$ or \mathbb{Z} . It is easily seen that the universal cover \tilde{X} of X has integral homology $H_1(\tilde{X}) = H_4(\tilde{X}) = 0$ and $H_3(\tilde{X}) \cong H^1(G; \mathbb{Z}G)$. We will prove (Theorem 3.4):

(A) *If $\chi(X) = 0$ then $H_2(\tilde{X}) \cong H^2(G; \mathbb{Z}G)$, the “second end-group” of G .* From this we get the result

(B) *If $H^1(G; \mathbb{Z}G) = H^2(G; \mathbb{Z}G) = 0$ then $\chi(X) = 0$ implies that \tilde{X} is contractible, whence $X = K(G, 1)$ and G is a PD^4 -group.*

These statements can be expressed in terms of the Hausmann-Weinberger invariant $q(G)$, see [H-W], for finitely presented groups G (Corollaries 2.5 and 3.6):

(C) *If G is infinite amenable then $q(G) \geq 0$. If $H^1(G; \mathbb{Z}G) = H^2(G; \mathbb{Z}G) = 0$ then $q(G) = 0$ implies that G is a PD^4 -group.*

In the context of these results it is of interest to look at 2-knot groups G since for these $q(G)$ is always $= 0$; see Section 4 below.

0.2. The proofs make use of (reduced and non-reduced) l_2 -cohomology of the infinite cell-complex \tilde{X} combined with the free cocompact action of G on \tilde{X} . The main tool then is a lemma of Cheeger-Gromov [Ch-G], see Section 2.2. We apply it not only to get the results for $\chi(X) = 0$ but also to give a new proof of the statement $\chi(X) \geq 0$ above. This is done in the more general context of a closed manifold of even dimension $n = 2k \geq 4$ which, if $k > 2$, is *aspherical* up to the middle dimension k ; for $n = 4$ there is no asphericity assumption.

These $2k$ -manifolds can be used to define a new invariant $\gamma_k(G)$ for groups G of type F_k , $k \geq 2$, generalizing the Hausmann-Weinberger invariant $q(G)$. For G of type F_2 (i.e., finitely presented) one has $\gamma_2(G) = q(G)$.

0.3. Section 1 contains various facts concerning l_2 -cohomology of \tilde{X} , ordinary cohomology of \tilde{X} , and G -cohomology of \tilde{X} for G -module coefficients such as l_2G and $\mathbb{Z}G$. They go a little beyond the minimum necessary for the following sections in view of later use.

0.4. Section 2 deals with $\chi(X) \geq 0$ for the $2k$ -manifolds as above and with $\gamma_k(G)$, Section 3 with the vanishing of $\chi(X)$ and the main results. Section 5 is an appendix on the “partial Euler characteristic” of groups G fulfilling certain finiteness conditions; the results appear already in [E] but are given new proofs by the l_2 -cohomology methods of the present paper.

0.5. Our results on 4-manifolds should be compared with some of those given by Hillman [H] for the case of “elementary amenable” groups, which constitute a special, but important class of amenable groups. The results of [H] are, however, more general in another sense, namely that G need only have a non-trivial normal subgroup which is elementary amenable.

0.6. Although this paper deals with amenable groups we want to emphasize that the results above on 4-manifolds and the invariant $q(G)$ are valid for other types of groups, in particular for all finitely presented groups with vanishing first l_2 -Betti number; see Section 6 below (Addendum).

1. Infinite cell-complexes and l_2 -cohomology

1.1. For a cell-complex X with $\pi_1X = G$ and a G -module A we consider cohomology with local coefficients $H^i(X; A)$; i.e., G -cohomology $H_G^i(\tilde{X}; A)$ of the universal cover, relative to the G -module A (G operates on the cell complex \tilde{X} and on A). A special situation occurs if X is a *finite* complex and G an *infinite* group, with regard to the coefficient modules $\mathbb{Z}G$ and l_2G (the Hilbert space of linear combinations $\sum_{x \in G} c_x x$, $c_x \in \mathbb{R}$, with $\sum_x c_x^2 < \infty$); G operates on $\mathbb{Z}G$ and on l_2G by left translations.

Namely, one has for the cochains $C^i(\tilde{X}; \mathbb{Z}G) = \text{Hom}_G(C_i(\tilde{X}), \mathbb{Z}G)$ and $C^i(\tilde{X}; l_2G) = \text{Hom}_G(C_i(\tilde{X}), l_2G)$ the isomorphisms

- (1) $C^i(\tilde{X}, \mathbb{Z}G) \cong C_{\text{fin}}^i(\tilde{X}; \mathbb{Z})$,
- (2) $C^i(\tilde{X}; l_2G) \cong C_{(2)}^i(\tilde{X}; \mathbb{R})$.

C_{fin}^i is the group of *finite cochains* of \tilde{X} , and $C_{(2)}^i$ the group of l_2 -cochains (functions $f(\sigma_i)$ of the cells σ_i of \tilde{X} with $\sum_{\sigma_i} f(\sigma_i)^2 < \infty$). The corresponding cohomology groups are respectively $H_{\text{comp}}^i(\tilde{X}; \mathbb{Z})$, cohomology with compact support; and $H_{(2)}^i(\tilde{X}; \mathbb{R})$, l_2 -cohomology of \tilde{X} .

1.2. For the convenience of the reader we recall the proof of (1) and (2).

We choose a (finite) $\mathbb{Z}G$ -basis $\{\tau_i\}$ of the chain group $C_i(\tilde{X})$ corresponding to the cells of X (one cell in each G -orbit). Given $f \in C^i(\tilde{X}; \mathbb{Z}G) = \text{Hom}_G(C_i(\tilde{X}), \mathbb{Z}G)$ we put $g(x\tau_i) = m_{x^{-1}} \in \mathbb{Z}$ where $f(\tau_i) = \sum_x m_x x$; clearly g is a finite cochain in \tilde{X} . Conversely, given $g \in C_{\text{fin}}^i(\tilde{X}; \mathbb{Z})$ we put $f(\tau_i) = \sum_x g(x^{-1}\tau_i)x \in \mathbb{Z}G$. The correspondence $f \mapsto g$ yields the isomorphism (1). Note that it is independent of the choice of basis $\{\tau_i\}$: Indeed if we replace τ_i by $y\tau_i$, $y \in G$, then $g(x\tau_i) = g(xy^{-1}y\tau_i) = m'_{yx^{-1}}$ where $f(y\tau_i) = \sum_x m_x yx = \sum_x m'_x x$, i.e., $m'_x = m_{y^{-1}x}$; thus $g(x\tau_i) = m'_{yx^{-1}} = m_{x^{-1}}$ as before.

Similarly, given $f \in C^i(\tilde{X}; l_2 G)$ we put $g(x\tau_i) = c_{x^{-1}}$ where $f(\tau_i) = \sum_x c_x x$ with $\sum_x c_x^2 < \infty$. Then

$$\sum_{\text{all } \sigma} g(\sigma)^2 = \sum_{\tau_i} \sum_x g(x\tau_i)^2 < \infty,$$

so g is an l_2 -cochain. This yields the isomorphism (2). We summarize:

PROPOSITION 1.1. *For a finite cell complex X (with infinite fundamental group G) the cohomology groups with local coefficients $H^i(X; \mathbb{Z}G)$ and $H^i(X; l_2 G)$ are isomorphic respectively to $H_{\text{comp}}^i(\tilde{X}; \mathbb{Z})$ and $H_{(2)}^i(\tilde{X}; \mathbb{R})$ of the universal cover \tilde{X} of X .*

Remark. Everything above holds if instead of \tilde{X} we take any free cocompact G -space (=cell complex) Y with $Y/G = X$; G is a factor group of $\pi_1 X$. The isomorphisms are of interest only if G is infinite.

1.3. We will also consider *reduced l_2 -cohomology* of \tilde{X} , denoted by $\tilde{H}^i(\tilde{X})$. It differs from $H_{(2)}^i(X; \mathbb{R})$ by $\delta C_{(2)}^{i-1}(\tilde{X}; \mathbb{R})$ being replaced by its l_2 -closure $\overline{\delta C_{(2)}^{i-1}}$. It imbeds equivariantly and isometrically in Z^i , the kernel of $\delta : C_{(2)}^i \rightarrow C_{(2)}^{i+1}$, and its von Neumann dimension relative to G is denoted by $\beta_i(\tilde{X} \text{ rel. } G)$, cf. [Ch-G].

There is an obvious map Φ of $H_{(2)}^i(\tilde{X}; \mathbb{R})$, i.e. the G -cohomology group $H_G^i(\tilde{X}; l_2 G)$ based on G -homomorphisms $C_i(\tilde{X}) \rightarrow l_2 G$, into the *ordinary* cohomology group $H^i(\tilde{X}; l_2 G)$ disregarding the G -action on \tilde{X} and $l_2 G$. Under that map Φ the closure of $\delta C^{i-1}(\tilde{X}; l_2 G)$ goes to 0. Indeed, the l_2 -limit f of a sequence of

i -coboundaries is $=0$ on the i -cycles; it thus defines $\varphi : \partial C_i(\tilde{X}) \rightarrow l_2 G$ which can be extended to all of C_{i-1} (since $l_2 G$ is divisible, i.e. \mathbb{Z} -injective), and $\delta\varphi = f$.

PROPOSITION 1.2. *The natural map $H_G^i(\tilde{X}; l_2 G) \rightarrow H^i(\tilde{X}; l_2 G)$ factors through the reduced l_2 -cohomology group $\bar{H}^i(\tilde{X})$.*

Of course $H^i(\tilde{X}; l_2 G)$ can be regarded as a $\mathbb{Z}G$ -module through the action of G on \tilde{X} and on $l_2 G$. The image of \varPhi lies in the invariant part $H^i(\tilde{X}; l_2 G)^G$.

1.4. The map $\Phi : H_G^n(\tilde{X}; l_2 G) \rightarrow H^n(\tilde{X}; l_2 G)^G$ occurs in a well-known exact sequence, available if \tilde{X} is $(n-1)$ -connected, i.e., if $\pi_i(X) = 0$ for $1 < i < n$ (deduced from the spectral sequence of the covering $\tilde{X} \rightarrow X$):

$$0 \rightarrow H^n(G; l_2 G) \rightarrow H_G^n(\tilde{X}; l_2 G) \xrightarrow{\Phi} H^n(\tilde{X}; l_2 G)^G \rightarrow H^{n+1}(G; l_2 G) \rightarrow H_G^{n+1}(\tilde{X}; l_2 G).$$

There is, of course, an analogous exact sequence for $\mathbb{Z}G$ -coefficients. The coefficient map $\mathbb{Z}G \rightarrow l_2 G$ by inclusion yields, in combination with Proposition 1.1, the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^n(G; \mathbb{Z}G) & \longrightarrow & H_{\text{comp}}^n(\tilde{X}; \mathbb{Z}) & \xrightarrow{\Phi'} & H^n(\tilde{X}; \mathbb{Z}G)^G \longrightarrow H^{n+1}(G; \mathbb{Z}G) \\ & & \downarrow & & \downarrow & & \downarrow \alpha \\ 0 & \longrightarrow & H^n(G; l_2 G) & \longrightarrow & H_{(2)}^n(\tilde{X}; \mathbb{R}) & \xrightarrow{\Phi} & H^n(\tilde{X}; l_2 G)^G \longrightarrow H^{n+1}(G; l_2 G) \end{array} \quad (4)$$

1.5. There is a further natural map $\Psi : H_{(2)}^i(\tilde{X}; \mathbb{R}) \rightarrow H^i(\tilde{X}; \mathbb{R})$; it clearly factors through $\bar{H}^i(\tilde{X})$ since the limit of a sequence of l_2 -coboundaries is an ordinary coboundary.

1.6. There is an l_2 -homology analogue of the above statements for l_2 -cohomology; we leave it to the reader. We just remark that it is based on the boundary operator $\partial : C_{(2)}^i \rightarrow C_{(2)}^{i-1}$ instead of the coboundary $\delta : C_{(2)}^i \rightarrow C_{(2)}^{i+1}$; and that the reduced homology groups $\bar{H}_i(\tilde{X})$ are isometrically isomorphic to the $\bar{H}^i(\tilde{X})$ – indeed, they are both isomorphic to the intersection $Z^i(\tilde{X}) \cap Z_i(\tilde{X})$ in $C_{(2)}^i$, where Z^i denotes the cocycle subspace, Z_i the cycle subspace of $C_{(2)}^i$, and $Z^i(\tilde{X}) \cap Z_i(\tilde{X})$ is (a) the orthogonal complement of $\delta C_{(2)}^{i-1}$ in Z^i , (b) the orthogonal complement of $\partial C_{(2)}^{i+1}$ in Z_i (Hodge-de Rham decomposition of $C_{(2)}^i$). We further remark that this yields a simple proof of l_2 -Poincaré duality for a closed n -manifold X by using (2) and ordinary Poincaré duality of X ; one gets $\bar{H}^i(\tilde{X}) \cong \bar{H}_{n-i}(\tilde{X}) \cong \bar{H}^{n-i}(\tilde{X})$ as Hilbert G -modules.

2. Closed manifolds of dimension $n = 2k$ and an invariant for groups of type F_k

2.1. We take for X a closed orientable (differentiable) n -manifold, $n = 2k \geq 4$ which if $k > 2$ is $(k-1)$ -aspherical; i.e., with $\pi_i(X) = 0$ for $1 < i < k$. We assume again $G = \pi_1(X)$ infinite.

We note that $H_i(\tilde{X}) = 0$ for $1 \leq i < k$, and that $H_{2k}(\tilde{X}) = 0$ since G is infinite (if in ordinary homology coefficients are not indicated they are meant to be \mathbb{Z}).

PROPOSITION 2.1. *For $k < i \leq 2k$ one has $H_i(\tilde{X}) \cong H^{2k-i}(G; \mathbb{Z}G)$.*

Proof. $H_i(\tilde{X}) \cong H_{\text{comp}}^{2k-i}(\tilde{X}) \cong H^{2k-i}(X; \mathbb{Z}G)$ by Poincaré duality. But since X is $(k-1)$ -aspherical $H^i(X; \mathbb{Z}G) \cong H^i(G; \mathbb{Z}G)$ for $0 \leq i < k$. If $n = 2k = 4$, there are no asphericity assumptions, and one simply has $H_3(\tilde{X}) \cong H^1(X; \mathbb{Z}G) \cong H^1(G; \mathbb{Z}G)$.

If the “end-groups” $H^i(G; \mathbb{Z}G)$ are 0 for $0 \leq i < k$ then $H_k(\tilde{X})$ is the only homology group of \tilde{X} which is possibly non-zero. If moreover $H_k(\tilde{X}) = 0$ then \tilde{X} is contractible, X is a $K(G, 1)$, and G is a PD^{2k} -group.

2.2. We now consider the Euler characteristic $\chi(X) = \sum_{i=0}^n (-1)^i \alpha_i = \sum_{i=0}^n (-1)^i \beta_i(X)$; α_i is the number of i -cells of a cell-decomposition of X , and $\beta_i(X) = \dim_{\mathbb{Q}} H_i(X; \mathbb{Q})$ the i -th Betti number. We recall ([Ch-G] and [E]) that $\chi(X)$ can also be expressed by the reduced Betti numbers $\bar{\beta}_i(\tilde{X} \text{ rel. } G)$ as

$$\chi(X) = \sum_{i=0}^n (-1)^i \bar{\beta}_i(\tilde{X} \text{ rel. } G).$$

$\bar{\beta}_i(\tilde{X} \text{ rel. } G)$ is the von Neumann dimension of $\bar{H}^i(\tilde{X})$ considered as a Hilbert G -module.

A lemma of Cheeger-Gromov [Ch-G] tells that if G is amenable then the natural map $\bar{H}^i(\tilde{X}) \rightarrow H^i(\tilde{X}; \mathbb{R})$ is *injective*. From our assumptions it follows that $H^i(\tilde{X}; \mathbb{R}) = 0$ for $0 < i < k$ whence $\bar{H}^i(\tilde{X}) = 0$ and $\bar{\beta}_i(\tilde{X} \text{ rel. } G) = 0$ for $0 \leq i < k$ ($\bar{\beta}_0 = 0$ since G is infinite). By Poincaré duality for the $\bar{\beta}_i$ (cf. 1.6, or [L-L], Proposition 4.2) it follows that $\bar{\beta}_i(\tilde{X} \text{ rel. } G) = 0$ for $k < i \leq 2k$. The Euler characteristic can thus be expressed by $\bar{\beta}_k$ alone:

THEOREM 2.2. *Let X be a closed orientable n -manifold, $n = 2k$, which for $k > 2$ is $(k-1)$ -aspherical, and with infinite amenable fundamental group G . Then*

$$\chi(X) = (-1)^k \bar{\beta}_k(\tilde{X} \text{ rel. } G).$$

COROLLARY 2.3. *For X as in Theorem 2.2 one has*

$$(-1)^k \chi(X) \geq 0.$$

This is due to the fact that $\bar{\beta}_k$ is a non-negative real number.

In the case $n = 4$ there are no asphericity assumptions and we get the result proved by a different method ("Følner sequence") in [E]:

THEOREM 2.4. *Let X be a closed orientable 4-manifold with infinite amenable fundamental group G . Then $\chi(X)$ is ≥ 0 .*

Or in terms of the Hausmann-Weinberger invariant $q(G)$:

COROLLARY 2.5. *For a finitely presented infinite amenable group G the invariant $q(G)$ is ≥ 0 .*

2.3. For manifolds X as considered in 2.1 the fundamental group $G = \pi_1(X)$ is of type F_k (finitely presented and of type FP_k). Indeed, the (finite) k -skeleton of a cell-decomposition of X can be extended to a $K(G, 1)$ by attaching cells of dimensions $>k$.

Conversely there exists for any group G of type F_k , $k \geq 2$, a closed orientable $2k$ -manifold with $\pi_1(X) = G$ and $\pi_i(X) = 0$ for $1 < i < k$. To find X one starts with any closed orientable differentiable $2k$ -manifold M with $\pi_1(M) = G$. For $k > 2$, type FP_k of G guarantees that $\pi_2(M) = H_2(\tilde{M})$ is finitely generated as a $\mathbb{Z}G$ -module. Thus $\pi_2(M)$ can be annihilated by a finite number of surgeries in M (see [M]), and there results a closed manifold M' with $\pi_1(M') = G$, $\pi_2(M') = 0$. If $k > 3$ then $\pi_3(M')$ is finitely generated over $\mathbb{Z}G$, and the procedure can be repeated until one has a manifold X as required.

Now we define for a group G of type F_k , $k \geq 2$, the invariant $\gamma_k(G)$ to be the minimum of $(-1)^k \chi(X)$ for all $2k$ -manifolds as above with $\pi_1(X) = G$, $\pi_i(X) = 0$ for $1 < i < k$. The minimum exists since

$$\begin{aligned} (-1)^k \chi(X) &= \beta_k(X) + 2 \sum_0^{k-1} (-1)^{i+k} \beta_i(X) \\ &= \beta_k(X) + 2 \sum_0^{k-1} (-1)^{i+k} \beta_i(G) \end{aligned}$$

and $\beta_k(X) \geq \beta_k(G)$.

Clearly $\gamma_2(G) = q(G)$.

COROLLARY 2.6. *For an infinite amenable group G of type F_k , $k \geq 2$, the invariant $\gamma_k(G)$ is ≥ 0 .*

3. The vanishing of $\chi(X)$

3.1. We return to a closed orientable manifold X of even dimension $n = 2k$ as in Section 2, aspherical up to the middle dimension k (if $k > 2$) and with infinite amenable fundamental group.

If $\chi(X) = 0$ then by Theorem 2.2 $\bar{\beta}_k(\tilde{X} \text{ rel. } G) = 0$, whence $\bar{H}^k(\tilde{X}) = 0$. We will show that this implies, in addition to Proposition 2.1, $H_k(\tilde{X}) \cong H^k(G; \mathbb{Z}G)$.

Since \tilde{X} is $(k-1)$ -connected we can use (part of) diagram (4) with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^k(G; \mathbb{Z}G) & \longrightarrow & H_{\text{comp}}^k(\tilde{X}; \mathbb{Z}) & \xrightarrow{\Phi'} & H^k(\tilde{X}; \mathbb{Z}G)^G \\ & & \downarrow & & \downarrow & & \downarrow \Omega \\ 0 & \longrightarrow & H^k(G; l_2 G) & \longrightarrow & H_{(2)}^k(\tilde{X}; \mathbb{R}) & \xrightarrow{\Phi} & H^k(\tilde{X}; l_2 G)^G \end{array}$$

Since Φ factors through $\bar{H}^k(\tilde{X})$ (see Proposition 1.2) which is 0 if $\chi(X) = 0$ the map

$$H_{\text{comp}}^k(\tilde{X}; \mathbb{Z}) \xrightarrow{\Phi'} H^k(\tilde{X}; \mathbb{Z}G)^G \xrightarrow{\Omega} H^k(\tilde{X}; l_2 G)$$

is 0. The coefficient map Ω is injective since $H^{k-1}(\tilde{X}; -) = 0$. Thus $\Phi' = 0$ and $H^k(G; \mathbb{Z}G) \cong H_{\text{comp}}^k(\tilde{X}; \mathbb{Z}) \cong H_k(\tilde{X})$.

THEOREM 3.1. *Let X be a compact orientable n -manifold, $n = 2k$, which for $k > 2$ is $(k-1)$ -aspherical, and with infinite amenable fundamental group G . If $\chi(X) = 0$ then*

$$H_k(\tilde{X}) \cong H^k(G; \mathbb{Z}G).$$

We recall that $H_i(\tilde{X}) = 0$ for $0 < i < k$, and that $H_i(\tilde{X}) \cong H^{2k-i}(G; \mathbb{Z}G)$ for $k < i < 2k$ (by Proposition 2.1); whence

COROLLARY 3.2. *Let X be as in Theorem 3.1. If $\chi(X) = 0$ and $H^i(G; \mathbb{Z}G) = 0$ for $0 \leq i \leq k$ then \tilde{X} is contractible, X a $K(G, 1)$, and G is a PD^{2k} -group.*

In terms of the invariant $\gamma_k(G)$ defined in 2.3:

COROLLARY 3.3. *If G is an infinite amenable group of type F_k , $k \geq 2$, with $H^i(G; \mathbb{Z}G) = 0$ for $0 \leq i \leq k$, then $\gamma_k(G) = 0$ implies that G is a PD^{2k} -group.*

3.2. Again $n = 2k = 4$ does not require any asphericity assumptions:

THEOREM 3.4. *Let X be a closed orientable 4-manifold with infinite amenable fundamental group G . If $\chi(X) = 0$ then $H_2(\tilde{X}) \cong H^2(G; \mathbb{Z}G)$.*

COROLLARY 3.5. *If for X as in Theorem 3.3, $H^1(G; \mathbb{Z}G) = H^2(G; \mathbb{Z}G) = 0$ and $\chi(X) = 0$ then X is a PD^4 -group.*

We recall that $H^1(G; \mathbb{Z}G)$ must be 0 or \mathbb{Z} ; it is $= \mathbb{Z}$ if and only if G is virtually infinite cyclic; whence

COROLLARY 3.6. *If G is a finitely presented infinite amenable group, not virtually infinite cyclic, with $H^2(G; \mathbb{Z}G) = 0$, then $q(G) = 0$ implies that G is PD^4 -group.*

4. Amenable 2-knot groups

4.1. A 2-knot, or a knot in dimension 4, is a differentiable embedding $f: S^2 \rightarrow S^4$ of the 2-sphere into the 4-sphere. The group G is called a 2-knot group if there is a 2-knot such that the fundamental group $\pi_1(S^4 - f(S^2))$ of the complement is $\cong G$. For such a group one has $H_1(G) = \mathbb{Z}$ and $H_2(G) = 0$ (cf. Kervaire [K]).

Let C be the closed complement of $f(S^2)$ in S^4 , obtained by removing an open tubular neighborhood of $f(S^2)$. Clearly $\pi_1 C = G$, and ∂C is homeomorphic to $f(S^2) \times S^1$. Attaching a handle $V^3 \times S^1$ to ∂C (“surgery along $f(S^2)$ ”) yields a closed 4-manifolds X , with $\pi_1 X = G$, $H_1 X = H_1 G = \mathbb{Z}$, and $H_2 X = 0$. The invariant $q(G)$ is $\geq 2 - 2\beta_1(G) + \beta_2(G) = 0$, and $q(G) \leq \chi(X) = 0$.

Thus one has quite generally $q(G) = 0$ for all 2-knot groups.

4.2. If the 2-knot group G is amenable then Theorem 3.3 can be applied, whence

THEOREM 4.1. *Let G be an amenable 2-knot group, not virtually \mathbb{Z} , and X the closed 4-manifold obtained by surgery from a 2-knot with fundamental group G . Then $H^2(G; \mathbb{Z}G) = H_2(\tilde{X})$.*

COROLLARY 4.2. *If G is an amenable 2-knot group with $H^1(G; \mathbb{Z}G) = H^2(G; \mathbb{Z}G) = 0$ then \tilde{X} is contractible, and G is a PD^4 -group.*

4.3. *Remark.* Since $H_1(G) = \mathbb{Z}$ for a 2-knot group (actually for any knot group) one can write G as an HNN extension over a finitely generated group; if G is amenable the HNN extension must be ascending, i.e. $G = H *_{H,p}$ (cf. [E], p. 389). Here H also being amenable is either finite or has one or two ends.

If H is finite then G is virtually infinite cyclic, i.e. G has two ends. If H has one end, and if we assume that H is almost finitely presented, then $H^1(G; \mathbb{Z}G) = H^2(G; \mathbb{Z}G) = 0$ by [B-G], thus G is a PD^4 -group. If H has 2 ends it must be infinite cyclic $= \langle a \rangle$; this yields $G = \langle a, p \mid pap^{-1} = a^k \rangle$ where $H_1(G) = \mathbb{Z}$ forces k to be $= 2$.

4.4. *Remark.* All 2-knot groups with 2 ends are determined by Hillman in [H2], Chapter 4. All *elementary amenable* 2-knot groups which are PD^4 -groups are virtually solvable (cf. [H-L]) and thus torsion-free virtually polycyclic; all such 2-knot groups have been determined in [H2], Chapter 6.

5. Partial Euler characteristic of groups

5.1. In this appendix we use the method of l_2 -cohomology to prove results concerning the “partial Euler characteristic” of an amenable group G which were already established earlier [E], partly by an entirely different method.

We assume that G is of type F_m ; i.e., G admits a $K(G, 1)$ which has a finite m -skeleton (G is of type FP_m and finitely presented if $m \geq 2$). We denote by X the m -skeleton of $K(G, 1)$ and consider its Euler characteristic $\chi(X)$. The minimum value of $(-1)^m \chi(X)$ for all such $K(G, 1)$ is written $q_m(G)$. The minimum exists since $\beta_i(X) = \beta_i(G)$ for $i < m$ and $\beta_m(X) \geq \beta_m(G)$.

5.2. Since $H_i(\tilde{X}) = 0$ for $0 < i < m$ the Cheeger-Gromov lemma yields, for amenable G , $\tilde{H}^i(\tilde{X}) = 0$ for $0 \leq i < m$, whence $\tilde{\beta}_i(\tilde{X} \text{ rel. } G) = 0$ for these i . Thus

$$\chi(X) = (-1)^m \tilde{\beta}_m(\tilde{X} \text{ rel. } G).$$

THEOREM 5.1. *For an infinite amenable group G of type F_m the group invariant $q_m(G)$ is ≥ 0 .*

We recall that this yields explicit results of the following type: If G is a finitely presented infinite amenable group then the defect $d(G)$ is ≤ 1 , cf. [E].

5.3. The vanishing of $q_m(G)$ is of special interest. It means that there is a certain $K(G, 1)$ – with finite m -skeleton X – such that $\chi(X) = 0$.

From 5.2 it follows that this implies $\bar{\beta}_m(\tilde{X} \text{ rel. } G) = 0$, whence $\bar{H}^m(\tilde{X}) = 0$. The map $\Psi : H_{(2)}^m(\tilde{X}; \mathbb{R}) \rightarrow H^m(\tilde{X}; \mathbb{R})$, see (5) in 1.5, factors through $\bar{H}^m(\tilde{X})$ and is therefore $= 0$.

We now consider an arbitrary *finite* subcomplex S of \tilde{X} . The restriction of \tilde{X} to S yields the commutative diagram

$$\begin{array}{ccc} H_{(2)}^m(\tilde{X}; \mathbb{R}) & \xrightarrow{\Psi = 0} & H^m(\tilde{X}; \mathbb{R}) \\ \downarrow & & \downarrow \\ H_{(2)}^m(S; \mathbb{R}) & \xrightarrow{=} & H^m(S; \mathbb{R}) \end{array}$$

The vertical maps are surjective due to exactness of the relative sequence of \tilde{X} modulo S , and to the fact that there are no $(m+1)$ -cells.

Thus $H^m(S; \mathbb{R}) = \text{Hom}(H_m(S), \mathbb{R}) = 0$. As $H_m(S)$ is \mathbb{Z} -free, it must be 0. Thus $H_m(\tilde{X}) = 0$, and \tilde{X} is contractible; i.e., we can take $X = K(G, 1)$.

THEOREM 5.2. *If for an infinite amenable group G of type F_m the group invariant $q_m(G) = 0$ then G admits a finite $K(G, 1)$ -complex of dimension $\leq m$; in particular the cohomology dimension cdG is $\leq m$.*

5.4. We finally remark that results such as Theorems 2.2 and 5.1 hold in the more general setting of [E], Section 5: namely for a group G of the appropriate type which need not be amenable, but is an *extension* $G/N = A$ of an infinite amenable group A by a normal subgroup N with $\beta_i(N)$ finite for the respective i . These results can be established by the l_2 -cohomology methods of the present paper. One takes, instead of \tilde{X} , the covering space Y corresponding to the subgroup N of G , which is a free cocompact A -space. Since $H^i(Y; \mathbb{R}) = H^i(N; \mathbb{R})$ has finite \mathbb{R} -dimension and $\bar{H}^i(Y) \rightarrow H^i(Y; \mathbb{R})$ is injective, $\bar{H}^i(Y)$ must be 0 ($\bar{H}^i(Y)$ is an invariant subspace of $C_{(2)}^i(Y; \mathbb{R})$ and cannot be of finite \mathbb{R} -dimension unless it is 0). Thus $\bar{\beta}_i(Y \text{ rel. } A) = 0$ and the arguments are as before. – These remarks, of course, do not apply to the “converse” statements concerning the vanishing of the Euler characteristic.

6. Addendum*) on groups with vanishing first l_2 -Betti number

6.1. For any finite complex X with fundamental group G , i.e., for any finitely presented group, $\bar{\beta}_1(\tilde{X} \text{ rel. } G)$ depends on G only; it can be written $\bar{\beta}_1(G)$. If X is a closed orientable 4-manifold with $\pi_1(X) = G$, and if $\bar{\beta}_1(G) = 0$, then

*)January 1994

$\chi(X) = \bar{\beta}_2(X \text{ rel. } G)$. Thus all arguments of Sections 2 and 3 concerning 4-manifolds can be carried through. Moreover, via the l_2 -signature theorem, one can obtain statements concerning the signature of X . We plan to return to these aspects in a separate paper.

6.2. Here we only note as an immediate consequence of Proposition 1.1 that finitely presented groups G with the Kazhdan (T) property have $\bar{\beta}_1(G) = 0$. Indeed, (T) implies $H^1(G; l_2 G) = 0$; but $H^1(G; l_2 G) = H^1(X; l_2 G) = H_{(2)}^1(\tilde{X})$, and since $H_{(2)}^1(\tilde{X})$ maps onto $\bar{H}^1(\tilde{X})$ it follows that $\bar{\beta}_1(\tilde{X} \text{ rel. } G) = \bar{\beta}_1(G) = 0$.

REFERENCES

- [Ch-G] J. CHEEGER and M. GROMOV, *L_2 -cohomology and group cohomology*, Topology 25 (1986), 189–215.
- [B-G] K. BROWN and R. GEOGHEGAN, *Cohomology with free coefficients of the fundamental group of a graph of groups*, Comment. Math. Helvetici 60 (1985), 31–45.
- [E] B. ECKMANN, *Amenable groups and Euler characteristic*, Comment. Math. Helvetici 67 (1992), 383–393.
- [F] E. FØLNER, *On groups with full Banach mean value*, Math. Scand. 3 (1955), 336–354.
- [G] K. W. GRUENBERG, *Partial Euler characteristics of finite groups and the decomposition of lattices*, Proc. London Math. Soc. (3), 48 (1984), 91–107.
- [H-W] J.-CL. HAUSMANN and S. WEINBERGER, *Caractéristique d'Euler et groupes fondamentaux des variétés de dimension 4*, Comment. Math. Helvetici 60 (1985), 139–144.
- [H-L] J. A. HILLMAN and PETER LINNELL, *Elementary amenable groups of finite Hirsch length are locally-finite by virtually-solvable*, J. Austral. Math. Soc. 52 (1992), 237–241.
- [H] J. A. Hillman, *Elementary amenable groups and 4-manifolds with Euler characteristic 0*, J. Austral. Math. Soc. Ser. A 50 (1991), 160–170.
- [H2] J. A. Hillman, *2-knots and their groups*, Cambridge University Press (1989).
- [K] M. KERVAIRE, *Les noeuds de dimensions supérieures*, Bull. Soc. Math. de France 93 (1965), 225–271.
- [L-L] J. LOTT and W. LÜCK, *l^2 -topological invariants of 3-manifolds*, Preprint 1992.
- [M] J. MILNOR, *A procedure for killing homotopy groups of differentiable manifolds*, Proc. Symposia in Pure Mathematics: AMS, vol. III (1961), 39–55.

Forschungsinstitut für Mathematik
E.T.H. Zürich,
8092 Zürich

Received April 21, 1993; May 10, 1994