

# On minimal annuli in a slab.

Autor(en): **Fang, Yi**

Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **69 (1994)**

PDF erstellt am: **29.04.2024**

Persistenter Link: <https://doi.org/10.5169/seals-52268>

## Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

## Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

# On minimal annuli in a slab

YI FANG\*

One of three beautiful theorems about a minimal annulus in a slab proved by Max Shiffman, [11], says that,

**THEOREM 1.** *If  $A$  is a minimal annulus in a slab  $S \subset \mathbb{R}^3$ , and  $\Gamma = \partial A$  is a pair of circles lying in the boundary planes of the slab, separated by the interior of  $A$ , then for every plane  $P$  which is contained in the slab  $S$ ,  $A \cap P$  is a circle. In particular,  $A$  is embedded.*

In [4], it is proved that the same conclusion is true if we replace the boundary circles in Theorem 1 by parallel straight lines and assume  $A$  is properly embedded. Furthermore, Toubiana has proved that if two straight lines lying in distinct parallel planes, but the two straight lines are not parallel to each other, then they cannot bound any proper minimal annulus in the slab bounded by the two parallel planes.

In this article we will give generalizations of the results stated above, with a unified proof.

We will denote  $P_t = \{(x, y, z): z = t\}$ , and assume that the boundary planes of the slab are parallel to  $P_t$ ; and we can further normalize the slab  $S$  such that  $S = S(-1, 1) = \{(x, y, z): -1 \leq t \leq 1\}$ . We will prove

**THEOREM 2.** *Suppose  $A \subset S(-1, 1)$  is a proper minimal annulus in a slab and  $A(1) = A \cap P_1$ ,  $A(-1) = A \cap P_{-1}$  are straight lines or circles.*

- (1) *If both  $A(1)$  and  $A(-1)$  are circles, then  $A(t) = A \cap P_t$  is a circle,  $-1 < t < 1$ . In particular,  $A$  is embedded.*
- (2) *If  $A(1)$  or  $A(-1)$  is a straight line and the other one is a circle, and  $A$  is embedded, then  $A(t) = A \cap P_t$  is a circle,  $-1 < t < 1$ .*
- (3) *If  $A(1)$  or  $A(-1)$  are both straight lines,  $A$  is embedded, then  $A(t) = A \cap P_t$  is a circle,  $-1 < t < 1$ .*

---

\*1991 *Mathematics Subject Classification* Primary 53A10; Secondary 35P99. The research described in this paper is supported by Australia Research Council grant A688 30148. The author would like to thank the referee for pointing out mistakes in the previous version.

REMARK 1. The first part of Theorem 2 is exactly the Theorem 1, we will give it a simpler, straightforward proof, essentially in the same way as the proof of the other cases. We will see that the third part of Theorem 2 implies the result in [4]. The second part of Theorem 2 is new.

Let  $A \subset S(-1, 1)$  be a proper minimal annulus such that  $A(1) = A \cap P_1$  and  $A(-1) = A \cap P_{-1}$  are straight lines or circles,  $\partial A = A(1) \cup A(-1)$ . In the case that there is only one straight line, we will always assume that  $A(1)$  is a straight line. Then the conformal structure of the interior of  $A$  is equivalent to the interior of

$$A_R = \{z \in \mathbb{C}: 1/R \leq |z| \leq R\},$$

for some  $1 < R < \infty$ . In fact the interior of  $A$  is conformally equivalent to

$$\{z \in \mathbb{C}: \rho < |z| < P, 0 \leq \rho < P \leq \infty\},$$

for some  $\rho$  and  $P$ . Since  $A$  has 1-dimensional boundary  $\partial A$  which is separated by the interior of  $A$ , so  $0 < \rho$  and  $P < \infty$ . Hence let  $R = \sqrt{P/\rho} > 1$ , then  $\text{Int}(A) \cong \text{Int}(A_R)$ .

There is a conformal harmonic immersion

$$X: A_R - C \hookrightarrow S(-1, 1),$$

where  $C$  is a subset of  $\partial A_R$  and  $X(\{|z| = R\} - C) = A(1)$ ,  $X(\{|z| = 1/R\} - C) = A(-1)$ . If  $A(1)$  and  $A(-1)$  are both circles, then  $C = \emptyset$ ; if only  $A(1)$  is a straight line, then  $C \subset \{|z| = R\}$ ; if  $A(1)$  and  $A(-1)$  are both straight lines, then  $C \cap \{|z| = R\} \neq \emptyset$ ,  $C \cap \{|z| = 1/R\} \neq \emptyset$ . When  $C \neq \emptyset$  we assume that  $X$  is an embedding. The Enneper-Weierstrass representation of  $A$  is

$$X(z) = \text{Re} \int_1^z (\omega_1, \omega_2, \omega_3) + V,$$

where  $V = (a, b, 0) \in \mathbb{R}^3$ , and

$$\begin{aligned} \omega_1 &= \frac{1}{2} (1 - g^2(z)) f(z) dz, \\ \omega_2 &= \frac{i}{2} (1 + g^2(z)) f(z) dz, \\ \omega_3 &= g(z) f(z) dz, \end{aligned} \tag{1}$$

where  $g$  is the Gauss map and  $f$  is a holomorphic function. Since  $X$  is proper, the third coordinate function  $X^3$ , which is harmonic,  $X^3|(\{|z|=1/R\}-C)=-1$  and  $X^3|(\{|z|=R\}-C)=1$ , and  $-1 < X^3| \text{Int}(A_R) < 1$ , can be extended to whole  $A_R$  such that  $X^3| \{|z|=1/R\}=-1$  and  $X^3| \{|z|=R\}=1$ . By the uniqueness of the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } \text{Int}(A_R) \\ u| \{|z|=1/R\} = -1, \quad u| \{|z|=R\} = 1, \end{cases}$$

where  $\text{Int}(A_R)$  is the interior of  $A_R$ , we have

$$X^3 = \frac{1}{\log R} \log |z|,$$

and

$$\omega_3 = f(z)g(z) dz = \frac{d}{dz} \left( \frac{1}{\log R} \log z \right) dz = \frac{1}{\log R} \frac{1}{z} dz.$$

Hence

$$f(z) = \frac{1}{\log R} \frac{1}{zg(z)}.$$

Thus by (1) we have

$$\omega_1 = \frac{1}{\log R} \frac{1}{2z} \left( \frac{1}{g} - g \right) dz$$

$$\omega_2 = \frac{1}{\log R} \frac{i}{2z} \left( \frac{1}{g} + g \right) dz$$

$$\omega_3 = \frac{1}{\log R} \frac{1}{z} dz,$$

and  $X$  can be represented as

$$X(p) = \frac{1}{\log R} \text{Re} \int_1^p \left( \frac{1}{2z} (1/g - g), \frac{i}{2z} (1/g + g), \frac{1}{z} \right) dz + V, \quad (2)$$



Let

$$g(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \quad \frac{1}{g(z)} = \sum_{n=-\infty}^{\infty} b_n z^n, \quad (3)$$

then (2) gives a minimal annulus if and only if

$$\operatorname{Im}(b_0) = \operatorname{Im}(a_0), \quad \operatorname{Re}(b_0) = -\operatorname{Re}(a_0). \quad (4)$$

The conformal factor of  $A$  is

$$\lambda^2 = \frac{1}{4(\log R)^2 |z|^2} \left( \frac{1}{|g|} + |g| \right)^2 \quad (5)$$

and the Gauss curvature is

$$K = - \left[ \frac{4 \log R |z| |g| |g'|}{(1 + |g|^2)^2} \right]^2. \quad (6)$$

The Gauss map  $g$  of the minimal annulus  $A$  has the features as stated in the following lemma.

**LEMMA 1.** *Let  $X : A(R) - C \rightarrow S(-1, 1)$  be a properly immersed minimal annulus,  $A = X(A(R) - C)$ . Then the Gauss map  $g$  of  $A$  has neither zero nor pole in the interior of  $A_R$ .*

*If  $A(1)$  and  $A(-1)$  are both straight lines, then  $C = \{p, q\}$ ,  $|p| = R$ ,  $|q| = 1/R$ ; if only  $A(1)$  is a straight line then  $C = \{p\}$ ; In both cases,  $g$  can be extended to a neighborhood of  $A_R$  as a meromorphic function.*

*Furthermore, the extended Gauss map  $\tilde{g} : A_R \rightarrow \mathbb{C} \cup \{\infty\}$  takes zero or  $\infty$  on  $C$ , and  $C$  is the only point set in  $A_R$ , such that at which  $\tilde{g}$  takes zero or  $\infty$ .*

*Proof.* If  $g$  achieves a zero or pole in the interior point  $z$  of  $A_R$ , then the tangent plane of  $A$  at  $X(p)$  is  $P_t$ ,  $t = \log |z| / \log R$ . It is well known that the preimage  $X^{-1}(A(t))$  contains an equiangular system at  $z$ , of at least order 4. This contradicts that  $X^{-1}(A(t)) = \{|z| = r = R^t\}$ , as seen easily from the Enneper–Weierstrass representation. This contradiction proves that  $g$  has neither zero nor pole in the interior of  $A_R$ . Since  $\partial A$  is analytic, by Hopf boundary maximum principle, we known that at each point  $p \in \partial A$ , the co-normal of  $A$  has non-zero third component, hence the normal vector of  $A$  at  $p$  is not vertical, hence for any  $z \in X^{-1}(p)$ ,  $g(z) \neq 0$  or  $\infty$ .

If  $A(1)$  is a straight line, we can assume that  $A(1)$  is parallel to the  $y$ -axis in  $\mathbb{R}^3$ , then the unit normal vector of  $A$  along  $A(1)$  is in the  $xz$ -plane. Let  $C_1 = C \cap \{|z| = R\}$ , we see that  $g$  and  $g^{-1}$  is real on  $\{|z| = R\} - C_1$ . Hence using (3),  $g$  and  $g^{-1}$  can be extended to  $\{R < |z| < R^3\}$  by

$$\tilde{g}(z) = \overline{g(R^2/\bar{z})} = \sum_{-\infty}^{\infty} \overline{a_n} R^{2n} z^{-n} = \sum_{-\infty}^{\infty} a'_n z^n,$$

$$\frac{1}{\tilde{g}(z)} = \frac{1}{\overline{g(R^2/\bar{z})}} = \sum_{-\infty}^{\infty} \overline{b_n} R^{2n} z^{-n} = \sum_{-\infty}^{\infty} b'_n z^n,$$

for  $R < |z| < R^3$ , where  $a'_n = \overline{a_{-n}} R^{-2n}$ ,  $b'_n = \overline{b_{-n}} R^{-2n}$ . By (4), we see that we can use  $\tilde{g}$  to substitute  $g$  in the Enneper–Weierstrass representation (2). Then we get a minimal surface

$$\bar{X} : \{1/R < |z| < R^3\} - C_1 \rightarrow S(-1, 3).$$

Since  $X$  is properly immersed, the surface  $\mathcal{S} = \bar{X}(\{1/R < |z| < R^3\} - C_1)$  is properly immersed and contains a complete minimal annular end. By Cone Lemma of [5], this annular end has finite total curvature. By a theorem of Osserman, ([10], Theorem 9.1 and Lemma 9.5, pages 81–2) this annular end has the conformal structure of a punctured disk, and the Gauss map of  $\mathcal{S}$  can be extended to the puncture. In particular,  $C_1 = \{p\}$  is a single point,  $g$  can be extended to  $p$ , and  $g(p)$  is either zero or  $\infty$  by the Enneper–Weierstrass representation, otherwise  $X(z)$  will remain bounded near  $p$ .

Similar treatment applies to the case that  $A(-1)$  is also a straight line. The proof of the lemma is complete.  $\square$

**REMARK 2.** Let  $R$  be the rotation of  $\pi$  angle around the straight line  $A(1)$  in  $\mathbb{R}^3$ , by Schwartz Reflection Principle,  $A \cup R(A)$  is a properly embedded minimal surface. By uniqueness it must be  $\mathcal{S} = A \cup R(A)$ , where  $\mathcal{S}$  is the surface we got in Lemma 1. In particular,

$$\int_A K dA = \frac{1}{2} \int_{\mathcal{S}} K dA, \quad (7)$$

where  $K$  is the Gauss curvature, and  $dA$  is the area element of  $A$ .

We would like to calculate the plane curvature of each  $A(t) = A \cap P_t$ ,  $-1 \leq t \leq 1$ . At any point of  $A(t)$ , draw a tangent vector to the curve  $A(t)$ , and let

$\psi$  be the angle made by this tangent vector with the positive  $x$ -axis.  $\psi$  maybe is a multivalued function, but we will see that  $\psi$  is a harmonic function. To see this, consider the normal vector of the curve  $A \cap P_\epsilon$ , and its angle with the positive  $x$ -axis,  $\phi$ . If we orient the surface such that the normal is inward to the unbounded component of  $S(-1 + \epsilon, 1 - \epsilon) - A$  in  $S(-1 + \epsilon, 1 - \epsilon)$ , for every  $\epsilon > 0$  small enough, then we have  $\psi = \phi + \pi/2$ . By Lemma 1,  $g \neq 0$  or  $\infty$  in the interior of  $A_R$ , hence the normal vector must be  $g/|g| \in \mathbb{R}^2$ , and so  $\phi = \arg g = \text{Im} \log g$ , thus  $\phi$  is harmonic, so is  $\psi$ . Now suppose that  $s$  is the arc length parameter of the curve  $A(t)$ , notice that  $X^{-1}(A(t)) = \{z: |z| = r = R'\}$ , write  $z = r e^{i\theta}$ , we can calculate the curvature of  $A(t)$  as follows:

$$\begin{aligned} \kappa &= \psi_s = \phi_s = \frac{d}{ds} (\text{Im} \log g) = \text{Im} \left( \frac{d}{ds} \log g \right) = \text{Im} \left( \frac{d}{dz} \log g \frac{dz}{ds} \right) \\ &= \text{Im} \left( \frac{g'}{g} \frac{dz}{d\theta} \frac{d\theta}{ds} \right) = \text{Im} \left( \frac{g'}{g} i z r^{-1} \Lambda^{-1} \right) = r^{-1} \Lambda^{-1} \text{Re} \left( z \frac{g'}{g} \right). \end{aligned}$$

Here we have used the facts that on the curve  $\{|z| = r = R'\}$ ,

$$\frac{dz}{d\theta} = i r e^{i\theta} = i z, \quad \text{and} \quad ds = \Lambda |dz| = \Lambda r d\theta.$$

Calculation shows that

$$r\kappa_\theta = \text{Im} \left[ \frac{1}{2} \frac{|g|^2 - 1}{|g|^2 + 1} \left( z \frac{g'}{g} \right)^2 - z \frac{g'}{g} - z^2 \frac{g''g - (g')^2}{g^2} \right].$$

Let

$$u = r\Lambda\kappa_\theta = \text{Im} \left[ \frac{1}{2} \frac{|g|^2 - 1}{|g|^2 + 1} \left( z \frac{g'}{g} \right)^2 - z \frac{g'}{g} - z^2 \frac{g''g - (g')^2}{g^2} \right], \quad (8)$$

then simple calculation shows that

$$\Delta u = -\frac{8|g'|^2}{(1 + |g|^2)^2} u. \quad (9)$$

By (5) and (6), we have

$$\frac{8|g'|^2}{(1+|g|^2)^2} = -\frac{1}{2}K \frac{(1+|g|^2)^2}{(\log R)^2|z|^2|g|^2} = -2K\lambda^2.$$

Hence  $u$  is a Jacobi field.

REMARK 3. The Jacobi field  $u$  is the same as in [11], page 79, formulas (8), (9), and (10), where it is denoted by  $\beta$  in  $\Psi$ ,  $H$  plane.

Remember that

$$\Delta_A = \lambda^{-2} \Delta,$$

where  $\Delta_A$  is the Laplacian under the metric  $ds^2 = \lambda^2|dz|^2$ . If  $\Gamma = A(1) \cup A(-1)$  consists of straight lines or circles, then  $\kappa_\theta \equiv 0$  on  $\partial A_R - C$ , hence on  $A_R - C$   $u$  satisfies

$$\begin{aligned} \Delta_A u - 2Ku &= 0, \\ u|(\partial A_R - C) &= 0. \end{aligned} \tag{10}$$

By the definition of  $u$ , we see that to prove Theorem 2 is equivalent to prove that  $u \equiv 0$ .

Now it is easy to prove the first part of Theorem 2. In fact since  $\kappa = \lambda^{-1}r^{-1} \operatorname{Re}(zg'/g)$  and  $\operatorname{Re}(zg'/g) > 0$  on  $\partial A_R$ ,  $\operatorname{Re}(zg'/g)$  is harmonic, hence  $\kappa$  is positive on all of  $A_R$ , thus each  $A(t)$  is strictly local convex. The total curvature of each  $A(t)$  must be  $2\pi$  by continuity, hence each  $A(t)$  is strictly convex. In particular,  $A$  is embedded. Now  $u|_{\partial A_R} \equiv 0$ ,  $u$  is an eigenfunction corresponding to the eigenvalue 0 to the operator  $L_A = \Delta_A - 2K$ . And since  $\kappa$  must have at least one maximum and one minimum on each  $A(t)$ ,  $\kappa_\theta$  and so  $u$  changes sign, if  $u \not\equiv 0$ . Since the eigenfunction corresponding to the first eigenvalue never changes sign, 0 is not the first eigenvalue of  $L_A$ , hence  $\lambda_2(\operatorname{Int}(A_R)) \leq 0$ . It is well known that  $g$  is a conformal diffeomorphism from the interior of  $A_R$  to  $C$  and  $\Omega = g(\operatorname{Int}(A_R))$  is a proper domain of  $S^2 = \mathbb{C} \cup \{\infty\}$  under the assumption that each  $A(t)$  is a strictly convex Jordan curve. (In fact it is a local diffeomorphism by  $\operatorname{Re}(zg'/g) > 0$  hence  $g' \neq 0$ , and a geometric, [8], or an analytic, [3], argument shows that  $g$  is one-to-one.) The eigenvalue problem of  $L_A$  is equivalent to the eigenvalue problem of the operator  $\Delta_{S^2} + 2$ , where  $\Delta_{S^2}$  is the sphere Laplacian. Therefore,  $\lambda_2(\operatorname{Int}(A_R)) + 2 = \lambda_2(\Omega)$ , the latter is the eigenvalue for  $\Delta_{S^2}$ . Since  $\Omega$  is a proper

subdomain,  $\lambda_2(\Omega) > 2$ , which contradicts to  $\lambda_2(\text{Int}(A_R)) \leq 0$ . This contradiction proves the first part of Theorem 2.

To complete the proof of Theorem 2, we need two more lemmas.

**LEMMA 2.** *Suppose that  $A \subset S(-1, 1)$  is a proper minimal annulus,  $A(1) = A \cap P_1$  is a straight line,  $A(-1) = A \cap P_{-1}$  is a circle or straight line, and  $\partial A = A(1) \cup A(-1)$ . Then*

$$\int_A K dA > -4\pi,$$

*if  $A(-1)$  is a circle; and*

$$\int_A K dA = -4\pi,$$

*if  $A(-1)$  is also a straight line.*

*Proof.* We will use the extended surface  $\mathcal{S}$  in the proof of Lemma 1 to calculate the total curvature of  $A$ . Notice that  $\mathcal{S}$  has an embedded flat annular end corresponding to the point  $p$ . Take  $Y_\rho = \mathcal{S} \cap S_\rho$ , where  $S_\rho$  is the sphere in  $\mathbb{R}^3$  centered at origin of radius  $\rho$ . Denote the compact ball bounded by  $S_\rho$  by  $B_\rho$  and let  $O_\rho = \mathcal{S} \cap (\mathbb{R}^3 - B_\rho)$ . If  $A(-1)$  is a circle, take  $\rho$  large enough, such that  $\partial \mathcal{S} \subset B_\rho$ , then using Remark 2 and the Gauss–Bonnet theorem.

$$\int_{\mathcal{S} - O_\rho} K dA = -2\pi - \int_{R(A(-1))} \kappa_g ds - \int_{A(-1)} \kappa_g ds - \int_{Y_\rho} \kappa_g ds.$$

By a theorem of Jorge and Meeks, [7], when  $\rho \rightarrow \infty$ ,  $X_\rho = Y_\rho/\rho$  approaches to a geodesic of the unit sphere  $S^2$  in a  $C^\infty$  way, thus

$$\int_{Y_\rho} \kappa_g ds \rightarrow 2\pi, \quad \rho \rightarrow \infty.$$

Notice that the other two integrals are larger than  $-2\pi$  because  $A(-1)$  and  $R(A(-1))$  are circles. We have

$$\int_{\mathcal{S}} K dA > -8\pi.$$

By (7), we conclude that the total curvature of  $A$  is larger than  $-4\pi$ .

For the proof of the case that  $A(-1)$  is also a straight line, we quote a lemma in [4] listed below, which implies that our surface has total curvature  $-4\pi$ , notice that  $A$  has genus 0.  $\square$

We only state the part of that lemma in [4] that is useful to us.

**LEMMA 3.** *Suppose  $A$  is a properly embedded minimal surface that is bounded by a pair of lines  $L = L_0 \cup L_1$  and lies in a slab between parallel planes,  $P = P_0 \cup P_1$  with  $L_i \subset P_i$ . Then  $A$  extends by Schwartz Reflection to a singly-periodic embedded minimal surface  $\mathcal{S}$ , invariant under a screw motion  $T$ , where  $T$  is  $R \circ R_0$ ,  $R_i$  being rotation by  $\pi$  about  $L_i$ . If  $A$  has genus  $k$ ,  $\mathcal{S}/T$  has genus  $(2k + 1)$ , two flat ends and total curvature  $-4\pi(2k + 2)$ .*

The strategy of the proof of Theorem 2 is then as follows. The zero set of  $u$  divides  $A_R$  into subdomains, called nodal domains. If  $u$  can be continuously extended to  $A_R$  such that  $u|_{\partial A_R} = 0$ , then  $u$  is an eigenfunction corresponding to the eigenvalue 0 for the operator  $L_A$  on these nodal domains. By a theorem of Barbosa and do Carmo, [1], the total curvature of each such nodal domain is less than or equal to  $-2\pi$ , hence the total curvature of  $A$  must be less than  $-2k\pi$ , if we have  $k$  nodal domains. If  $k \geq 2$ , we get a contradiction to Lemma 2. Thus we must have  $u \equiv 0$ , which is equivalent to each  $A(t)$  is a circle, for  $-1 < t < 1$ .

By the four-vertex-theorem, see [6], which says that the zero set of  $\kappa_\theta$  divides each  $A(t)$  into at least four components,  $-1 < t < 1$ , if  $\kappa_\theta \not\equiv 0$ . Hence there are at least four nodal domains.

Lemma 4 below will prove that  $u$  is actually continuous on  $A_R$ , with  $u|_{\partial A_R} = 0$ . Thus we will get the anticipated contradiction.

**LEMMA 4.** *Let  $A$  be as in Theorem 2,  $p, q$  be as in Lemma 1, and  $u$  be as defined in (8). Then  $u$  is continuous on  $A_R$  and  $u|_{\partial A_R} = 0$ .*

*Proof.* Without loss of generality, we can assume that  $p = R$ . By Lemma 1, we can assume that the Gauss map  $g$  has limit zero at  $p = R$ , then  $g$  can be extended to the disk  $D_\rho$  centered at  $z = R$ , as a holomorphic function  $\tilde{g}$ , and let  $\zeta = z - R$ ,

$$\tilde{g}(z) = (z - R)^n \tilde{h}(z) = \zeta^n h(\zeta),$$

where  $h$  is a holomorphic function and  $h(0) \neq 0$ . Since  $R$  corresponds to an embedded flat end, and that end intersects  $P_1$  at a straight line, we have  $n = 2$ . Please see [12] for the argument or just see it from Lemma 3, since  $\mathcal{S}$  has total curvature  $-8\pi$ , so the Gauss map should be a degree 2 map.

For convenience, we will write  $g$  instead of  $\tilde{g}$ . Then

$$z \frac{g'(z)}{g(z)} = \frac{2R}{z-R} + 2 + z \frac{h'(z)}{h(z)},$$

and

$$z \frac{g'(z)}{g(z)} = \frac{a_{-1}}{\zeta} + \sum_{k=0}^{\infty} a_k \zeta^k,$$

$a_1 = 2R$ . Then

$$\begin{aligned} \left( z \frac{g'(z)}{g(z)} \right)^2 &= \frac{a_{-1}^2}{\zeta^2} + \frac{2a_{-1}a_0}{\zeta} + \sum_{k=0}^{\infty} b_k \zeta^k. \\ z \frac{d}{dz} \left( z \frac{g'(z)}{g(z)} \right) &= -\frac{a_{-1}R}{\zeta^2} - \frac{a_{-1}}{\zeta} + (\zeta + R) \sum_{k=1}^{\infty} k a_k \zeta^{k-1}. \end{aligned}$$

Since  $a_{-1} = 2R$ ,

$$\frac{1}{2} a_{-1}^2 - a_{-1}R = 0.$$

Since

$$\begin{aligned} z \frac{h'(z)}{h(z)} &= (\zeta + R) \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{h'(z)}{h(z)} \right)^{(k)}_{z=R} \zeta^k = R \frac{h'(R)}{h(R)} + \sum_{k=1}^{\infty} d_k \zeta^k, \\ a_0 &= 2 + R \frac{h'(R)}{h(R)}. \end{aligned}$$

We would like to calculate  $a_0$ . The Weierstrass representation for the extended surface  $\mathcal{S}$  is

$$\begin{cases} \omega_1 = \frac{1}{\log R} \frac{1}{2z} \left( \frac{1}{\tilde{g}} - \tilde{g} \right) dz \\ \omega_2 = \frac{1}{\log R} \frac{i}{2z} \left( \frac{1}{\tilde{g}} + g \right) dz \\ \omega_3 = \frac{1}{\log R} \frac{1}{z} dz, \end{cases}$$

as commented in the proof of Lemma 1. Again we will write  $g$  instead  $\tilde{g}$ . Let  $C$  be a loop around  $z = R$  in a small disk. Then since  $X : \{z : 1/R < |z| < R^3\} - \{R\} \rightarrow \mathbb{R}^3$  is well defined and

$$X(z) = \operatorname{Re} \int_{p_0}^z (\omega_1, \omega_2, \omega_3),$$

we must have

$$\begin{aligned} \operatorname{Re} \int_C \frac{1}{2z} \left( \frac{1}{g(z)} - g(z) \right) dz &= 0, \\ -\operatorname{Im} \int_C \frac{1}{2z} \left( \frac{1}{g(z)} + g(z) \right) dz &= 0, \\ \int_C \frac{1}{zg} dz &= \overline{\int_C \frac{d}{z} dz} = 0, \end{aligned}$$

since  $g(z)/z$  is holomorphic at  $z = R$ . Hence we know that the residue of  $1/zg(z)$  at  $z = R$  is zero. Then we have

$$\begin{aligned} 0 &= \lim_{z \rightarrow R} \left( \frac{(z - R)^2}{zg(z)} \right)' = \lim_{z \rightarrow R} \left( \frac{1}{zh(z)} \right)' \\ &= \lim_{z \rightarrow R} \left( -\frac{1}{z^2 h(z)} - \frac{h'(z)}{zh^2(z)} \right) \\ &= -\frac{1}{R^2 h(R)} - \frac{h'(R)}{Rh^2(R)}. \end{aligned}$$

Hence

$$\frac{H'(R)}{h(R)} = -\frac{1}{R},$$

and

$$a_0 = 2 + R \frac{h'(R)}{h(R)} = 1.$$



Then  $a_{-1}a_0 - a_{-1} = 0$  and

$$\begin{aligned} & -\frac{1}{2} \left( z \frac{g'(z)}{g(z)} \right)^2 - z \frac{d}{dz} \left( z \frac{g'(z)}{g(z)} \right) \\ &= -\frac{1}{2} \frac{a_{-1}^2 - 2a_{-1}R}{\zeta^2} - \frac{a_{-1}a_0 - a_{-1}}{\zeta} - \frac{1}{2} \sum_{k=0}^{\infty} b_k \zeta^k - (\zeta + R) \sum_{k=1}^{\infty} k a_k \zeta^{k-1} \\ &= \sum_{k=0}^{\infty} c_k \zeta^k, \end{aligned}$$

hence

$$\Phi(z) = -\frac{1}{2} \left( z \frac{g'(z)}{g(z)} \right)^2 - z \frac{d}{dz} \left( z \frac{g'(z)}{g(z)} \right)$$

is holomorphic near  $z = R$ . Now consider the function

$$U(z) = \frac{|g|^2 - 1}{2(1 + |g|^2)} \left( z \frac{g'(z)}{g(z)} \right) - z \frac{d}{dz} \left( z \frac{g'(z)}{g(z)} \right),$$

note that our function  $u$  is  $\text{Im } U$ .  $U(z)$  can be rewritten as

$$\begin{aligned} U(z) &= -\frac{1}{2} \left( z \frac{g'(z)}{g(z)} \right)^2 - z \frac{d}{dz} \left( z \frac{g'(z)}{g(z)} \right) + \left( 1 - \frac{1}{1 + |g|^2} \right) \left( z \frac{g'(z)}{g(z)} \right)^2 \\ &= \Phi(z) + \Psi(z). \end{aligned}$$

Note that

$$\zeta^2 \left( z \frac{g'(z)}{g(z)} \right)^2$$

is holomorphic and since  $|g|^2 = |z - R|^4 |h(z)|^2 = |\zeta|^4 |h(z)|^2$ ,

$$\frac{1}{\zeta^2} \left( 1 - \frac{1}{1 + |g|^2} \right) = \frac{1}{\zeta^2} \sum_{k=1}^{\infty} (-1)^{k+1} |g|^{2k} = \overline{\zeta^2} \sum_{k=1}^{\infty} (-1)^{k+1} |\zeta|^{4(k-1)} |h(z)|^{2k}$$

is a  $C^\infty$  complex function in a neighborhood of  $R$ . Thus

$$\Psi(z) = \frac{1}{\zeta^2} \left( 1 - \frac{1}{1 + |g|^2} \right) \zeta^2 \left( z \frac{g'(z)}{g(z)} \right)^2$$

is a  $C^\infty$  complex function near  $z = R$ , so is  $U(z)$ . In particular,  $u(z) = \operatorname{Im} U(z)$  is  $C^\infty$  near  $z = R$ . Hence  $u(R) = 0$ , since on  $|z| = R$ , if  $z \neq R$ , then  $u(z) = 0$  by (10). Hence  $u$  can be continuously extended to  $p = R$  such that  $u(p) = 0$ . The proof for  $q$  is exactly the same, if we note that either  $g$  or  $g^{-1}$  can be regarded as the Gauss map, therefore we can always assume  $g$  has a double zero at  $q$ .  $\square$

The proof of Theorem 2 is complete.

From the proof we see immediately that

**COROLLARY 1.** *Suppose that  $A \subset S(-1, 1)$  is a proper minimal annulus. If  $A(1) = A \cap P_1$  is a straight line and  $A$  is embedded in a neighborhood of  $A(1)$ ,  $A(-1) = A \cap P_{-1}$  is a circle, and the total curvature of  $A$  is larger than or equal to  $-8\pi$ . Then each  $A(t) = A \cap P_t$ ,  $-1 < t < 1$ , is a circle. In particular,  $A$  is embedded.*

*Proof.* We only need to point out that we can still use the four-vertex theorem, even some level sets  $A(t)$  may not be Jordan curves. But it is shown in [6], that all curves which have exactly two vertices are curves which have exactly two simple loops, on each loop the curvature is positive or negative, hence its total curvature must be 0.  $\square$

The minimal annuli which can be foliated by circles or straight lines, but not all straight lines, were classified by Enneper and B. Riemann in the 19th century, given by elliptic integrals. See [9], pages 85–90 for detail, or see [4] or [12] for a representation using elliptic functions. The facts are, all of those circles are in parallel planes; if any two circles are coaxial, the surface is a piece of a catenoid, the only rotational minimal surface; if the circles are not coaxial, then the surface has two parallel straight lines lying in parallel planes, as boundary. Based on these facts, we have proved that

**COROLLARY 2.** *Let  $L_1 \subset P_1$ ,  $L_{-1} \subset P_{-1}$  be two parallel straight lines. If  $\Gamma = L_1 \cup L_{-1}$  is the boundary of a properly embedded minimal annulus  $A$  in  $S(-1, 1)$ , then  $A$  is one of the Riemann's examples.*

Finally, we have a non-existence theorem.

**COROLLARY 3.** *Let  $L_1 \subset P_1$ ,  $L_{-1} \subset P_{-1}$  be two straight lines, and they are not parallel to each other. Then  $\Gamma = L_1 \cup L_{-1}$  cannot bound an properly embedded minimal annulus in  $S(-1, 1)$ .*

Corollary 2 is the main theorem of [4], in which it is proved via the elliptic function theory. Corollary 3 is a result of Toubiana [12].

## REFERENCES

- [1] L. BARBOSA and M. DO CARMO, *On the size of a stable minimal surface in  $\mathbb{R}^3$* . Amer. J. of Math., 98: 515–28, 1976.
- [2] I. CHAVEL, *Eigenvalues in Riemannian Geometry*. Academic Press, Inc. Orlando, San Diego, New York, London, Montreal, Sydney, Tokyo, 1984.
- [3] Y. FANG, *Lectures on Minimal Surface of Annular Type*. Lecture Notes in A.N.U. Canberra, Australia.
- [4] D. HOFFMAN, H. KARCHER and H. ROSENBERG, *Embedded minimal annuli in  $\mathbb{R}^3$  bounded by a pair of straight lines*. Comment. Math. Helvetici, 66: 599–617, 1991.
- [5] D. HOFFMAN and W. MEEKS III, *The asymptotic behavior of properly embedded minimal surfaces of finite topology*. Journal of the Amer. Math. Soc., 2(4): 667–82, 1989.
- [6] S. B. JACKSON, *Vertices for plane curves*. Bull. of Amer. Math. Soc., 50: 564–78, 1944.
- [7] L. P. JORGE and W. MEEKS III, *The topology of complete minimal surfaces of finite total Gaussian curvature*. Topology, 20(2): 203–21, 1983.
- [8] W. MEEKS III and B. WHITE, *Minimal surfaces bounded by convex curves in parallel planes*. Comment. Math. Helvetici, 66: 263–78, 1991.
- [9] J. C. C. NITSCHKE, *Lectures on Minimal Surfaces, Vol. I*. Cambridge University Press, Cambridge, New York, New Rochelle, Melbourne, Sydney, 1989.
- [10] R. OSSERMAN, *A Survey of Minimal Surfaces*, Dover Publishers, New York, 2nd edition, 1986.
- [11] M. SHIFFMAN, *On surfaces of stationary area bounded by two circles, or convex curves, in parallel planes*. Annals of Math., 63: 77–90, 1956.
- [12] E. TOUBIANA, *On the minimal surfaces of Riemann*. Comment. Math. Helvetici, 67: 546–70, 1992.

*Department of Mathematics*  
*School of Mathematical Sciences*  
*The Australian National University*  
*Canberra ACT 0200*  
*Australia*

Received August 12, 1993