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p-Nilpotence, classifying space indecomposability, and other properties of almost all finite groups

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0. Introduction

In this paper we establish several properties holding for almost all finite groups. For example we show that a random group is p-nilpotent and that the completed reduced double Burnside ring of a random p-group is local or, what is the same thing, that its classifying space is stably indecomposable. It follows that a random group has the same mod-p cohomology ring as its Sylow p-subgroups. Here the notions of almost all and random are those of U. Martin [Mn] who showed that almost all p-groups have automorphism group a p-group. Our results were partly inspired by her work and partly by indecomposability questions of classifying spaces. However, in order to make this paper more accessible to algebraists, we derive our topological results as corollaries of their algebraic counterparts.

Use of the double Burnside ring also leads to a generalization of Swan's Theorem which computes the mod-p cohomology of a finite group G with abelian Sylow p-subgroup P as

$$H^*(G; \mathbb{F}_p) = H^*(P; \mathbb{F}_p)^{W_G(P)},$$

the invariants under $W_G(P) = N_G(P)/P \cdot C_G(P)$, where $N_G(P)$, resp. $C_G(P)$, is the normalizer, resp. centralizer, of P in G. Our version shows the result continues to hold for non-abelian P provided "taking commutators reduces orders" in P (see Definition 1.6 and Theorem 1.9).

The paper is organized as follows: In §1, after the requisite definitions we state the main result Theorem 1.3 (the completed reduced double Burnside ring $\tilde{A}(P, P)^\wedge$ is local for almost all p-groups) and its corollaries. We observe this follows from Theorems 1.7 and 1.8. In Theorem 1.7 we describe three group theoretic conditions on P, which imply $\tilde{A}(P, P)^\wedge$ is local. Theorem 1.8 states that almost all p-groups satisfy these conditions. Related results are described for general finite groups. Section 2 is devoted to the proof of Theorem 1.7. In §3 we recall Martin's theory concerning almost all p-groups. Section 4 contains the proof of Theorem 1.8. In §5 we derive group cohomological results, in particular Theorem 1.9, related to certain of our conditions. Explicit examples are given in §6 of p-groups with $\tilde{A}(P, P)^\wedge$ local and BP stably indecomposable.

Finally we remark that our results are somewhat surprising in that previously, the only known p-groups with $\tilde{A}(P, P)^\wedge$ local were the cyclic groups $\mathbb{Z}/2^n$ and they were thought to be exceptional rather than typical as we show.

1. Statement of results

All groups in this paper are assumed to be finite unless otherwise stated.
We begin by recalling

DEFINITION 1.1. The *Frattini series* of a group G

$$G_1 \rhd G_2 \rhd \cdots \rhd G_n \rhd G_{n+1} \rhd \cdots$$

is given by $G_1 = G$, $G_{n+1} = G_n^p[G, G_n]$, $n \geq 1$. A group is said to have *Frattini length* n if $G_{n+1} = 1 \neq G_n$.

Then $\Phi G = G_2$ is the *Frattini subgroup* of G and $G/G_2 = H_1(G; \mathbb{F}_p)$ is an elementary abelian p-group. If G is a p-group then d, the minimum number of generators of G, is the rank of G/G_2.

Following Martin [Mn] we defined "almost all".

DEFINITION 1.2. Let $A_{d,n}$ be the (finite) set of isomorphism classes of p-groups of Frattini length n generated by a minimum of d elements. Given a property S of groups (invariant under isomorphisms) let $D_{d,n} \subset A_{d,n}$ be the subset of elements having property S. Then we say *almost all* p-groups of Frattini length n have property S if

$$\lim_{d \to \infty} \frac{|S_{d,n}|}{|A_{d,n}|} = 1.$$

It is known [Mn] that $|A_{d,n}| \to \infty$ as soon as $n \geq 2$.

To state our first result we recall the definition of the Burnside category $\mathscr{A}$ ([AGM, p. 454–5]). The objects are the finite groups and $\mathrm{mor}_{\mathscr{A}}(G_1, G_2)$ is the Grothendieck group (under disjoint union) of isomorphism classes of finite $G_1 \times G_2$-sets where we assume that G_1 acts from the left and G_2 acts freely from the right. Then $\mathrm{mor}_{\mathscr{A}}(G_1, G_2)$ is easily seen to be the free abelian group on isomorphism classes of transitive such $G_1 \times G_2$-sets, representatives of which are given by the orbit sets $G_1 \times G_2/H_\rho$ where H is a subgroup of G_1, ρ is a homomorphism from H to G_2 and $H_\rho = \{(h, \rho(h)): h \in H\}$. Composition $\mathrm{mor}_{\mathscr{A}}(G_1, G_2) \times \mathrm{mor}_{\mathscr{A}}(G_2, G_3) \to \mathrm{mor}_{\mathscr{A}}(G_1, G_3)$ is bilinear and induced by taking Cartesian products over G_2.

Equivalently $\mathrm{mor}_{\mathscr{A}}(G_1, G_2)$ could be defined as the free abelian group on equivalence classes of pairs (H, ρ) with $(H, \rho) \sim (K, \tau)$ if H_ρ and K_τ are conjugate in $G_1 \times G_2$. If one thinks of the pair (H, ρ) as a formal composition of an abstract transfer from G_1 to H followed by the homomorphism ρ then the composition law as defined above is equivalent to the double coset formula (cf. [N, §3]). Therefore mod p cohomology (with trivial coefficients) can be thought of as a contravariant functor from $\mathscr{A}$ to abelian groups. Such functors are also called global Mackey functors. Now the double Burnside ring $A(G, G)$ is defined to be the endomorphism ring $\mathrm{mor}_{\mathscr{A}}(G, G)$.

A reduced global Mackey functor is one which vanishes on the trivial group, e.g. reduced cohomology with trivial coefficients. Accordingly we define the reduced double Burnside ring $\tilde{A}(G, G)$ as the quotient of $A(G, G)$ by the two-sided ideal of morphisms which factor through the trivial group. Finally we define the reduced completed double Burnside ring $\tilde{A}(P, P)^\wedge$ of a finite p-group P as $\tilde{A}(P, P) \otimes \mathbb{Z}_p$ where $\mathbb{Z}_p$ denotes the p-adic integers.

THEOREM 1.3. *$\tilde{A}(P, P)^\wedge$ is a local ring for almost all p-groups P of Frattini length $n \geq 2$.*

Using G. Carlsson's solution of the Segal Conjecture [C], Lewis, May, and McClure [LMM] have established a ring isomorphism

$$\tilde{A}(P, P)^\wedge \approx \{BP, BP\} \tag{1.4}$$

for p-groups. Here BP is the classifying space of P and $\{BP, BP\}$ denotes the ring of stable self maps under composition. The topological translation of Theorem 1.3 is

COROLLARY 1.5. *BP is stably indecomposable for almost all p-groups P of Frattini length $n \geq 2$.*

Theorem 1.3 follows directly from Theorems 1.7, 1.8 below.

DEFINITION 1.6. We say P satisfies

Condition (a) if $[P, \Omega_{k+1}P] \leq \Omega_k P$ for all $k \geq 0$, where $\Omega_k P$ is the subgoup of P generated by elements of order p^k or less.

Condition (b) if $P_j = \Omega_{n+1-j}P$ for all $j \geq 1$, where P_j is the j-th term in the Frattini series and n is the Frattini length of P.

Obviously (a) holds if P is abelian. Also (b) $\Rightarrow$ (a) since $[P, \Omega_{k+1}P] = [P, P_{n-k}] \leq P_{n+1-k} = \Omega_k P$. Informally (a) means taking commutators reduces the orders of elements in P. More precisely it means that conjugation by elements of P is trivial on the quotients $\Omega_{k+1}P/\Omega_k P$, $k \geq 0$. This implies in particular that the p-th power map takes $\Omega_{k+1}P$ to $\Omega_k P$ for all k and by induction we see that $x^{p^{k+1}} = 1$ for each $x \in \Omega_{k+1}P$.

Examples illustrating these conditions are discussed in Remarks 2, 3 below; see also Example 6.2. For odd primes, condition (a) is implied by the assumption for $k = 0$. We heartily thank the referee for this important observation.

PROPOSITION 1.6.1. *Let P be a p-group, $p > 2$, all of whose elements of order p lie in the center. Then P satisfies condition (a).*

Proof. The proof proceeds by induction on the order of P and follows Blackburn's proof of a theorem of Thompson [Hu; III.12.2].

Consider $G = P/\Omega_1 P$. We will show that $\Omega_1 G$ is central in G, hence G satisfies condition (a) by inductive hypothesis. In order to deduce (a) for P it then suffices to show that the preimage of $\Omega_{k-1}(P/\Omega_1 P) = \Omega_{k-1}G$ under the projection map from P to G is $\Omega_k P$. Now each element $y \in \Omega_{k-1}G$ has order at most p^{k-1} because G satisfies (a). So if x is a preimage of y then $x^{p^{k-1}} \in \Omega_1 P$ which is elementary abelian by assumption and we are done.

So we have to show that $\Omega_1 G$ is central in G. Let $A/\Omega_1 P$ be maximal among the normal abelian subgroups of G of exponent p. Then $[A, A] \leq \Omega_1 P \leq Z(p)$ by the inductive assumption. Hence $[A, A]$ has exponent p and A has class 2. If $a \in A$, $g \in P$ then $a^g = ab$, $b \in A$ and $a^p \in \Omega_1 P$. Hence

$$a^p = (a^p)^g = (a^g)^p = (ab)^p = a^p b^p [b, a]^{\binom{p}{2}} = a^p b^p$$

and $b^p = 1$. Thus $b \in \Omega_1 P$ and $A/\Omega_1 P \leq Z(G)$. It follows from Alperin's theorem [Hu; III.12.1] that every element of order p of G is in $A/\Omega_1 P$ and hence central. □

THEOREM 1.7. *Suppose P satisfies condition (a). If* Out (P) *is a p-group and P has no non-trivial retracts then $\tilde{A}(P, P)^\wedge$ is a local ring.*

We recall that $Q \leq P$ is a *retract* if there is a homomorphism $r : P \to Q$ such that $r\,|_Q = \mathrm{id}_Q$

Let $K_{d,n} \subset A_{d,n}$ consist of p-groups P satisfying

(i) Out (P) is a p-group,

(ii) P has no non-trivial retracts, and

(iii) condition (b).

THEOREM 1.8. *Almost all p-groups of Frattini length $n \geq 2$ satisfy* (i), (ii), *and* (iii), *i.e.*

$$\lim_{d\to\infty} \frac{|K_{d,n}|}{|A_{d,n}|} = 1$$

Next we examine some group cohomological consequences of these ideas. If H is a subgroup of G, then we denote the restriction homomorphism from $H^*(G; \mathbb{F}_p)$ to $H^*(H; \mathbb{F}_p)$ by $i^*_{H,G}$.

THEOREM 1.9. *If P satisfies condition* (*a*) *then*

$$i^*_{N,G} : H^*(G; \mathbb{F}_p) \xrightarrow{\approx} H^*(N_G P; \mathbb{F}_p) = H^*(P; \mathbb{F}_p)^{W_G(P)}$$

for any finite group G with P as Sylow p-subgroup. ($W_G P = N_G P / P \cdot C_G P$).

REMARKS.

1. This result generalizes Swan's Theorem [S] which is the assertion for P abelian. Swan's proof relies essentially on the method of stable elements ([CE]) and does not carry over directly. Recently, J. Thévenaz has given a completely group theoretic proof of 1.9 using Alperin's fusion theory. Also, there are "classical proofs" of the H^1-version of 1.9, e.g. by using Grün's Second Theorem [Hu], [S] and an easy induction on the length of the Ω-filtration of P. A result of this type was first shown to us by A. Brandis and it is a pleasure to acknowledge helpful discussions with him on this point. We wish to emphasise, however, that knowledge of the structure of the double Burnside ring was crucial in our discovery of the result.
2. If $P = Q_{2^m}$, the generalized quaternion group of order 2^m, then P satisfies condition (a) iff $m = 3$. If $m > 3$ then Out (P) is a 2-group and hence $W_G(P) = 1$ acts trivially on $H^*(P; \mathbb{F}_2)$. For a suitable odd prime power q, P is a Sylow 2-subgroup of $G = SL_2(\mathbb{F}_q)$ and $i^*_{N,G}$ is not an isomorphism. Thus one sees that the conclusion of Theorem 1.9 holds iff $m = 3$.

3. An example satisfying condition (a) but not (b) is given by the semidirect product

$$P = \langle x, y \mid x^8 = 1, y^4 = 1, yxy^{-1} = x^5 \rangle.$$

The next result is folklore; we include it for completeness.

PROPOSITION 1.10. *If* Out (P) *is a p-group then*

$$i^*_{P,N} : H^*(N_G P; \mathbb{F}_p) \xrightarrow{\approx} H^*(P; \mathbb{F}_p).$$

for any group G with P as Sylow p-subgroup.

Finally we consider a random finite group. To make this precise, we say two groups G, G' are *Sylow p-equivalent* if they have isomorphic Sylow p-subgroups $P \approx P'$. We may then consider the equivalence classes of Sylow p-equivalent groups, one for each p-group P.

We recall that a group G is called *p-nilpotent* if a Sylow p-subgroup possesses a normal p-complement, i.e., a normal p'-subgroup K such that $G = K \cdot P$. Tate [T] has shown that G is p-nilpotent iff $i^*_{P,G} : H^*(G; \mathbb{F}_p) \to H^*(P; \mathbb{F}_p)$ is an isomorphism in dimension 1 (and hence in all dimensions). Then we have

COROLLARY 1.11. *For almost all p-groups of Frattini length $n \geq 2$ each group in the associated Sylow p-equivalence class satisfies the following equivalent conditions*:

(i) *G is p-nilpotent*
(ii) $i^*_{P,G}: H^*(G; \mathbb{F}_p) \xrightarrow{\approx} H^*(P; \mathbb{F}_p)$
(iii) *$BP \simeq BG$ stably, localized at p.*

This follows from 1.8, 1.9 and 1.10.

2. Proof of Theorem 1.7

We recall from [P; Th. 1.5] that if $\tilde{A}(P, P)^\wedge$ fails to be a local ring then one of the following must hold

(α) Out (P) is not a p-group
(β) there are subgroups, $Q \leq P' \leq P$, $1 \neq Q \neq P$, a retraction $g : P' \to Q$ and a primitive idempotent $e \in \mathbb{F}_p$ Out Q which contains the following element $\bar{W}_g$

as a factor: if $N(Q, P')$ denotes the set of elements $x \in P$ such that $xQx^{-1} \leq P'$ then P' acts on $N(Q, P')$ from the left and we let $\bar{W}_g = \Sigma_x g \circ c_x$ where $c_x(\cdot) = x(\cdot)x^{-1}$ and the sum runs over all those cosets in $P' \backslash N(Q, P')$ for which $g \circ c_x$ is an automorphism of Q.

In [P] these criteria are given in topological terms but the translation is direct from (1.4). We also note that in case $P' = P$ then $\bar{W}_g = 1$ and so (β) simply asserts the existence of a retraction $g : P \to Q$.

LEMMA 2.1. *Suppose* (β) *holds for* P. *Then condition* (a) *implies* $P = P'$ *in* (β).

Proof. Because $\Omega_n P = P$ for n large, it suffices to show by induction that $\Omega_k P \leq P'$ for $k = 0, 1, 2, \ldots, n$. This is trivial for $k = 0$ since $\Omega_0 P = 1$. Now suppose $\Omega_k P \leq P'$ for some k, $0 \leq k < n$. By (a) we have $[Q, \Omega_{k+1} P] \leq \Omega_k P \leq P'$ which implies $\Omega_{k+1} P \leq N(Q, P')$. Moreover, $\Omega_{k+1} P$ acts on the right of $P' \backslash N(Q, P')$ by right multiplication. To see this let $a \in \Omega_{k+1} P$, $x \in N(Q, P')$, $q \in Q$ then

$$(xa)q(xa)^{-1} = (x[a, q]x^{-1})(xqx^{-1}) \tag{1}$$

where $x[a, q]x^{-1} \in \Omega_k P \leq P'$ since $\Omega_k P$ is normal in P.

SUBLEMMA 2.2. *The action of* $\Omega_{k+1} P$ *on* $P' \backslash N(Q, P')$ *has a fixed point.*

If follows that $\Omega_{k+1} P \leq P'$ which completes the induction. □

Proof of sublemma 2.2. Suppose $q \in \Omega_{j+1} Q$, $j \geq 0$. Then

$$[a, q] \in [\Omega_{k+1} P, \Omega_{j+1} Q] \leq \Omega_{\min\{k,j\}} P \leq P'$$

by induction and so $[a, q] \in \Omega_{\min\{k,j\}} P' \leq \Omega_j P'$ since $\Omega_l P \leq P'$ implies $\Omega_l P \leq \Omega_l P'$. By (1)

$$c_{xa}(q) = c_x([a, q]) \cdot c_x(q) = c_x(q) \bmod \Omega_j P'.$$

Hence

$$g \circ c_{xa}(q) = g \circ c_x(q) \bmod \Omega_j Q \tag{2}$$

since $g : P' \to Q$ restricts to $g : \Omega_j P' \to \Omega_j Q$. Thus $g \circ c_{xa}$ and $g \circ c_x$ induce the same endomorphisms of $\Omega_{j+1} Q/\Omega_j Q$. It follows that $g \circ c_{xa}$ is an automorphism iff $g \circ c_x$ is one.

Now to prove the Sublemma we consider the natural homomorphism

$$\alpha\colon \mathrm{Out}\,(Q) \to \prod_j \mathrm{Out}\,(\Omega_{j+1} Q/Q_j Q).$$

An automorphism of a p-group which stabilizes a normal series must have order a power of p [G; Cor 5.3.3]. Hence $\ker \alpha$ is a p-group and α induces an algebra homomorphism

$$\mathbb{F}_p(\alpha) : \mathbb{F}_p\,\mathrm{Out}\,(Q) \to \mathbb{F}_p\left(\prod \mathrm{Out}\,(\Omega_{j+1} Q/\Omega_j Q)\right)$$

with nilpotent kernel [HK]. However, if $\Omega_{k+1} P$ acts on $P'\backslash N(Q, P')$ without fixed points then (2) shows that each orbit contributes 0 to $\mathbb{F}_p(\alpha)\,(\bar{W}_g)$. Hence $\bar{W}_g$ is in the kernel of $\mathbb{F}_p(\alpha)$ contradicting $\bar{W}_g$ being a factor in a nonzero idempotent. □

Proof of Theorem 1.7. By Lemma 2.1, either (α) holds or (β) holds with $P' = P$ contradicting our assumption that Out (P) is a p-group and P has no non-trivial retracts. □

3. Martin's theory

In this section we recall some methods and results of U. Martin [Mn], which we use in the proof of Theorem 1.8.

If P is a p-group generated by d elements then an automorphism of P induces an automorphism of P/P_2, i.e. an element of $GL_d(\mathbb{F}_p)$. The group of automorphisms arising in this way we denote by $A(P)$. Thus there is a short exact sequence

$$1 \longrightarrow K(P) \longrightarrow \mathrm{Aut}\,(P) \longrightarrow A(P) \longrightarrow 1$$

where $K(P)$ is the subgroup of automorphisms inducing the identity on P/P_2. By a result of P. Hall [Hu; III.3.17], $K(P)$ is a p-group.

Now let $F = F(d)$ denote the free group in d generators and let $H = F/F_{n+1}$ for $n \geq 1$. Then H satisfies the following universal property: For a group G of Frattini length n, homomorphisms $H \to G$ are specified by a choice of d elements of G. From this it follows that a p-group $P \in A_{d,n}$ (see Def. 1.2) corresponds to an orbit or normal subgroups $K \lhd H$ under the action of Aut (H), i.e. $P \approx H/K$ and K's in the same orbit give rise to isomorphic P's. In fact such subgroups K are easily seen to lie in H_2.

Upon passing to the limit $d \to \infty$, more can be said. Let $B_{d,n} \subset A_{d,n}$ be the subset corresponding to subgroups of H which lie in H_n, $C_{d,n} \subset B_{d,n}$ be the subset consisting of isomorphism classes of groups with $A(P) = 1$. By definition $GL_d(\mathbb{F}_p)$ acts on the vector space H_1/H_2. This extends to an action on H_i/H_{i+1} by use of the standard commutator formulas.

THEOREM 3.1. ([Mn], Th. 2.2). *$C_{d,n}$ bijects with the set of regular orbits of $GL_d(\mathbb{F}_p)$ on subspaces of H_n.*

THEOREM 3.2. ([Mn], Th. 3.4).

$$\lim_{d \to \infty} \frac{|B_{d,n}|}{|A_{d,n}|} = 1 \tag{i}$$

$$\lim_{d \to \infty} \frac{|C_{d,n}|}{|B_{d,n}|} = 1 \tag{ii}$$

4. Proof of Theorem 1.8

We will show that for most subspaces $K \leq H_n$ the group $P = H/K$ satisfies conditions (ii) and (iii) in Theorem 1.8. Of course, these conditions are independent of the member K of a $GL_d(\mathbb{F}_p)$ orbit of subspaces of H_n and, as most orbits are regular by Theorems 3.1 and 3.2(ii), we will then see that for most orbits we have that P satisfies (ii), (iii) as well as (i) by Theorem 3.1. Then Theorem 3.2(i) will complete the proof.

We start our proof by recalling the following facts

(1) The dimension $\omega(n, d)$ of H_n (H with d-generators) is a polynomial in d of degree n with leading coefficient $1/n$. [HB, Chap. VIII, Thms. 1.9, 11.5].

(2) The number $v(k, l)$ of all l-dimensional subspaces of a k-dimensional $\mathbb{F}_p$-vector space is given by

$$v(k, l) = \frac{(p^k - 1) \cdots (p^k - p^{l-1})}{(p^l - 1) \cdots (p^l - p^{l-1})}$$

As a lower bound for the cardinality of the set $R(n, d)$ of all subspaces of H_n we use

(3) $|R(n, d)| \geq \nu(\omega(n, d), [\omega(n, d)/2])$ where $[m/2]$ denotes the greatest integer $\leq m/2$.

Now we deal with conditions (ii) and (iii) separately.

Condition (ii). We consider the set $N(n, d)$ of all subspaces $K \leq H_n$ such that $P = H/K$ has non-trivial retracts, or equivalently, such that there is a non-trivial idempotent endomorphism ρ of P with $\rho \neq \mathrm{id}$.

Given such a ρ we perform the following construction. Choose a lift of ρ to an endomorphism $\bar{\rho}$ of H which leaves K invariant.

$$\begin{array}{ccccc} K & \longrightarrow & H & \longrightarrow & P \\ \downarrow \bar{\rho} & & \downarrow \bar{\rho} & & \downarrow \rho \\ K & \longrightarrow & H & \longrightarrow & P \end{array}$$

Then $\bar{\rho}$ is also a lift of the retraction r on $V = P/P_2 \cong H/H_2$ induced by ρ. Furthermore $\bar{\rho}\,|_{H_n}$ depends only on r and is also a retraction.

To say that K is invariant under $\bar{\rho}$ is equivalent to $K = K_1 \oplus K_2$ with $K_1 \leq \bar{\rho}(H_n)$ and $K_2 \leq (\mathrm{id} - \bar{\rho})(H_n)$. Hence we get

$N(n, d) := \{K \leq H_n \mid$ there is a retraction r of V, $r \neq 0$, $r \neq \mathrm{id}$, such that $K = K_1 \oplus K_2$ with $K_1 \leq \bar{\rho}(H_n)$, $K_2 \leq (\mathrm{id} - \bar{\rho})(H_n)$ where $\bar{\rho}$ is the unique endomorphism of H_n obtained as restriction of some lift of $r\}$

To estimate $|N(n, d)|$ we use the following notation.

Denote $\dim r(V)$ by d', $\dim \bar{\rho}(H_n) = \omega(n, d')$ by ω' and $\omega - \omega'$ by ω''. The number of retractions of V with $\dim r(V) = d'$ will be denoted by $\rho(d, d')$.

Then we have

$$|N(n, d)| \leq \sum_{d'=1}^{d-1} \rho(d, d') \sum_{k_1=0}^{\omega'} \sum_{k_2=0}^{\omega''} \nu(\omega', k_1)\nu(\omega'', k_2) \tag{4}$$

Next we use

$$\begin{aligned} \rho(d, d') &= |\{\text{Retractions to a fixed subspace of dimension } d'\}| \cdot \nu(d, d') \\ &= p^{d'(d-d')}\nu(d, d') \end{aligned} \tag{5}$$

$$\nu(k, l) \leq \nu\left(k, \left[\frac{k}{2}\right]\right) \qquad \text{for all } 0 \leq l \leq k \tag{6}$$

to get

$$|N(n, d)| \leq \sum_{d'=1}^{d-1} \rho(d, d')\omega'\omega''\nu\left(\omega', \left[\frac{\omega'}{2}\right]\right)\nu\left(\omega'', \left[\frac{\omega''}{2}\right]\right)$$

$$\leq \omega^2 \sum_{d'=1}^{d-1} p^{d'(d-d')}\nu(d, d')\nu\left(\omega', \left[\frac{\omega'}{2}\right]\right)\nu\left(\omega'', \left[\frac{\omega''}{2}\right]\right)$$

$$\leq \omega^2(d-1) \max_{1 \leq d' \leq d-1} p^{d'(d-d')}\nu(d, d')\nu\left(\omega', \left[\frac{\omega'}{2}\right]\right)\nu\left(\omega'', \left[\frac{\omega''}{2}\right]\right) \tag{7}$$

To proceed we need the following

LEMMA. *For each primp p and each $\varepsilon > 0$ there is a constant $C = C(p, \varepsilon)$ such that for all positive integers $l \leq k$ we have*

$$p^{l(k-l)} \leq \nu(k, l) \leq Cp^{l(k-l)+\varepsilon k}$$

Proof. Write

$$\nu(k, l) = p^{l(k-l)} \prod_{1 \leq r \leq l} \frac{1-p^{-k+r-1}}{1-p^{-r}}$$

$$\leqslant p^{l(k-l)} \prod_{1 \leq r \leq r_0} \frac{1}{1-p^{-r}} \cdot \prod_{r_0 < r \leq l} \frac{1}{1-p^{-r}}.$$

Now choose r_0 large enough such that $1/(1-p^{-r}) \leq p^{\varepsilon}$ for all $r \geq r_0$ and put $C = \Pi_{1 \leq r \leq r_0} 1/(1-p^{-r})$. This gives the upper estimate for $\nu(k, l)$, the lower estimate is immediate. □

From (7) and the Lemma we get with a suitable new constant $C' = C'(p, \varepsilon)$

$$\log_p |N(n, d)| \leq C' + 2\log_p \omega + \log_p d$$

$$+ \max_{1 \leq d' \leq d-1} \left\{2d'(d-d') + \frac{1}{4}\omega'^2 + \frac{1}{4}\omega''^2 + \varepsilon(d + \omega' + \omega'')\right\}$$

$$\leq C' + 2\log_p \omega + \log_p d + \varepsilon(d + \omega)$$

$$+ \max_{1 \leq d' \leq d-1} \left\{\frac{1}{4}\omega^2 + \frac{1}{2}\omega'(\omega' - \omega) + 2d'(d-d')\right\}$$

while (3) and the Lemma yield

$$\log_p |R(n, d)| \geq \frac{1}{4}\omega^2.$$

Hence it suffices to show that

$$\lim_{d\to\infty} \Big\{ C' + 2\log_p \omega + \log_p d + \varepsilon(d+\omega)$$

$$+ \max_{1\leq d'\leq d-1} \Big\{ \frac{\omega'}{2}(\omega - \omega')\Big(\frac{4d'(d-d')}{\omega'(\omega-\omega')} - 1\Big)\Big\}\Big\} = -\infty. \tag{8}$$

Now we use that ω resp. ω' are polynomials in d resp. d' of degree $n \geq 2$ with positive leading coefficients. Therefore we find

$$\lim_{d\to\infty} \max_{1\leq d'\leq d-1} \frac{1}{\omega}\Big\{\frac{\omega'}{2}(\omega-\omega')\Big(\frac{4d'(d-d')}{\omega'(\omega-\omega')} - 1\Big)\Big\} = -\frac{1}{2}$$

and (8) holds if we choose $0 < \varepsilon < 1/2$. □

Condition (*iii*). We consider only $\Omega_{n+1-j}P \leq P_j$; the opposite inequality always holds since $(P_j)^{p^{n+1-j}} \leq P_{n+1} = 1$. Thus we are reduced to showing that for almost all subspaces $K \leq H_n$, if $x \in H$ and $x^{p^{n+1-j}} \in K$, then $x \in H_j$ $(1 \leq j \leq n)$.

Now if $n > 2$ there is an isomorphism of vector spaces

$$H^p \cap H_n \oplus \frac{\gamma_n(H)}{\gamma_n(H)^p \cdot \gamma_{n+1}(H)} \xrightarrow{\approx} \frac{H_n}{H_{n+1}} = H_n$$

where H^p is the subgroup generated by p-th powers and $\gamma_i(H)$ is the i-th term of the lower central series of H (this follows from [HB; Chap VIII, Th. 1.9, Lemma 1.1]; see also [BK; proof of Th. 3]). It also follows from these references that $\Omega_{n+1-j}H = H_j$. Thus it suffices to show that almost all $K \leq H_n$ satisfy $K \cap H^p = 0$.

Now we estimate the number $\mu(b, a)$ of subspaces of a b-dimensional $\mathbb{F}_p$-vector space B which intersect a given a-dimensional subspace A nontrivially by

$$\mu(b, a) \leq \nu(a, 1) \cdot \sum_{b'=0}^{b-1} \nu(b-1, b') \quad .$$

and hence by (6)

$$\mu(b, a) \leq b \cdot v(a, 1) \cdot v\left(b-1, \left[\frac{b-1}{2}\right]\right).$$

In our case we have $b = \omega$ and $a = \dim (H^p \cap H_n)$ is polynomial in d of degree $n-1$ (see [HB, Chap VIII, Thms. 1.9, 11.15).

By using the Lemma we find (again with a suitable constant $C' = C'(p, \varepsilon)$)

$$\begin{aligned}\lim_{d\to\infty} \log_p \frac{\mu(b, a)}{|R(n, d)|} &\leq \lim\left\{\log_p b + a + \frac{1}{4}(b-1)^2 + \varepsilon(b-1) - \frac{1}{4}b^2 + C'\right\}\\ &= \lim_{d\to\infty}\left\{\log_p b + a - \frac{b}{2} + \frac{1}{4} + \varepsilon(b-1) + C'\right\}\end{aligned}$$

Choosing $0 < \varepsilon < 1/2$ we see that this limit is $-\infty$ and we are done in case $n > 2$.

In case $n = 2$ any P which satisfies (ii) also satisfies (iii). That is, if $\Omega_1 P$ is not contained in P_2 then P has a nontrivial retract and we are done in this case, too. $\square$

5. Proofs of Theorem 1.9 and Proposition 1.10

Throughout this section we abbreviate $N_G(P)$ by N and $H^*(\,; \mathbb{F}_p)$ by $H^*()$.

Proof of Theorem 1.9. The composition of $i^*_{N,G}$ with the transfer $tr^*_{G,N}\colon H^*G \to H^*N$ is multiplication by $[G:N]$ and hence is mono because $[G:N]$ is prime to p.

To show surjectivity it suffices to show that the compositions

$$e^*_N\colon H^*P \xrightarrow{tr^*_{N,P}} H^*N \xrightarrow{i^*_{P,N}} H^*P$$

and

$$e^*_G\colon H^*P \xrightarrow{tr^*_{G,P}} H^*G \xrightarrow{i^*_{P,G}} H^*P$$

have the same image (note that both transfers are onto and both restrictions are mono). Clearly

$$Im e^*_G \leq Im e^*_N. \tag{1}$$

Now we consider in $A = \tilde{A}(P, P) \otimes \mathbb{F}_p$ the elements

$$e_N = tr_{N,P} \circ i_{P,N}$$

and

$$e_G = tr_{G,P} \circ i_{P,G}$$

By the double coset formula we have

$$e_N = \sum_{x \in W} c_x, \qquad (W = N/P),$$

and

$$e_G = e_N + l$$

where l is a linear combination of terms of the form $\phi \circ tr_{P,P'}$ where $P \subsetneqq P'$ and ϕ is a suitable homomorphism $P \to P'$. We will see below that l is in the nilpotent radical Rad A.

Now let $e'_G = e_G/|W|$, $e'_N = e_N/|W|$, and $l' = l/|W|$. Then because e'_N is clearly idempotent

$$(e'_N e'_G e'_N)^{p^k} = e'_N$$

if k is large. Using the idempotency of e'_N again and (1) we conclude $Im e'^*_N = Im e'^*_G$.

It remains to show that l is in Rad A. If not, then decomposing A as a direct sum of indecomposable A-modules and passing to the semisimple quotient $A/$ Rad A shows that l gives rise to a map $M_1 \to M_2$ of suitable indecomposable summands which is nontrivial on the corresponding simple modules and hence is an isomorphism. Moreover l involves only transfers to proper subgroups of P and hence by the proof of [P; Thm. 1.5] condition (β) of §2 must hold for subgroups $Q \leq P' \leq P$, $P' \neq P$ (using the language of [P], M corresponds to a dominant summand of BQ for some $Q \subsetneqq P$). However, by Lemma 2.1, (a) implies $P' = P$ in (β). This contradiction completes the proof.

Proof of Proposition 1.10. Since $W = N/P$ is a p'-group we have, by Zassenhaus' Theorem, a split short exact sequence

$$1 \longrightarrow P \longrightarrow N \longrightarrow W \longrightarrow 1$$

Since Out(P) is a p-group, W acts trivially on P. Thus $N \cong P \times W$ and so $i^*_{P,N}: H^*N \xrightarrow{\cong} H^*P$ is an isomorphism by the Künneth Theorem.

6. Examples

In this section we give two examples to illustrate the phenomena discussed in this paper.

EXAMPLE 6.1. Let p be an odd prime. Let P be the group of Frattini length 2 generated by x_1, x_2, x_3 subject to the relations

$$x_1^p = [x_1, x_2], \qquad x_2^p = [x_1, x_3][x_2, x_3], \qquad x_3^p = [x_2, x_3]$$

Then $|P| = p^6$. Heineken and Liebeck [HL, §6] showed that Aut P is a p-group (condition i)). Their argument also shows that P has no non-trivial retracts (condition ii)). By inspection $\Omega_{3-j}P = P_j$, $j = 1, 2$ (condition iii)). Thus by Theorem 1.7, $\tilde{A}(P, P)^\wedge$ is a local ring and BP is stably indecomposable.

The next example shows (a) and (b) are not necessary conditions for indecomposability.

EXAMPLE 6.2. J. Dietz [D] has shown that BP is stably indecomposable for P a non-split metacyclic p-group (i.e. P is an extension of a cyclic p-group by a cyclic p-group, but no such extension for P splits) unless P is a generalized quaternion group. The smallest such group is

$$P = \langle x, y \mid x^{p^3} = 1, y^{p^3} = x^{p^2}, yxy^{-1} = x^{p+1} \rangle$$

of order p^6 if p is odd and

$$P = \langle x, y \mid x^8 = 1, y^4 = x^4, yxy^{-1} = x^3 \rangle$$

of order 32. These groups do not satisfy condition (a).

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