

Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 69 (1994)

Artikel: The genus of the Barnes-Wall lattice.
Autor: Scharlau, Rudolf / Venkov, Boris B.
DOI: <https://doi.org/10.5169/seals-52264>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 06.12.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

The genus of the Barnes-Wall lattice

RUDOLF SCHARLAU AND BORIS B. VENKOV

1. Introduction

In this paper, we combine the general concept of the root system of a Euclidean lattice as studied in [B–S] with the “main lemma” Proposition 1 of [Ve] to obtain the full classification of all 2-elementary totally even 16-dimensional lattices of determinant 2^8 . It turns out that this genus of lattices consists of 24 isometry classes. Like in the case of 24-dimensional, even unimodular lattices, 23 of them are *reflective* in the sense that the root system has maximal rank 16. We derive a list of ‘possible’ root systems, which are subject to two essential restrictions. It turns out that for each root system in our list, there exists a unique lattice.

2. Results

By ‘Euclidean lattice’, or ‘lattice’ for short, we mean as usual a \mathbb{Z} -lattice of full rank in a rational vector space with a positive definite scalar product (x, y) . The lattice is called integral if $(x, y) \in \mathbb{Z}$ for all $x, y \in L$, and *p*-elementary, for some prime number *p*, if $pL^\# \subseteq L$. Here $L^\# = \{y \in \mathbb{Q}L \mid (x, y) \in \mathbb{Z} \text{ for all } x \in L\}$ denotes the dual lattice of *L*.

By a left upper index $\alpha \in \mathbb{Q}^+$, we denote scaling: ${}^\alpha L$ equals *L* as a lattice, but the form is multiplied by α . If *L* is *p*-elementary, then ${}^p(L^\#)$ is again (integral and) *p*-elementary. In this paper, we specialize to the case $p = 2$, and we write ${}^2(L^\#) = L^\#$ for short. We say that *L* is *totally even* if both *L* and $L^\#$ are even, i.e. $(x, x) \in 2\mathbb{Z}$ for all $x \in L$ and $(y, y) \in \mathbb{Z}$ for all $y \in L^\#$.

For a given dimension *n* and determinant $\det L = |L^\# / L| = 2^r$, all 2-elementary totally even lattices form one genus. They exist if and only if either $r \in \{0, n\}$ and $n \equiv 0(8)$, or $0 < r < n$, $r \equiv 0(2)$, $n \equiv 0(4)$. If $n = 4m$ and $r = 2m$, the mapping $L \rightarrow L^\#$ is an involution on (the isometry classes of) this genus. A $\#$ -invariant representative is $mD_4 = D_4 \perp \cdots \perp D_4$, where D_4 as usual denoted the D_4 -root lattice. Another, prominent, lattice in the genus of $4D_4$ is the Barnes-Wall lattice

$BW = BW_{16}$ which has no vector of norm 2 and again satisfies $BW \cong BW^*$. From [B-S] we recall the definition

$$R(L) := \{v \in L \mid v \text{ primitive, } 2(v, L)/(v, v) \subseteq \mathbb{Z}\}$$

of the *root system* of an arbitrary lattice L . The quadratic form is part of the structure of such root systems R . Therefore, scaling aR makes sense, and the irreducible ones are ${}^aA_l, {}^bB_l, {}^cC_l, \dots$. The lattice generated by a root system A, B, C, \dots, R is denoted by A, B, C, \dots, R . Specializing again to 2-elementary lattices, we observe, that

$$R(L) = \{v \in L \mid (v, v) = 2\} \cup \{v \in 2L^\# \mid (v, v) = 4\} =: R_{sh} \cup R_{lo}$$

(‘short’ and ‘long’ roots). Thus, the possible irreducible components of $R(L)$ are $A_l, {}^2A_l, {}^2B_l, C_l, D_l, {}^2D_l, F_4, E_6, {}^2E_6, E_7, {}^2E_7, E_8, {}^2E_8$.

By “Niemeier lattice” we mean a 24-dimensional even unimodular lattice. A full classification of this genus of lattices into 24 isometry classes has been given by Niemeier in [Nie], and was simplified in [Ve]. The purpose of this note is to announce and partially prove a classification result for the genus of $4D_4$ which is completely analogous to the classification of Niemeier lattices in the form presented in [Ve]. There are, again, precisely 24 isometry classes. We do not know if this happens incidentally. There is a natural map from our genus into the genus of Niemeier lattices which we shall introduce below, after Proposition 2. But it will become clear after the proof of Theorem 1 that this map is not bijective. In the following, h denotes the Coxeter number of an irreducible root system R . It is characterized by the following formula:

$$\sum_{v \in R} \frac{(v, x)^2}{(v, v)} = h \cdot (x, x) \quad \text{for all } x \in \mathbb{Q}R. \quad (1)$$

See e.g. [Bou], Chap. V, §6.2, p. 121. By \mathcal{G} we denote the genus consisting of all integral, 2-elementary, totally even lattices of dimensions 16 and determinant 2^8 .

THEOREM 1. *Let L be a lattice in \mathcal{G} and $R = R(L)$ its root system. Then the following properties hold:*

- (a) $R = \emptyset$ or $\text{rank } R = 16$.
- (b) If $R \neq \emptyset$, then

- (i) the Coxeter number h is the same for all irreducible components of R ,
- (ii) $|R_{sh}| = |R_{lo}|$,
- (iii) The determinant $\det R$ is a square.

THEOREM 2. *Let R be a 16-dimensional root system consisting of roots of norm 2 and 4 only. If properties (i), (ii) and (iii) of Theorem 1, part (b) hold, then there exists a lattice L in \mathcal{G} with $R(L) = R$. For a given R , the lattice L is determined uniquely up to isometry.*

The explicit classification of \mathcal{G} follows immediately from Theorems 1 and 2 and the following elementary proposition which is obtained simply by inspecting all 16-dimensional combinations of irreducible roots systems with fixed Coxeter number $h = 2, 3, 4, \dots, 16, 18, 30$.

PROPOSITION 1. *The complete list of all root systems in dimension 16, consisting of roots only of norm 2 and 4, and satisfying properties (i) and (ii) is the following:*

$$\begin{array}{llllll}
 8A_1 8^2 A_1 & 4A_2 4^2 A_2 & 8C_2 & 2A_3 2^2 A_3 2C_2 & 2A_4 2^2 A_4 & 2D_4 2^2 D_4 \\
 D_4 4^2 B_3 & 2^2 D_4 4C_3 & A_5^2 A_5 C_3^2 B_3 & 2C_4 2^2 B_4 & A_7^2 D_5^2 B_4 & 2^2 A_7 D_5 C_4 \\
 A_8^2 A_8 & D_6 2^2 B_5 & 2^2 D_6 2C_5 & 4F_4 & C_6^2 B_6 F_4 & E_6^2 E_6 F_4 \\
 D_8^2 D_8 & C_8^2 B_8 & C_9^2 E_7 & 2^2 B_9 E_7 & E_8^2 E_8 & A_3^2 A_3 5C_2.
 \end{array}$$

We observe that the additional property $|R_{sh}| = 8h$ holds for all reflective lattices in Proposition 1. This is readily derived from properties (i) and (ii).

Now recall that the Barnes-Wall lattice is the unique lattice in \mathcal{G} with minimum 4 (for a proof, see [Que], Theorem 4), and observe that only the root system $A_3^2 A_3 5C_2$ in the proposition above has a non-square determinant. The next theorem combines the results stated so far.

THEOREM 3. *The class number of the genus \mathcal{G} of $4D_4$ is equal to 24. Representatives of \mathcal{G} are the Barnes-Wall lattice and 23 lattices having the first 23 root systems listed in Proposition 1.*

The orthogonal group $O(L)$ of a reflective lattice L (i.e. $\text{rank } R(L) = \dim L$) is of the form $O(L) = W(L) \rtimes A(L)$, where $W(L)$ is the Weyl group of (the root system of) L , and $A(L)$ can be identified with a subgroup of the outer automorphism group (group of diagram automorphisms) of the root system. In the following table, we list all our lattices together with the Coxeter number h and the order of $O(L)$.

lattice	h	root system	order of $O(L)$
$L_1 = BW$	–	empty	$2^{21} \cdot 3^5 \cdot 5^2 \cdot 7$
L_2	2	$8A_1 8^2 A_1$	$2^{25} \cdot 3 \cdot 7$
L_3	3	$4A_2 4^2 A_2$	$2^{14} \cdot 3^9$
L_4	4	$8C_2$	$2^{30} \cdot 3 \cdot 7$
L_5	4	$2A_3 2^2 A_3 2C_2$	$2^{22} \cdot 3^4$
L_6	5	$2A_4 2^2 A_4$	$2^{15} \cdot 3^4 \cdot 5^4$
L_7	6	$2D_4 2^2 D_4$	$2^{27} \cdot 3^5$
L_8	6	$D_4 4^2 B_3$	$2^{25} \cdot 3^6$
$L_9 = L_8^\#$	6	$^2 D_4 4C_3$	$2^{25} \cdot 3^6$
L_{10}	6	$A_5 ^2 A_5 C_3 ^2 B_3$	$2^{17} \cdot 3^6 \cdot 5^2$
L_{11}	8	$2C_4 2^2 B_4$	$2^{30} \cdot 3^4$
L_{12}	8	$A_7 ^2 D_5 ^2 B_4$	$2^{22} \cdot 3^4 \cdot 5^2 \cdot 7$
$L_{13} = L_{12}^\#$	8	$^2 A_7 D_5 C_4$	$2^{22} \cdot 3^4 \cdot 5^2 \cdot 7$
L_{14}	9	$A_8 ^2 A_8$	$2^{15} \cdot 3^8 \cdot 5^2 \cdot 7^2$
L_{15}	10	$D_6 2^2 B_5$	$2^{26} \cdot 3^4 \cdot 5^3$
$L_{16} = L_{15}^\#$	10	$^2 D_6 2C_5$	$2^{26} \cdot 3^4 \cdot 5^3$
L_{17}	12	$4F_4$	$2^{31} \cdot 3^9$
L_{18}	12	$C_6 ^2 B_6 F_4$	$2^{27} \cdot 3^6 \cdot 5^2$
L_{19}	12	$E_6 ^2 E_6 F_4$	$2^{22} \cdot 3^{10} \cdot 5^2$
L_{20}	14	$D_8 ^2 D_8$	$2^{28} \cdot 3^4 \cdot 5^2 \cdot 7^2$
L_{21}	16	$C_8 ^2 B_8$	$2^{30} \cdot 3^4 \cdot 5^2 \cdot 7^2$
L_{22}	18	$C_9 ^2 E_7$	$2^{26} \cdot 3^8 \cdot 5^2 \cdot 7^2$
$L_{23} = L_{22}^\#$	18	$^2 B_9 E_7$	$2^{26} \cdot 3^8 \cdot 5^2 \cdot 7^2$
L_{24}	30	$E_8 ^2 E_8$	$2^{28} \cdot 3^{10} \cdot 5^4 \cdot 7^2$

Using the orders of the orthogonal groups, one can readily confirm our enumeration by the mass formula. The mass of \mathcal{G} is equal to

$$M(\mathcal{G}) = \frac{691}{2^{30} \cdot 3^{10} \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13} \cdot \frac{1}{2} \cdot 11 \cdot 13 \cdot 17^2 \cdot 31 \cdot 43 \cdot 127.$$

The first factor is the mass of even unimodular 16-dimensional lattices which can be looked up for instance in [Se], Chap. V, §2. The second factor is a correction factor coming from the determinant times the quotient of the 2-adic densities of the genera in question. It is readily derived e.g. from [Pf], Satz 1, Hilfssatz 8. The mass can also be obtained from [C-S2]. In the notation of that paper, our genus reads $1_{II}^{8+} 2_{II}^{8+}$.

3. Some proofs

If L is any integral lattice, we denote by $T(L) := L^*/L$ its discriminant group. It carries a \mathbb{Q}/\mathbb{Z} -valued symmetric bilinear form, the so-called discriminant (bilinear) form $(x + L, y + L) := (x, y) + \mathbb{Z}$, where $x, y \in L^*$. If L is even, we also have the *discriminant quadratic form* $q(y + L) = \frac{1}{2}(y, y) + \mathbb{Z}$. The integral even over-lattices M of L correspond bijectively to the subgroups $\mathcal{M} (:= M/L)$ of $T(L)$ which are totally isotropic with respect to q . The discriminant group (and form) of M can be identified with (the \mathbb{Q}/\mathbb{Z} -valued quadratic form induced by q_L on) $\mathcal{M}^\perp/\mathcal{M}$. Here, \mathcal{M}^\perp denotes the orthogonal complement of \mathcal{M} with respect to q_L (or rather its associated bilinear form). The following proposition is an immediate consequence of the remarks made so far.

PROPOSITION 2. *Let L_1, L_2 be even, integral lattices, and*

$$\varphi: (T(L_1), q_1) \rightarrow (T(L_2), -q_2)$$

be an isometry. Then

$$L_1 \times_\varphi L_2 := \{(x_1, x_2) \in L_1^* \perp L_2^* \mid \varphi \bar{x}_1 = \bar{x}_2\}$$

is an integral, even, unimodular lattice. Conversely, any even unimodular over-lattice of $L_1 \perp L_2$ can be obtained in this way.

Indeed, $\mathcal{L} := \{(\bar{x}, \varphi \bar{x}) \mid \bar{x} \in T(L_1)\} \subseteq T(L_1 \perp L_2)$ is obviously totally isotropic, and for reasons of group orders, it must coincide with \mathcal{L}^\perp . The converse is also easy.

We shall use Proposition 2 in the following special case: $L_1 = L$ is a arbitrary lattice in the genus \mathcal{G} , and $L_2 = {}^2E_8$. Then, φ as above actually exists (take for instance $L = E_8 \perp {}^2E_8$). We shall, by the way, not use the fact that the discriminant form determines the genus (see e.g. [Nik]), but only the trivial fact that it depends only on the genus. In our particular case, the “glueing map” φ is essentially unique (i.e., two possible φ ’s can be transformed into each other by an automorphism of $L_1 \times L_2$ preserving the components). This is a trivial consequence of the well-known fact that the orthogonal group $O(E_8)$ maps surjectively onto the orthogonal group of the \mathbb{F}_2 -valued quadratic form $\frac{1}{2}(x, x) + 2\mathbb{Z}$ on $E_8/2E_8$. Thus, we have a mapping $L \mapsto \tilde{L} := L \times_\varphi {}^2E_8$ from \mathcal{G} to the Niemeier lattices which is well defined on isometry classes.

Proof of Theorem 1. We begin with the equality $|R_{sh}| = |R_{lo}|$. It is readily derived from the basic observation that the long roots are in one-to-one correspondence with the short roots of L^* , taking into account the following fact: the theta

series ϑ_L and ϑ_{L^*} are modular forms for $\Gamma_0(2)$ of weight 8; the corresponding space of cusp forms is one-dimensional (see e.g. [Ra] Chap. 1.5, 7.1); the transformation formula for theta series gives a linear transformation mapping coefficients of ϑ_L onto those of ϑ_{L^*} . (Using a slightly more refined argument, Quebbemann shows that $\vartheta_L = \vartheta_{L^*}$ for any L in \mathcal{G} .)

Now assume that $R = R(L) \neq \emptyset$. Then also $R_{sh} \neq \emptyset$, and thus $\tilde{R} := R(\tilde{L}) \neq \emptyset$. Let us describe $\tilde{R} = \{v \in \tilde{L} \mid (v, v) = 2\}$ explicitly in terms of L . The vectors \tilde{v} in \tilde{L} of norm 2 are of two kinds: either $\tilde{v} \in L$, or \tilde{v} is of the form $\tilde{v} = (v_1 + v_2)$, $v_1 \in L^\#$, $v_2 \in \frac{1}{2}E_8$, $(v_1, v_1) = (v_2, v_2) = 1$, $\varphi v_1 = \bar{v}_2$.

We now have to recall the following well-known property of the E_8 -root-lattice: for any element in $E_8/2E_8$ which is anisotropic with respect to the quadratic form $\frac{1}{2}(x, x) + 2\mathbb{Z}$, there exists a representative in E_8 of norm $\frac{1}{2}(x, x) = 1$. It is unique up to sign. It follows that the roots of \tilde{L} of the second kind are of the shape $v_1 \pm v_2$, where $2v_1$ runs through the long roots of L , and $v_2 = \varphi v_1$ is a function of v_1 (and a root of E_8). We see that

$$|\tilde{R}| = |R_{sh}| + 2|R_{lo}| = 3|R_{sh}|. \quad (2)$$

Now we use the following formula from [Ve], Proposition 1:

$$\sum_{v \in R} (v, x)^2 = \frac{1}{12} (x, x) |\tilde{R}| \quad \text{for all } x \in \tilde{L}. \quad (3)$$

The above description of \tilde{R} gives

$$\sum_{v \in \tilde{R}} (v, x)^2 = \sum_{v \in R_{sh}} (v, x)^2 + 2 \sum_{v \in R_{lo}} \left(\frac{v}{2}, x\right)^2 = 2 \sum_{v \in R} \frac{(v, x)^2}{(v, v)} \quad \text{for all } x \in \tilde{L}. \quad (4)$$

Now specify an irreducible component R_i of R and choose x non-zero in R_i . Substituting (4) into (3) and comparing with (1) gives

$$h_i = \frac{1}{24} |\tilde{R}|$$

for the Coxeter number h_i of R_i . This proves (i). (In view of (2), it also proves the additional property $|R_{sh}| = 8h_i$.) \square

We observe that the Coxeter number h is equal to the Coxeter number of any component of \tilde{R} . This in particular shows that the above map from \mathcal{G} to the Niemeier lattices is not bijective. For instance, the Niemeier lattices with root

systems $4A_6$, $2A_{12}$, $2D_{12}$, A_{24} , D_{24} are not in the image, since their Coxeter numbers do not occur in our table, whereas the unique Niemeier lattice with Coxeter number 8 (its roots system is $2A_7 2D_5$) contains 3 different lattices from \mathcal{G} .

SOME CASES OF THEOREM 2. *The lattice L with $R(L) = 2C_2 2A_3 2^2 A_3$.*

The root lattice is $L_0 = 4A_1 2A_3 2^2 A_3$. Its discriminant group is of the form

$$T(L_0) = \langle g_1, \dots, g_4; g_5, g_6; \tfrac{1}{2}g_7, \tfrac{1}{2}g_8 \rangle \cong Z_2^4 \times Z_4^2 \times Z_8^2,$$

where g_1, \dots, g_4 are generators of the four copies of $T(A_1)$, and g_5, \dots, g_8 are generators of the four copies of $T(A_3)$, and $Z_m := \mathbb{Z}/m\mathbb{Z}$. By a general result of [B-S], the desired code \mathcal{L} is contained in the subgroup of $T(L_0)$ which is obtained by forgetting the scaling (observe that ${}^\alpha M^\# = \tfrac{1}{\alpha} M^\beta$, and hence $T(M) \subseteq T({}^\alpha M)$, for any M and α):

$$\mathcal{L} \subseteq T = \langle g_1, \dots, g_4; g_5, g_6; g_7, g_8 \rangle \cong Z_2^4 \times Z_4^2 \times Z_4^2.$$

Since $\det L_0 = 2^{18}$, we look for an \mathcal{L} with $|\mathcal{L}| = 2^5$. The discriminant form on T is the diagonal form

$$\text{diag} \langle \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}; \tfrac{3}{4}, \tfrac{3}{4}; \tfrac{3}{2}, \tfrac{3}{2} \rangle,$$

where the entries are actually the values of $\ell(g_i)$ of the *length function* on T which is defined as

$$\begin{aligned} \ell: T(L_0) &\rightarrow \mathbb{Q} \\ \ell(x + L_0) &= \min\{(y, y) \mid y \in x + L_0\} \quad \text{where } x \in L_0^\#. \end{aligned}$$

We shall also use the values

$$\ell(2g_5) = \ell(2g_6) = 1, \quad \ell(2g_7) = \ell(2g_8) = 2.$$

We now observe the following conditions on \mathcal{L} , where $L = L(\mathcal{L})$ denotes the inverse image of \mathcal{L} in $L_0^\#$:

$$\begin{aligned} &\text{the length function } \ell \text{ takes integral, even values on } \mathcal{L}, \\ &\text{and } l(e) > 2 \text{ for all } e \in \mathcal{L} \setminus \{0\}. \end{aligned} \tag{5}$$

This is the condition that L should be even, and all vectors of norm 2 in L should

be contained in L_0 . Since $\frac{1}{2}\ell \bmod \mathbb{Z}$ gives back the discriminant quadratic form q , the previous general condition that \mathcal{L} should be totally isotropic with respect to q , is included in condition (5).

the transpositions (12) and (34) preserve \mathcal{L} , but (13), (14), (23) do not. (6)

This comes from the fact that we want to root system $4A_1 = \{\pm a_1, \pm a_2, \pm a_3, \pm a_4\}$ to be enlarged to $2C_2 = 4A_1 \cup \{\pm a_1 \pm a_2, \pm a_3 \pm a_4\}$.

If we furthermore arrange \mathcal{L} in such a way that L becomes 2-elementary and 2-even, it follows from Theorem 1 (b) (ii) that L not only has no “new roots” of norm 2, but also no “new roots” of norm 4. In this connection observe that $L^\# / L \cong \mathcal{L}^\perp / \mathcal{L}$. For the actual calculation of $L^\#$, split L_0 as

$$L_0 = L_{0,1} \perp L_{0,2} \perp L_{0,3} = 4A_1 \perp 2A_3 \perp 2^2A_3.$$

Compute \mathcal{L}^\perp with respect to the bilinear form

$$\text{diag} \langle \frac{1}{2}, \dots, \frac{1}{2}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4} \rangle$$

on T . Then

$$L^\# = \{y_1 + y_2 + \frac{1}{2}y_3 \mid y_i \in \mathcal{L}_{0,i}^\#, \bar{y}_1 + \bar{y}_2 + \bar{y}_3 \in \mathcal{L}^\perp\},$$

and L is 2-elementary if and only if $2\bar{y}_1 + 2\bar{y}_2 + \bar{y}_3 \in \mathcal{L}$ for all $\bar{y} = \bar{y}_1 + \bar{y}_2 + \bar{y}_3 \in \mathcal{L}^\perp$. In particular, $pr_3(\mathcal{L}^\perp) \subseteq \mathcal{L}$ (projection onto the third factor).

Having collected all these conditions, it is now relatively easy to see that the following \mathcal{L} is a solution to our problem, and is unique up to outer automorphisms of the root system $2C_2 2A_3 2^2A_3$

$$\begin{aligned} \mathcal{L} = \langle e_1, e_2, e_3 \rangle, \quad \text{where } e_1 &= g_1 + g_2 + g_5 + g_6 + g_7 \\ e_2 &= g_3 + g_4 + g_5 - g_6 + g_8 \\ e_3 &= 2g_5 + g_7 - g_8. \end{aligned}$$

We finally calculate the orthogonal group $O(L) = W(L) \rtimes A(L)$. Recall that $A(L)$ can be identified with either the subgroup of $\text{Aut } T(R(L))$ preserving \mathcal{L} , or a certain subgroup of the diagram automorphisms of $R(L)$. The latter description shows that

$$A(L) \hookrightarrow Z_2 \times Di_8 \times Di_8,$$

where Di_8 is the dihedral group of order 8. We represent the automorphisms of $T \cong Z_2^2 \times Z_4^2 \times Z_4^2$ by triples of 2×2 -matrices; then the diagram automorphisms correspond to the monomial matrices with entries ± 1 . We use the notation

$$\sigma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and if π is a matrix in $\langle \sigma_0, \tau_0 \rangle \cong Di_8$, then π_i , ($i = 1, 2, 3$) denotes the corresponding element acting on the i -th factor of T . Direct verification shows that the following elements preserve \mathcal{L} :

$$\sigma = \text{id}_1 \sigma_2 \pi_3, \quad \tau = \sigma_1 \tau_2 \sigma_3, \quad \rho = \dot{\sigma}_1 (-\text{id})_2 \text{id}_3.$$

They clearly generate a group isomorphic to $Di_8 \times Z_2$. In order to show that this is all of $A = A(L)$, it suffices to show that the projection p_{12} onto the first two components of $Z_2 \times Di_8 \times Di_8$ is injective. Its kernel $H = \{(\text{id}, \text{id}, -)\} \cap A$ maps under p_3 onto a normal subgroup of $p_3 A = Di_8$. Assuming H nontrivial, we have $(\text{id}, \text{id}, -\text{id}) =: \pi \in H$. This is not the case, since $\pi e_1 - e_1 \notin \mathcal{L}$.

The lattice L with $R(L) = A_7^2 D_5^2 B_4$

We start with the necessary information about the discriminant group of A_7 . It is cyclic of order 8; in the standard coordinates $A_7 = \{(x_1, \dots, x_8) \in \mathbb{Z}^8 \mid \sum x_i = 0\}$ it consists of the residue classes

$$\begin{aligned} g_1 &\equiv \frac{1}{8}(-7, 1, 1, 1, 1, 1, 1, 1), & 2g_1 &\equiv \frac{1}{8}(-6, -6, 2, 2, 2, 2, 2, 2), \\ 3g_1 &\equiv \frac{1}{8}(-5, -5, -5, 3, 3, 3, 3, 3), & 4g_1 &\equiv \frac{1}{8}(-4, -4, -4, -4, 4, 4, 4, 4) \end{aligned}$$

and their negatives. These vectors are in fact shortest coset representatives, thus the length function has the values

$$\ell(g_1) = \frac{7}{8}, \quad \ell(2g_1) = \frac{3}{2}, \quad \ell(3g_1) = \frac{15}{8}, \quad \ell(4g_1) = 2.$$

The generator g_1 is by the way equal to the fundamental highest weight usually called $\bar{\omega}_1$; in a root basis, it equals $\frac{1}{8}(7, 6, 5, 4, 3, 2, 1)$. Writing our complete root lattice as $L_0 = A_7^2 D_5^2 4A_1$, its discriminant group is

$$T(L_0) = \langle g_1; \frac{1}{2}g_2; g_3, g_4, g_5, g_6 \rangle \cong Z_8 \times Z_8 \times Z_2^4,$$

(where g_2 is the canonical generator of $T(D_5) \cong Z_4$). As above, the desired group \mathcal{L} is contained in

$$T := \langle g_1, g_2, g_3, g_4, g_5, g_6 \rangle \cong Z_8 \times Z_4 \times Z_2^4.$$

Notice that $\ell(g_2) = \frac{5}{2}$, $\ell(2g_2) = 2$. Since $\det L_0 = 2^{14}$, we want $|\mathcal{L}| = 2^3$. Since we want $4A_1$ to be enlarged to 2B_4 , the group \mathcal{L} must be invariant under arbitrary permutations of g_3, g_4, g_5, g_6 . Having in mind this fact and the fundamental restriction $\ell(g) \equiv 0 \pmod{2}$, $\ell(g) > 2$ for all $g \in \mathcal{L} \setminus \{0\}$, we easily arrive at the group

$$\mathcal{L} = \langle 2g_1 + g_2, 2g_2 + g_3 + g_4 + g_5 + g_6 \rangle.$$

Arguing as above, one sees that $T(L)$ consists of all $\{y_1g_1 + y_2\frac{g_2}{2} + y_3g_3 + \cdots + y_6g_6\}$, where $y_1 - y_2 \equiv 0 \pmod{4}$ and $y_2 + y_3 + \cdots + y_6 \equiv 0 \pmod{2}$. It readily follows that $(v, v) \in \mathbb{Z}$ for all $v \in L^*$, and $2L^* \subseteq L$. Thus L is 2-elementary and 2-even.

The determination of the orthogonal group is trivial in this case, since $W(L)$ is only of index 4 in $\text{Aut } R(L)$, and $-\text{id} \notin W(L)$. We must have

$$O(L) = \langle -\text{id} \rangle \times W(L).$$

Indeed $O(L)$ cannot be all of $\text{Aut } R(L)$, since in this case 3D_5 would enlarge to 2B_5 .

4. Further results

A remarkable property of the genus \mathcal{G} is that, for any possible root system, the corresponding lattice does exist, and is unique (Theorem 2). This phenomenon is not completely understood. This is so even in the widely studied case of the genus of 24-dimensional unimodular lattices. In view of the fact that the proof of Theorem 2 goes case by case, we find it necessary to present, in a continuation of this paper, the explicit construction of each lattice, and a sketch of each uniqueness proof. We shall also present a different proof of the existence (and uniqueness) of each lattice L in \mathcal{G} making further use of the associated Niemeier lattice $\tilde{L} \supset L$. If we define an involution σ on $\mathbb{Q}\tilde{L}$ by $\sigma|_L = \text{id}$, $\sigma|_{L^\perp} = -\text{id}$, then σ is an isometry of \tilde{L} . We shall describe all (conjugacy classes of) involutions on Niemeier lattices such that the fixed lattice \tilde{L}^σ is 16-dimensional, 2-elementary, totally even, and $(\tilde{L}^\sigma)^\perp$ contains no roots.

We shall also list all other 2-elementary totally even lattices of dimensions $n \leq 16$ and determinant 2^{2k} , $k \leq n/4$. It is known from the work of Esselmann [Es], though not explicitly stated there, that all these genera except for the one treated here, are totally reflective (i.e., consist of reflective lattices only).

For $n \leq 12$, the lattices can be easily obtained in an ad hoc way, starting from the smallest possible k ($=0$ or 1), without paying much attention to the possible root systems. The class numbers $h(n, k)$ are:

$$h(4, 1) = h(8, 0) = h(8, 1) = h(8, 2) = 1$$

$$h(12, 1) = 2 \quad h(12, 2) = 2 \quad h(12, 3) = 3.$$

In the last case, the root systems are $3F_4$, 2B_6C_6 , $E_6{}^2E_6$.

Of course, this classification can also be derived as an immediate corollary of Theorem 3, simply by adding an orthogonal summand D_4 (with root system F_4). It is, by the way, a general fact that a root system aF_4 in an arbitrary lattice always splits off; see [B-S]. This allows us to reduce the existence and uniqueness proof for those root systems in Proposition 1 containing an F_4 -component to the 12-dimensional case.

Results completely analogous to Theorems 1 and 2 hold for the 12-dimensional genus of $6A_2$, or of the Coxeter-Todd lattice. The number of short roots equals $6h$ (in the reflective case), the root systems are $6A_16^3A_1$, $3A_23^3A_2$, $2A_32^3A_3$, $D_4{}^3D_42G_2$, $6G_2$, $A_5{}^3A_5G_2$, $A_6{}^3A_6$, $E_6{}^3E_6$, the class number is 10.

We unify and extend our results by treating the other genera of 'extremal modular' lattices considered by Quebbemann in [Que]. In that paper, a lattice L is called *modular* (of level p) if it is isometric to its p -scaled dual: $L \cong {}^p(L^\#)$. Such a lattice is in particular p -elementary of determinant $p^{n/2}$.

THEOREM 4. *Let p be one of the prime numbers 1, 2, 3, 5, 7, 11, 23 (i.e., $p+1 \mid 24$), let the dimension be equal to $n = \frac{48}{1+p} = 24, 16, 12, 8, 6, 4, 2$. For each of these pairs (p, n) , there exist a unique p -elementary even lattice of determinant at most $p^{n/2}$ and with minimum 4. All other lattices in the respective genera are reflective. For $p \geq 3$, all these lattices are modular.*

The 8-dimensional lattice with determinant 5^4 and minimum 4 is the lattice $Q_8(1)$ described in [C-S1]. The 6-dimensional lattice with determinant 7^3 and minimum 4 is the lattice $Q_6(1) = P_6$ first found by Barnes. The class numbers for $p = 5, 7, 11, 23$ are 5, 3, 3, 2, respectively. All 'possible' root systems (i.e., $|R_{sh}| = |R_{lo}|$) do occur.

Acknowledgements

Most of this work was done during a stay of the second author at the Sonderforschungsbereich 343 of the Faculty of Mathematics, University of Bielefeld,

in November and December 1992. B. Venkov thanks this institution for its hospitality. R. Scharlau's thanks go to the Max-Planck-Institut für Mathematik, Bonn, for a short invitation in May 1993 which allowed him to finish this work. Both authors are indebted to H.-G. Quebbemann for his interest and advice.

REFERENCES

- [B-S] B. BLASCHKE, R. SCHARLAU: *Reflective integral lattices*. In preparation.
- [Bou] N. BOURBAKI: *Groupes et Algèbre de Lie, Chap. IV, V et VI.*: Hermann, Paris (1968).
- [C-S1] J. H. CONWAY, N. J. A. SLOANE: *Low-dimensional lattices. II. Subgroups of $GL(n, \mathbb{Z})$* . Proc. R. Soc. Lond. **A419** (1988), 29–68.
- [C-S2] J. H. CONWAY, N. J. A. SLOANE: *Low-dimensional lattices. IV. The mass formula*. Proc. R. Soc. Lond. **A419** (1988), 259–286.
- [Es] F. ESSELMANN: *Über die maximale Dimension von Lorentz-Gittern mit coendlicher Spiegelungsgruppe*. Universität Bielefeld, Sonderforschungsbereich 343, Preprint 92-023.
- [Nie] H.-V. NIEMEIER: *Definite quadratische Formen der Dimension 24 und Diskriminate 1*. J. Number Theory **5** (1973), 142–178.
- [Nik] V. V. NIKULIN: *Integral symmetric bilinear forms and some of their geometric applications*. Math. USSR Izvestiya **14** (1980), 103–167.
- [Pf] H. PFEUFFER: *Einklassige Geschlechter totalpositiver quadratischer Formen in totalreellen algebraischen Zahlkörpern*. J. Number Theory **4** (1971), 371–411.
- [Que] H.-G. QUEBBEMANN: *Modular Lattices in Euclidean Spaces*. Universität Bielefeld, Sonderforschungsbereich 343, Preprint 93-031.
- [Ra] R. A. RANKIN: *Modular forms and functions*. Cambridge University Press (1977).
- [Se] J.-P. SERRE: *Cours d'Arithmétique*. Presses Universitaires de France (1970).
- [Ve] B. B. VENKOV: *On the classification of integral even unimodular 24-dimensional quadratic forms*. Proc. Steklov Inst. Math. **4** (1980), 63–74, also reprinted as chapter 18 in J. H. CONWAY, N. J. A. SLOANE, *Sphere packings, lattices and groups*, Springer, New York 1988.

*Fachbereich Mathematik
Universität Dortmund
D-44221 Dortmund
Germany*

*St. Petersburg's department
of the Steklov mathematical institute
Fontanka 27, St. Petersburg
Russia*

and

*Fakultät für Mathematik
Universität Bielefeld
Postfach 10 01 31
D-33501 Bielefeld
Germany*

Received September 23, 1993; March 18, 1994

Buchanzeigen

MALCOLM CLARK, **A Plain TEX Primer**, Oxford University Press Inc., 1992, 481 p.p., £19.50.

1. Introduction – 2. Getting started – 3. Do it yourself – 4. Beginning mathematics – 5. Continuing mathematics – 6. More words – 7. Commands – 8. More maths – 9. Boxing – 10. Commands #1 – 11. Matrix manipulations – 12. Pages – 13. Tables by tabs – 14. Tables again – 15. Rules – 16. Further rules – 17. Graphics – 18. Fonts – 19. More detailed fonts – 20. Making pages – 21. Breaking up – 22. Delays and deferments – 23. Collections – 24. Last words – A. Fonts – B. Annotated bibliography and references – C. Resources – D. Solutions to the exercises – Index.

ANATOLIJ A. KARATSUBA, **Basic Analytic Number Theory**, Springer Verlag, 1993, 222 p.p., DM 148.-.

I. Integer points – II. Entire functions of finite order – III. The Euler gamma function – IV. The Riemann zeta function – V. The connection between the sum of the coefficients of a Dirichlet series and the function defined by this series – VI. The method of I. M. Vinogradov in the theory of the zeta function – VII. The density of the zero of the zeta function and the problem of the distribution of prime numbers in short intervals – VIII. Dirichlet L-functions – IX. Prime numbers in arithmetic progressions – X. The Goldbach conjecture – XI. Waring's problem – Hints for the solution of the exercises – Table prime number < 4070 and their smallest primitive roots – Bibliography – Subject index.

M. WALDSCHMIDT, P. MOUSSA, J.-M. LUCK and C. ITZYKSON, editors, **From Number Theory to Physics**, Springer Verlag, 1992, 690 p.p., DM 128.-.

1. An Introduction to Zeta Functions, P. Cartier – 2. Introduction to Compact Riemann Surfaces, Jacobians and Abelian Varieties, J.-B. Bost – 3. Elliptic Curves, H. Cohen – 4. Introduction to Modular Forms, J. Zagier – 5. Decorated Elliptic Curves: Modular Aspects, R. Gergondey – 6. Galois Theory, Algebraic Number Theory and Zeta Functions, H. M. Stark – 7. Galois Theory for Coverings and Riemann Surfaces, E. Reyssat – 8. Differential Galois Theory, F. Beukers – 9. p -adic Numbers and Ultrametricity, G. Christol – 10. Introduction to Lattice Geometry, M. Senechal – 11. A Short Introduction to Quasicrystallography, A. Katz – 12. Gap Labelling Theorems for Schrödinger Operators, J. Bellissard – 13. Circle Maps: Irrationally Winding, P. Cvitanovic – 14. An Introduction to Small Divisors Problems, J. C. Yoccoz – Index.

O. D. TSERETELI, editor, **Integral Operators and Boundary Properties of Functions**. Fourier Series, Nova Science Publishers, Inc, 1992, \$84.-.

Two-Weight Norm Inequalities for Fractional Maximal Functions, M. A. Gabidzashvili – Holomorphic Functions with a Maximum Modulus of Arbitrarily Slow Growth, M. I. Gvaradze – On the Mixed Partial Derivatives of Poisson Integral, O. P. Dzagnidze – On the Differentiation of Integrals by Bases of Convex Sets, T. Sh. Zerekidze – On a Relation Between Strong Differentiability of Integrals and Convergence of Multiple Fourier-Harr Series, T. S. Zerekidze – Weighted Norm Inequalities for Maximal Functions and Fractional Integrals in Lorentz Spaces, V. M. Kokilashvili – On the Divergence of Fourier Series with Respect to General Multiplicative Systems, Sh. V. Khkadze – A Modification of the Hardy-Littlewood Maximal Function, L. N. Chikvinidze – Topological Characterization of a Set of Continuous Functions Whose Conjugates are Continuous and of Bonded Variation, G. T. Oniani – Subject Index.

BERNARD DACOROGNA, **Introduction au Calcul des Variations**, Presses Polytechniques et Universitaires Romandes, 1992, 213 p.p..

0. Introduction – 0.1 Bref historique – 0.2 Problème modèle et des exemples – 0.3 Méthodes de résolution – 1. Préliminaires – 1.0 Introduction – 1.1 Espaces de fonctions continues et hölderiennes –