

**Zeitschrift:** Commentarii Mathematici Helvetici  
**Herausgeber:** Schweizerische Mathematische Gesellschaft  
**Band:** 69 (1994)  
  
**Artikel:** Liouville integrability of geometric variational problems.  
**Autor:** Langer, J. / Singer, D.  
**DOI:** <https://doi.org/10.5169/seals-52260>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 05.02.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

# Liouville integrability of geometric variational problems

J. LANGER AND D. SINGER

## §0. Introduction

The Betchov–Da Rios equation, otherwise known as the “localized induction equation” (LIE),  $\partial\gamma/\partial t = \partial\gamma/\partial s \times \partial^2\gamma/\partial s^2$ , is a soliton equation for space curves  $\gamma(s, t)$ , best known as a model for the behavior of thin vortex tubes in an incompressible, inviscid, three-dimensional fluid ([2], [3], [6], [17], [27]). As a completely integrable Hamiltonian system, LIE possesses infinitely many conserved quantities, all of them integrals involving the curvature  $k$  and torsion  $\tau$  of  $\gamma$ . The first five integrals in the LIE hierarchy are:  $\mathcal{J}_1 = \int ds$ ,  $\mathcal{J}_2 = \int \tau ds$ ,  $\mathcal{J}_3 = \int \frac{1}{2}k^2 ds$ ,  $\mathcal{J}_4 = \int k^2\tau ds$ , and  $\mathcal{J}_5 = \int \frac{1}{2}(k')^2 + \frac{1}{2}k^2\tau^2 - \frac{1}{8}k^4 ds$ . These integrals may be used to define variational problems for curves  $\gamma(s)$ , and the resulting equilibria provide initial conditions  $\gamma(s, 0) = \gamma(s)$  for soliton solutions to LIE.

Of course, equilibria for  $\mathcal{J}_1$  are just geodesics; it turns out that equilibria for linear combinations of  $\mathcal{J}_1$  and  $\mathcal{J}_2$  are helices, and equilibria for combinations of  $\mathcal{J}_1$ ,  $\mathcal{J}_2$ , and  $\mathcal{J}_3$  are “Kirchhoff elastic rods” (as shown in [21]). The connection between elastic curves and solitons for LIE was the original discovery of Hasimoto [9, 10], which sparked a whole series of papers on LIE, including [5], [8], [12], [15], [16], [18], [25], and [28].

Here we derive classical Hamiltonian systems with  $n = 6, 7$ , or  $8$  degrees of freedom from variational problems for  $\mathcal{J}_3$ ,  $\mathcal{J}_4$ , and  $\mathcal{J}_5$ , and show that these Hamiltonian systems are Liouville integrable in the sense that there exist  $n$  constants of motion in involution. Our results are for curves  $\gamma$  in  $M$ , any of the three-dimensional space forms,  $\mathbb{R}^3$ ,  $S^3$ , or  $\mathbb{H}^3$ . In each case, five constants exist “automatically” due to the nature of the underlying spaces and the fact that each  $\mathcal{J}_j$  is a “geometric” functional; the proof then comes down to exhibiting one, two, or three additional constants arising from the special nature of the integrals  $\mathcal{J}_j$ . (The full result for the elastic rod was first proved in [21].)

We conclude the introduction with a few comments:

(i) In the general theory of Hamiltonian systems, Liouville integrability tends to be regarded as the most satisfactory possible situation, since it implies the dynamics can be described simply as “linear flow on a torus” and – at least in principle – the

trajectories can be represented explicitly in terms of quadratures and algebraic operations (see [1]).

However, the integrability results presented here are hardly the end of the story; in fact, we regard them more as “existence results” – invitations to actually integrate the equations for these problems using their special structure, not attempting to use the general theory. We believe this view is amply justified by numerous papers on elastic curves and rods, e.g., [4], [11], [13], [19], [20], [21], [23], [24].

(ii) Our results are related to some general phenomena in soliton theory. For example, it is shown in [26] that complete integrability of the KdV equation implies Liouville integrability of the associated ‘stationary problems for higher order KdV’ regarded as finite-dimensional Hamiltonian systems. A more abstract approach to this result is given in [29, 30], where the key ingredient is the existence of a one-parameter family of Bäcklund transformations. Our own approach is “from the bottom up”, emphasizing the geometry of curves in space forms, rather than the general machinery of soliton theory, and it is not obvious how our concrete computations are related to the above-mentioned arguments. It should be noted that the geometric setting of LIE leads to some interesting special features not shared by KdV, including the fact that singular problems abound in the LIE hierarchy, and the fact that a fundamental role is played by non-canonical coordinates on the symplectic manifold  $T^*F$ .

(iii) We have tried to keep the “set-up” as brief as possible; more background is given in [22]. Following Jurdjevic [11], we have obtained our Hamiltonian systems with the aid of the Pontrjagin Maximum Principle. Another approach to such variational problems with non-holonomic constraints is the theory of exterior differential systems, as explained in [4].

## §1. Geometric Hamiltonian systems

Though the constructions of this section are naturally formulated in the general Riemannian setting ([14], [31]), it will be convenient to specialize at once to the case of a three dimensional Riemannian manifold  $M$ . Since the orthonormal frame bundle  $F$  of  $M$  is six-dimensional, our ‘core’ symplectic manifold  $T^*F$ , the cotangent bundle of  $F$ , is twelve-dimensional. We will construct Hamiltonians on  $T^*F$  (or  $T^*(F \times R^k)$ ) from geometric variational problems for curves  $\gamma(t)$  in  $M$ .

Since  $SO(3)$  acts on  $F$  by rotation of frames, the standard basis for  $so(3)$  determines three ‘fundamental vectorfields’  $A_j$  on  $F$ , the corresponding infinitesimal generators. Complementary to these are three ‘basic vectorfields’  $B_i$  defined, at each point (frame)  $f = (f_1, f_2, f_3) \in F$ , using the Riemannian connection on  $M$ :  $B_i$  is the unique horizontal vector projecting to  $f_i$ . The vertical vectorfields  $A_j$  and horizontal

vectorfields  $B_i$  display  $TF$  as a trivial bundle, and the corresponding ‘linear Hamiltonians’ provide half of a (non-canonical) coordinate system on  $T^*F$ :  $\mathcal{A}_j(p) = p(A_j)$ ,  $\mathcal{B}_i(p) = p(B_i)$ ,  $p$  any covector on  $F$

The (generalized) Frenet System for lifts of curves in  $M$  to curves  $f(t)$  in  $F$  may now be written:

$$\frac{df}{dt} = B_1(f) + k_1 A_1(f) - k_3 A_2(f) + k_2 A_3(f). \quad (\text{FS})$$

A solution  $f(t)$  to (FS) represents an ‘adapted frame’ along  $\gamma$  in the sense that  $\gamma' = T = f_1$ . To mention two useful special cases,  $f(t)$  corresponds to the ‘standard’ (Frenet) frame  $f = (T, N, B)$  in case  $k_3 = 0$  (and consequently  $k_1 = \tau$  and  $k_2 = k$ ), while  $f = (T, f_2, f_3)$  is a ‘natural frame’ if  $k_1 = 0$ . The relationship between standard and natural frames and invariants can be written:  $k_2 + ik_3 = k e^{i\vartheta}$ ,  $f_2 + if_3 = (N + iB) e^{i\vartheta}$ , where  $\vartheta' = \tau$ . While the standard frame and invariants are uniquely determined along a given curve  $\gamma$  (with non-vanishing curvature), any  $\gamma$  has a circle’s worth of natural frames and invariants associated to it.

In most of our applications of the Pontrjagin Maximum Principle, the invariants  $\kappa = (k_1, k_2, k_3)$  will play the roles of the controls – so (FS) will be regarded as a control system – and the cost function will be a geometric Lagrangian of the form  $\mathcal{L}(\kappa)$ . It is then an optimal control problem is to minimize the cost,  $\int \mathcal{L}(\kappa) ds$ , among solutions to (FS) satisfying given endpoint conditions  $f(0) = f_0, f(\ell) = f_\ell$ .

In this context, the recipe of the maximum principle (for finding smooth, regular, optimal trajectories, in the non-singular case) can now be summarized roughly as follows:

(i) Define a time-dependent Hamiltonian on  $T^*F$ , depending on controls  $\kappa(s)$ , by subtracting the cost function  $\mathcal{L}(\kappa(s))$  from the linear (time-dependent) Hamiltonian associated with the right hand side of (FS) (i.e., just replace  $A, B$  by  $\mathcal{A}, \mathcal{B}$ ):

$$\mathcal{H}(p, \kappa) = \mathcal{B}_1(f) + k_1 \mathcal{A}_1(f) - k_3 \mathcal{A}_2(f) + k_2 \mathcal{A}_3(f) - \mathcal{L}(\kappa),$$

(ii) Obtain a time-independent Hamiltonian  $\mathcal{H}(p) = \mathcal{H}(p, \kappa(p))$  on  $T^*F$  by maximizing  $\mathcal{H}(p, \kappa)$  with respect to the controls; for our purposes, this will amount to solving the equations  $\partial \mathcal{H} / \partial k_j = 0$  for uniquely determined  $\kappa = \kappa(p)$ .

(iii) Solve the Hamiltonian system on  $T^*F$  determined by  $\mathcal{H}(p)$ ; the trajectories  $p(t)$  of this system project to the solutions  $f(t)$  to the optimal control problem (for various boundary conditions).

We illustrate the procedure using the Lagrangian for the Kirchhoff elastic rod:  $\mathcal{L}(\kappa) = \alpha/2((k_2)^2 + (k_3)^2) + \beta/2(k_1)^2$ , where  $\alpha$  and  $\beta$  are positive constants. The first term here corresponds to the “bending energy” of the rod, while the second term



gives the “twisting energy”. (There is no “stretching energy” term, since the rod is assumed to be inextensible.) According to the Kirchhoff model, the rod in equilibrium solves the variational problem: minimize  $\int \mathcal{L}(\kappa) ds$  among rods of length  $\ell$  with clamped ends,  $f(0) = f_0, f(\ell) = f_\ell$  (note that  $f(s)$  includes the position  $\gamma(s)$ ). Thus, we take as our time-dependent Hamiltonian

$$\mathcal{H}(p, \kappa) = \mathcal{B}_1 + k_1 \mathcal{A}_1 - k_3 \mathcal{A}_2 + k_2 \mathcal{A}_3 - \left( \frac{\alpha}{2} ((k_2)^2 + (k_3)^2) + \frac{\beta}{2} (k_1)^2 \right).$$

Setting  $0 = \partial \mathcal{H} / \partial k_i$ ,  $i = 1, 2, 3$ , we read off  $k_1 = \mathcal{A}_1 / \beta$ ,  $k_2 = \mathcal{A}_3 / \alpha$ , and  $k_3 = -\mathcal{A}_2 / \alpha$ . Substitution gives the time independent Hamiltonian

$$\mathcal{H}(p) = \mathcal{B}_1 + \frac{(\mathcal{A}_2)^2 + (\mathcal{A}_3)^2}{2\alpha} + \frac{(\mathcal{A}_1)^2}{2\beta}.$$

The integrability of this Hamiltonian system is discussed in the next section.

## §2. Integrable Hamiltonian systems

To discuss integrability, we now specialize to three-dimensional space forms,  $M = \mathbb{R}^3$ ,  $M = S^3$ , or  $M = \mathbb{H}^3$ , and let  $\sigma$  be the curvature of  $M$ :  $\sigma = 0, 1$ , or  $-1$ . In this case, the orthonormal frame bundle  $F$  may be identified with the isometry group  $G$  of  $M$ :  $G = E(3)$ ,  $G = SO(4)$ , or  $G = SO(3, 1)$ .

Further, the  $A_j$  and  $B_i$  make up a basis of left invariant vectorfields on  $G$  satisfying the standard Lie bracket relations for the Lie algebra  $\mathfrak{g}$  of  $G$ :  $[A_i, A_j] = \epsilon_{ijk} A_k$ ,  $[A_i, B_j] = \epsilon_{ijk} B_k$ , and  $[B_i, B_j] = \sigma \epsilon_{ijk} A_k$ , where  $\epsilon_{123} = 1$ , and  $\epsilon_{ijk}$  is antisymmetric in  $i, j, k$ . The corresponding Poisson brackets of left-invariant linear Hamiltonians,  $\mathcal{A}_j, \mathcal{B}_i$ , can now be read off, using the following general fact: if  $V$  and  $W$  are two vectorfields on a Riemannian manifold  $G$ , and if  $\{ , \}$  denotes the canonical Poisson bracket on  $T^*G$ , then the bracket of the corresponding linear Hamiltonians  $\mathcal{H}_V$  and  $\mathcal{H}_W$  is given by  $\{\mathcal{H}_V, \mathcal{H}_W\} = \mathcal{H}_{[V, W]}$ . Thus, part of the Poisson structure of  $T^*G$  – the only part we will ever need in our computations – is given by:

$$\{\mathcal{A}_i, \mathcal{A}_j\} = -\epsilon_{ijk} \mathcal{A}_k, \quad \{\mathcal{A}_i, \mathcal{B}_j\} = -\epsilon_{ijk} \mathcal{B}_k, \quad \text{and} \quad \{\mathcal{B}_i, \mathcal{B}_j\} = -\sigma \epsilon_{ijk} \mathcal{A}_k.$$

Using these bracket formulas, it is an easy computation to verify the following key

**LEMMA.** *The quadratic Hamiltonians  $\mathcal{P} = \mathcal{A}_1 \mathcal{B}_1 + \mathcal{A}_2 \mathcal{B}_2 + \mathcal{A}_3 \mathcal{B}_3$  and  $\mathcal{Q} = \mathcal{B}_1^2 + \mathcal{B}_2^2 + \mathcal{B}_3^2 + \sigma(\mathcal{A}_1^2 + \mathcal{A}_2^2 + \mathcal{A}_3^2)$  lie in the center of the Poisson–Lie algebra  $\mathcal{L}G$  of left-invariant functions on  $T^*G$ .*

Recall that if  $\mathcal{H}$  defines a Hamiltonian system, and if  $\mathcal{K}$  is any other Hamiltonian, then the time evolution of  $\mathcal{K}$  is given by  $d\mathcal{K}/dt = \{\mathcal{K}, \mathcal{H}\}$ . Using this, the above lemma will provide two “free” constants of motion,  $\mathcal{P}$ ,  $\mathcal{Q}$ , for all of our Hamiltonian systems. Two more constants will come from the symmetry of  $M$ , a la Noether’s Theorem. Specifically, if  $X$  is a right-invariant vectorfield on  $G$  (such a vectorfield projects to a Killing field on  $M$ ), and if  $\mathcal{R} = \mathcal{H}_X$  is the corresponding right-invariant linear Hamiltonian, then  $\{\mathcal{R}, \mathcal{A}_j\} = -\mathcal{H}_{[X, \mathcal{A}_j]} = 0$  and  $\{\mathcal{R}, \mathcal{B}_j\} = 0$ ,  $j = 1, 2, 3$ . Therefore  $\mathcal{R}$  Poisson-commutes with any element of  $\mathcal{L}G$ . Note that we can always choose two such right-invariant linear Hamiltonians  $\mathcal{R}_1, \mathcal{R}_2$ , satisfying  $\{\mathcal{R}_1, \mathcal{R}_2\} = 0$ . Thus, any  $\mathcal{H} \in \mathcal{L}G$  which is independent of  $\mathcal{P}$  and  $\mathcal{Q}$  determines a Hamiltonian system on  $T^*G$  with five constants of motion in involution. This is precisely the situation in the first three of the following examples; for Liouville integrability, it will therefore suffice, in each case, to discover one additional constant of motion  $\mathcal{C} \in \mathcal{L}G$ .

#### EXAMPLE 1. Elastic rods and curves

For rods, we have already obtained the Hamiltonian

$$\mathcal{H}(p) = \mathcal{B}_1 + \frac{(\mathcal{A}_2)^2 + (\mathcal{A}_3)^2}{2\alpha} + \frac{(\mathcal{A}_1)^2}{2\beta}.$$

One readily checks that  $\{\mathcal{A}_1, \mathcal{H}\} = 0$ , so  $\mathcal{C} = \mathcal{A}_1$  is a constant of motion (which reflects the symmetry of  $\mathcal{H}(p)$  relative to the coordinates  $\mathcal{A}_2$  and  $\mathcal{A}_3$ ). Therefore the Kirchhoff system is Liouville integrable (for more details see [21], [22]).

The Lagrangian for elastic curves is defined by the “bending energy”  $\mathcal{L} = k^2/2$ ; in other words, it is the special case  $\beta = 0$  (and  $\alpha = 1$ ) of the rod Lagrangian. However, the above optimal control problem for the rod is singular in the case  $\beta = 0$  – one cannot solve for  $k_1$  in the equations  $\partial\mathcal{H}/\partial k_i = 0$ . This technical difficulty can be finessed by working with natural frames, thus reducing the number of controls to two. Setting  $k_1 = 0$  in (FS) leads to the time-dependent Hamiltonian

$$\mathcal{H}(p, \kappa) = \mathcal{B}_1 + k_2 \mathcal{A}_3 - k_3 \mathcal{A}_2 - \frac{(k_2)^2 + (k_3)^2}{2}.$$

The usual procedure now gives

$$k_2 = \mathcal{A}_3, \quad k_3 = -\mathcal{A}_2, \quad \mathcal{H}(p) = \mathcal{B}_1 + \frac{(\mathcal{A}_2)^2 + (\mathcal{A}_3)^2}{2},$$

and again  $\mathcal{C} = \mathcal{A}_1$  is constant of motion. (Actually, it turns out that only the solutions with  $\mathcal{C} = 0$  give elastic curves; solutions with  $\mathcal{C} \neq 0$  are byproducts of the fact that  $f(0) = f_0, f(\ell) = f_\ell$  “overdetermines” a natural framing.)

EXAMPLE 2.  $\mathcal{J}_4 = \int k^2 \tau \, ds$ .

For this problem we work with standard frames, since general frames and natural frames both lead to singular problems (on a higher dimensional space). Thus,  $\mathcal{H}(p, \kappa) = \mathcal{B}_1 + k\mathcal{A}_3 + \tau\mathcal{A}_1 - k^2\tau$ . This time, maximizing  $\mathcal{H}$  with respect to the controls,  $k$  and  $\tau$ , gives  $\mathcal{A}_1 = k^2, \tau = \mathcal{A}_3/2k$ . Eliminating  $k$  and  $\tau$  results in a rather surprising Hamiltonian:  $\mathcal{H}(p) = \mathcal{B}_1 + \mathcal{A}_3\sqrt{\mathcal{A}_1}$ . Using the Poisson bracket formulas, it can be checked that  $\mathcal{C} = \mathcal{A}_1^2 + \mathcal{A}_2^2 + \mathcal{A}_3^2 - 4\sqrt{\mathcal{A}_1}\mathcal{B}_3 - 4\sigma\mathcal{A}_1$  is a constant of motion.

EXAMPLE 3.  $\tau$ -elastic curves

Here we consider the Lagrangian  $\mathcal{L} = k^2/2$  and the control system of example 2, except that we constrain the torsion to equal a fixed constant  $\tau = c$  (so there is only one control  $k$ ). This leads to the Hamiltonian  $\mathcal{H}(p) = \mathcal{B}_1 + c\mathcal{A}_1 + (\mathcal{A}_3)^2/2$ , and it can be checked that  $\mathcal{C} = \mathcal{B}_3 - c\mathcal{A}_3$  is a constant of motion, in the special case  $\sigma = 1, c = \pm 1$ .

EXAMPLE 4.  $\mathcal{J}_5 = \int \frac{1}{2}(k')^2 + \frac{1}{2}k^2\tau^2 - \frac{1}{8}k^4 \, ds$  with natural frames.

In terms of the natural frame, this functional can be rewritten as

$$\mathcal{J}_5 = \int \frac{1}{2}\dot{k}_2^2 + \frac{1}{2}\dot{k}_3^2 - \frac{1}{8}(k_2^2 + k_3^2)^2 \, ds.$$

In order to define a control problem, we enlarge our space to  $T^*(E \times \mathbb{R}^2)$ . The curvatures  $k_2$  and  $k_3$  are now space variables, with corresponding momenta  $\mathcal{K}_2$  and  $\mathcal{K}_3$ . The controls,  $u_2$  and  $u_3$ , correspond to the derivatives of the curvatures. The Poisson brackets in the new space are simple to compute; the old functions lift to functions on the product, which Poisson commute with the functions  $k_i$  and  $\mathcal{K}_i$ . We also need the brackets  $\{k_i, \mathcal{K}_j\} = \delta_{ij}$ .

Following the usual procedure, we write the Hamiltonian

$$\mathcal{H}(p, u) = \mathcal{B}_1 + k_2 \mathcal{A}_3 - k_3 \mathcal{A}_2 + u_2 \mathcal{K}_2 + u_3 \mathcal{K}_3 - \frac{1}{2}(u_2^2 + u_3^2) + \frac{1}{8}(k_2^2 + k_3^2)^2$$

Set  $\partial \mathcal{H} / \partial u_2 = 0 = \mathcal{K}_2 - u_2$ ,  $\partial \mathcal{H} / \partial u_3 = 0 = \mathcal{K}_3 - u_3$ , and get

$$\mathcal{H}(p) = \mathcal{B}_1 + k_2 \mathcal{A}_3 - k_3 \mathcal{A}_2 + \frac{1}{2}(\mathcal{K}_2^2 + \mathcal{K}_3^2) + \frac{1}{8}(k_2^2 + k_3^2)^2.$$

Note that  $\mathcal{H}$  is no longer a left-invariant Hamiltonian; however, it still commutes with  $\mathcal{P}$ ,  $\mathcal{Q}$ ,  $\mathcal{R}_1$ , and  $\mathcal{R}_2$ . Note also,  $\mathcal{H}$  defines a system with eight degrees of freedom, so we must find three more first integrals, and they all must Poisson-commute with each other (and with the other functions).

One can check that the following functions satisfy these conditions:

$$\mathcal{C}_1 = \mathcal{A}_1 + k_3 \mathcal{K}_2 - k_2 \mathcal{K}_3.$$

$$\mathcal{C}_2 = k_3 \mathcal{B}_2 - k_2 \mathcal{B}_3 + \mathcal{A}_2 \mathcal{K}_2 + \mathcal{A}_3 \mathcal{K}_3 + \frac{1}{2}(k_2^2 + k_3^2) \mathcal{A}_1 - \sigma \mathcal{A}_1$$

$$\mathcal{C}_3 = \mathcal{B}_2 \mathcal{K}_2 + \mathcal{B}_3 \mathcal{K}_3 + \frac{1}{2}(k_2^2 + k_3^2 - 2\sigma) \mathcal{B}_1 - \sigma(k_2 \mathcal{A}_3 - k_3 \mathcal{A}_2)$$

The required computations can be organized using the notation,  $\mathcal{B}_j = \varphi \mathcal{A}_j$ ,  $\varphi^2 = \sigma$ , which allows us to regard the equations for the Poisson brackets  $\{\mathcal{A}_j, \mathcal{B}_i\}$  and  $\{\mathcal{B}_j, \mathcal{B}_i\}$  as formal consequences of the equations for  $\{\mathcal{A}_i, \mathcal{A}_j\}$ . In fact, we can write everything in terms of the  $\mathcal{A}_j$ 's (no  $\mathcal{B}_j$ 's), and then  $\mathcal{C}_2$  and  $\mathcal{C}_3$  can be written more compactly as  $\mathcal{C}_2 = \mathcal{G}_j \mathcal{A}_j$ ,  $\mathcal{C}_3 = \varphi \mathcal{C}_2$ , where we have introduced "complex coefficients":  $\mathcal{G}_1 = k^2/2 - \sigma$ ,  $\mathcal{G}_2 = \mathcal{K}_2 + \varphi k_3$ , and  $\mathcal{G}_3 = \mathcal{K}_3 - \varphi k_2$ . With this notation, e.g., one can make the trivial computation  $\{\mathcal{C}_2, \mathcal{C}_3\} = \{\mathcal{C}_2, \varphi \mathcal{C}_2\} = \varphi \{\mathcal{C}_2, \mathcal{C}_2\} = 0$ ; further, once one has checked that  $\mathcal{C}_2$  is a constant of motion, it follows at once that  $\mathcal{C}_3$  is also constant. We caution the reader that although  $\varphi$  may be treated as a scalar in the Lie algebra of linear Hamiltonians with "complex coefficients" (to which  $\mathcal{C}_2$  and  $\mathcal{C}_3$  belong), it may not be treated as such in the Poisson–Lie algebra  $\mathcal{LG}$  of functions.

Finally, we note the relationship between the  $\mathcal{G}_i$  and the vectorfields in the LIE hierarchy (see [18]). In terms of natural frames, the first and second vectorfields in LIE may be written:

$$X_1 = -k_3 f_2 + k_2 f_3, \quad \text{and} \quad X_2 = (k^2/2 - \sigma)T + \mathcal{K}_2 f_2 + \mathcal{K}_3 f_3.$$

Then

$$X_2 - \varphi X_1 = \mathcal{G}_i f_i.$$

EXAMPLE 5.  $\mathcal{J}_5 = \int \frac{1}{2}(k')^2 + \frac{1}{2}k^2\tau^2 - \frac{1}{8}k^4 ds$  with standard frames.

The same variational problem may be set up as a nonsingular control problem using the standard frame. In this case we must introduce one new space variable  $k$  and the corresponding momentum  $\mathcal{K}$ , with  $\{k, \mathcal{K}\} = 1$ . This gives rise to a Hamiltonian system with seven degrees of freedom on the space  $T^*(E \times \mathbb{R})$ . The Hamiltonian is:

$$\mathcal{H}(p) = \mathcal{B}_1 + k\mathcal{A}_3 + \frac{\mathcal{A}_1^2}{2k^2} + \frac{1}{2}\mathcal{K}^2 + \frac{1}{8}\mathcal{K}^4$$

In this set-up we require two constants of motion in involution for integrability. One can check that the desired functions are:

$$\mathcal{C} = \frac{k^2}{2}\mathcal{A}_1 + \frac{1}{k}\mathcal{A}_1\mathcal{A}_3 - k\mathcal{B}_3 + \mathcal{A}_2\mathcal{K} - \sigma\mathcal{A}_1$$

$$\mathcal{D} = \frac{k^2}{2}\mathcal{B}_1 + \frac{1}{k}\mathcal{A}_1\mathcal{B}_3 - \sigma k\mathcal{A}_3 + \mathcal{B}_2\mathcal{K} - \sigma\mathcal{B}_1$$

## REFERENCES

- [1] V. ARNOL'D, *Mathematical Methods of Classical Mechanics*, Springer-Verlag, 1978.
- [2] G. K. BATCHELOR, *An Introduction to Fluid Dynamics*, Cambridge University Press, 1967.
- [3] R. BETCHOV, *On the curvature and torsion of an isolated vortex filament*, J. Fluid Mech. 22 (1965), 471–479.
- [4] R. BRYANT and P. GRIFFITHS, *Reduction for constrained variational problems and  $\frac{1}{2}\int k^2 ds$* , Am. J. Math. 108 (1986), 525–570.
- [5] J. CIESLINSKI, P. GRAGERT, and A. SYM, *Exact solution to localized-induction-approximation equation modeling smoke ring motion*, Phys. Rev. Lett. 57 (1986), 1507–1510.
- [6] L. S. DA RIOS, *On the motion of an unbounded fluid with a vortex filament of any shape* (Italian), Rend. Circ. Mat. Palermo 22 (1906), 117.
- [7] X. GENG, *A new hierarchy of nonlinear evolution equations and corresponding finite-dimensional completely integrable systems*, Phys. Lett. A. 162 (1992), 375–380.
- [8] R. GOLDSTEIN and D. PETRICH, *The Korteweg–de Vries hierarchy as dynamics of closed curves in the plane*, Phys. Rev. Lett. 67 (1991), 3203–3206.
- [9] H. HASIMOTO, *Motion of a vortex filament and its relation to elastica*, J. Phys. Soc. Japan 31 (1971), 293–294.
- [10] H. HASIMOTO, *A soliton on a vortex filament*, J. Fluid Mech. 51 (1972), 477–485.
- [11] V. JURDJEVIC, *Non-Euclidean elastica*, (preprint).
- [12] S. KIDA, *A vortex filament moving without change of form*, J. Fluid Mech. 112 (1981), 397–409.

- [13] P. S. KRISHNAPRASAD and J. E. MARSDEN, *Hamiltonian structures and stability for rigid bodies with flexible attachments*, Arch. Rat. Mech. Anal. 98 (1987), 71–93.
- [14] S. KOBAYASHI and K. NOMIZU, *Foundations of Differential Geometry*, vol. 1, Wiley-Interscience, 1963.
- [15] G. L. LAMB, JR., *Solitons and the motion of Helical curves*, Phys. Rev. Lett. 37 (1976), 235–237.
- [16] G. L. LAMB, JR., *Solitons on moving space curves*, J. Math. Phys. 18 (1977), 1654–1661.
- [17] T. LEVI-CIVITA, *On the attraction produced by a material line at neighboring points* (Italian), Rend. R. Acc. Lincei 17 (1908), 3.
- [18] J. LANGER and R. PERLINE, *Poisson geometry of the filament equation*, J. Nonlinear Sci. 1 (1991), 71–93.
- [19] J. LANGER and D. SINGER, *On the total squared curvature of closed curves*, J. Diff. Geometry 20 (1984), 1–22.
- [20] J. LANGER and D. SINGER, *Knotted elastic curves in  $\mathbb{R}^3$* , J. London Math. Soc. 16 (1984), 512–520.
- [21] J. LANGER and D. SINGER, *Lagrangian aspects of the Kirchhoff elastic rod*, (1993) preprint.
- [22] J. LANGER and D. SINGER, *Hamiltonian Aspects of the Kirchhoff Elastic Rod*, (1993) preprint.
- [23] LI, HSU-HSIANG, *Geometrically constrained curves and the calculus of variations* (thesis), Rice University, 1986.
- [24] W. S. MASSEY, *Elasticae in  $\mathbb{R}^3$* , (preprint).
- [25] K. NAKAYAMA, H. SEGUR, and M. WADATI, *Integrability and the motion of curves*, Phys. Rev. Lett. 67 (1992), 2603–2605.
- [26] S. NOVIKOV, S. V. MANAKOV, L. P. PITAEVSKII, and V. E. ZAKHAROV, *Theory of Solitons*, Consultants Bureau, New York 1984.
- [27] R. L. RICCA, *Rediscovery of the Da Rios equations*, Nature 352 (1991), 561–562.
- [28] R. L. RICCA, *Physical interpretation of certain invariants for vortex filament motion under LIA*, Phys. Fluids. A. 4 (1992), 938–944.
- [29] W. SHADWICK, *Bäcklund transformations and integrable Hamiltonian systems*, Proc. NSF-CBMS Conf. on Nonlinear Waves and Integrable Systems (1982), 234–241.
- [30] W. SHADWICK, *The Hamilton–Cartan formalism for  $r$ th-order Lagrangian and the integrability of the KdV and modified KdV equations*, Lett. Math. Phys. 5 (1981), 137–141.
- [31] M. SPIVAK, *A Comprehensive Introduction to Differential Geometry*, vol. 2., Publish or Perish, (1970).
- [32] Y. ZENG, *An approach to the deduction of the finite-dimensional integrability from the infinite-dimensional integrability*, Phys. Lett. A. 160 (1991), 541–547.

Department of Mathematics  
Case Western Reserve University  
Cleveland  
Ohio 44106-7058  
USA

Received June 10, 1993