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Foliations transverse to fibers of Seifert manifolds*†

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Abstract. In this paper we prove the conjecture of Jankins and Neumann [JN2] about rotation numbers of products of circle homeomorphisms, which together with other results of [EHN] and [JN2] (mentioned below) implies that a Seifert manifold admits foliations transverse to its fibers only if it admits such foliations with a projective transverse structure.

1. History

The question of existence of foliations transverse to fibers (to foliate with dimension 2 a 3-manifold means to write it as a disjoint union of surfaces, called leaves of the foliation, which locally look like disk × interval) was originally answered for (locally trivial) circle bundles by Milnor [M], Wood [W], Thurston [T], and Levitt [L]. Then Eisenbud, Hirsch, and Neumann [EHN] asked this question for the more general case of non-locally trivial circle bundles, i.e. Seifert manifolds. For the case when the base space is not S^2 they fully answered the question in terms of arithmetic criteria involving the Seifert invariants of the manifold. For the case when the base space is S^2, however, they obtained only a partial solution. This solution was later improved in two papers by Jankins and Neumann, [JN1], [JN2]. However, the solution was still incomplete; they found conditions sufficient to prove existence of foliations, and also conditions sufficient to prove their nonexistence, and for manifolds satisfying neither conditions, conjectured nonexistence of foliations.

As a corollary of proving the conjecture, together with results of Brittenham [B] and Claus [C] we get that these manifolds (in the conjecture) do not admit essential laminations. More precisely,

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COROLLARY. *Let $p, q > 1$ be relatively prime integers, and let p' and q' be the unique integers satisfying*:

$$pp' = 1 \bmod q, \qquad 0 < p' < q,$$

$$qq' = 1 \bmod p, \qquad 0 < q' < p.$$

If M is a Seifert manifold with Seifert invariants $(0; (1, -1), (\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3))$ (*i.e. three exceptional fibers over* S^2), *such that, up to permuting indices,* $\beta_1/\alpha_1 \geq (p - q')/p$, $\beta_2/\alpha_2 \geq (q - p')/q$, $\beta_3/\alpha_3 \geq 1/(p + q)$, *then M admits no essential laminations. In particular, it does not admit any foliations without compact leaves* (*or without Reeb components*).

Theorem 1 (below) implies similar results for Seifert manifolds with more than three exceptional fibers over S^2 (see [JN2]).

2. Definitions, and statement of the conjecture

Let homeo(**R**) denote the group of self-homeomorphisms of **R** under composition, and $\mathrm{sh}(\gamma) \in \mathrm{homeo}(\mathbf{R})$ the shift function $\mathrm{sh}(\gamma)(x) = x + \gamma$, $x, \gamma \in \mathbf{R}$. By abuse of notation, we also let $\mathrm{sh}(\gamma) \in \mathrm{homeo}(S^1)$ denote the circle homeomorphism which rotates every point of S^1 by γ; so γ can be reduced mod **Z**. We say that $f \in \mathrm{homeo}(S^1)$ is a *shift conjugate* if it is conjugate in the group $\mathrm{homeo}(S^1)$ to $\mathrm{sh}(\gamma)$ for some γ.

In [EHN] it is proven that:

THEOREM 3.5 [EHN]. *A Seifert manifold M admits a transverse foliation if and only if there exists a homomorphism* $\phi : \pi_1(M) \to \mathrm{homeo}(\mathbf{R})$ *with* $\phi(z) = \mathrm{sh}(1)$, where $z \in \pi_1(M)$ *is the class of a regular fiber of M.*

Let **R** cover $S^1 \cong \mathbf{R}/\mathbf{Z}$ by the map $x \mapsto x \bmod \mathbf{Z}$. Then given $f \in \mathrm{homeo}(S^1)$, and a lift $\tilde{f} \in \mathrm{homeo}(\mathbf{R})$ of f, we define the rotation number of $\tilde{f}$ as:

$$\mathrm{rot}(\tilde{f}) = \lim_{n \to \infty} \frac{1}{n} (\tilde{f}^n(x) - \tilde{f}(x))$$

where $x \in \mathbf{R}$ is arbitrary. It is easy to check that this is well-defined, i.e. the limit exists, and is independent of x. We also define $\mathrm{rot}(f) = \mathrm{rot}(\tilde{f}) \bmod \mathbf{Z}$. Note that rotation number of a circle homeomorphism does not change under conjugation.

If M is a Seifert manifold over S^2 with n singular fibers, then we can write its Seifert invariants as $(g = 0; (1, -b), (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n))$, where $g = 0$ is the genus of the base space S^2, $\alpha_i, \beta_i \in \mathbf{Z}^+$, $0 < \beta_i/\alpha_i < 1$, and $b \in \mathbf{Z}$ (the pair $(1, -b)$ does not represent a singular fiber, but is rather used to normalize β_i/α_i mod $\mathbf{Z}$; it plays the role of the Euler Class; see [JN3] or [S]). Then Theorem 3.5 of [EHN] (above) implies that M admits a transverse foliation if and only if $(b; \beta_1/\alpha_1, \ldots, \beta_n/\alpha_n)$ is realizable (defined below).

DEFINITION. Let $n \in \mathbf{Z}^+$, $J \subset \{1, \ldots, n\}$, $b \in \mathbf{Z}$, and for $i = 1, \ldots, n$, $\gamma_i \in \mathbf{R}$. Then we say $(J; b; \gamma_1, \ldots, \gamma_n)$ is **realizable** if $\exists f_i \in \text{homeo}(S^1)$ with some lift $\tilde{f}_i$ such that $\text{rot}(\tilde{f}_i) = \gamma_i$, f_i is a shift conjugate for $i \in J$, and $\tilde{f}_n \circ \cdots \circ \tilde{f}_1 = \text{sh}(b)$.

Note: When we omit J and simply write $(b; \gamma_1, \ldots, \gamma_n)$, it is to be understood that $J = \{1, \ldots, n\}$.

Deciding when $(J; b; \gamma_1, \ldots, \gamma_n)$ is realizable is trivial for $n = 2$ (when $\gamma_1 + \gamma_2 = b$). The case of $n \geq 4$ is inductively reduced to the case of $n = 3$. When $n = 3$, $(J; b; \gamma_1, \ldots, \gamma_n)$ is not realizable unless $b = 1$ or 2. The $b = 2$ case is easily reduced to the $b = 1$ case by replacing γ_i by $1 - \gamma_i$ (for proofs of these facts see [JN2]). Now it is easy to check that realizing $(J; 1; \gamma_1, \gamma_2, \gamma_3)$ is equivalent to finding $f_i \in \text{homeo}(S^1)$ such that $\text{rot}(f_i) = \gamma_i$, f_i is a shift conjugate for $i \in J$, and $f_3 \circ f_2 \circ f_1 = \text{id}$, where "id" denotes identity on S^1.

In [EHN] it was shown that if $\Sigma_{i=1}^3 \gamma_i \leq 1$, then $(1; \gamma_1, \gamma_2, \gamma_3)$ is realizable. Then in [JN1] it was shown that in fact $(1; \gamma_1, \gamma_2, \gamma_3)$ is realizable by Mobius maps if and only if $\Sigma_{i=1}^3 \gamma_i \leq 1$. A beautiful and simple proof of this fact using hyperbolic geometry on the unit disk (which was shown to me by Eric Klassen) is as follows. Given three elliptic Mobius maps f_i, $i = 1, 2, 3$, let α_i be the angles of the hyperbolic triangle formed by the fixed points of f_i. Then it is easy to show that the product of f_i is equal to identity if and only if $\alpha_i = \pi\gamma_i$, where $\gamma_i = \text{rot}\,(f_i)$. And there exists a (possibly singular) triangle with angles $\pi\gamma_i$ if and only if $\Sigma_{i=1}^3 \gamma_i \leq 1$.

Then in [JN2] more solutions were found using the following great idea. Given $f_3 \circ f_2 \circ f_1 = \text{id}$, let g_i be a lift of f_i to the m-th cyclic cover of S^1, for some fixed $m \in \mathbf{Z}^+$. Then for any $k_1, k_2, k_3 \in \mathbf{Z}$ whose sum is a multiple of m, $\text{sh}(k_3/m) \circ g_3 \circ \cdots \circ \text{sh}(k_1/m) \circ g_1 = \text{id}$. Let $\gamma_i = \text{rot}(\text{sh}(k_i/m) \circ g_i)$. Then it turns out that by picking k_i appropriately we can get the sum of γ_i to be larger than 1 (by as much as $1/m$). Using this idea they showed:

THEOREM 3 [JN2]. *$(J; 1; \gamma_1, \gamma_2, \gamma_3)$ is realizable if there exist integers $0 < a < m$ such that for some permutation of μ_i we have: $\gamma_i < \mu_i$ for $i \in J$, $\gamma_i \leq \mu_i$ for $i \notin J$, where $(\mu_1, \mu_2, \mu_3) = (a/m, (m - a)/m, 1/m)$.*

In [JN2] they also showed that most of the $(J; 1; \gamma_1, \gamma_2, \gamma_3)$'s (most in the sense of Euclidean volume in $[0, 1]^3$) which do not satisfy the hypothesis of the above theorem are not realizable, and conjectured that in fact all such $(J; 1; \gamma_1, \gamma_2, \gamma_3)$'s are not realizable.

CONJECTURE [JN2]. *If* $(J; 1; \gamma_1, \gamma_2, \gamma_3)$ *does not satisfy the hypothesis of Theorem* 3 *above, then it is not realizable.*

3. Proof of the conjecture

In [JN2], though not stated explicitly, it is shown (in Section 6) that the following non-realizability statement is equivalent to the conjecture.

THEOREM 1. *Let* $p, q > 1$ *be relatively prime integers, and let* p' *and* q' *be the unique integers satisfying*:

$$pp' = 1 \bmod q, \qquad 0 < p' < q,$$

$$qq' = 1 \bmod p, \qquad 0 < q' < p.$$

Let $\gamma_1 = (p - q')/p$, $\gamma_2 = (q - p')/q$, *and* $\gamma_3 \geq 1/(p + q)$. *Then given* $J \subset \{1, 2, 3\}$ *and* (μ_1, μ_2, μ_3) *such that* $\mu_i \geq \gamma_i$ *for* $i \in J$, $\mu_i > \gamma_i$ *for* $i \notin J$, $(J; 1; \mu_1, \mu_2, \mu_3)$ *is not realizable.*

The proof follows immediately from the following two lemmas.

MAIN LEMMA. *Let* p, q, p', q' *be as in Theorem* 1 *above, and let* $\gamma_1 = (p - q')/p$, $\gamma_2 = (q - p')/q$. *Then for all* $\gamma_3 \geq 1/(p + q)$, $(1; \gamma_1, \gamma_2, \gamma_3)$ *is not realizable.*

Proof. Let $f, g \in \text{homeo}(S^1)$ be shift conjugates whose rotation numbers are γ_1 and γ_2 respectively, and suppose h is also a shift conjugate, satisfying $h \circ g \circ f = \text{id}$. Then we want to show $\text{rot}(h) < 1/(p + q)$. We can assume $p < q$. Fix $x_0 \in S^1$, and let $\{x_0, x_1, \ldots, x_{q-1}\}$ be its orbit under g, such that on S^1 $x_{i-1} < x_i < x_{i+1}$, where of course the index of x is always mod q. (To be rigorous, we could lift everything to the universal cover, but for the sake of simplicity, we do not. To make sense of $a < b < c$ however, it is enough to fix an orientation on S^1; then "b is between a and c" means it is on the arc from a to c).

CLAIM. $x_{p'-1} < f(x_0) < x_{p'}$, *and* $x_0 < h(x_0) < x_1$.

Proof of Claim. By definition $g(x_{p'}) = x_{p'+(q-p')} = x_0 = hgf(x_0)$, and $\mathrm{rot}(h) > 0$, so clearly $x_0 < f(x_0) < x_{p'}$. So $\forall x \in S^1$, in going from x to $f(x)$ we "jump over" at most p' x_i's, i.e. the cardinality of the set $\{x_i \mid x \le x_i < f(x)\}$ is $\le p'$. So from x to $f^{p-1}(x)$ we jump over at most $(p-1)p'$ x_i's. But from x to $f^p(x) = x$ we go around S^1 $p-q'$ times, so we jump over exactly $(p-q')q$ x_i's. Therefore from $f^{p-1}(x)$ to $f^p(x)$ we must jump over at least $(p-q')q-(p-1)p' = pq-qq'-pp'+p' = (p'-1)$ x_i's (since $pp'+qq' = pq+1$). This shows that *for exactly one* $j \in \{0, \ldots, p-1\}$ *there are* $(p'-1)$ x_i*'s between* $f^j(x)$ *and* $f^{j+1}(x)$, *and for all other* j *there are* p' x_i*'s.*

Now by a symmetrical argument we see that $x_{q-p'} < f^{-1}(x_0) < x_0$, i.e. between $f^{p-1}(x_0)$ and $f^p(x_0)$ there are only $(p'-1)$ x_i's (since we defined "between" to be left inclusive, right exclusive), so by above, $\forall j \ne p'-1 \bmod p$, there must be p' x_i's between $f^j(x_0)$ and $f^{j+1}(x_0)$. Therefore $x_{p'-1} < f(x_0) < x_{p'}$.

Now $h(x_0) = f^{-1}g^{-1}(x_0) = f^{-1}(x_{p'})$, and by above we can check that $f(x_0) < x_{p'} < f(x_1)$, so $x_0 < h(x_0) < x_1$. □(Claim)

So between each x_i and x_{i+1} we can "fit an h". To prove the lemma we will show that for at least $p-1$ distinct i's we can "fit an extra nonoverlapping h" between x_i and x_{i+1} (i.e. $x_i < h^2(x_i) \le x_{i+1}$). And then we will "fit one more h" somewhere else, as explained later, so that in the end $p+q$ nonoverlapping h's will fit on S^1, showing that $\mathrm{rot}(h) < 1/(p+q)$.

Let $y_0 = h(x_0)$, and as with x_i, let $\{y_0, y_1, \ldots, y_{q-1}\}$ be the orbit of y_0 under g, so that $x_0 < y_0 < x_1 < y_1 < \cdots$. $f(y_0) = fh(x_0) = g^{-1}(x_0) = x_{p'}$. So between y_0 and $f(y_0)$ there are only $(p'-1)$ x_i's, which implies that for $i = 2, \ldots, p-1$, $x_{ip'-1} < f^i(y_0) < x_{ip'}$. Similarly, $y_{q-p'} < f^{-1}(y_0) < y_0$, so between $f^{-1}(y_0)$ and y_0 there are only $(p'-1)$ y_i's. It follows that for $i = 1, \ldots, p-1$, $y_{ip'-1} < f^i(y_0) < y_{ip'}$. So we get:

$$y_{ip'-1} < f^i(y_0) < x_{ip'}, \quad i = 2, \ldots, p-1; \qquad f(y_0) = x'_p$$

Now for $i = 1, \ldots, p-1$, we have: $g^{-1}(y_{ip'-1}) = y_{(i+1)p'-1} \le f^{i+1}(y_0)$ (with equality iff $i = p-1$), so $f^{-1}(y_{(i+1)p'-1}) \le f^i(y_0) \le x_{ip'}$ ($f^i(y_0) = x_{ip'}$ iff $i = 1$), so $h(y_{ip'-1}) = f^{-1}g^{-1}(y_{ip'-1}) \le x_{ip'}$, so:

$$y_{ip'-1} < h(y_{ip'-1}) \le x_{ip'}, \qquad i = 1, \ldots, p-1 \tag{1}$$

We can assume that $x_1 < h(y_0) < x_2$, since otherwise $\forall i, x_i < h^2(x_i) \le x_{i+1}$, so $\mathrm{rot}(h) \le 1/2q < 1/(p+q)$, and we are done.

Let $z = h(y_0)$. $g^{-1}(y_0) = y_{p'}$, so $f(z) = y_{p'}$, so $x_{p'} < f(z) < x_{p'+1}$. And $y_0 < x_1 < z < y_1 < x_2$, so by a "counting" argument as above, for $i = 1, \ldots, p$,

$x_{ip'} < f^i(z) < x_{ip'+1}$, and for $i = 2, \ldots, p$, $y_{ip'-1} < f^i(z) < y_{ip'}$. So:

$$x_{ip'} < f^i(z) < y_{ip'}, \quad i = 2, \ldots, p; \qquad x_{p'} < f(z) = y_{p'} < x_{p'+1}$$

For $i = 1, \ldots, p-1$, $g^{-1}(x_{ip'}) = x_{(i+1)p'} < f^{i+1}(z)$, so $f^{-1}g^{-1}(x_{ip'}) < f^i(z) \leq y_{ip'}$ (equality iff $i = 1$), so:

$$x_{ip'} < h(x_{ip'}) < y_{ip'}, \qquad i = 1, \ldots, p-1 \tag{2}$$

Equations (1) and (2) imply:

$$y_{ip'-1} < h^2(y_{ip'-1}) < y_{ip'}, \qquad i = 1, \ldots, p-1 \tag{3}$$

Let $i_0 \in \{1, \ldots, p-1\}$ be such that for $i \in \{1, \ldots, p-1\}$, $y_0 < y_{ip'-1} \leq y_{i_0p'-1}$ implies $i = i_0 \bmod q$. Since $y_i < h(y_i) < y_{i+1}$, $y_0 < h^{i_0p'-1}(y_0) < y_{i_0p'-1}$. Furthermore, $h(x_0) = y_0$, and by (1) $y_{i_0p'-1} < h(y_{i_0p'-1}) \leq x_{i_0p'}$, therefore $x_0 < h^{i_0p'+1}(x_0) < x_{i_0p'}$. But x_0 was arbitrary (to prove equations (2) and (3) x_0 was not arbitrary, but for (1), and hence in this paragraph, it is), so:

$$y_{-1} < h^{i_0p'+1}(y_{-1}) < y_{i_0p'-1} \tag{4}$$

Now p' and q are also relatively prime, so $y_{ip'} \neq y_0$ for any $i \neq 0 \bmod q$, and by assumption $p < q$, so:

$$\text{for } i, j \in \{1, \ldots, p-1\}, \qquad i \neq j \text{ implies } y_{ip'} \neq y_{jp'} \tag{5}$$

and similarly

$$y_{ip'-1} \neq y_{-1}, \qquad i = 1, \ldots, p-1 \tag{6}$$

Equation (3) gives $p-1$ "extra h's", and (5) says we are not counting any of them more than once. (4) gives "one more h", and it was not already counted in (3) because of (6) and the way i_0 was chosen. So we get the desired $p+q$ nonoverlapping h's on S^1. □(Main Lemma)

LEMMA (Weak Monotonicity). *Fix $\gamma_1, \gamma_2, \gamma_3 \in (0, 1)$, and suppose $\forall \gamma \geq \gamma_3$, $(1; \gamma_1, \gamma_2, \gamma)$ is not realizable. Then given $J \subset \{1, 2, 3\}$ and (μ_1, μ_2, μ_3) such that $\mu_i \geq \gamma_i$ for $i \in J$, $\mu_i > \gamma_i$ for $i \notin J$, $(J; 1; \mu_1, \mu_2, \mu_3)$ is not realizable.*

Remark. A stronger lemma (which follows after having proved The Conjecture, but which we could not prove "directly") would be obtained by weakening the hypothesis to only "$(1; \gamma_1, \gamma_2, \gamma_3)$ not realizable". Hence *Weak* Monotonicity.

Proof. In the following, we repeatedly use the fact that rotation number is continuous ([H], Chapter II, Proposition 2.7).

Suppose towards contradiction, that $\exists \phi_i \in \text{homeo}(S^1)$ such that $\text{rot}(\phi_i) = \mu_i$ and $\phi_3 \circ \phi_2 \circ \phi_1 = \text{id}$. Write $\theta = \phi_3^{-1} = \phi_2 \circ \phi_1$.

CASE 1. There is no i with $\mu_i = \gamma_i$. So in particular, $\text{rot}(\theta) < 1 - \gamma_3$.

Step 1. For $i = 1, 2$ perturb ϕ_i slightly, if necessary, so that: (1) ϕ_i is now smooth, (2) $\text{rot}(\phi_i)$ is still $> \gamma_i$, and (3) $\text{rot}(\theta) = \text{rot}(\phi_2 \circ \phi_1)$ is still $< 1 - \gamma_3$.

Step 2. For $i = 1, 2$ replace ϕ_i by $\text{sh}(-\epsilon_i) \circ \phi_i, \epsilon_i \geq 0$, so that $\text{rot}(\phi_i)$ is now irrational, but still $> \gamma_i$. Clearly $\text{rot}(\theta)$ is still $< 1 - \gamma_3$ (even if ϵ_i is not small, which it may not be).

Now by Denjoy's Theorem ([CFS], section 3.4), since ϕ_i is smooth (C^2 is enough in fact) with irrational rotation number, it must be a shift conjugate, i.e. $\phi_i = f_i \,\text{sh}(\rho_i) f_i^{-1}$ for some $f_i \in \text{homeo}(S^1)$, where $\rho_i = \text{rot}(\phi_i)$. Now, Denjoy's Theorem does not guarantee that f_i will be smooth, so we perturb it slightly if necessary, so that it is smooth, and $\text{rot}(\theta) < 1 - \gamma_3$ still holds.

Step 3. $\gamma_i < \rho_i$, so $f_i \,\text{sh}(\gamma_i) f_i^{-1} < f_i \,\text{sh}(\rho_i) f_i^{-1}$ ($f < g$ means $\forall x \in S^1, x \leq f(x) < g(x)$), so now we replace ϕ_i by $f_i \,\text{sh}(\gamma_i) f_i^{-1}$, and we still have $\text{rot}(\theta) < 1 - \gamma_3$.

Now we perturb f_i slightly if necessary, by replacing it by $\epsilon \cdot \text{id} + (1 - \epsilon) \cdot f_i$, so that $\text{rot}(\theta)$ becomes irrational, but still $< 1 - \gamma_3$. So now θ too is a shift conjugate, with $\text{rot}(\theta^{-1}) > \gamma_3$, a contradiction.

CASE 2. There is exactly one i with $\mu_i = \gamma_i$.

Say $i = 3$ (so $\mu_1 > \gamma_1, \mu_2 > \gamma_2$). Then by hypothesis, $3 \in J$, i.e., $\phi_3 = f \,\text{sh}(\gamma_3) f^{-1}$. Write $\phi_1^{-1} = \phi_3 \circ \phi_2$, and replace ϕ_3 by $f \,\text{sh}(\gamma_3 + \epsilon) f^{-1}, \epsilon > 0$ small enough so that $\text{rot}(\phi_1)$ is still $> \gamma_1$. Then we are in Case 1 again.

CASE 3. There are exactly 2 i's with $\mu_i = \gamma_i$.

Say $i = 2, 3$ (so $\mu_1 > \gamma_1$). Then by hypothesis, $\{2, 3\} \in J$. So $\phi_3 = f \,\text{sh}(\gamma_3) f^{-1}$. Write $\phi_1^{-1} = \phi_3 \circ \phi_2$, and replace ϕ_3 by $f \,\text{sh}(\gamma_3 + \epsilon) f^{-1}, \epsilon > 0$ small enough so that $\text{rot}(\phi_1)$ is still $> \gamma_1$. Then we are in Case 2 again.

And of course when $\mu_i = \gamma_i$ for all i, we have nonrealizability by hypothesis. □(Weak Monotonicity Lemma)

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