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## Foliations transverse to fibers of Seifert manifolds\*†

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*Abstract.* In this paper we prove the conjecture of Jankins and Neumann [JN2] about rotation numbers of products of circle homeomorphisms, which together with other results of [EHN] and [JN2] (mentioned below) implies that a Seifert manifold admits foliations transverse to its fibers only if it admits such foliations with a projective transverse structure.

### 1. History

The question of existence of foliations transverse to fibers (to foliate with dimension 2 a 3-manifold means to write it as a disjoint union of surfaces, called leaves of the foliation, which locally look like  $\text{disk} \times \text{interval}$ ) was originally answered for (locally trivial) circle bundles by Milnor [M], Wood [W], Thurston [T], and Levitt [L]. Then Eisenbud, Hirsch, and Neumann [EHN] asked this question for the more general case of non-locally trivial circle bundles, i.e. Seifert manifolds. For the case when the base space is not  $S^2$  they fully answered the question in terms of arithmetic criteria involving the Seifert invariants of the manifold. For the case when the base space is  $S^2$ , however, they obtained only a partial solution. This solution was later improved in two papers by Jankins and Neumann, [JN1], [JN2]. However, the solution was still incomplete; they found conditions sufficient to prove existence of foliations, and also conditions sufficient to prove their nonexistence, and for manifolds satisfying neither conditions, conjectured nonexistence of foliations.

As a corollary of proving the conjecture, together with results of Brittenham [B] and Claus [C] we get that these manifolds (in the conjecture) do not admit essential laminations. More precisely,

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**COROLLARY.** *Let  $p, q > 1$  be relatively prime integers, and let  $p'$  and  $q'$  be the unique integers satisfying:*

$$pp' = 1 \bmod q, \quad 0 < p' < q,$$

$$qq' = 1 \bmod p, \quad 0 < q' < p.$$

*If  $M$  is a Seifert manifold with Seifert invariants  $(0; (1, -1), (\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3))$  (i.e. three exceptional fibers over  $S^2$ ), such that, up to permuting indices,  $\beta_1/\alpha_1 \geq (p - q')/p$ ,  $\beta_2/\alpha_2 \geq (q - p')/q$ ,  $\beta_3/\alpha_3 \geq 1/(p + q)$ , then  $M$  admits no essential laminations. In particular, it does not admit any foliations without compact leaves (or without Reeb components).*

Theorem 1 (below) implies similar results for Seifert manifolds with more than three exceptional fibers over  $S^2$  (see [JN2]).

## 2. Definitions, and statement of the conjecture

Let  $\text{homeo}(\mathbf{R})$  denote the group of self-homeomorphisms of  $\mathbf{R}$  under composition, and  $\text{sh}(\gamma) \in \text{homeo}(\mathbf{R})$  the shift function  $\text{sh}(\gamma)(x) = x + \gamma$ ,  $x, \gamma \in \mathbf{R}$ . By abuse of notation, we also let  $\text{sh}(\gamma) \in \text{homeo}(S^1)$  denote the circle homeomorphism which rotates every point of  $S^1$  by  $\gamma$ ; so  $\gamma$  can be reduced mod  $\mathbf{Z}$ . We say that  $f \in \text{homeo}(S^1)$  is a *shift conjugate* if it is conjugate in the group  $\text{homeo}(S^1)$  to  $\text{sh}(\gamma)$  for some  $\gamma$ .

In [EHN] it is proven that:

**THEOREM 3.5 [EHN].** *A Seifert manifold  $M$  admits a transverse foliation if and only if there exists a homomorphism  $\phi : \pi_1(M) \rightarrow \text{homeo}(\mathbf{R})$  with  $\phi(z) = \text{sh}(1)$ , where  $z \in \pi_1(M)$  is the class of a regular fiber of  $M$ .*

Let  $\mathbf{R}$  cover  $S^1 \cong \mathbf{R}/\mathbf{Z}$  by the map  $x \mapsto x \bmod \mathbf{Z}$ . Then given  $f \in \text{homeo}(S^1)$ , and a lift  $\tilde{f} \in \text{homeo}(\mathbf{R})$  of  $f$ , we define the rotation number of  $\tilde{f}$  as:

$$\text{rot}(\tilde{f}) = \lim_{n \rightarrow \infty} \frac{1}{n} (\tilde{f}^n(x) - \tilde{f}(x))$$

where  $x \in \mathbf{R}$  is arbitrary. It is easy to check that this is well-defined, i.e. the limit exists, and is independent of  $x$ . We also define  $\text{rot}(f) = \text{rot}(\tilde{f}) \bmod \mathbf{Z}$ . Note that rotation number of a circle homeomorphism does not change under conjugation.

If  $M$  is a Seifert manifold over  $S^2$  with  $n$  singular fibers, then we can write its Seifert invariants as  $(g = 0; (1, -b), (\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n))$ , where  $g = 0$  is the genus of the base space  $S^2$ ,  $\alpha_i, \beta_i \in \mathbb{Z}^+$ ,  $0 < \beta_i/\alpha_i < 1$ , and  $b \in \mathbb{Z}$  (the pair  $(1, -b)$  does not represent a singular fiber, but is rather used to normalize  $\beta_i/\alpha_i \bmod \mathbb{Z}$ ; it plays the role of the Euler Class; see [JN3] or [S]). Then Theorem 3.5 of [EHN] (above) implies that  $M$  admits a transverse foliation if and only if  $(b; \beta_1/\alpha_1, \dots, \beta_n/\alpha_n)$  is realizable (defined below).

**DEFINITION.** Let  $n \in \mathbb{Z}^+$ ,  $J \subset \{1, \dots, n\}$ ,  $b \in \mathbb{Z}$ , and for  $i = 1, \dots, n$ ,  $\gamma_i \in \mathbb{R}$ . Then we say  $(J; b; \gamma_1, \dots, \gamma_n)$  is **realizable** if  $\exists f_i \in \text{homeo}(S^1)$  with some lift  $\tilde{f}_i$  such that  $\text{rot}(\tilde{f}_i) = \gamma_i$ ,  $f_i$  is a shift conjugate for  $i \in J$ , and  $\tilde{f}_n \circ \dots \circ \tilde{f}_1 = \text{sh}(b)$ .

*Note:* When we omit  $J$  and simply write  $(b; \gamma_1, \dots, \gamma_n)$ , it is to be understood that  $J = \{1, \dots, n\}$ .

Deciding when  $(J; b; \gamma_1, \dots, \gamma_n)$  is realizable is trivial for  $n = 2$  (when  $\gamma_1 + \gamma_2 = b$ ). The case of  $n \geq 4$  is inductively reduced to the case of  $n = 3$ . When  $n = 3$ ,  $(J; b; \gamma_1, \dots, \gamma_n)$  is not realizable unless  $b = 1$  or  $2$ . The  $b = 2$  case is easily reduced to the  $b = 1$  case by replacing  $\gamma_i$  by  $1 - \gamma_i$  (for proofs of these facts see [JN2]). Now it is easy to check that realizing  $(J; 1; \gamma_1, \gamma_2, \gamma_3)$  is equivalent to finding  $f_i \in \text{homeo}(S^1)$  such that  $\text{rot}(f_i) = \gamma_i$ ,  $f_i$  is a shift conjugate for  $i \in J$ , and  $f_3 \circ f_2 \circ f_1 = \text{id}$ , where “id” denotes identity on  $S^1$ .

In [EHN] it was shown that if  $\sum_{i=1}^3 \gamma_i \leq 1$ , then  $(1; \gamma_1, \gamma_2, \gamma_3)$  is realizable. Then in [JN1] it was shown that in fact  $(1; \gamma_1, \gamma_2, \gamma_3)$  is realizable by Mobius maps if and only if  $\sum_{i=1}^3 \gamma_i \leq 1$ . A beautiful and simple proof of this fact using hyperbolic geometry on the unit disk (which was shown to me by Eric Klassen) is as follows. Given three elliptic Mobius maps  $f_i$ ,  $i = 1, 2, 3$ , let  $\alpha_i$  be the angles of the hyperbolic triangle formed by the fixed points of  $f_i$ . Then it is easy to show that the product of  $f_i$  is equal to identity if and only if  $\alpha_i = \pi\gamma_i$ , where  $\gamma_i = \text{rot}(f_i)$ . And there exists a (possibly singular) triangle with angles  $\pi\gamma_i$  if and only if  $\sum_{i=1}^3 \gamma_i \leq 1$ .

Then in [JN2] more solutions were found using the following great idea. Given  $f_3 \circ f_2 \circ f_1 = \text{id}$ , let  $g_i$  be a lift of  $f_i$  to the  $m$ -th cyclic cover of  $S^1$ , for some fixed  $m \in \mathbb{Z}^+$ . Then for any  $k_1, k_2, k_3 \in \mathbb{Z}$  whose sum is a multiple of  $m$ ,  $\text{sh}(k_3/m) \circ g_3 \circ \dots \circ \text{sh}(k_1/m) \circ g_1 = \text{id}$ . Let  $\gamma_i = \text{rot}(\text{sh}(k_i/m) \circ g_i)$ . Then it turns out that by picking  $k_i$  appropriately we can get the sum of  $\gamma_i$  to be larger than 1 (by as much as  $1/m$ ). Using this idea they showed:

**THEOREM 3** [JN2].  $(J; 1; \gamma_1, \gamma_2, \gamma_3)$  is realizable if there exist integers  $0 < a < m$  such that for some permutation of  $\mu_i$  we have:  $\gamma_i < \mu_i$  for  $i \in J$ ,  $\gamma_i \leq \mu_i$  for  $i \notin J$ , where  $(\mu_1, \mu_2, \mu_3) = (a/m, (m - a)/m, 1/m)$ .



In [JN2] they also showed that most of the  $(J; 1; \gamma_1, \gamma_2, \gamma_3)$ 's (most in the sense of Euclidean volume in  $[0, 1]^3$ ) which do not satisfy the hypothesis of the above theorem are not realizable, and conjectured that in fact all such  $(J; 1; \gamma_1, \gamma_2, \gamma_3)$ 's are not realizable.

**CONJECTURE [JN2].** *If  $(J; 1; \gamma_1, \gamma_2, \gamma_3)$  does not satisfy the hypothesis of Theorem 3 above, then it is not realizable.*

### 3. Proof of the conjecture

In [JN2], though not stated explicitly, it is shown (in Section 6) that the following non-realizability statement is equivalent to the conjecture.

**THEOREM 1.** *Let  $p, q > 1$  be relatively prime integers, and let  $p'$  and  $q'$  be the unique integers satisfying:*

$$pp' = 1 \pmod{q}, \quad 0 < p' < q,$$

$$qq' = 1 \pmod{p}, \quad 0 < q' < p.$$

*Let  $\gamma_1 = (p - q')/p$ ,  $\gamma_2 = (q - p')/q$ , and  $\gamma_3 \geq 1/(p + q)$ . Then given  $J \subset \{1, 2, 3\}$  and  $(\mu_1, \mu_2, \mu_3)$  such that  $\mu_i \geq \gamma_i$  for  $i \in J$ ,  $\mu_i > \gamma_i$  for  $i \notin J$ ,  $(J; 1; \mu_1, \mu_2, \mu_3)$  is not realizable.*

The proof follows immediately from the following two lemmas.

**MAIN LEMMA.** *Let  $p, q, p', q'$  be as in Theorem 1 above, and let  $\gamma_1 = (p - q')/p$ ,  $\gamma_2 = (q - p')/q$ . Then for all  $\gamma_3 \geq 1/(p + q)$ ,  $(1; \gamma_1, \gamma_2, \gamma_3)$  is not realizable.*

*Proof.* Let  $f, g \in \text{homeo}(S^1)$  be shift conjugates whose rotation numbers are  $\gamma_1$  and  $\gamma_2$  respectively, and suppose  $h$  is also a shift conjugate, satisfying  $h \circ g \circ f = \text{id}$ . Then we want to show  $\text{rot}(h) < 1/(p + q)$ . We can assume  $p < q$ . Fix  $x_0 \in S^1$ , and let  $\{x_0, x_1, \dots, x_{q-1}\}$  be its orbit under  $g$ , such that on  $S^1$   $x_{i-1} < x_i < x_{i+1}$ , where of course the index of  $x$  is always mod  $q$ . (To be rigorous, we could lift everything to the universal cover, but for the sake of simplicity, we do not. To make sense of  $a < b < c$  however, it is enough to fix an orientation on  $S^1$ ; then “ $b$  is between  $a$  and  $c$ ” means it is on the arc from  $a$  to  $c$ ).

**CLAIM.**  $x_{p'-1} < f(x_0) < x_{p'}$ , and  $x_0 < h(x_0) < x_1$ .

*Proof of Claim.* By definition  $g(x_{p'}) = x_{p' + (q - p')} = x_0 = hgf(x_0)$ , and  $\text{rot}(h) > 0$ , so clearly  $x_0 < f(x_0) < x_{p'}$ . So  $\forall x \in S^1$ , in going from  $x$  to  $f(x)$  we “jump over” at most  $p'$   $x_i$ 's, i.e. the cardinality of the set  $\{x_i \mid x \leq x_i < f(x)\}$  is  $\leq p'$ . So from  $x$  to  $f^{p-1}(x)$  we jump over at most  $(p-1)p'$   $x_i$ 's. But from  $x$  to  $f^p(x) = x$  we go around  $S^1$   $p - q'$  times, so we jump over exactly  $(p - q')q$   $x_i$ 's. Therefore from  $f^{p-1}(x)$  to  $f^p(x)$  we must jump over at least  $(p - q')q - (p-1)p' = pq - qq' - pp' + p' = (p' - 1)x_i$ 's (since  $pp' + qq' = pq + 1$ ). This shows that for exactly one  $j \in \{0, \dots, p-1\}$  there are  $(p' - 1)x_i$ 's between  $f^j(x)$  and  $f^{j+1}(x)$ , and for all other  $j$  there are  $p'$   $x_i$ 's.

Now by a symmetrical argument we see that  $x_{q-p'} < f^{-1}(x_0) < x_0$ , i.e. between  $f^{p-1}(x_0)$  and  $f^p(x_0)$  there are only  $(p' - 1)x_i$ 's (since we defined “between” to be left inclusive, right exclusive), so by above,  $\forall j \neq p' - 1 \pmod p$ , there must be  $p'$   $x_i$ 's between  $f^j(x_0)$  and  $f^{j+1}(x_0)$ . Therefore  $x_{p'-1} < f(x_0) < x_{p'}$ .

Now  $h(x_0) = f^{-1}g^{-1}(x_0) = f^{-1}(x_{p'})$ , and by above we can check that  $f(x_0) < x_{p'} < f(x_1)$ , so  $x_0 < h(x_0) < x_1$ .  $\square$ (Claim)

So between each  $x_i$  and  $x_{i+1}$  we can “fit an  $h$ ”. To prove the lemma we will show that for at least  $p-1$  distinct  $i$ 's we can “fit an extra nonoverlapping  $h$ ” between  $x_i$  and  $x_{i+1}$  (i.e.  $x_i < h^2(x_i) \leq x_{i+1}$ ). And then we will “fit one more  $h$ ” somewhere else, as explained later, so that in the end  $p+q$  nonoverlapping  $h$ 's will fit on  $S^1$ , showing that  $\text{rot}(h) < 1/(p+q)$ .

Let  $y_0 = h(x_0)$ , and as with  $x_i$ , let  $\{y_0, y_1, \dots, y_{q-1}\}$  be the orbit of  $y_0$  under  $g$ , so that  $x_0 < y_0 < x_1 < y_1 < \dots$ .  $f(y_0) = fh(x_0) = g^{-1}(x_0) = x_{p'}$ . So between  $y_0$  and  $f(y_0)$  there are only  $(p' - 1)x_i$ 's, which implies that for  $i = 2, \dots, p-1$ ,  $x_{ip'-1} < f^i(y_0) < x_{ip'}$ . Similarly,  $y_{q-p'} < f^{-1}(y_0) < y_0$ , so between  $f^{-1}(y_0)$  and  $y_0$  there are only  $(p' - 1)y_i$ 's. It follows that for  $i = 1, \dots, p-1$ ,  $y_{ip'-1} < f^i(y_0) < y_{ip'}$ . So we get:

$$y_{ip'-1} < f^i(y_0) < x_{ip'}, \quad i = 2, \dots, p-1; \quad f(y_0) = x_{p'}$$

Now for  $i = 1, \dots, p-1$ , we have:  $g^{-1}(y_{ip'-1}) = y_{(i+1)p'-1} \leq f^{i+1}(y_0)$  (with equality iff  $i = p-1$ ), so  $f^{-1}(y_{(i+1)p'-1}) \leq f^i(y_0) \leq x_{ip'}$  ( $f^i(y_0) = x_{ip'}$  iff  $i = 1$ ), so  $h(y_{ip'-1}) = f^{-1}g^{-1}(y_{ip'-1}) \leq x_{ip'}$ , so:

$$y_{ip'-1} < h(y_{ip'-1}) \leq x_{ip'}, \quad i = 1, \dots, p-1 \tag{1}$$

We can assume that  $x_1 < h(y_0) < x_2$ , since otherwise  $\forall i, x_i < h^2(x_i) \leq x_{i+1}$ , so  $\text{rot}(h) \leq 1/2q < 1/(p+q)$ , and we are done.

Let  $z = h(y_0)$ .  $g^{-1}(y_0) = y_{p'}$ , so  $f(z) = y_{p'}$ , so  $x_{p'} < f(z) < x_{p'+1}$ . And  $y_0 < x_1 < z < y_1 < x_2$ , so by a “counting” argument as above, for  $i = 1, \dots, p$ ,

$x_{ip'} < f^i(z) < x_{ip'+1}$ , and for  $i = 2, \dots, p$ ,  $y_{ip'-1} < f^i(z) < y_{ip'}$ . So:

$$x_{ip'} < f^i(z) < y_{ip'}, \quad i = 2, \dots, p; \quad x_{p'} < f(z) = y_{p'} < x_{p'+1}$$

For  $i = 1, \dots, p-1$ ,  $g^{-1}(x_{ip'}) = x_{(i+1)p'} < f^{i+1}(z)$ , so  $f^{-1}g^{-1}(x_{ip'}) < f^i(z) \leq y_{ip'}$  (equality iff  $i = 1$ ), so:

$$x_{ip'} < h(x_{ip'}) < y_{ip'}, \quad i = 1, \dots, p-1 \quad (2)$$

Equations (1) and (2) imply:

$$y_{ip'-1} < h^2(y_{ip'-1}) < y_{ip'}, \quad i = 1, \dots, p-1 \quad (3)$$

Let  $i_0 \in \{1, \dots, p-1\}$  be such that for  $i \in \{1, \dots, p-1\}$ ,  $y_0 < y_{ip'-1} \leq y_{i_0p'-1}$  implies  $i = i_0 \bmod q$ . Since  $y_i < h(y_i) < y_{i+1}$ ,  $y_0 < h^{i_0p'-1}(y_0) < y_{i_0p'-1}$ . Furthermore,  $h(x_0) = y_0$ , and by (1)  $y_{i_0p'-1} < h(y_{i_0p'-1}) \leq x_{i_0p'}$ , therefore  $x_0 < h^{i_0p'+1}(x_0) < x_{i_0p'}$ . But  $x_0$  was arbitrary (to prove equations (2) and (3)  $x_0$  was not arbitrary, but for (1), and hence in this paragraph, it is), so:

$$y_{-1} < h^{i_0p'+1}(y_{-1}) < y_{i_0p'-1} \quad (4)$$

Now  $p'$  and  $q$  are also relatively prime, so  $y_{ip'} \neq y_0$  for any  $i \neq 0 \bmod q$ , and by assumption  $p < q$ , so:

$$\text{for } i, j \in \{1, \dots, p-1\}, \quad i \neq j \text{ implies } y_{ip'} \neq y_{jp'} \quad (5)$$

and similarly

$$y_{ip'-1} \neq y_{-1}, \quad i = 1, \dots, p-1 \quad (6)$$

Equation (3) gives  $p-1$  “extra  $h$ ’s”, and (5) says we are not counting any of them more than once. (4) gives “one more  $h$ ”, and it was not already counted in (3) because of (6) and the way  $i_0$  was chosen. So we get the desired  $p+q$  nonoverlapping  $h$ ’s on  $S^1$ .  $\square$ (Main Lemma)

**LEMMA (Weak Monotonicity).** Fix  $\gamma_1, \gamma_2, \gamma_3 \in (0, 1)$ , and suppose  $\forall \gamma \geq \gamma_3$ ,  $(1; \gamma_1, \gamma_2, \gamma)$  is not realizable. Then given  $J \subset \{1, 2, 3\}$  and  $(\mu_1, \mu_2, \mu_3)$  such that  $\mu_i \geq \gamma_i$  for  $i \in J$ ,  $\mu_i > \gamma_i$  for  $i \notin J$ ,  $(J; 1; \mu_1, \mu_2, \mu_3)$  is not realizable.

*Remark.* A stronger lemma (which follows after having proved The Conjecture, but which we could not prove “directly”) would be obtained by weakening the hypothesis to only “ $(1; \gamma_1, \gamma_2, \gamma_3)$  not realizable”. Hence *Weak Monotonicity*.

*Proof.* In the following, we repeatedly use the fact that rotation number is continuous ([H], Chapter II, Proposition 2.7).

Suppose towards contradiction, that  $\exists \phi_i \in \text{homeo}(S^1)$  such that  $\text{rot}(\phi_i) = \mu_i$  and  $\phi_3 \circ \phi_2 \circ \phi_1 = \text{id}$ . Write  $\theta = \phi_3^{-1} = \phi_2 \circ \phi_1$ .

**CASE 1.** There is no  $i$  with  $\mu_i = \gamma_i$ . So in particular,  $\text{rot}(\theta) < 1 - \gamma_3$ .

*Step 1.* For  $i = 1, 2$  perturb  $\phi_i$  slightly, if necessary, so that: (1)  $\phi_i$  is now smooth, (2)  $\text{rot}(\phi_i)$  is still  $> \gamma_i$ , and (3)  $\text{rot}(\theta) = \text{rot}(\phi_2 \circ \phi_1)$  is still  $< 1 - \gamma_3$ .

*Step 2.* For  $i = 1, 2$  replace  $\phi_i$  by  $\text{sh}(-\epsilon_i) \circ \phi_i$ ,  $\epsilon_i \geq 0$ , so that  $\text{rot}(\phi_i)$  is now irrational, but still  $> \gamma_i$ . Clearly  $\text{rot}(\theta)$  is still  $< 1 - \gamma_3$  (even if  $\epsilon_i$  is not small, which it may not be).

Now by Denjoy’s Theorem ([CFS], section 3.4), since  $\phi_i$  is smooth ( $C^2$  is enough in fact) with irrational rotation number, it must be a shift conjugate, i.e.  $\phi_i = f_i \text{sh}(\rho_i) f_i^{-1}$  for some  $f_i \in \text{homeo}(S^1)$ , where  $\rho_i = \text{rot}(\phi_i)$ . Now, Denjoy’s Theorem does not guarantee that  $f_i$  will be smooth, so we perturb it slightly if necessary, so that it is smooth, and  $\text{rot}(\theta) < 1 - \gamma_3$  still holds.

*Step 3.*  $\gamma_i < \rho_i$ , so  $f_i \text{sh}(\gamma_i) f_i^{-1} < f_i \text{sh}(\rho_i) f_i^{-1}$  ( $f < g$  means  $\forall x \in S^1, x \leq f(x) < g(x)$ ), so now we replace  $\phi_i$  by  $f_i \text{sh}(\gamma_i) f_i^{-1}$ , and we still have  $\text{rot}(\theta) < 1 - \gamma_3$ .

Now we perturb  $f_i$  slightly if necessary, by replacing it by  $\epsilon \cdot \text{id} + (1 - \epsilon) \cdot f_i$ , so that  $\text{rot}(\theta)$  becomes irrational, but still  $< 1 - \gamma_3$ . So now  $\theta$  too is a shift conjugate, with  $\text{rot}(\theta^{-1}) > \gamma_3$ , a contradiction.

**CASE 2.** There is exactly one  $i$  with  $\mu_i = \gamma_i$ .

Say  $i = 3$  (so  $\mu_1 > \gamma_1, \mu_2 > \gamma_2$ ). Then by hypothesis,  $3 \in J$ , i.e.,  $\phi_3 = f \text{sh}(\gamma_3) f^{-1}$ . Write  $\phi_1^{-1} = \phi_3 \circ \phi_2$ , and replace  $\phi_3$  by  $f \text{sh}(\gamma_3 + \epsilon) f^{-1}$ ,  $\epsilon > 0$  small enough so that  $\text{rot}(\phi_1)$  is still  $> \gamma_1$ . Then we are in Case 1 again.

**CASE 3.** There are exactly 2  $i$ ’s with  $\mu_i = \gamma_i$ .

Say  $i = 2, 3$  (so  $\mu_1 > \gamma_1$ ). Then by hypothesis,  $\{2, 3\} \in J$ . So  $\phi_3 = f \text{sh}(\gamma_3) f^{-1}$ . Write  $\phi_1^{-1} = \phi_3 \circ \phi_2$ , and replace  $\phi_3$  by  $f \text{sh}(\gamma_3 + \epsilon) f^{-1}$ ,  $\epsilon > 0$  small enough so that  $\text{rot}(\phi_1)$  is still  $> \gamma_1$ . Then we are in Case 2 again.

And of course when  $\mu_i = \gamma_i$  for all  $i$ , we have nonrealizability by hypothesis. □(Weak Monotonicity Lemma)

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