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Foliations transverse to fibers of Seifert manifolds*†

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Abstract. In this paper we prove the conjecture of Jankins and Neumann [JN2] about rotation numbers of products of circle homeomorphisms, which together with other results of [EHN] and [JN2] (mentioned below) implies that a Seifert manifold admits foliations transverse to its fibers only if it admits such foliations with a projective transverse structure.

1. History

The question of existence of foliations transverse to fibers (to foliate with dimension 2 a 3-manifold means to write it as a disjoint union of surfaces, called leaves of the foliation, which locally look like $\text{disk} \times \text{interval}$) was originally answered for (locally trivial) circle bundles by Milnor [M], Wood [W], Thurston [T], and Levitt [L]. Then Eisenbud, Hirsch, and Neumann [EHN] asked this question for the more general case of non-locally trivial circle bundles, i.e. Seifert manifolds. For the case when the base space is not S^2 they fully answered the question in terms of arithmetic criteria involving the Seifert invariants of the manifold. For the case when the base space is S^2 , however, they obtained only a partial solution. This solution was later improved in two papers by Jankins and Neumann, [JN1], [JN2]. However, the solution was still incomplete; they found conditions sufficient to prove existence of foliations, and also conditions sufficient to prove their nonexistence, and for manifolds satisfying neither conditions, conjectured nonexistence of foliations.

As a corollary of proving the conjecture, together with results of Brittenham [B] and Claus [C] we get that these manifolds (in the conjecture) do not admit essential laminations. More precisely,

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COROLLARY. *Let $p, q > 1$ be relatively prime integers, and let p' and q' be the unique integers satisfying:*

$$pp' = 1 \pmod{q}, \quad 0 < p' < q,$$

$$qq' = 1 \pmod{p}, \quad 0 < q' < p.$$

If M is a Seifert manifold with Seifert invariants $(0; (1, -1), (\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3))$ (i.e. three exceptional fibers over S^2), such that, up to permuting indices, $\beta_1/\alpha_1 \geq (p - q')/p$, $\beta_2/\alpha_2 \geq (q - p')/q$, $\beta_3/\alpha_3 \geq 1/(p + q)$, then M admits no essential laminations. In particular, it does not admit any foliations without compact leaves (or without Reeb components).

Theorem 1 (below) implies similar results for Seifert manifolds with more than three exceptional fibers over S^2 (see [JN2]).

2. Definitions, and statement of the conjecture

Let $\text{homeo}(\mathbf{R})$ denote the group of self-homeomorphisms of \mathbf{R} under composition, and $\text{sh}(\gamma) \in \text{homeo}(\mathbf{R})$ the shift function $\text{sh}(\gamma)(x) = x + \gamma$, $x, \gamma \in \mathbf{R}$. By abuse of notation, we also let $\text{sh}(\gamma) \in \text{homeo}(S^1)$ denote the circle homeomorphism which rotates every point of S^1 by γ ; so γ can be reduced mod \mathbf{Z} . We say that $f \in \text{homeo}(S^1)$ is a *shift conjugate* if it is conjugate in the group $\text{homeo}(S^1)$ to $\text{sh}(\gamma)$ for some γ .

In [EHN] it is proven that:

THEOREM 3.5 [EHN]. *A Seifert manifold M admits a transverse foliation if and only if there exists a homomorphism $\phi : \pi_1(M) \rightarrow \text{homeo}(\mathbf{R})$ with $\phi(z) = \text{sh}(1)$, where $z \in \pi_1(M)$ is the class of a regular fiber of M .*

Let \mathbf{R} cover $S^1 \cong \mathbf{R}/\mathbf{Z}$ by the map $x \mapsto x \pmod{\mathbf{Z}}$. Then given $f \in \text{homeo}(S^1)$, and a lift $\tilde{f} \in \text{homeo}(\mathbf{R})$ of f , we define the rotation number of \tilde{f} as:

$$\text{rot}(\tilde{f}) = \lim_{n \rightarrow \infty} \frac{1}{n} (\tilde{f}^n(x) - \tilde{f}(x))$$

where $x \in \mathbf{R}$ is arbitrary. It is easy to check that this is well-defined, i.e. the limit exists, and is independent of x . We also define $\text{rot}(f) = \text{rot}(\tilde{f}) \pmod{\mathbf{Z}}$. Note that rotation number of a circle homeomorphism does not change under conjugation.

If M is a Seifert manifold over S^2 with n singular fibers, then we can write its Seifert invariants as $(g = 0; (1, -b), (\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n))$, where $g = 0$ is the genus of the base space S^2 , $\alpha_i, \beta_i \in \mathbf{Z}^+$, $0 < \beta_i/\alpha_i < 1$, and $b \in \mathbf{Z}$ (the pair $(1, -b)$ does not represent a singular fiber, but is rather used to normalize $\beta_i/\alpha_i \pmod{\mathbf{Z}}$; it plays the role of the Euler Class; see [JN3] or [S]). Then Theorem 3.5 of [EHN] (above) implies that M admits a transverse foliation if and only if $(b; \beta_1/\alpha_1, \dots, \beta_n/\alpha_n)$ is realizable (defined below).

DEFINITION. Let $n \in \mathbf{Z}^+$, $J \subset \{1, \dots, n\}$, $b \in \mathbf{Z}$, and for $i = 1, \dots, n$, $\gamma_i \in \mathbf{R}$. Then we say $(J; b; \gamma_1, \dots, \gamma_n)$ is **realizable** if $\exists f_i \in \text{homeo}(S^1)$ with some lift \tilde{f}_i such that $\text{rot}(\tilde{f}_i) = \gamma_i$, f_i is a shift conjugate for $i \in J$, and $\tilde{f}_n \circ \dots \circ \tilde{f}_1 = \text{sh}(b)$.

Note: When we omit J and simply write $(b; \gamma_1, \dots, \gamma_n)$, it is to be understood that $J = \{1, \dots, n\}$.

Deciding when $(J; b; \gamma_1, \dots, \gamma_n)$ is realizable is trivial for $n = 2$ (when $\gamma_1 + \gamma_2 = b$). The case of $n \geq 4$ is inductively reduced to the case of $n = 3$. When $n = 3$, $(J; b; \gamma_1, \dots, \gamma_n)$ is not realizable unless $b = 1$ or 2 . The $b = 2$ case is easily reduced to the $b = 1$ case by replacing γ_i by $1 - \gamma_i$ (for proofs of these facts see [JN2]). Now it is easy to check that realizing $(J; 1; \gamma_1, \gamma_2, \gamma_3)$ is equivalent to finding $f_i \in \text{homeo}(S^1)$ such that $\text{rot}(f_i) = \gamma_i$, f_i is a shift conjugate for $i \in J$, and $f_3 \circ f_2 \circ f_1 = \text{id}$, where “id” denotes identity on S^1 .

In [EHN] it was shown that if $\sum_{i=1}^3 \gamma_i \leq 1$, then $(1; \gamma_1, \gamma_2, \gamma_3)$ is realizable. Then in [JN1] it was shown that in fact $(1; \gamma_1, \gamma_2, \gamma_3)$ is realizable by Mobius maps if and only if $\sum_{i=1}^3 \gamma_i \leq 1$. A beautiful and simple proof of this fact using hyperbolic geometry on the unit disk (which was shown to me by Eric Klassen) is as follows. Given three elliptic Mobius maps f_i , $i = 1, 2, 3$, let α_i be the angles of the hyperbolic triangle formed by the fixed points of f_i . Then it is easy to show that the product of f_i is equal to identity if and only if $\alpha_i = \pi\gamma_i$, where $\gamma_i = \text{rot}(f_i)$. And there exists a (possibly singular) triangle with angles $\pi\gamma_i$ if and only if $\sum_{i=1}^3 \gamma_i \leq 1$.

Then in [JN2] more solutions were found using the following great idea. Given $f_3 \circ f_2 \circ f_1 = \text{id}$, let g_i be a lift of f_i to the m -th cyclic cover of S^1 , for some fixed $m \in \mathbf{Z}^+$. Then for any $k_1, k_2, k_3 \in \mathbf{Z}$ whose sum is a multiple of m , $\text{sh}(k_3/m) \circ g_3 \circ \dots \circ \text{sh}(k_1/m) \circ g_1 = \text{id}$. Let $\gamma_i = \text{rot}(\text{sh}(k_i/m) \circ g_i)$. Then it turns out that by picking k_i appropriately we can get the sum of γ_i to be larger than 1 (by as much as $1/m$). Using this idea they showed:

THEOREM 3 [JN2]. $(J; 1; \gamma_1, \gamma_2, \gamma_3)$ is realizable if there exist integers $0 < a < m$ such that for some permutation of μ_i we have: $\gamma_i < \mu_i$ for $i \in J$, $\gamma_i \leq \mu_i$ for $i \notin J$, where $(\mu_1, \mu_2, \mu_3) = (a/m, (m - a)/m, 1/m)$.

In [JN2] they also showed that most of the $(J; 1; \gamma_1, \gamma_2, \gamma_3)$'s (most in the sense of Euclidean volume in $[0, 1]^3$) which do not satisfy the hypothesis of the above theorem are not realizable, and conjectured that in fact all such $(J; 1; \gamma_1, \gamma_2, \gamma_3)$'s are not realizable.

CONJECTURE [JN2]. *If $(J; 1; \gamma_1, \gamma_2, \gamma_3)$ does not satisfy the hypothesis of Theorem 3 above, then it is not realizable.*

3. Proof of the conjecture

In [JN2], though not stated explicitly, it is shown (in Section 6) that the following non-realizability statement is equivalent to the conjecture.

THEOREM 1. *Let $p, q > 1$ be relatively prime integers, and let p' and q' be the unique integers satisfying:*

$$pp' = 1 \pmod{q}, \quad 0 < p' < q,$$

$$qq' = 1 \pmod{p}, \quad 0 < q' < p.$$

Let $\gamma_1 = (p - q')/p$, $\gamma_2 = (q - p')/q$, and $\gamma_3 \geq 1/(p + q)$. Then given $J \subset \{1, 2, 3\}$ and (μ_1, μ_2, μ_3) such that $\mu_i \geq \gamma_i$ for $i \in J$, $\mu_i > \gamma_i$ for $i \notin J$, $(J; 1; \mu_1, \mu_2, \mu_3)$ is not realizable.

The proof follows immediately from the following two lemmas.

MAIN LEMMA. *Let p, q, p', q' be as in Theorem 1 above, and let $\gamma_1 = (p - q')/p$, $\gamma_2 = (q - p')/q$. Then for all $\gamma_3 \geq 1/(p + q)$, $(1; \gamma_1, \gamma_2, \gamma_3)$ is not realizable.*

Proof. Let $f, g \in \text{homeo}(S^1)$ be shift conjugates whose rotation numbers are γ_1 and γ_2 respectively, and suppose h is also a shift conjugate, satisfying $h \circ g \circ f = \text{id}$. Then we want to show $\text{rot}(h) < 1/(p + q)$. We can assume $p < q$. Fix $x_0 \in S^1$, and let $\{x_0, x_1, \dots, x_{q-1}\}$ be its orbit under g , such that on S^1 $x_{i-1} < x_i < x_{i+1}$, where of course the index of x is always mod q . (To be rigorous, we could lift everything to the universal cover, but for the sake of simplicity, we do not. To make sense of $a < b < c$ however, it is enough to fix an orientation on S^1 ; then “ b is between a and c ” means it is on the arc from a to c).

CLAIM. $x_{p'-1} < f(x_0) < x_{p'}$, and $x_0 < h(x_0) < x_1$.

Proof of Claim. By definition $g(x_{p'}) = x_{p'+(q-p')} = x_0 = hgf(x_0)$, and $\text{rot}(h) > 0$, so clearly $x_0 < f(x_0) < x_{p'}$. So $\forall x \in S^1$, in going from x to $f(x)$ we “jump over” at most p' x_i 's, i.e. the cardinality of the set $\{x_i \mid x \leq x_i < f(x)\}$ is $\leq p'$. So from x to $f^{p-1}(x)$ we jump over at most $(p-1)p'$ x_i 's. But from x to $f^p(x) = x$ we go around S^1 $p - q'$ times, so we jump over exactly $(p - q')q$ x_i 's. Therefore from $f^{p-1}(x)$ to $f^p(x)$ we must jump over at least $(p - q')q - (p - 1)p' = pq - qq' - pp' + p' = (p' - 1)x_i$'s (since $pp' + qq' = pq + 1$). This shows that for exactly one $j \in \{0, \dots, p-1\}$ there are $(p' - 1)$ x_i 's between $f^j(x)$ and $f^{j+1}(x)$, and for all other j there are p' x_i 's.

Now by a symmetrical argument we see that $x_{q-p'} < f^{-1}(x_0) < x_0$, i.e. between $f^{p-1}(x_0)$ and $f^p(x_0)$ there are only $(p' - 1)$ x_i 's (since we defined “between” to be left inclusive, right exclusive), so by above, $\forall j \neq p' - 1 \pmod p$, there must be p' x_i 's between $f^j(x_0)$ and $f^{j+1}(x_0)$. Therefore $x_{p'-1} < f(x_0) < x_{p'}$.

Now $h(x_0) = f^{-1}g^{-1}(x_0) = f^{-1}(x_{p'})$, and by above we can check that $f(x_0) < x_{p'} < f(x_1)$, so $x_0 < h(x_0) < x_1$. \square (Claim)

So between each x_i and x_{i+1} we can “fit an h ”. To prove the lemma we will show that for at least $p - 1$ distinct i 's we can “fit an extra nonoverlapping h ” between x_i and x_{i+1} (i.e. $x_i < h^2(x_i) \leq x_{i+1}$). And then we will “fit one more h ” somewhere else, as explained later, so that in the end $p + q$ nonoverlapping h 's will fit on S^1 , showing that $\text{rot}(h) < 1/(p + q)$.

Let $y_0 = h(x_0)$, and as with x_i , let $\{y_0, y_1, \dots, y_{q-1}\}$ be the orbit of y_0 under g , so that $x_0 < y_0 < x_1 < y_1 < \dots < f(y_0) = fh(x_0) = g^{-1}(x_0) = x_{p'}$. So between y_0 and $f(y_0)$ there are only $(p' - 1)$ x_i 's, which implies that for $i = 2, \dots, p - 1$, $x_{ip'-1} < f^i(y_0) < x_{ip'}$. Similarly, $y_{q-p'} < f^{-1}(y_0) < y_0$, so between $f^{-1}(y_0)$ and y_0 there are only $(p' - 1)$ y_i 's. It follows that for $i = 1, \dots, p - 1$, $y_{ip'-1} < f^i(y_0) < y_{ip'}$. So we get:

$$y_{ip'-1} < f^i(y_0) < x_{ip'}, \quad i = 2, \dots, p - 1; \quad f(y_0) = x'_p$$

Now for $i = 1, \dots, p - 1$, we have: $g^{-1}(y_{ip'-1}) = y_{(i+1)p'-1} \leq f^{i+1}(y_0)$ (with equality iff $i = p - 1$), so $f^{-1}(y_{(i+1)p'-1}) \leq f^i(y_0) \leq x_{ip'}$ ($f^i(y_0) = x_{ip'}$ iff $i = 1$), so $h(y_{ip'-1}) = f^{-1}g^{-1}(y_{ip'-1}) \leq x_{ip'}$, so:

$$y_{ip'-1} < h(y_{ip'-1}) \leq x_{ip'}, \quad i = 1, \dots, p - 1 \tag{1}$$

We can assume that $x_1 < h(y_0) < x_2$, since otherwise $\forall i, x_i < h^2(x_i) \leq x_{i+1}$, so $\text{rot}(h) \leq 1/2q < 1/(p + q)$, and we are done.

Let $z = h(y_0)$. $g^{-1}(y_0) = y_{p'}$, so $f(z) = y_{p'}$, so $x_{p'} < f(z) < x_{p'+1}$. And $y_0 < x_1 < z < y_1 < x_2$, so by a “counting” argument as above, for $i = 1, \dots, p$,

$x_{ip'} < f^i(z) < x_{ip'+1}$, and for $i = 2, \dots, p$, $y_{ip'-1} < f^i(z) < y_{ip'}$. So:

$$x_{ip'} < f^i(z) < y_{ip'}, \quad i = 2, \dots, p; \quad x_{p'} < f(z) = y_{p'} < x_{p'+1}$$

For $i = 1, \dots, p-1$, $g^{-1}(x_{ip'}) = x_{(i+1)p'} < f^{i+1}(z)$, so $f^{-1}g^{-1}(x_{ip'}) < f^i(z) \leq y_{ip'}$ (equality iff $i = 1$), so:

$$x_{ip'} < h(x_{ip'}) < y_{ip'}, \quad i = 1, \dots, p-1 \quad (2)$$

Equations (1) and (2) imply:

$$y_{ip'-1} < h^2(y_{ip'-1}) < y_{ip'}, \quad i = 1, \dots, p-1 \quad (3)$$

Let $i_0 \in \{1, \dots, p-1\}$ be such that for $i \in \{1, \dots, p-1\}$, $y_0 < y_{ip'-1} \leq y_{i_0p'-1}$ implies $i = i_0 \bmod q$. Since $y_i < h(y_i) < y_{i+1}$, $y_0 < h^{i_0p'-1}(y_0) < y_{i_0p'-1}$. Furthermore, $h(x_0) = y_0$, and by (1) $y_{i_0p'-1} < h(y_{i_0p'-1}) \leq x_{i_0p'}$, therefore $x_0 < h^{i_0p'+1}(x_0) < x_{i_0p'}$. But x_0 was arbitrary (to prove equations (2) and (3) x_0 was not arbitrary, but for (1), and hence in this paragraph, it is), so:

$$y_{-1} < h^{i_0p'+1}(y_{-1}) < y_{i_0p'-1} \quad (4)$$

Now p' and q are also relatively prime, so $y_{ip'} \neq y_0$ for any $i \neq 0 \bmod q$, and by assumption $p < q$, so:

$$\text{for } i, j \in \{1, \dots, p-1\}, \quad i \neq j \text{ implies } y_{ip'} \neq y_{jp'} \quad (5)$$

and similarly

$$y_{ip'-1} \neq y_{-1}, \quad i = 1, \dots, p-1 \quad (6)$$

Equation (3) gives $p-1$ “extra h ’s”, and (5) says we are not counting any of them more than once. (4) gives “one more h ”, and it was not already counted in (3) because of (6) and the way i_0 was chosen. So we get the desired $p+q$ nonoverlapping h ’s on S^1 . \square (Main Lemma)

LEMMA (Weak Monotonicity). Fix $\gamma_1, \gamma_2, \gamma_3 \in (0, 1)$, and suppose $\forall \gamma \geq \gamma_3$, $(1; \gamma_1, \gamma_2, \gamma)$ is not realizable. Then given $J \subset \{1, 2, 3\}$ and (μ_1, μ_2, μ_3) such that $\mu_i \geq \gamma_i$ for $i \in J$, $\mu_i > \gamma_i$ for $i \notin J$, $(J; 1; \mu_1, \mu_2, \mu_3)$ is not realizable.

Remark. A stronger lemma (which follows after having proved The Conjecture, but which we could not prove “directly”) would be obtained by weakening the hypothesis to only “ $(1; \gamma_1, \gamma_2, \gamma_3)$ not realizable”. Hence *Weak Monotonicity*.

Proof. In the following, we repeatedly use the fact that rotation number is continuous ([H], Chapter II, Proposition 2.7).

Suppose towards contradiction, that $\exists \phi_i \in \text{homeo}(S^1)$ such that $\text{rot}(\phi_i) = \mu_i$ and $\phi_3 \circ \phi_2 \circ \phi_1 = \text{id}$. Write $\theta = \phi_3^{-1} = \phi_2 \circ \phi_1$.

CASE 1. There is no i with $\mu_i = \gamma_i$. So in particular, $\text{rot}(\theta) < 1 - \gamma_3$.

Step 1. For $i = 1, 2$ perturb ϕ_i slightly, if necessary, so that: (1) ϕ_i is now smooth, (2) $\text{rot}(\phi_i)$ is still $> \gamma_i$, and (3) $\text{rot}(\theta) = \text{rot}(\phi_2 \circ \phi_1)$ is still $< 1 - \gamma_3$.

Step 2. For $i = 1, 2$ replace ϕ_i by $\text{sh}(-\epsilon_i) \circ \phi_i$, $\epsilon_i \geq 0$, so that $\text{rot}(\phi_i)$ is now irrational, but still $> \gamma_i$. Clearly $\text{rot}(\theta)$ is still $< 1 - \gamma_3$ (even if ϵ_i is not small, which it may not be).

Now by Denjoy’s Theorem ([CFS], section 3.4), since ϕ_i is smooth (C^2 is enough in fact) with irrational rotation number, it must be a shift conjugate, i.e. $\phi_i = f_i \text{sh}(\rho_i) f_i^{-1}$ for some $f_i \in \text{homeo}(S^1)$, where $\rho_i = \text{rot}(\phi_i)$. Now, Denjoy’s Theorem does not guarantee that f_i will be smooth, so we perturb it slightly if necessary, so that it is smooth, and $\text{rot}(\theta) < 1 - \gamma_3$ still holds.

Step 3. $\gamma_i < \rho_i$, so $f_i \text{sh}(\gamma_i) f_i^{-1} < f_i \text{sh}(\rho_i) f_i^{-1}$ ($f < g$ means $\forall x \in S^1, x \leq f(x) < g(x)$), so now we replace ϕ_i by $f_i \text{sh}(\gamma_i) f_i^{-1}$, and we still have $\text{rot}(\theta) < 1 - \gamma_3$.

Now we perturb f_i slightly if necessary, by replacing it by $\epsilon \cdot \text{id} + (1 - \epsilon) \cdot f_i$, so that $\text{rot}(\theta)$ becomes irrational, but still $< 1 - \gamma_3$. So now θ too is a shift conjugate, with $\text{rot}(\theta^{-1}) > \gamma_3$, a contradiction.

CASE 2. There is exactly one i with $\mu_i = \gamma_i$.

Say $i = 3$ (so $\mu_1 > \gamma_1, \mu_2 > \gamma_2$). Then by hypothesis, $3 \in J$, i.e., $\phi_3 = f \text{sh}(\gamma_3) f^{-1}$. Write $\phi_1^{-1} = \phi_3 \circ \phi_2$, and replace ϕ_3 by $f \text{sh}(\gamma_3 + \epsilon) f^{-1}$, $\epsilon > 0$ small enough so that $\text{rot}(\phi_1)$ is still $> \gamma_1$. Then we are in Case 1 again.

CASE 3. There are exactly 2 i ’s with $\mu_i = \gamma_i$.

Say $i = 2, 3$ (so $\mu_1 > \gamma_1$). Then by hypothesis, $\{2, 3\} \in J$. So $\phi_3 = f \text{sh}(\gamma_3) f^{-1}$. Write $\phi_1^{-1} = \phi_3 \circ \phi_2$, and replace ϕ_3 by $f \text{sh}(\gamma_3 + \epsilon) f^{-1}$, $\epsilon > 0$ small enough so that $\text{rot}(\phi_1)$ is still $> \gamma_1$. Then we are in Case 2 again.

And of course when $\mu_i = \gamma_i$ for all i , we have nonrealizability by hypothesis. □(Weak Monotonicity Lemma)

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