Polynomial invariants of representations of quivers

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Let k be an algebraically closed field of arbitrary characteristic. Recall that if an affine algebraic group G over k acts on an affine variety Z then we get an induced action of G on the coordinate algebra $k[Z]$, given by $(x \cdot f)(z) = f(x^{-1}z)$, for $x \in G$, $f \in k[Z]$ and $z \in Z$. We consider here the space $R(Q, \alpha)$ of all k-representations of a quiver Q with given dimension vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. There is a natural action of the product of general linear groups $\mathrm{GL}(\alpha) = \mathrm{GL}(\alpha_1, k) \times \mathrm{GL}(\alpha_2, k) \times \cdots \times \mathrm{GL}(\alpha_n, k)$ on $R(Q, \alpha)$ and the purpose of this note is to describe generators for the algebra of polynomial invariants $k[R(Q, \alpha)]^{\mathrm{GL}(\alpha)_\theta}$ of the coordinate algebra $k[R(Q, \alpha)]$, where $\mathrm{GL}(\alpha)_\theta$ is the centralizer in $\mathrm{GL}(\alpha)$ of an element θ. In particular we show that $k[R(Q, \alpha)]^{\mathrm{GL}(\alpha)}$ is generated by the coefficients of the characteristic polynomials of products over oriented cycles. In characteristic zero this is a result of Le Bruyn and Procesi, [4], Theorem 1. I am very grateful to Dr. W. W. Crawley-Boevey for bringing this problem to my attention.

By a quiver we mean a quadruple $Q = (V, A, h, t)$, consisting of the vertex set $V = \{1, 2, \ldots, n\}$, a finite set A of arrows and maps $h : A \to V$, $t : A \to V$ which assign to an arrow $a \in A$ its head, $h(a)$, and tail, $t(a)$.

Let $E_1, E_2, \ldots, E_n$ be finite dimensional k-vector spaces and let $E = E_1 \oplus E_2 \oplus \cdots \oplus E_n$. Let $\alpha_i = \dim_k E_i$, $1 \le i \le n$ and let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. We write $\mathrm{GL}(\alpha) = \mathrm{GL}(E_1) \times \mathrm{GL}(E_2) \times \cdots \times \mathrm{GL}(E_n)$ and identify $\mathrm{GL}(\alpha)$ with a subgroup of $\mathrm{GL}(E)$. Thus $\mathrm{GL}(\alpha)$ is the centralizer in $\mathrm{GL}(E)$ of a linear endomorphism σ of E which acts as multiplication by c_i on E_i, $1 \le i \le n$, for distinct scalars $c_1, c_2, \ldots, c_n \in k$. Then $R(Q, \alpha) = \Pi_{a \in A} \mathrm{Hom}_k(E_{t(a)}, E_{h(a)})$ is the space of all k-representations of Q on the spaces $E_1, E_2, \ldots, E_n$. Now $\mathrm{GL}(\alpha)$ acts rationally on $R(Q, \alpha)$ by $g \cdot (y_a)_{a \in A} = (g_{h(a)} y_a g_{t(a)}^{-1})_{a \in A}$, for $g = (g_1, g_2, \ldots, g_n) \in \mathrm{GL}(\alpha)$ and $(y_a)_{a \in A} \in R(Q, \alpha)$. For an endomorphism z of a k-vector space E of finite dimension d and non-negative integer $s \le d$ we let $\chi_s(z)$ denote $(-1)^s$ times the coefficient of t^{d-s} in the characteristic polynomial $\det(tI - z)$ of z (where I is the identity map on E). In the case in which Q has only one vertex the following becomes the description of generators of matrix invariants given in [3], §2, Theorem 1.

PROPOSITION. *The algebra of invariants* $k[R(Q, \alpha)]^{\mathrm{GL}\,(\alpha)}$ *is generated by the functions* $(y_a)_{a \in A} \mapsto \chi_s(y_{a_r} \cdots y_{a_2} y_{a_1})$, *where* $(a_1, a_2, \ldots, a_r)$ *is an oriented cycle and* $s \geq 0$.

Proof. First suppose that Q is the quiver which has m arrows between each pair of vertices, i.e. there is a positive integer m such that for each $(p, q) \in V \times V$ there are precisely m arrows $a \in A$ with $t(a) = p$ and $h(a) = q$. We write A as a disjoint union $A = A_1 \cup A_2 \cup \cdots \cup A_m$ in such a way that for each $(p, q) \in V \times V$ and $1 \leq r \leq m$ there is exactly one element $a \in A_r$ with $t(a) = p$ and $h(a) = q$. We regard $\mathrm{End}_k (E)$ as a $\mathrm{GL}\,(\alpha)$-module via conjugation and $\mathrm{End}_k (E)^m$ as the direct sum $\mathrm{End}_k (E) \oplus \mathrm{End}_k (E) \oplus \cdots \oplus \mathrm{End}_k (E)$. Then we have an isomorphism of $\mathrm{GL}\,(\alpha)$-modules (and varieties) $\phi : R(Q, \alpha) \to \mathrm{End}_k (E)^m$ given by $\phi((y_a)_{a \in A}) = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_m)$, for $(y_a)_{a \in A} \in R(Q, \alpha)$, where $\bar{y}_r = \Sigma_{a \in A_r} y_a$, for $1 \leq r \leq m$. Now the comorphism $\phi^* : k[\mathrm{End}_k (E)^m] \to k[R(Q, \alpha)]$ induces an isomorphism $k[\mathrm{End}_k (E)^m]^{\mathrm{GL}\,(\alpha)} \to k[R(Q, \alpha)]^{\mathrm{GL}\,(\alpha)}$ on invariants. By [3], §2 Theorem 2, we have that $k[\mathrm{End}_k (E)^m]^{\mathrm{GL}\,(\alpha)}$ is generated by the functions $(\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_m) \mapsto \chi_u(\sigma^{q_1} \bar{y}_{i_1} \sigma^{q_2} \bar{y}_{i_2} \cdots \sigma^{q_r} \bar{y}_{i_r})$, for $r \geq 1$, $q_1, q_2, \ldots, q_r \geq 0$, $u \geq 0$ and $(i_1, i_2, \ldots, i_r)$ an r-tuple with entries in $\{1, 2, \ldots, m\}$. Therefore $k[R(Q, \alpha)]^{\mathrm{GL}\,(\alpha)}$ is generated by functions of the form

$$(y_a)_{a \in A} \mapsto \chi_u\left(\sigma^{q_1}\left(\sum_{a \in A_{i_1}} y_a\right)\sigma^{q_2}\left(\sum_{a \in A_{i_2}} y_a\right) \cdots \sigma^{q_r}\left(\sum_{a \in A_{i_r}} y_a\right)\right)$$

with r, $(i_1, i_2, \ldots, i_r)$, and $q_1, q_2, \ldots, q_r$ as above. However, $\sigma y_a = c_{h(a)} y_a$ for $a \in A$, so the above function is

$$(y_a)_{a \in A} \mapsto \chi_u\left(\left(\sum_{a \in A_{i_1}} c_{h(a)}^{q_1} y_a\right)\left(\sum_{a \in A_{i_2}} c_{h(a)}^{q_2} y_a\right) \cdots \left(\sum_{a \in A_{i_r}} c_{h(a)}^{q_r} y_a\right)\right).$$

However, as is well known, a signed coefficient $\chi_u(z_1 + z_2)$, of the characteristic polynomial of a sum of endomorphisms z_1, z_2, is a linear combination of products of the coefficients of the characteristic coefficients in monomials in z_1 and z_2. (Also, this follows from the main result of [3], since the function $(z_1, z_2) \mapsto \chi_u(z_1 + z_2)$ is a polynomial invariant for the action of the general linear group by simultaneous conjugation on pairs on endomorphisms.) Hence the above function is a linear combination of products of functions of the form

$$(y_a)_{a \in A} \mapsto \chi_s(y_{a_r} \cdots y_{a_2} y_{a_1})$$

for $a_1, a_2, \ldots, a_r \in A$, $s \geq 0$. Now, for $a, b \in A$, we have $y_a \in \mathrm{Hom}_k (E_{t(a)}, E_{h(a)})$ and $y_b \in \mathrm{Hom}_k (E_{t(b)}, E_{h(b)})$ so that $y_a y_b$ is zero unless $h(a) = t(b)$. Thus $y_{a_r} \cdots y_{a_2} y_{a_1}$ is

zero unless $h(a_1) = t(a_2)$, $h(a_2) = t(a_3), \ldots, h(a_r) = t(a_1)$. Moreover, $y_{a_r} \cdots y_{a_2} y_{a_1}$ belongs to $\mathrm{Hom}_k (E_{t(a_1)}, E_{h(a_r)}) \le \mathrm{End}_k (E)$ and, for an element z of $\mathrm{Hom}_k (E_i, E_j)$, we have $\chi_s(z) = 0$ for all $s > 0$ unless $i = j$. Hence $k[R(Q, \alpha)]^{\mathrm{GL} (\alpha)}$ is generated by the functions $(y_a)_{a \in A} \mapsto \chi_s(y_{a_r} \cdots y_{a_2} y_{a_1})$, where $(a_1, a_2, \ldots, a_r)$ is an oriented cycle and $s \ge 0$.

To conclude we make use of the elementary remark that if $L = M \oplus N$ is a direct sum decomposition of a finite dimensional rational H-module L, where H is an affine algebraic group over k, then the restriction map $k[L] \to k[M]$ is a split surjection of H-modules and hence the induced map $k[L]^H \to k[M]^H$ is surjective. So now let Q be arbitrary. Let m be a positive integer such that for each $(p, q) \in V \times V$ the number of arrows $a \in A$ satisfying $t(a) = p$ and $h(a) = q$ is at most m. Let $\hat{Q}$ be a (complete) quiver on the same vertex set V with set of arrows $\hat{A}$ containing A such that for each $(p, q) \in V \times V$ there are exactly m arrows $a \in \hat{A}$ with $t(a) = p$ and $h(a) = q$. Let Q' be the complement of Q in $\hat{Q}$, i.e. the quiver on vertex set V with arrows $A' = \hat{A} \backslash A$. We identify $R(Q, \alpha)$ with the subspace of $R(\hat{Q}, \alpha)$ consisting of the elements $(y_a)_{a \in \hat{A}}$ such that $y_a = 0$ for $a \notin A$. We similarly identify $R(Q', \alpha)$ with a subspace of $R(\hat{Q}, \alpha)$. Then $R(\hat{Q}, \alpha) = R(Q, \alpha) \oplus R(Q', \alpha)$ is a decomposition of GL (α)-modules. Hence the map $k[R(\hat{Q}, \alpha)]^{\mathrm{GL} (\alpha)} \to k[R(Q, \alpha)]^{\mathrm{GL} (\alpha)}$ is surjective. By the case already considered $k[R(\hat{Q}, \alpha)]^{\mathrm{GL} (\alpha)}$ is generated by the functions $(y_a)_{a \in \hat{A}} \mapsto \chi_s(y_{a_r} \cdots y_{a_2} y_{a_1})$, where $(a_1, a_2, \ldots, a_r)$ is an oriented cycle in $\hat{A}$, and $s \ge 0$. By restricting these functions to $R(Q, \alpha)$ we get that $k[R(Q, \alpha)]^{\mathrm{GL} (\alpha)}$ is generated by the functions $(y_a)_{a \in A} \mapsto \chi_s(y_{a_r} \cdots y_{a_2} y_{a_1})$, where $(a_1, a_2, \ldots, a_r)$ is an oriented cycle in A, and $s \ge 0$.

We now generalize this result to give generators for the invariants of $k[R(Q, \alpha)]$, for the action of a centralizer in GL (α). In the case $[V] = 1$ this is [3], §2, Theorem 2, and in general follows from the Proposition above in the same way that [3], §2, Theorem 2 follows from [3], §2, Theorem 1.

Let $\theta_i \in \mathrm{End}_k (E_i)$ and let GL $(E_i)_{\theta_i}$ be the centralizer of θ_i in GL (E_i), $1 \le i \le n$. Let $\theta = (\theta_1, \theta_2, \ldots, \theta_n) \in \mathrm{End}_k (E_1) \oplus \mathrm{End}_k (E_2) \oplus \cdots \oplus \mathrm{End}_k (E_n)$ and let GL $(\alpha)_\theta = \mathrm{GL} (E_1)_{\theta_1} \times \mathrm{GL} (E_2)_{\theta_2} \times \cdots \times \mathrm{GL} (E_n)_{\theta_n}$.

We recall, from [2], the notion of a good pair of varieties. Let G be a reductive affine algebraic group over k. By a *good filtration* of a rational G-module M we mean an ascending filtration $0 = M_0 \le M_1 \le M_2 \le \cdots$ such that, for each $i > 0$, the section M_i/M_{i+1} is either zero or isomorphic to a module induced from a one dimensional module for a Borel subgroup of G. We call Z a good G-variety if the coordinate algebra $k[Z]$ admits a good G-module filtration. By a *good pair* of G varieties we mean a pair (Z, T), where Z is an affine G-variety, T is a closed G-stable subvariety of Z and the G-modules $k[Z]$ and I_T admit good G-module filtrations, where $I_T \le k[Z]$ is the ideal of T. Recall from [2], §1.3, that if (Z, T) is

a good pair of G-varieties then Z and T are good G-varieties. The main point about good pairs, as far as invariant theory is concerned, is that the restriction map on fixed points $k[Z]^G \to k[T]^G$ is surjective (by [2], Proposition 1.4a).

THEOREM. *The algebra of invariants* $k[R(Q, \alpha)]^{GL\,(\alpha)_\theta}$ *is generated by the functions* $(y_a)_{a \in A} \mapsto \chi_s(\theta^{q_r}_{h(a_r)} y_{a_r} \cdots \theta^{q_2}_{h(a_2)} y_{a_2} \theta^{q_1}_{h(a_1)} y_{a_1})$, *for* $(a_1, a_2, \ldots, a_r)$ *an oriented cycle,* $q_1, q_2, \ldots, q_r \geq 0$ *and* $s \geq 0$.

Proof. Let Q^+ be the quiver (V^+, A^+, t^+, h^+) obtained from Q by adding n extra loops, one at each vertex. Thus we have $V^+ = V$, $A^+ = A \cup L$, the disjoint union of A and $L = \{l_1, l_2, \ldots, l_n\}$, $t^+|_A = t$, $h^+|_A = h$, and $t^+(l_i) = h^+(l_i) = i$, for $1 \leq i \leq n$. We let $\alpha^+ = \alpha$ and take

$$R(Q^+, \alpha^+) = \prod_{a \in A^+} \mathrm{Hom}_k\,(E_{t(a)}, E_{h(a)}) = R(Q, \alpha) \times S$$

where $S = \Pi_{l \in L}\,\mathrm{Hom}_k\,(E_{t(l)}, E_{h(l)}) = \bigoplus_{i=1}^n \mathrm{End}_k\,(E_i)$.
For any affine $GL\,(\alpha)$-variety X we have

$$k[X]^{GL\,(\alpha)_\theta} \cong (k[X] \otimes \mathrm{Ind}^{GL\,(\alpha)}_{GL\,(\alpha)_\theta} k)^{GL\,(\alpha)}$$

by Frobenius reciprocity and the tensor identity for induction (see also [5], (1.4)). But we have the natural isomorphism $\mathrm{Ind}^{GL\,(\alpha)}_{GL\,(\alpha)_\theta} k \cong k[\overline{C(\theta)}]$, by [2], Theorem 2.2a (iii), where $\overline{C(\theta)}$ is the Zariski closure of the $GL\,(\alpha)$ conjugacy class $C(\theta)$ of θ. Thus we get $k[X]^{GL\,(\alpha)_\theta} \cong k[GL\,(\alpha) \times \overline{C(\theta)}]^{GL\,(\alpha)}$. Explicitly, we have the isomorphism $\xi : k[X \times \overline{C(\theta)}]^{GL\,(\alpha)} \to k[X]^{GL\,(\alpha)}$, $\xi(f)(x) = f(x, \theta)$, for $f \in k[X \times \overline{C(\theta)}]^{GL\,(\alpha)}$, $x \in X$. Assume now that X is a good G-variety. Then $(S, \overline{C(\theta)})$ is a good pair of $GL\,(\alpha)$-varieties by [2], Theorem 2.2a (ii), so that $(X \times S, X \times \overline{C(\theta)})$ is a good pair of $GL\,(\alpha)$-varieties, by [2], Proposition 1.3c (i). Hence the restriction $k[X \times S]^{GL\,[\alpha]} \to k[X \times \overline{C(\theta)}]$ is surjective, by [2], Lemma 2.3a (or Proposition 1.4a). Thus, for a good $GL\,(\alpha)$-variety X, we have the surjective map $\eta : k[X \times S]^{GL\,(\alpha)} \to k[X]^{GL\,(\alpha)_\theta}$, $\eta(f)(x) = f(x, \theta)$, for $f \in k[X \times \mathrm{End}_k\,(E)^n]^{GL\,(\alpha)}$, $x \in X$.

Now take $X = R(Q, \alpha)$. Then $X \times S = R(Q^+, \alpha^+)$ so every $GL\,(\alpha)_\theta$ invariant of $k[R(Q, \alpha)]$ has the form $f|_{y_l = \theta_{h(l)}, l \in L}$, for some $f \in k[R(Q^+, \alpha^+)]^{GL\,(\alpha)}$. But by the Proposition $k[R(Q^+, \alpha^+)]^{GL\,(\alpha)}$ is generated by the functions $(y_b)_{b \in A^+} \mapsto \chi_s(y_{b_t} \cdots y_{b_2} y_{b_1})$, with $(b_1, b_2, \ldots, b_t)$ an oriented cycle. Specializing y_l to $\theta_{h(l)}$, for $l \in L$, gives generators of the form described in the theorem.

Remark. We now deduce that the Proposition is valid over $\mathbb{Z}$ and stable under base change. The arguments are entirely analogous to those of [3], §3,1 so we shall

be brief about the details. Let $E_{i,\mathbb{Z}}$ be a free $\mathbb{Z}$-module of rank α_i, $1 \le i \le n$. Let $GL(\alpha)$ be the product of general linear group schemes corresponding to $E_{1,\mathbb{Z}}, E_{2,\mathbb{Z}}, \ldots, E_{n,\mathbb{Z}}$. Let $R(Q, \alpha)_{\mathbb{Z}} = \prod_{a \in A} \mathrm{Hom}_{\mathbb{Z}}(E_{t(a), \mathbb{Z}}, E_{h(a), \mathbb{Z}})$ and let $\mathbb{Z}[R(Q, \alpha)]$ be the symmetric algebra on the dual abelian group $\mathrm{Hom}_{\mathbb{Z}}(R(Q, \alpha)_{\mathbb{Z}}, \mathbb{Z})$. Let $J = \mathbb{Z}[R(Q, \alpha)]^{GL(\alpha)}$ and let J' be the subring generated by the coefficients of the characteristic polynomials of products, taken over oriented cycles, of elements of $\mathrm{Hom}_{\mathbb{Z}}(E_{t(a), \mathbb{Z}}, E_{h(a), \mathbb{Z}})$, $a \in A$. Let $A = \{a_1, a_2, \ldots, a_m\}$ have cardinality m. Then there is a natural multigrading $\mathbb{Z}[R(Q, \alpha)] = \bigoplus_{\omega \in \mathbb{N}_0^m} \mathbb{Z}[R(Q, \alpha)]_\omega$, such that a non-zero element of the dual of $\mathrm{Hom}_{\mathbb{Z}}(E_{t(a_r), \mathbb{Z}}, E_{h(a_r), \mathbb{Z}})$ has degree $(0, \ldots, 0, 1, 0, \ldots, 0)$ (1 in the rth position), for $1 \le r \le m$. This induces multigradings on $J, J', k[R(Q, \alpha)]$ and $k[R(Q, \alpha)]^{GL(\alpha)}$. For $\omega \in \mathbb{N}_0^m$, the component $k[R(Q, \alpha)]_\omega$ has a good $GL(\alpha)$-filtration (e.g. by [1], Corollary 3.2.6 and the fact that $R(Q, \alpha)$ is a good $GL(\alpha)$-variety). Furthermore, the formal character of $k[R(Q, \alpha)]_\omega$ is independent of the field k and determines $\dim_k k[R(Q, \alpha)]_\omega^{GL(\alpha)}$, which is therefore also independent of the field k. By the Proposition, the natural map $k \otimes_{\mathbb{Z}} J'_\omega \to k[R(Q, \alpha)]_\omega^{GL(\alpha)}$ is surjective for every algebraically closed field k. It follows that $J = J'$ and that the natural map $k \otimes_{\mathbb{Z}} J \to k[R(Q, \alpha)]^{GL(\alpha)}$ is an isomorphism, for every algebraically closed field k.

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