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A regularity criterion for positive weak solutions of $-\Delta u = u^\alpha$

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1. Introduction

Let Ω be an open of $\mathbb{R}^n$, in this paper we want to study the regularity of positive weak solutions of

$$-\Delta u = u^\alpha, \tag{1}$$

where $\alpha > 1$ and $u \in L^\alpha(\Omega)$.

We only assume that u is a solution of (1) in the sense of distributions, i.e. for every $\phi \in \mathscr{C}^\infty(\Omega)$ with compact support in Ω, we have

$$\int_\Omega \Delta\phi(x)u(x)\,dx = -\int_\Omega \phi(x)u^\alpha(x)\,dx.$$

The fact that we have assumed that the solution u is positive is crucial. Obviously, weak solutions of (1) have no reason to be regular on all of Ω and examples of singular solutions are given in [1], [2] and [5].

Define S to be the set of points $x \in \Omega$ for which u is not bounded in any neighborhood V of x in Ω. Let us notice that if u, solution of (1), is bounded in a neighborhood of a point $x_0 \in \Omega$, then the classical theory of regularity shows us that u is in fact regular in a neighborhood of x_0. With this definition, S the set of singularities of u, is a closed subset of Ω.

The problem is to determine the structure of S. This structure can be very complicated as the recent work of R. Schoen and S. T. Yau [8] shows in the case of the critical exponent $\alpha = (n+2)/(n-2)$.

A reasonable conjecture seems to be the following:

The Hausdorff dimension of the set of singularities is less than or equal to

$$n - \frac{2\alpha}{\alpha - 1}, \qquad \text{if } \alpha \geq \frac{n}{n-2}.$$

Let us notice that in the case where $\alpha < n/(n-2)$, a classical bootstrap argument shows that weak solutions of (1) are in fact regular.

For $u \in L^1(\Omega)$, we define the map $I_{n-2}u(x) : \Omega \to \mathbb{R} \cup \{+\infty\}$ by

$$I_{n-2}u(x) = \int_\Omega \frac{u(y)}{|x-y|^{n-2}}\, dy.$$

Multiplied by a suitable constant, $I_{n-2}u$ is nothing else than the Poisson kernel of u.

We can now give the principal result of our paper:

THEOREM 1. *For $\alpha \geq n/(n-2)$, let u be a positive weak solution of* (1) *and suppose that the map $I_{n-2}u^{\alpha-1} : \Omega \to \mathbb{R} \cup \{+\infty\}$ defined as above is continuous from Ω into $\mathbb{R} \cup \{+\infty\}$. Then the Hausdorff dimension of the singular set of u is less than or equal to $n - 2\alpha/(\alpha-1)$.*

Let us emphasize that we allow $I_{n-2}u^{\alpha-2}$ to take infinite values.

2. Intermediate results

The result given in the first part is an easy corollary of some stronger results that we give just after this definition:

DEFINITION 1. Let $f : \Omega \to \mathbb{R} \cup \{+\infty\}$. We define the jump of f at the point $x \in \Omega$ by

$$S(f)(x) = \overline{\lim_{y \to x}} f(y) - \underline{\lim_{y \to x}} f(y).$$

We add the following convention: If $\underline{\lim}_{y \to x} f(y) = +\infty$, then $S(f)(x) = 0$.

We can now state our ϵ-regularity result:

PROPOSITION 1. *Let $\alpha \geq n/(n-2)$. There exists a constant $\epsilon_0 > 0$ such that for any positive weak solution u of* (1) *the following holds*:

If

$$S(I_{n-2}u^{\alpha-1})(x) \leq \epsilon_0,$$

$$I_{n-2}u^{\alpha-1}(x) < +\infty,$$

and if

$$\overline{\lim_{R \to 0}} \frac{1}{R^\lambda} \int_{B(x, R)} u^\alpha(y)\, dy < \epsilon_0,$$

then u is regular in a neighborhood of x.

Using this proposition we prove:

COROLLARY 1. *Let $\alpha \geq n/(n-2)$ and let $\epsilon_0 > 0$ be the constant given in Proposition 1. Assume that for all $x \in \Omega$ there holds $S(I_{n-2} u^{\alpha-1})(x) \leq \epsilon_0$. Then the Hausdorff dimension of the singular set of u is less than or equal to $n - 2\alpha/(\alpha - 1)$.*

Notice that if we assume, as in Theorem 1, that the map

$$I_{n-2} u^{\alpha-1} : \Omega \to \mathbb{R} \cup \{+\infty\}$$

is continuous from Ω into $\mathbb{R} \cup \{+\infty\}$, this implies that for all $x \in \Omega$, there holds $S(I_{n-2} u^{\alpha-1})(x) = 0$. Thus Theorem 1 is a consequence of Corollary 1.

3. Proof of the results

The proof of the results is divided in a series of lemmas in order to simplify the reading.

The first lemma is an easy estimate that has already been used in [6]:

LEMMA 1. *Let u be a weak solution of (1) on Ω. Then for almost every $x \in \Omega$ we have the estimate*

$$u(x) \leq \frac{1}{\omega_n r^n} \int_{B(x, r)} u(y)\, dy + \frac{1}{n(n-2)\omega_n} \int_{B(x, r)} \frac{u^\alpha(y)}{|x - y|^{n-2}}\, dy,$$

where ω_n is the volume of the unit ball of $\mathbb{R}^n$ and $r < \operatorname{dist}(x, \partial\Omega)$.

Using the fact that u is a solution of (1), we can write for almost every $x \in \Omega$

$$\frac{d}{ds}\left(\frac{1}{s^{n-1}} \int_{\partial B(x, s)} u(y)\, d\sigma + \frac{1}{n-2} \int_0^s (t^{2-n} - s^{2-n}) \left(\int_{\partial B(x, t)} u^\alpha(y)\, d\sigma\right) dt\right) = 0.$$

Integrating from s to s' we derive the following formula

$$\frac{1}{s^{n-1}}\int_{\partial B(x,s)} u(y)\,d\sigma + \frac{1}{n-2}\int_0^s (t^{2-n} - s^{2-n})\left(\int_{\partial B(x,t)} u^\alpha(y)\,d\sigma\right)dt$$
$$= \frac{1}{s'^{n-1}}\int_{\partial B(x,s')} u(y)\,d\sigma + \frac{1}{n-2}\int_0^{s'} (t^{2-n} - s'^{2-n})\left(\int_{\partial B(x,t)} u^\alpha(y)\,d\sigma\right)dt.$$

Passing to the limit when s' goes to 0 we obtain the estimate

$$s^{n-1}u(x) \leq \frac{1}{n\omega_n}\int_{\partial B(x,s)} u(y)\,d\sigma + \frac{s^{n-1}}{n(n-2)\omega_n}\int_{B(x,s)} \frac{u^\alpha(y)}{|x-y|^{n-2}}\,dy.$$

Then we integrate this inequality on $(0, r)$ in order to obtain the inequality of Lemma 1.

Multiplying the inequality obtained in the last lemma by $u^{\alpha-1}(x)$ and integrating on the ball of center x and radius r we obtain the lemma:

LEMMA 2. *Let u be a positive weak solution of* (1) *on Ω, then there exists a constant $c_0 > 0$ such that for any $x \in \Omega$ and for any sufficiently small number $r > 0$ we have*

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} u^\alpha(y)\,dy \leq c_0\left\{\left(\frac{1}{|B(x,2r)|}\int_{B(x,2r)} u^{\alpha-1}(y)\,dy\right)^{\alpha/(\alpha-1)}\right.$$
$$\left. + \frac{1}{|B(x,2r)|}\int_{B(x,2r)} u^\alpha(y)\left(\int_{B(y,2r)} \frac{u^{\alpha-1}(z)}{|z-y|^{n-2}}\,dz\right)dy\right\}.$$

If we apply now the Proposition 1.1, page 122 of [4], we obtain the following reverse Hölder inequality:

LEMMA 3. *Let u be a positive weak solution of* (1) *on Ω and assume that there exists some $R_0 > 0$ such that for all $x \in \Omega$ with* $\operatorname{dist}(x, \partial\Omega) < R_0$ *we have*

$$\int_{B(x,R_0)} \frac{u^{\alpha-1}(y)}{|x-y|^{n-2}}\,dy < \frac{1}{2c_0},$$

where c_0 is the constant given in the last lemma. Then there exist $\beta > \alpha$ and a constant $c_1 > 0$ such that for all $x \in \Omega$ and for all $r < R_0/2$ we have

$$\left\{\frac{1}{|B(x,r)|}\int_{B(x,r)} u^\beta(y)\,dy\right\}^{1/\beta} \leq c_1\left\{\frac{1}{|B(x,2r)|}\int_{B(x,2r)} u^\alpha(y)\,dy\right\}^{1/\alpha}.$$

We now make the following assumption on solutions u of (1):

(H) There exists some $R_0 > 0$ for which

$$\int_{B(x, R_0)} \frac{u^{\alpha - 1}(y)}{|x - y|^{n-2}} \, dy < \frac{1}{2c_0},$$

for all $x \in \Omega$.

Under the hypothesis (H) we can prove the lemma:

LEMMA 4. *There are some constants $\theta \in (0, 1)$ and $\epsilon_0 > 0$ such that, for any positive weak solution u of* (1) *satisfying* (H), *any $x \in \Omega$ and any $R < R_0$ for which*

dist $(x, \partial\Omega) > 2R_0$

the following holds. If

$$\int_{B(x, R)} u^\alpha(y) \, dy < \epsilon_0^\alpha R^\lambda,$$

where $\lambda = n - 2\alpha/(\alpha - 1)$, then

$$\frac{1}{(\theta R)^\lambda} \int_{B(x, \theta R)} u^\alpha(y) \, dy \leq \frac{1}{2} \frac{1}{R^\lambda} \int_{B(x, R)} u^\alpha(y) \, dy.$$

We prove this lemma by contradiction. Let us assume that, for some suitably chosen $\theta > 0$, there exists a sequence $\epsilon_n > 0$ going to 0, a sequence u_n of positive weak solutions of (1) satisfying (H), a sequence of points $x_n \in \Omega$ and a sequence of radii $R_n < R_0$ such that

dist $(x_n, \partial\Omega) < 2R_0$,

$$\frac{1}{(\theta R_n)^\lambda} \int_{B(x_n, R_n \theta)} u_n^\alpha(y) \, dy \geq \epsilon_n^\alpha / 2$$

and

$$\frac{1}{R_n^\lambda} \int_{B(x, R_n)} u_n^\alpha(y) \, dy = \epsilon_n^\alpha.$$

Define $v_n(x) = R_n^{2/(\alpha-1)} u_n(x_n + R_n x)$ and notice that v_n is a weak positive solution of (1) on $B(0, 2)$.

Moreover, the following estimates hold

$$\frac{1}{\theta^{\lambda}} \int_{B(0,\theta)} v_n^{\alpha}(y)\, dy \geq \epsilon_n^{\alpha}/2$$

and

$$\int_{B(0,1)} v_n^{\alpha}(y)\, dy = \epsilon_n^{\alpha}.$$

In addition, from (H), for all $x \in B(0, 1)$, we have the inequality

$$\int_{B(x,1)} \frac{v_n^{\alpha-1}(y)}{|x-y|^{n-2}}\, dy < \frac{1}{2c_0}.$$

Thus, the reverse Hölder inequality that has been proved in Lemma 3 holds for the sequence v_n on $B(0, 1)$. We deduce from this that the sequence $w_n = v_n/\epsilon_n$ is solution of the equation $-\Delta w_n = \epsilon_n^{\alpha-1} w_n^{\alpha}$ and satisfies

$$\left(\int_{B(0,1/2)} w_n^{\beta}(y)\, dy\right)^{1/\beta} \leq c_1 \left(\int_{B(0,1)} w_n^{\alpha}(y)\, dy\right)^{1/\alpha},$$

$$\frac{1}{\theta^{\lambda}} \int_{B(0,\theta)} w_n^{\alpha}(y)\, dy \geq 1/2$$

and

$$\int_{B(0,1)} w_n^{\alpha}(y)\, dy = 1.$$

The sequence w_n being bounded in $L^{\beta}(B(0, 1/2))$ and in $L^{\alpha}(B(0, 1))$, we can take a subsequence, that we will still denote by w_n, such that

$w_n \to w$ strongly in $L^1(B(0, 1))$,

$w_n \to w$ almost everywhere in $B(0, 1)$,

$w_n \rightharpoonup w$ weakly in $L^{\alpha}(B(0, 1))$,

$w_n \to w$ strongly in $L^{\alpha}(B(0, 1/2))$.

Let us notice that, passing to the limit in the equation satisfied by w_n, we get $\Delta w = 0$ in $B(0, 1)$ and also $w \geq 0$.

Passing to the weak limit we finally derive the estimate

$$\int_{B(0,1)} w^\alpha(x)\, dx \leq 1.$$

w being harmonic, we deduce from this information that for all $x \in B(0, 1/2)$ we can write

$$w(x) = \frac{1}{|B(x, 1/2)|} \int_{B(x, 1/2)} w(y)\, dy,$$

whence we get the inequality

$$\frac{1}{\theta^\lambda} \int_{B(0,\theta)} w^\alpha(y)\, dy \leq c_2 \theta^{n-\lambda} \left(\int_{B(0,1)} w(y)\, dy \right)^\alpha.$$

Holder's inequality allows us to conclude that

$$\frac{1}{\theta^\lambda} \int_{B(0,\theta)} w^\alpha(y)\, dy \leq c_3 \theta^{n-\lambda} \int_{B(0,1)} w^\alpha(y)\, dy \leq c_3 \theta^{n-\lambda}.$$

If at the beginning we choose θ such that $c_3 \theta^{n-\lambda} < 1/2$ we obtain a contradiction. Hence with this choice the hypothesis cannot be true and this proves the lemma.

We are now able to state a partial regularity result:

LEMMA 5. *Any u positive weak solution of* (1) *satisfying* (H) *is regular on Ω except for a closed set whose Hausdorff dimension is less than or equal to $n - 2\alpha/(\alpha - 1)$.*

Choose $\Omega' \subset\subset \Omega$. In assumption (H), up to a reduction of R_0, we can assume that $R_0 < \operatorname{dist}(\Omega', \partial\Omega)$. Let $\epsilon_0 > 0$ be the constant obtained in the former lemma and define

$$S = \left\{ x \in \Omega' / \forall R < R_0 \int_{B(x,R)} u^\alpha(y)\, dy \geq \epsilon_0^\alpha R^\lambda \right\}.$$

The set S is closed in Ω' and has Hausdorff dimension less than or equal to $n - 2\alpha/(\alpha - 1)$.

Take some point x_0 in $\Omega'\backslash S$. By definition of S, there exists some $R_1 < R_0$ such that

$$\int_{B(x, R_1)} u^\alpha(y)\, dy < \epsilon_0^\alpha R_1^\lambda,$$

for all x in some neighborhood of x_0.

The assumptions of Lemma 4 are satisfied in some neighborhood of x_0, so we can conclude that in some neighborhood of x_0, we have

$$\frac{1}{(\theta R_1)^\lambda} \int_{B(x, \theta R_1)} u^\alpha(y)\, dy \le \frac{1}{2} \frac{1}{R_1^\lambda} \int_{B(x, R_1)} u^\alpha(y)\, dy.$$

As in the proof of Theorem 1.1, page 95 of [4], we claim that there exist some constants $\mu > \lambda$ and $c > 0$ for which

$$\int_{B(x, R)} u^\alpha(y)\, dy < cR^\mu,$$

for all x in some neighborhood of x_0 and for all $R < R_0$.

In fact we obtain by induction that, in some neighborhood of x_0, we have

$$\frac{1}{(\theta^k R_1)^\lambda} \int_{B(x, \theta^k R_1)} u^\alpha(y)\, dy \le 2^{-k} \frac{1}{R_1^\lambda} \int_{B(x, R_1)} u^\alpha(y)\, dy,$$

for all $k \in \mathbb{N}$. Choosing $\mu > \lambda$ such that $\theta^{\mu - \lambda} > \frac{1}{2}$ we derive that for some constant $c > 0$ we have

$$\int_{B(x, \theta^k R_1)} u^\alpha(y)\, dy \le c(\theta^k R_1)^\mu,$$

for all $k \in \mathbb{N}$, from which we derive the claim.

Therefore there exists a neighborhood $\omega \subset \Omega'\backslash S$ of x_0 such that $u \in L^{\alpha, \mu}(\omega)$. In a previous paper [7] we had obtained the following regularity criterion for weak solutions of (1):

THEOREM 2. *If* $u \in L^{\alpha, \mu}(\Omega)$ *is a weak solution of* (1) *and if* $\mu > n - 2\alpha/(\alpha - 1)$ *then* u *is regular in all* $\Omega' \subset\subset \Omega$.

For a definition of $L^{\alpha, \mu}(\Omega)$ see [3] or [4].

Using this result we can conclude that u is regular in a neighborhood of x_0. This finishes the proof of the lemma.

We can now derive the results stated in the second part of this paper.

Proof of Proposition 1. Proposition 1 is a simple consequence of Lemma 5.

On one hand, assume that the hypotheses of the proposition are satisfied at $x_0 \in \Omega$. Therefore there exists some $R_0 > 0$ such that

$$\int_{B(x_0, R_0)} \frac{u^{\alpha-1}(y)}{|x_0 - y|^{n-2}}\, dy < \epsilon_0.$$

On the other hand, for all x, x' in some neighborhood of x_0 we have

$$\left| \int_{B(x, R_0)} \frac{u^{\alpha-1}(y)}{|x - y|^{n-2}}\, dy - \int_{B(x', R_0)} \frac{u^{\alpha-1}(y)}{|x' - y|^{n-2}}\, dy \right| \leq 2\epsilon_0.$$

Finally the map

$$x \to \int_{\Omega \setminus B(x, R_0)} \frac{u^{\alpha-1}(y)}{|x - y|^{n-2}}\, dy,$$

is continuous in some neighborhood of x_0. We deduce from all this the existence of a neighborhood $\omega \subset \Omega$ of x_0 such that, for all $x \in \omega$

$$\int_{B(x, R_0)} \frac{u^{\alpha-1}(y)}{|x - y|^{n-2}}\, dy < 4\epsilon_0.$$

Choosing ϵ_0 small enough, the conclusion of the proposition is then a simple application of the proof of Lemma 5.

Remark. In the case where $\alpha \geq 2$ we can drop the assumption $\int_{B(x, R)} u^\alpha(x)\, dx < \epsilon_0^\alpha R^\lambda$. In fact if $I_{n-2}u^{\alpha-1}(x) < +\infty$ then for all $\epsilon > 0$ there exist some $R > 0$ such that

$$\int_{B(x, 2R)} \frac{u^{\alpha-1}(y)}{|x - y|^{n-2}}\, dy < \epsilon.$$

So, we derive the estimate

$$\int_{B(x, 2R)} u^{\alpha-1}(x)\, dx < \epsilon(2R)^{n-2}.$$

Since $\alpha \geq 2$, Hölder's inequality gives us

$$\int_{B(x,2R)} u(y)\,dy \leq \left(\int_{B(x,2R)} u^{\alpha-1}(y)\,dy\right)^{1/(\alpha-1)} |B(x,2R)|^{1-1/(\alpha-1)}.$$

Therefore

$$\int_{B(x,2R)} u(y)\,dy \leq c_4 \epsilon^{1/(\alpha-1)} R^{\lambda+2}.$$

Now, in a previous paper [7] we have proved that there exists some constant $c_5 > 0$, depending only on the dimension of the space such that

$$R^2 \int_{B(x,R)} u^{\alpha}(y)\,dy \leq c_5 \int_{B(x,2R)} u(y)\,dy,$$

for every positive weak solution of (1). The last two inequalities allow us to estimate

$$\int_{B(x,R)} u^{\alpha}(y)\,dy \leq c_6 \epsilon^{1/(\alpha-1)} R^{\lambda},$$

for some constant $c_6 > 0$ depending only on the dimension of the space. Choosing $\epsilon > 0$ such that $\epsilon_0^{\alpha} > c_6 \epsilon^{1/(\alpha-1)}$ we get the desired estimate.

We are now left with the proof of Corollary 1.

Proof of Corollary 1. It is sufficient to show that the set of points x in Ω where $I_{n-2}u^{\alpha-1}(x) = +\infty$ forms a set of Hausdorff dimension less than or equal to $n - 2\alpha/(\alpha-1)$. Denote by E this set, $u^{\alpha-1} \in L^{\alpha/(\alpha-1)}(\Omega)$, using the definition of the Riesz capacity, we deduce from this [9] that $R_{2,\alpha/(\alpha-1)}(E) = 0$, thus the Hausdorff dimension of E is less than or equal to $n - 2\alpha/(\alpha-1)$. The result of Corollary 1 is then a consequence of Proposition 1.

4. General remarks

In order to find a regularity criterion for weak positive solutions of (1) one could be tempted to consider the natural quantity

$$\frac{1}{R^{\lambda}} \int_{B(x,R)} u^{\alpha}(y)\,dy,$$

where $\lambda = n - 2\alpha/(\alpha - 1)$, and conjecture that if this quantity is small enough then u is regular in some neighborhood of x. Unfortunately this conjecture does not hold in general as can be shown using the examples given in [5]. In the last pages of their paper the authors display all the radial positive solutions of (1) in $\mathbb{R}^n$, and if

$$\alpha \in \left(\frac{n}{n-2}, \frac{n+2}{n-2}\right)$$

then they show that there exists a positive radial solution u of (1) which is singular at 0 (i.e. $u(x)$ behaves like $C/|x|^{2/(\alpha-1)}$ near $x = 0$) and regular at ∞ (i.e. $u(x)$ behaves like $c/|x|^{n-2}$ near ∞). For some parameter δ we consider the family $u_\delta(x) = \delta^{2/(\alpha-1)}u(\delta x)$. It is easy to see that u_δ is a weak positive solution of (1) having a singularity at the origin and that the quantity

$$\frac{1}{R^\lambda}\int_{B(0,R)} u_\delta^\alpha(y)\,dy,$$

can be made as small as we want if δ is chosen large enough.

We finish this paper by giving some open question:

If u if a positive weak solution of (1) and if, for some $x_0 \in \Omega$, the following condition is satisfied

$$I_{n-2}u^{\alpha-1}(x_0) < +\infty,$$

is $I_{n-2}u^{\alpha-1}(x)$ continuous at x_0?

Let us observe that a positive answer to this conjecture would prove the conjecture stated in the introduction.

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