

**Zeitschrift:** Commentarii Mathematici Helvetici  
**Herausgeber:** Schweizerische Mathematische Gesellschaft  
**Band:** 68 (1993)  
  
**Artikel:** A regularity criterion for positive weak solutions of  $-\Delta u = u \dots$   
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**DOI:** <https://doi.org/10.5169/seals-51761>

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# A regularity criterion for positive weak solutions of $-\Delta u = u^\alpha$

F. PACARD

## 1. Introduction

Let  $\Omega$  be an open of  $\mathbb{R}^n$ , in this paper we want to study the regularity of positive weak solutions of

$$-\Delta u = u^\alpha, \tag{1}$$

where  $\alpha > 1$  and  $u \in L^\alpha(\Omega)$ .

We only assume that  $u$  is a solution of (1) in the sense of distributions, i.e. for every  $\phi \in \mathcal{C}^\infty(\Omega)$  with compact support in  $\Omega$ , we have

$$\int_{\Omega} \Delta \phi(x) u(x) \, dx = - \int_{\Omega} \phi(x) u^\alpha(x) \, dx.$$

The fact that we have assumed that the solution  $u$  is positive is crucial. Obviously, weak solutions of (1) have no reason to be regular on all of  $\Omega$  and examples of singular solutions are given in [1], [2] and [5].

Define  $S$  to be the set of points  $x \in \Omega$  for which  $u$  is not bounded in any neighborhood  $V$  of  $x$  in  $\Omega$ . Let us notice that if  $u$ , solution of (1), is bounded in a neighborhood of a point  $x_0 \in \Omega$ , then the classical theory of regularity shows us that  $u$  is in fact regular in a neighborhood of  $x_0$ . With this definition,  $S$  the set of singularities of  $u$ , is a closed subset of  $\Omega$ .

The problem is to determine the structure of  $S$ . This structure can be very complicated as the recent work of R. Schoen and S. T. Yau [8] shows in the case of the critical exponent  $\alpha = (n+2)/(n-2)$ .

A reasonable conjecture seems to be the following:

The Hausdorff dimension of the set of singularities is less than or equal to

$$n - \frac{2\alpha}{\alpha - 1}, \quad \text{if } \alpha \geq \frac{n}{n-2}.$$

Let us notice that in the case where  $\alpha < n/(n-2)$ , a classical bootstrap argument shows that weak solutions of (1) are in fact regular.

For  $u \in L^1(\Omega)$ , we define the map  $I_{n-2}u(x) : \Omega \rightarrow \mathbb{R} \cup \{+\infty\}$  by

$$I_{n-2}u(x) = \int_{\Omega} \frac{u(y)}{|x-y|^{n-2}} dy.$$

Multiplied by a suitable constant,  $I_{n-2}u$  is nothing else than the Poisson kernel of  $u$ .

We can now give the principal result of our paper:

**THEOREM 1.** *For  $\alpha \geq n/(n-2)$ , let  $u$  be a positive weak solution of (1) and suppose that the map  $I_{n-2}u^{\alpha-1} : \Omega \rightarrow \mathbb{R} \cup \{+\infty\}$  defined as above is continuous from  $\Omega$  into  $\mathbb{R} \cup \{+\infty\}$ . Then the Hausdorff dimension of the singular set of  $u$  is less than or equal to  $n - 2\alpha/(\alpha - 1)$ .*

Let us emphasize that we allow  $I_{n-2}u^{\alpha-1}$  to take infinite values.

## 2. Intermediate results

The result given in the first part is an easy corollary of some stronger results that we give just after this definition:

**DEFINITION 1.** Let  $f : \Omega \rightarrow \mathbb{R} \cup \{+\infty\}$ . We define the jump of  $f$  at the point  $x \in \Omega$  by

$$S(f)(x) = \overline{\lim}_{y \rightarrow x} f(y) - \underline{\lim}_{y \rightarrow x} f(y).$$

We add the following convention: If  $\underline{\lim}_{y \rightarrow x} f(y) = +\infty$ , then  $S(f)(x) = 0$ .

We can now state our  $\epsilon$ -regularity result:

**PROPOSITION 1.** *Let  $\alpha \geq n/(n-2)$ . There exists a constant  $\epsilon_0 > 0$  such that for any positive weak solution  $u$  of (1) the following holds:*

*If*

$$S(I_{n-2}u^{\alpha-1})(x) \leq \epsilon_0,$$

$$I_{n-2}u^{\alpha-1}(x) < +\infty,$$

and if

$$\lim_{R \rightarrow 0} \frac{1}{R^\lambda} \int_{B(x, R)} u^\alpha(y) dy < \epsilon_0,$$

then  $u$  is regular in a neighborhood of  $x$ .

Using this proposition we prove:

**COROLLARY 1.** *Let  $\alpha \geq n/(n-2)$  and let  $\epsilon_0 > 0$  be the constant given in Proposition 1. Assume that for all  $x \in \Omega$  there holds  $S(I_{n-2}u^{\alpha-1})(x) \leq \epsilon_0$ . Then the Hausdorff dimension of the singular set of  $u$  is less than or equal to  $n - 2\alpha/(\alpha - 1)$ .*

Notice that if we assume, as in Theorem 1, that the map

$$I_{n-2}u^{\alpha-1} : \Omega \rightarrow \mathbb{R} \cup \{+\infty\}$$

is continuous from  $\Omega$  into  $\mathbb{R} \cup \{+\infty\}$ , this implies that for all  $x \in \Omega$ , there holds  $S(I_{n-2}u^{\alpha-1})(x) = 0$ . Thus Theorem 1 is a consequence of Corollary 1.

### 3. Proof of the results

The proof of the results is divided in a series of lemmas in order to simplify the reading.

The first lemma is an easy estimate that has already been used in [6]:

**LEMMA 1.** *Let  $u$  be a weak solution of (1) on  $\Omega$ . Then for almost every  $x \in \Omega$  we have the estimate*

$$u(x) \leq \frac{1}{\omega_n r^n} \int_{B(x, r)} u(y) dy + \frac{1}{n(n-2)\omega_n} \int_{B(x, r)} \frac{u^\alpha(y)}{|x-y|^{n-2}} dy,$$

where  $\omega_n$  is the volume of the unit ball of  $\mathbb{R}^n$  and  $r < \text{dist}(x, \partial\Omega)$ .

Using the fact that  $u$  is a solution of (1), we can write for almost every  $x \in \Omega$

$$\frac{d}{ds} \left( \frac{1}{s^{n-1}} \int_{\partial B(x, s)} u(y) d\sigma + \frac{1}{n-2} \int_0^s (t^{2-n} - s^{2-n}) \left( \int_{\partial B(x, t)} u^\alpha(y) d\sigma \right) dt \right) = 0.$$



Integrating from  $s$  to  $s'$  we derive the following formula

$$\begin{aligned} & \frac{1}{s^{n-1}} \int_{\partial B(x, s)} u(y) d\sigma + \frac{1}{n-2} \int_0^s (t^{2-n} - s^{2-n}) \left( \int_{\partial B(x, t)} u^\alpha(y) d\sigma \right) dt \\ &= \frac{1}{s'^{n-1}} \int_{\partial B(x, s')} u(y) d\sigma + \frac{1}{n-2} \int_0^{s'} (t^{2-n} - s'^{2-n}) \left( \int_{\partial B(x, t)} u^\alpha(y) d\sigma \right) dt. \end{aligned}$$

Passing to the limit when  $s'$  goes to 0 we obtain the estimate

$$s^{n-1} u(x) \leq \frac{1}{n\omega_n} \int_{\partial B(x, s)} u(y) d\sigma + \frac{s^{n-1}}{n(n-2)\omega_n} \int_{B(x, s)} \frac{u^\alpha(y)}{|x-y|^{n-2}} dy.$$

Then we integrate this inequality on  $(0, r)$  in order to obtain the inequality of Lemma 1.

Multiplying the inequality obtained in the last lemma by  $u^{\alpha-1}(x)$  and integrating on the ball of center  $x$  and radius  $r$  we obtain the lemma:

**LEMMA 2.** *Let  $u$  be a positive weak solution of (1) on  $\Omega$ , then there exists a constant  $c_0 > 0$  such that for any  $x \in \Omega$  and for any sufficiently small number  $r > 0$  we have*

$$\begin{aligned} \frac{1}{|B(x, r)|} \int_{B(x, r)} u^\alpha(y) dy &\leq c_0 \left\{ \left( \frac{1}{|B(x, 2r)|} \int_{B(x, 2r)} u^{\alpha-1}(y) dy \right)^{\alpha/(\alpha-1)} \right. \\ &\quad \left. + \frac{1}{|B(x, 2r)|} \int_{B(x, 2r)} u^\alpha(y) \left( \int_{B(y, 2r)} \frac{u^{\alpha-1}(z)}{|z-y|^{n-2}} dz \right) dy \right\}. \end{aligned}$$

If we apply now the Proposition 1.1, page 122 of [4], we obtain the following reverse Hölder inequality:

**LEMMA 3.** *Let  $u$  be a positive weak solution of (1) on  $\Omega$  and assume that there exists some  $R_0 > 0$  such that for all  $x \in \Omega$  with  $\text{dist}(x, \partial\Omega) < R_0$  we have*

$$\int_{B(x, R_0)} \frac{u^{\alpha-1}(y)}{|x-y|^{n-2}} dy < \frac{1}{2c_0},$$

where  $c_0$  is the constant given in the last lemma. Then there exist  $\beta > \alpha$  and a constant  $c_1 > 0$  such that for all  $x \in \Omega$  and for all  $r < R_0/2$  we have

$$\left\{ \frac{1}{|B(x, r)|} \int_{B(x, r)} u^\beta(y) dy \right\}^{1/\beta} \leq c_1 \left\{ \frac{1}{|B(x, 2r)|} \int_{B(x, 2r)} u^\alpha(y) dy \right\}^{1/\alpha}.$$

We now make the following assumption on solutions  $u$  of (1):

(H) There exists some  $R_0 > 0$  for which

$$\int_{B(x, R_0)} \frac{u^{\alpha-1}(y)}{|x-y|^{n-2}} dy < \frac{1}{2c_0},$$

for all  $x \in \Omega$ .

Under the hypothesis (H) we can prove the lemma:

LEMMA 4. *There are some constants  $\theta \in (0, 1)$  and  $\epsilon_0 > 0$  such that, for any positive weak solution  $u$  of (1) satisfying (H), any  $x \in \Omega$  and any  $R < R_0$  for which*

$$\text{dist}(x, \partial\Omega) > 2R_0$$

*the following holds. If*

$$\int_{B(x, R)} u^\alpha(y) dy < \epsilon_0^\alpha R^\lambda,$$

*where  $\lambda = n - 2\alpha/(\alpha - 1)$ , then*

$$\frac{1}{(\theta R)^\lambda} \int_{B(x, \theta R)} u^\alpha(y) dy \leq \frac{1}{2} \frac{1}{R^\lambda} \int_{B(x, R)} u^\alpha(y) dy.$$

We prove this lemma by contradiction. Let us assume that, for some suitably chosen  $\theta > 0$ , there exists a sequence  $\epsilon_n > 0$  going to 0, a sequence  $u_n$  of positive weak solutions of (1) satisfying (H), a sequence of points  $x_n \in \Omega$  and a sequence of radii  $R_n < R_0$  such that

$$\text{dist}(x_n, \partial\Omega) < 2R_0,$$

$$\frac{1}{(\theta R_n)^\lambda} \int_{B(x_n, R_n \theta)} u_n^\alpha(y) dy \geq \epsilon_n^\alpha / 2$$

and

$$\frac{1}{R_n^\lambda} \int_{B(x_n, R_n)} u_n^\alpha(y) dy = \epsilon_n^\alpha.$$

Define  $v_n(x) = R_n^{2/(\alpha-1)} u_n(x_n + R_n x)$  and notice that  $v_n$  is a weak positive solution of (1) on  $B(0, 2)$ .

Moreover, the following estimates hold

$$\frac{1}{\theta^\lambda} \int_{B(0, \theta)} v_n^\alpha(y) dy \geq \epsilon_n^\alpha / 2$$

and

$$\int_{B(0, 1)} v_n^\alpha(y) dy = \epsilon_n^\alpha.$$

In addition, from (H), for all  $x \in B(0, 1)$ , we have the inequality

$$\int_{B(x, 1)} \frac{v_n^{\alpha-1}(y)}{|x-y|^{n-2}} dy < \frac{1}{2c_0}.$$

Thus, the reverse Hölder inequality that has been proved in Lemma 3 holds for the sequence  $v_n$  on  $B(0, 1)$ . We deduce from this that the sequence  $w_n = v_n / \epsilon_n$  is solution of the equation  $-\Delta w_n = \epsilon_n^{\alpha-1} w_n^\alpha$  and satisfies

$$\left( \int_{B(0, 1/2)} w_n^\beta(y) dy \right)^{1/\beta} \leq c_1 \left( \int_{B(0, 1)} w_n^\alpha(y) dy \right)^{1/\alpha},$$

$$\frac{1}{\theta^\lambda} \int_{B(0, \theta)} w_n^\alpha(y) dy \geq 1/2$$

and

$$\int_{B(0, 1)} w_n^\alpha(y) dy = 1.$$

The sequence  $w_n$  being bounded in  $L^\beta(B(0, 1/2))$  and in  $L^\alpha(B(0, 1))$ , we can take a subsequence, that we will still denote by  $w_n$ , such that

$$w_n \rightarrow w \text{ strongly in } L^1(B(0, 1)),$$

$$w_n \rightarrow w \text{ almost everywhere in } B(0, 1),$$

$$w_n \rightharpoonup w \text{ weakly in } L^\alpha(B(0, 1)),$$

$$w_n \rightarrow w \text{ strongly in } L^\alpha(B(0, 1/2)).$$

Let us notice that, passing to the limit in the equation satisfied by  $w_n$ , we get  $\Delta w = 0$  in  $B(0, 1)$  and also  $w \geq 0$ .

Passing to the weak limit we finally derive the estimate

$$\int_{B(0, 1)} w^\alpha(x) dx \leq 1.$$

$w$  being harmonic, we deduce from this information that for all  $x \in B(0, 1/2)$  we can write

$$w(x) = \frac{1}{|B(x, 1/2)|} \int_{B(x, 1/2)} w(y) dy,$$

whence we get the inequality

$$\frac{1}{\theta^\lambda} \int_{B(0, \theta)} w^\alpha(y) dy \leq c_2 \theta^{n-\lambda} \left( \int_{B(0, 1)} w(y) dy \right)^\alpha.$$

Holder's inequality allows us to conclude that

$$\frac{1}{\theta^\lambda} \int_{B(0, \theta)} w^\alpha(y) dy \leq c_3 \theta^{n-\lambda} \int_{B(0, 1)} w^\alpha(y) dy \leq c_3 \theta^{n-\lambda}.$$

If at the beginning we choose  $\theta$  such that  $c_3 \theta^{n-\lambda} < 1/2$  we obtain a contradiction. Hence with this choice the hypothesis cannot be true and this proves the lemma.

We are now able to state a partial regularity result:

**LEMMA 5.** *Any  $u$  positive weak solution of (1) satisfying (H) is regular on  $\Omega$  except for a closed set whose Hausdorff dimension is less than or equal to  $n - 2\alpha/(\alpha - 1)$ .*

Choose  $\Omega' \subset \subset \Omega$ . In assumption (H), up to a reduction of  $R_0$ , we can assume that  $R_0 < \text{dist}(\Omega', \partial\Omega)$ . Let  $\epsilon_0 > 0$  be the constant obtained in the former lemma and define

$$S = \left\{ x \in \Omega' / \forall R < R_0 \int_{B(x, R)} u^\alpha(y) dy \geq \epsilon_0^\alpha R^\lambda \right\}.$$

The set  $S$  is closed in  $\Omega'$  and has Hausdorff dimension less than or equal to  $n - 2\alpha/(\alpha - 1)$ .

Take some point  $x_0$  in  $\Omega' \setminus S$ . By definition of  $S$ , there exists some  $R_1 < R_0$  such that

$$\int_{B(x, R_1)} u^\alpha(y) dy < \epsilon_0^\alpha R_1^\lambda,$$

for all  $x$  in some neighborhood of  $x_0$ .

The assumptions of Lemma 4 are satisfied in some neighborhood of  $x_0$ , so we can conclude that in some neighborhood of  $x_0$ , we have

$$\frac{1}{(\theta R_1)^\lambda} \int_{B(x, \theta R_1)} u^\alpha(y) dy \leq \frac{1}{2} \frac{1}{R_1^\lambda} \int_{B(x, R_1)} u^\alpha(y) dy.$$

As in the proof of Theorem 1.1, page 95 of [4], we claim that there exist some constants  $\mu > \lambda$  and  $c > 0$  for which

$$\int_{B(x, R)} u^\alpha(y) dy < cR^\mu,$$

for all  $x$  in some neighborhood of  $x_0$  and for all  $R < R_0$ .

In fact we obtain by induction that, in some neighborhood of  $x_0$ , we have

$$\frac{1}{(\theta^k R_1)^\lambda} \int_{B(x, \theta^k R_1)} u^\alpha(y) dy \leq 2^{-k} \frac{1}{R_1^\lambda} \int_{B(x, R_1)} u^\alpha(y) dy,$$

for all  $k \in \mathbb{N}$ . Choosing  $\mu > \lambda$  such that  $\theta^{\mu-\lambda} > \frac{1}{2}$  we derive that for some constant  $c > 0$  we have

$$\int_{B(x, \theta^k R_1)} u^\alpha(y) dy \leq c(\theta^k R_1)^\mu,$$

for all  $k \in \mathbb{N}$ , from which we derive the claim.

Therefore there exists a neighborhood  $\omega \subset \Omega' \setminus S$  of  $x_0$  such that  $u \in L^{\alpha, \mu}(\omega)$ . In a previous paper [7] we had obtained the following regularity criterion for weak solutions of (1):

**THEOREM 2.** *If  $u \in L^{\alpha, \mu}(\Omega)$  is a weak solution of (1) and if  $\mu > n - 2\alpha/(\alpha - 1)$  then  $u$  is regular in all  $\Omega' \subset \subset \Omega$ .*

For a definition of  $L^{\alpha, \mu}(\Omega)$  see [3] or [4].

Using this result we can conclude that  $u$  is regular in a neighborhood of  $x_0$ . This finishes the proof of the lemma.

We can now derive the results stated in the second part of this paper.

*Proof of Proposition 1.* Proposition 1 is a simple consequence of Lemma 5.

On one hand, assume that the hypotheses of the proposition are satisfied at  $x_0 \in \Omega$ . Therefore there exists some  $R_0 > 0$  such that

$$\int_{B(x_0, R_0)} \frac{u^{\alpha-1}(y)}{|x_0 - y|^{n-2}} dy < \epsilon_0.$$

On the other hand, for all  $x, x'$  in some neighborhood of  $x_0$  we have

$$\left| \int_{B(x, R_0)} \frac{u^{\alpha-1}(y)}{|x - y|^{n-2}} dy - \int_{B(x', R_0)} \frac{u^{\alpha-1}(y)}{|x' - y|^{n-2}} dy \right| \leq 2\epsilon_0.$$

Finally the map

$$x \rightarrow \int_{\Omega \setminus B(x, R_0)} \frac{u^{\alpha-1}(y)}{|x - y|^{n-2}} dy,$$

is continuous in some neighborhood of  $x_0$ . We deduce from all this the existence of a neighborhood  $\omega \subset \Omega$  of  $x_0$  such that, for all  $x \in \omega$

$$\int_{B(x, R_0)} \frac{u^{\alpha-1}(y)}{|x - y|^{n-2}} dy < 4\epsilon_0.$$

Choosing  $\epsilon_0$  small enough, the conclusion of the proposition is then a simple application of the proof of Lemma 5.

*Remark.* In the case where  $\alpha \geq 2$  we can drop the assumption  $\int_{B(x, R)} u^\alpha(x) dx < \epsilon_0^\alpha R^\lambda$ . In fact if  $I_{n-2} u^{\alpha-1}(x) < +\infty$  then for all  $\epsilon > 0$  there exist some  $R > 0$  such that

$$\int_{B(x, 2R)} \frac{u^{\alpha-1}(y)}{|x - y|^{n-2}} dy < \epsilon.$$

So, we derive the estimate

$$\int_{B(x, 2R)} u^{\alpha-1}(x) dx < \epsilon (2R)^{n-2}.$$

Since  $\alpha \geq 2$ , Hölder's inequality gives us

$$\int_{B(x, 2R)} u(y) dy \leq \left( \int_{B(x, 2R)} u^{\alpha-1}(y) dy \right)^{1/(\alpha-1)} |B(x, 2R)|^{1-1/(\alpha-1)}.$$

Therefore

$$\int_{B(x, 2R)} u(y) dy \leq c_4 \epsilon^{1/(\alpha-1)} R^{\lambda+2}.$$

Now, in a previous paper [7] we have proved that there exists some constant  $c_5 > 0$ , depending only on the dimension of the space such that

$$R^2 \int_{B(x, R)} u^\alpha(y) dy \leq c_5 \int_{B(x, 2R)} u(y) dy,$$

for every positive weak solution of (1). The last two inequalities allow us to estimate

$$\int_{B(x, R)} u^\alpha(y) dy \leq c_6 \epsilon^{1/(\alpha-1)} R^\lambda,$$

for some constant  $c_6 > 0$  depending only on the dimension of the space. Choosing  $\epsilon > 0$  such that  $\epsilon_0^\alpha > c_6 \epsilon^{1/(\alpha-1)}$  we get the desired estimate.

We are now left with the proof of Corollary 1.

*Proof of Corollary 1.* It is sufficient to show that the set of points  $x$  in  $\Omega$  where  $I_{n-2} u^{\alpha-1}(x) = +\infty$  forms a set of Hausdorff dimension less than or equal to  $n - 2\alpha/(\alpha - 1)$ . Denote by  $E$  this set,  $u^{\alpha-1} \in L^{\alpha/(\alpha-1)}(\Omega)$ , using the definition of the Riesz capacity, we deduce from this [9] that  $R_{2, \alpha/(\alpha-1)}(E) = 0$ , thus the Hausdorff dimension of  $E$  is less than or equal to  $n - 2\alpha/(\alpha - 1)$ . The result of Corollary 1 is then a consequence of Proposition 1.

#### 4. General remarks

In order to find a regularity criterion for weak positive solutions of (1) one could be tempted to consider the natural quantity

$$\frac{1}{R^\lambda} \int_{B(x, R)} u^\alpha(y) dy,$$

where  $\lambda = n - 2\alpha/(\alpha - 1)$ , and conjecture that if this quantity is small enough then  $u$  is regular in some neighborhood of  $x$ . Unfortunately this conjecture does not hold in general as can be shown using the examples given in [5]. In the last pages of their paper the authors display all the radial positive solutions of (1) in  $\mathbb{R}^n$ , and if

$$\alpha \in \left( \frac{n}{n-2}, \frac{n+2}{n-2} \right)$$

then they show that there exists a positive radial solution  $u$  of (1) which is singular at 0 (i.e.  $u(x)$  behaves like  $C/|x|^{2/(\alpha-1)}$  near  $x = 0$ ) and regular at  $\infty$  (i.e.  $u(x)$  behaves like  $c/|x|^{n-2}$  near  $\infty$ ). For some parameter  $\delta$  we consider the family  $u_\delta(x) = \delta^{2/(\alpha-1)}u(\delta x)$ . It is easy to see that  $u_\delta$  is a weak positive solution of (1) having a singularity at the origin and that the quantity

$$\frac{1}{R^\lambda} \int_{B(0, R)} u_\delta^\alpha(y) dy,$$

can be made as small as we want if  $\delta$  is chosen large enough.

We finish this paper by giving some open question:

If  $u$  is a positive weak solution of (1) and if, for some  $x_0 \in \Omega$ , the following condition is satisfied

$$I_{n-2} u^{\alpha-1}(x_0) < +\infty,$$

is  $I_{n-2} u^{\alpha-1}(x)$  continuous at  $x_0$ ?

Let us observe that a positive answer to this conjecture would prove the conjecture stated in the introduction.

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Received November 13, 1991