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Exponents for extraordinary homology groups

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Introduction

If $E_*(-)$ is the extraordinary homology theory associated with a CW-spectrum E and X a CW-complex, then it is in general quite difficult to compute $E_*(X)$. The classical way to do it is to work with the Atiyah–Hirzebruch spectral sequence $H_s(X; \pi_t E) \Rightarrow E_{s+t}(X)$, but, as usual with spectral sequences, one must first understand its *differentials*, and secondly solve the *extension problems* given by its E^∞-term (see [Ar3] for some general results on the differentials).

The purpose of this paper is to introduce a new method which does not determine exactly $E_*(X)$, but which produces, in a very general way, a good *approximation* of the extraordinary homology groups $E_n(X)$. The argument presented here explains actually the relationships between $E_n(X)$ and the groups on the line $s+t=n$ of the E^2-term of the Atiyah–Hirzebruch spectral sequence, but avoids the two difficulties which occur in the spectral sequence calculations.

Remember that it is sufficient to compute the reduced homology groups $\tilde{E}_n(X)$ since $E_n(X) \cong \tilde{E}_n(X) \oplus \pi_n E$ for all n. Our comparison of an extraordinary homology group with ordinary homology may be formulated as follows (see Theorems 2.2 and 2.4).

Let E be a bounded below spectrum (with $c \in \mathbb{Z}$ such that $\pi_i E = 0$ for $i < c$), then for any connected CW-complex X and for any integer $n \geq b + c + 1$ (where b is the greatest positive integer for which $\tilde{H}_i(X; \mathbb{Z}) = 0$ for $i < b$):

(a) *there exist homomorphisms*

$$\Phi_n : \tilde{E}_n(X) \to \bigoplus_{t=c}^{n-b} H_{n-t}(X; \pi_t E)$$

and

$$\Psi_n : \bigoplus_{t=c}^{n-b} H_{n-t}(X; \pi_t E) \to \tilde{E}_n(X)$$

with the property that the composition $\Psi_n \Phi_n : \tilde{E}_n(X) \to \tilde{E}_n(X)$ is multiplication *by $\rho_{n-b}\rho_{n-b-1}\cdots\rho_{c+1}$, where the ρ_i's ($i \geq c$) are integers depending on the spectrum E;*

(b) *for any integer t with $c \leq t \leq n-b$ there exist homomorphisms*

$$\Theta_{n,t} : H_{n-t}(X; \pi_t E) \to \tilde{E}_n(X)$$

and

$$\Lambda_{n,t} : \tilde{E}_n(X) \to H_{n-t}(X; \pi_t E)$$

such that the composition $\Lambda_{n,t}\Theta_{n,t} : H_{n-t}(X; \pi_t E) \to H_{n-t}(X; \pi_t E)$ is multiplication by $\rho_{n-b}\rho_{n-b-1}\cdots\rho_t$.

The integers ρ_i are in fact the order of the k-invariants $k^{i+1}(E)$ of the spectrum E (in particular $\rho_c = 1$) and may be replaced by integers which are *independent of* E (see Remarks 1.6 and 2.6). Notice that the homomorphisms mentioned above are induced by maps of spectra such that their compositions are suitable power maps (see Section 1 for the construction of these maps).

It is then possible to deduce from this result interesting information on the groups $\tilde{E}_n(X)$, for instance lower and upper bounds for their *exponent*, depending on the integers ρ_i and on the exponents of the ordinary homology groups of X. This is the main application of our argument and is formulated in general in Theorem 3.1. But in order to illustrate this, let us consider for the moment the special case of the *Brown–Peterson homology* theory $BP_*(-)$ for a given prime p: we obtain the following assertion (see Corollary 4.5).

Let X be a connected CW-complex, b the greatest positive integer for which $\tilde{H}_i(X; \mathbb{Z}) = 0$ for $i < b$, and assume that all homology groups $H_i(X; \mathbb{Z}_{(p)})$ are of finite exponent p^{e_i} for $i \geq b$. If n is any integer $\geq b+1$ and r the positive integer such that $2r(p-1) \leq n-b < 2(r+1)(p-1)$, then the exponent of $\widetilde{BP}_n(X)$ is p^{ε_n}, where

$$\varepsilon_n \leq \max\{e_{n-2t(p-1)} \mid 0 \leq t \leq r\} + \frac{r(r+1)}{2}$$

and

$$\varepsilon_n \geq \max\left\{e_{n-2t(p-1)} - \frac{(r+t)(r-t+1)}{2} \,\middle|\, 0 \leq t \leq r\right\}.$$

These inequalities enable us to perform some computations on the Brown–Peterson homology of the classifying spaces of certain special linear groups.

The second application of our method is the study of the relationships between the p-torsion (where p is a given prime) in the ordinary integral homology of a CW-complex X and the p-torsion in the extraordinary homology $\tilde{E}_*(X)$ associated with connective spectra E such that $\pi_0 E$ is not p-divisible. For instance, we consider this question when X is the classifying space BG of a p-torsion-free group G. For certain primes p, the congruence subgroups Γ_q of odd prime level $q \neq p$ in the infinite special linear group $SL(\mathbb{Z})$ turn out to be examples of torsion-free groups having p-torsion in $\tilde{E}_*(B\Gamma_q)$ when $\tilde{E}_*(-)$ runs over the reduced homology theories associated with all connective spectra E such that $\pi_0 E$ is not p-divisible (see Corollary 5.4); we say that such groups have *universally strange p-torsion*.

Let us remark that our main results also hold if the spectrum E is not bounded below or if we consider an extraordinary *cohomology* theory (instead of homology), assuming that the CW-complex X is finite dimensional.

The paper is organized as follows. In Section 1, we construct maps between a given spectrum and the corresponding Eilenberg–MacLane spectra, which we control on the homotopy level. Section 2 shows how these maps induce the homomorphisms Φ_n, Ψ_n, $\Theta_{n,t}$ and $\Lambda_{n,t}$ presented above. The consequences on the investigation of the exponent of the groups $\tilde{E}_n(X)$ are explained in Section 3. We examine the special case of the Brown–Peterson homology in Section 4: in particular, we determine the integers ρ_i for BP, prove the inequalities involving the exponent of the Brown–Peterson homology groups and calculate some examples. Section 5 is devoted to the study of the relationships between the torsion in the ordinary integral homology of a CW-complex (or of a group) and the torsion in its extraordinary homology theories. Finally, our main results are formulated in Section 6 for cohomology theories applied to CW-complexes of finite dimension.

1. Maps between a given spectrum and the corresponding Eilenberg–MacLane spectra

If E is a spectrum and n an integer, let us call $\alpha_n : E \to E[n]$ its n-th Postnikov section: $E[n]$ is a spectrum with $\pi_i E[n] = 0$ for $i > n$ and α_n is a map of spectra inducing an isomorphism on π_i for $i \leq n$. The purpose of this section is to consider bounded below spectra E (i.e., for which there exists an integer c with $\pi_i E = 0$ for $i < c$) and investigate the relationships between $E[n]$ and the wedge of the corresponding Eilenberg–MacLane spectra $\bigvee_{i=c}^{n} \Sigma^i H(\pi_i E)$.

For an Ω-spectrum E and a positive integer ρ, let us denote by $\chi^\rho : E \to E$ the ρ-th power map; if E is an arbitrary spectrum, we may also consider the self-map χ^ρ, because E is homotopy equivalent to an Ω-spectrum. The map χ^ρ induces multiplication by ρ on all homotopy groups of E. Notice that for any n, $\chi^\rho : E \to E$

extends to a map $E[n] \to E[n]$ which is also the ρ-th power map and which will also be denoted by χ^ρ. Our first goal is to show that for a given integer n, if ρ is big enough in comparison with n, then the map $\chi^\rho : E[n] \to E[n]$ factors through $\bigvee_{t=\ell}^{n} \Sigma^t H(\pi_t E)$ in the case of a bounded below spectrum E.

In order to formulate our first result, let us consider the Postnikov k-invariants of a spectrum E: they are homotopy classes of maps of spectra $k^{n+1}(E)$: $E[n-1] \to \Sigma^{n+1} H(\pi_n E)$ and therefore cohomology classes in $H^{n+1}(E[n-1]; \pi_n E)$.

THEOREM 1.1. *Let E be any spectrum, n an integer, and assume that the k-invariant $k^{n+1}(E)$ is an element of finite order ρ_n in $H^{n+1}(E[n-1]; \pi_n E)$. Then there exist maps of spectra*

$$\zeta_n : E[n] \to E[n-1] \vee \Sigma^n H(\pi_n E)$$

and

$$\eta_n : E[n-1] \vee \Sigma^n H(\pi_n E) \to E[n]$$

such that the composition $\eta_n \zeta_n$ is homotopic to the power map $\chi^{\rho_n} : E[n] \to E[n]$.

Proof. Let us denote by β_{n-1} the Postnikov section $E[n] \to E[n-1]$. The map $\chi^{\rho_n} : E[n] \to E[n]$ has an extension $\bar{\chi}^{\rho_n}$ on $E[n-1]$ and restricts to $\tilde{\chi}^{\rho_n}$ on the fibre $\Sigma^n H(\pi_n E)$ of β_{n-1}; these three self-maps induce multiplication by ρ_n on all homotopy groups. Look at the commutative diagram

$$\begin{array}{ccccc}
\Sigma^n H(\pi_n E) & \xrightarrow{\tilde{\chi}^{\rho_n}} & \Sigma^n H(\pi_n E) & \xrightarrow{\simeq} & \Sigma^n H(\pi_n E) \\
\downarrow & & \downarrow & & \downarrow \\
E[n] & \xrightarrow{\chi^{\rho_n}} & E[n] & \xrightarrow{\xi} & * \\
\downarrow{\scriptstyle \beta_{n-1}} & & \downarrow{\scriptstyle \beta_{n-1}} & & \downarrow \\
E[n-1] & \xrightarrow{\bar{\chi}^{\rho_n}} & E[n-1] & \xrightarrow{k^{n+1}(E)} & \Sigma^{n+1} H(\pi_n E),
\end{array}$$

where the vertical arrows are (co)fibrations, the right one being the path fibration over the Eilenberg–MacLane spectrum $\Sigma^{n+1} H(\pi_n E)$, and the bottom right square is a pull-back diagram. Now, write F for the homotopy fibre of the composition $k^{n+1}(E)\bar{\chi}^{\rho_n}$ and σ for the inclusion map $F \hookrightarrow E[n-1]$; observe that the homotopy fibre of σ is $\Sigma^n H(\pi_n E)$. The cohomology class corresponding to $k^{n+1}(E)\bar{\chi}^{\rho_n}$ is actually the image of $k^{n+1}(E) \in H^{n+1}(E[n-1]; \pi_n E)$ under the induced homomorphism $(\bar{\chi}^{\rho_n})^* : H^{n+1}(E[n-1]; \pi_n E) \to H^{n+1}(E[n-1]; \pi_n E)$. By naturality of the k-invariants with respect to the map χ^{ρ_n} (see [Wh], p. 424, Theorem 2.6), this is exactly the image of $k^{n+1}(E)$ under the homomorphism $H^{n+1}(E[n-1]; \pi_n E) \to H^{n+1}(E[n-1]; \pi_n E)$ induced by $(\chi^{\rho_n})_* : \pi_n E \to \pi_n E$, in other words induced by multiplication by ρ_n. Thus, we may deduce from the hypothesis $\rho_n k^{n+1}(E) = 0$ that $(\bar{\chi}^{\rho_n})^*(k^{n+1}(E)) = 0$, or that the composition $k^{n+1}(E)\bar{\chi}^{\rho_n}$ is homotopic to the trivial map. It then follows that there is a homotopy equivalence $F \simeq E[n-1] \vee \Sigma^n H(\pi_n E)$.

This construction provides clearly a map of spectra $\zeta_n : E[n] \to F$ with the property that $\sigma\zeta_n = \beta_{n-1}$: thus, the induced homomorphism $(\zeta_n)_* : \pi_i E[n] \to \pi_i F$ is an isomorphism for $i \leq n-1$ and the commutativity of the above diagram shows that $(\zeta_n)_* : \pi_n E[n] \to \pi_n F$ is multiplication by ρ_n (up to an automorphism).

On the other hand, $\bar{\chi}^{\rho_n} : E[n-1] \to E[n-1]$ induces a map η_n from the fibre F of $k^{n+1}(E)\bar{\chi}^{\rho_n}$ to the fibre $E[n]$ of $k^{n+1}(E)$ such that $\beta_{n-1}\eta_n = \bar{\chi}^{\rho_n}\sigma$. It turns out that the homomorphism $(\eta_n)_* : \pi_i F \to \pi_i E[n]$ induced by η_n is multiplication by ρ_n (up to an automorphism) for $i \leq n-1$ and an isomorphism for $i = n$.

Finally, observe that the composition $\eta_n\zeta_n : E[n] \to E[n]$ and the map $\chi^{\rho_n} : E[n] \to E[n]$ satisfy $\beta_{n-1}\eta_n\zeta_n = \bar{\chi}^{\rho_n}\sigma\zeta_n = \bar{\chi}^{\rho_n}\beta_{n-1}$ and $\beta_{n-1}\chi^{\rho_n} = \bar{\chi}^{\rho_n}\beta_{n-1}$. This means that both are the restriction of $\bar{\chi}^{\rho_n} : E[n-1] \to E[n-1]$ to $E[n]$: consequently, $\eta_n\zeta_n$ and χ^{ρ_n} are homotopic. This completes the proof. Notice that the maps ζ_n and η_n given by this argument are not uniquely determined.

COROLLARY 1.2. *Let E be any spectrum, m and n two integers with $m \leq n$, and assume that for $m \leq i \leq n$ the k-invariant $k^{i+1}(E)$ of E is a cohomology class of finite order ρ_i in $H^{i+1}(E[i-1]; \pi_i E)$. Then there exist maps of spectra*

$$\varphi_{n,m} : E[n] \to E[m-1] \vee \left(\bigvee_{t=m}^{n} \Sigma^t H(\pi_t E) \right)$$

and

$$\psi_{n,m} : E[m-1] \vee \left(\bigvee_{t=m}^{n} \Sigma^t H(\pi_t E) \right) \to E[n]$$

such that the composition $\psi_{n,m}\varphi_{n,m}$ is homotopic to $\chi^{\rho_n\rho_{n-1}\cdots\rho_m} : E[n] \to E[n]$.

Proof. We proceed inductively by using Theorem 1.1. First, if $m = n$, take $\varphi_{n,n} = \zeta_n$ and $\psi_{n,n} = \eta_n$. Now, suppose that $\varphi_{n,m+1} : E[n] \to E[m] \vee (\bigvee_{t=m+1}^{n} \Sigma^t H(\pi_t E))$ and $\psi_{n,m+1} : E[m] \vee (\bigvee_{t=m+1}^{n} \Sigma^t H(\pi_t E)) \to E[n]$ are constructed with the property that $\psi_{n,m+1}\varphi_{n,m+1} \simeq \chi^{\rho_n\rho_{n-1}\cdots\rho_{m+1}}$, then define $\varphi_{n,m}$ and $\psi_{n,m}$ as follows:

$$\varphi_{n,m} = (\zeta_m \vee \mathrm{id})\varphi_{n,m+1} : E[n] \to E[m-1] \vee \left(\bigvee_{t=m}^{n} \Sigma^t H(\pi_t E) \right)$$

and

$$\psi_{n,m} = \psi_{n,m+1}(\eta_m \vee \chi^{\rho_m}) : (E[m-1] \vee \Sigma^m H(\pi_m E)) \vee \left(\bigvee_{t=m+1}^{n} \Sigma^t H(\pi_t E) \right) \to E[n],$$

where $\zeta_m : E[m] \to E[m-1] \vee \Sigma^m H(\pi_m E)$ and $\eta_m : E[m-1] \vee \Sigma^m H(\pi_m E) \to E[m]$ are the maps given by Theorem 1.1, id denotes the identity of $\bigvee_{t=m+1}^{n} \Sigma^t H(\pi_t E)$ and χ^{ρ_m} the ρ_m-th power map on $\bigvee_{t=m+1}^{n} \Sigma^t H(\pi_t E)$. The composition $\psi_{n,m}\varphi_{n,m} : E[n] \to E[n]$ is actually

$$\psi_{n,m+1}(\eta_m \vee \chi^{\rho_m})(\zeta_m \vee \mathrm{id})\varphi_{n,m+1} = \psi_{n,m+1}(\eta_m\zeta_m \vee \chi^{\rho_m})\varphi_{n,m+1}.$$

According to Theorem 1.1, $\eta_m\zeta_m \vee \chi^{\rho_m}$ is the ρ^m-th power map on $E[m] \vee (\bigvee_{t=m+1}^{n} \Sigma^t H(\pi_t E))$, and finally, $\psi_{n,m}\varphi_{n,m}$ is homotopic to $\chi^{\rho_n\rho_{n-1}\cdots\rho_{m+1}}\chi^{\rho_m}$: $E[n] \to E[n]$ by the induction hypothesis.

REMARK 1.3. In fact, the homomorphisms induced by $\varphi_{n,m}$ and $\psi_{n,m}$ on homotopy act as follows: $(\varphi_{n,m})_*$ is multiplication by ρ_t on $\pi_t E$ (up to an automorphism) for $m \le t \le n$ and an isomorphism for $t < m$; $(\psi_{n,m})_*$ is multiplication by $(\rho_n\rho_{n-1}\cdots\rho_m)/\rho_t$ on $\pi_t E$ (up to an automorphism) for $m \le t \le n$ and multiplication by $\rho_n\rho_{n-1}\cdots\rho_m$ (up to an automorphism) for $t < m$.

Now, if E is a bounded below spectrum, the hypothesis that the k-invariants of E have finite order is always fulfilled (see Section 1 of [Ar3]) and the previous corollary produces the following factorization of a suitable power map $E[n] \to E[n]$ through the wedge of the corresponding Eilenberg–MacLane spectra.

COROLLARY 1.4. *Let E be a bounded below spectrum (with $c \in \mathbb{Z}$ such that $\pi_i E = 0$ for $i < c$) and for $i \ge c+1$ let ρ_i denote the order of the k-invariant $k^{i+1}(E)$ in $H^{i+1}(E[i-1]; \pi_i E)$. Then for any integer $n \ge c+1$ there exist maps of spectra*

$$\varphi_n : E[n] \to \bigvee_{t=c}^{n} \Sigma^t H(\pi_t E)$$

and

$$\psi_n : \bigvee_{t=c}^{n} \Sigma^t H(\pi_t E) \to E[n]$$

such that the composition $\psi_n\varphi_n$ is homotopic to the map $\chi^{\rho_n\rho_{n-1}\cdots\rho_{c+1}} : E[n] \to E[n]$.

Proof. This is a direct consequence of Corollary 1.2 with $m = c + 1$, because $E[c] = \Sigma^c H(\pi_c E)$. Note that the statement of the corollary is trivial if $n = c$.

In the next theorem, we show that it is also possible to get a similar result in the other direction: more precisely, if we consider a suitable power map $\Sigma^t H(\pi_t E) \to \Sigma^t H(\pi_t E)$ for a given integer $t \le n$, then we can factor it through $E[n]$.

THEOREM 1.5. *Let E be any sprectum, t and n two integers with $t \le n$, and assume that for $t \le i \le n$ the k-invariant $k^{i+1}(E)$ of E is a cohomology class of finite order ρ_i in $H^{i+1}(E[i-1]; \pi_i E)$. Then there exist maps of spectra*

$$\theta_{n,t} : \Sigma^t H(\pi_t E) \to E[n]$$

and

$$\lambda_{n,t} : E[n] \to \Sigma^t H(\pi_t E)$$

such that the composition $\lambda_{n,t}\theta_{n,t}$ is homotopic to the map $\chi^{\rho_n \rho_{n-1} \cdots \rho_t} : \Sigma^t H(\pi_t E) \to \Sigma^t H(\pi_t E)$ which induces multiplication by $\rho_n \rho_{n-1} \cdots \rho_t$ on $\pi_t E$.

Proof. The Eilenberg–MacLane spectrum $\Sigma^t H(\pi_t E)$ is the fibre of the Postnikov section $\beta_t : E[t] \to E[t-1]$. If $t = n$, define $\theta_{n,n}$ just as the inclusion $\Sigma^n H(\pi_n E) \hookrightarrow E[n]$. If $t < n$, then $\Sigma^t H(\pi_t E)$ (which is a subspectrum of $E[t]$) may be viewed as a subspectrum of the wedge $E[t] \vee (\bigvee_{s=t+1}^{n} \Sigma^s H(\pi_s E))$ and we denote by $\theta_{n,t}$ the composition of that inclusion with the map $\psi_{n,t+1}$ given by Corollary 1.2. Then, let $\lambda_{n,t}$ be the composition of the map $\varphi_{n,t}$ of Corollary 1.2 with the projection of $E[t-1] \vee (\bigvee_{s=t}^{n} \Sigma^s H(\pi_s E))$ onto the summand $\Sigma^t H(\pi_t E)$. According to Remark 1.3, $\theta_{n,t}$ and $\lambda_{n,t}$ induce multiplication by $\rho_n \rho_{n-1} \cdots \rho_{t+1}$, and by ρ_t respectively, on $\pi_t E$, eventually up to an automorphism of $\pi_t E$: this implies $\lambda_{n,t}\theta_{n,t} \simeq \chi^{\rho_n \rho_{n-1} \cdots \rho_t}$.

Again, if E is bounded below (with $c \in \mathbb{Z}$ such that $\pi_i E = 0$ for $i < c$), the hypothesis of Theorem 1.5 is always satisfied for any integer t such that $c \le t \le n$.

REMARK 1.6. For bounded below spectra we have actually proved in Section 1 of [Ar3] the existence of integers R_j ($j \ge 1$), *independent of E*, which have the property that $R_{i-c+1} k^{i+1}(E) = 0$ for all $i \ge c + 1$. It is important to notice that a prime number q divides R_j if and only if $q \le j/2 + 1$ and consequently that if q divides ρ_i, then $q \le (i - c + 3)/2$ since R_{i-c+1} is a positive multiple of ρ_i. In the formulation of Theorems 1.1 and 1.5 and Corollaries 1.2 and 1.4, we can of course replace ρ_i by any positive multiple of ρ_i, in particular by R_{i-c+1}. Now, if we define

$T_{u,v} := \Pi_{j=u}^{v} R_j$ for integers u and v with $1 \le u \le v$, then our argument produces the following assertions.

(a) There are maps φ_n and ψ_n as in the statement of Corollary 1.4 such that $\psi_n \varphi_n \simeq \chi^{T_{2,n-c+1}} : E[n] \to E[n]$.

(b) For each integer t with $c \le t \le n$, there exist maps $\theta_{n,t}$ and $\lambda_{n,t}$ as in the statement of Theorem 1.5 such that $\lambda_{n,t}\theta_{n,t} \simeq \chi^{T_{t-c+1,n-c+1}}$: $\Sigma^t H(\pi_t E) \to \Sigma^t H(\pi_t E)$.

Observe again that a prime q divides $T_{2,n-c+1}$, respectively $T_{t-c+1,n-c+1}$, if and only if $q \le (n-c+3)/2$.

2. Homomorphisms between extraordinary and ordinary homology groups

We consider here the homomorphisms induced on homotopy by the maps of spectra introduced in the previous section, after taking the smash product with a given CW-complex X.

LEMMA 2.1. *Let E be a spectrum, X a CW-complex, $\tilde{E}_*(X)$ the reduced E-homology of X, and ρ an integer. The ρ-th power map $\chi^\rho : E \to E$ induces multiplication by $\rho : \tilde{E}_*(X) \to \tilde{E}_*(X)$.*

Proof. It is sufficient to establish the assertion for an Ω-spectrum E. Let us work on the space level: if we write E_l for the l-th space of the spectrum E, the ρ-th power map $\chi^\rho \in [E_l, E_l]$ is defined as the map corresponding to

$$\hat{\chi}^\rho : \Sigma E_l \xrightarrow{\text{pinch}} \bigvee_\rho \Sigma E_l \xrightarrow{\vee i} \bigvee_\rho E_{l+1} \xrightarrow{\text{fold}} E_{l+1}$$

under the isomorphism $[E_l, E_l] \cong [E_l, \Omega E_{l+1}] \cong [\Sigma E_l, E_{l+1}]$ (here, $i : \Sigma E_l \to E_{l+1}$ corresponds to the identity in $[E_l, E_l]$). Now, $\hat{\chi}^\rho$ induces $\hat{\chi}^\rho \wedge \mathrm{id} : \Sigma E_l \wedge X \to E_{l+1} \wedge X$, where id is the identity $X \to X$, but the commutative diagram

$$\begin{array}{ccccccc}
\Sigma E_l \wedge X & \xrightarrow{\text{pinch} \wedge \mathrm{id}} & \left(\bigvee_\rho \Sigma E_l\right) \wedge X & \xrightarrow{(\vee i) \wedge \mathrm{id}} & \left(\bigvee_\rho E_{l+1}\right) \wedge X & \xrightarrow{\text{fold} \wedge \mathrm{id}} & E_{l+1} \wedge X \\
\| & & \| & & \| & & \| \\
\Sigma(E_l \wedge X) & \xrightarrow{\text{pinch}} & \bigvee_\rho (\Sigma E_l \wedge X) & \xrightarrow{\vee (i \wedge \mathrm{id})} & \bigvee_\rho (E_{l+1} \wedge X) & \xrightarrow{\text{fold}} & E_{l+1} \wedge X,
\end{array}$$

where the first line is $\hat{\chi}^\rho \wedge \mathrm{id}$, shows that this map corresponds in fact to the ρ-th power map in $[E_l \wedge X, \Omega(E_{l+1} \wedge X)]$. Consequently, it induces multiplication by ρ on homotopy.

We are now able to prove the main results of the paper which provide a new way to obtain interesting approximations of the E-homology groups of a CW-complex X; as mentioned in the introduction, it is sufficient to look at the reduced homology groups $\tilde{E}_n(X)$. Throughout the paper, a CW-complex X is called $(b-1)$-*homologically-connected*, $b \geq 1$, if it is connected and $\tilde{H}_i(X; \mathbb{Z}) = 0$ for $i \leq b-1$. Observe first that if E is a bounded below spectrum (with $c \in \mathbb{Z}$ such that $\pi_i E = 0$ for $i < c$) and X a $(b-1)$-homologically-connected CW-complex, then $\tilde{E}_n(X) = 0$ for $n < b + c$ and $\tilde{E}_{b+c}(X) \cong H_b(X; \pi_c E)$. Therefore, the interesting dimensions are $n \geq b + c + 1$.

THEOREM 2.2. *Let E be a bounded below spectrum (with $c \in \mathbb{Z}$ such that $\pi_i E = 0$ for $i < c$) and for $i \geq c+1$ let ρ_i denote the order of the k-invariant $k^{i+1}(E)$ in $H^{i+1}(E[i-1]; \pi_i E)$. Then for any $(b-1)$-homologically-connected CW-complex X and for any integer $n \geq b + c + 1$ there exist homomorphisms*

$$\Phi_n : \tilde{E}_n(X) \to \bigoplus_{i=c}^{n-b} H_{n-i}(X; \pi_i E)$$

and

$$\Psi_n : \bigoplus_{i=c}^{n-b} H_{n-i}(X; \pi_i E) \to \tilde{E}_n(X)$$

with the property that the composition $\Psi_n \Phi_n : \tilde{E}_n(X) \to \tilde{E}_n(X)$ is multiplication by $\rho_{n-b}\rho_{n-b-1} \cdots \rho_{c+1}$.

Proof. We proved in Lemma 4.1 of [Ar3] that $\tilde{E}_n(X) \cong \tilde{E}[n-b]_n(X)$. Then, the homomorphisms Φ_n and Ψ_n are induced by the maps φ_n and ψ_n constructed in Corollary 1.4 and the assertion follows from Lemma 2.1.

COROLLARY 2.3. *Let E be a bounded below spectrum (with $c \in \mathbb{Z}$ such that $\pi_i E = 0$ for $i < c$) and for $i \geq c+1$ let ρ_i denote the order of the k-invariant $k^{i+1}(E)$ in $H^{i+1}(E[i-1]; \pi_i E)$. Then for any $(b-1)$-homologically-connected CW-complex X and for any integer $n \geq b + c + 1$ one has:*

(a) $\rho_{n-b}\rho_{n-b-1} \cdots \rho_{c+1} \cdot \tilde{E}_n(X)$ *is a subquotient of* $\bigoplus_{i=c}^{n-b} H_{n-i}(X; \pi_i E)$.

(b) *There exists a homomorphism*

$$\Xi_n : \tilde{E}_n(X) \oplus \ker \Psi_n \to \bigoplus_{i=c}^{n-b} H_{n-i}(X; \pi_i E)$$

such that $\rho_{n-b}\rho_{n-b-1} \cdots \rho_{c+1} \cdot \ker \Xi_n = 0$ *and* $\rho_{n-b}\rho_{n-b-1} \cdots \rho_{c+1} \cdot \operatorname{coker} \Xi_n = 0$.

Proof. Since $\rho_{n-b}\rho_{n-b-1} \cdots \rho_{c+1} \cdot \tilde{E}_n(X) = \Psi_n \Phi_n(\tilde{E}_n(X))$, it is a subgroup of the image of Ψ_n and thus a subgroup of a quotient of $\bigoplus_{i=c}^{n-b} H_{n-i}(X; \pi_i E)$. It is easy to verify the second assertion if one defines Ξ_n as follows: $\Xi_n(g, h) = \Phi_n(g) + h$ for $g \in \tilde{E}_n(X)$ and $h \in \ker \Psi_n$.

Similarly, the next result follows from Theorem 1.5.

THEOREM 2.4. *Let E be a bounded below spectrum (with $c \in \mathbb{Z}$ such that $\pi_i E = 0$ for $i < c$) and for $i \geq c$ let ρ_i denote the order of the k-invariant $k^{i+1}(E)$ in $H^{i+1}(E[i-1]; \pi_i E)$ (observe that $\rho_c = 1$). Then for any $(b-1)$-homologically-connected CW-complex X and for any pair of integers t and n with $c \leq t \leq n - b$ there exist homomorphisms*

$$\Theta_{n,t} : H_{n-t}(X; \pi_t E) \to \tilde{E}_n(X)$$

and

$$\Lambda_{n,t} : \tilde{E}_n(X) \to H_{n-t}(X; \pi_t E)$$

such that the composition $\Lambda_{n,t}\Theta_{n,t} : H_{n-t}(X; \pi_t E) \to H_{n-t}(X; \pi_t E)$ is multiplication by $\rho_{n-b}\rho_{n-b-1} \cdots \rho_t$.

COROLLARY 2.5. *Let E be a bounded below spectrum (with $c \in \mathbb{Z}$ such that $\pi_i E = 0$ for $i < c$) and for $i \geq c$ let ρ_i denote the order of the k-invariant $k^{i+1}(E)$ in $H^{i+1}(E[i-1]; \pi_i E)$. Then for any $(b-1)$-homologically-connected CW-complex X and for any pair of integers t and n with $c \leq t \leq n - b$ one has:*

(a) *$\rho_{n-b}\rho_{n-b-1} \cdots \rho_t \cdot H_{n-t}(X; \pi_t E)$ is a subquotient of $\tilde{E}_n(X)$.*

(b) *There exists a homomorphism*

$$\Omega_{n,t} : H_{n-t}(X; \pi_t E) \oplus \ker \Lambda_{n,t} \to \tilde{E}_n(X)$$

such that $\rho_{n-b}\rho_{n-b-1} \cdots \rho_t \cdot \ker \Omega_{n,t} = 0$ *and* $\rho_{n-b}\rho_{n-b-1} \cdots \rho_t \cdot \operatorname{coker} \Omega_{n,t} = 0$.

REMARK 2.6. As mentioned in Remark 1.6, it is sometimes useful to replace $\rho_{n-b}\rho_{n-b-1} \cdots \rho_{c+1}$ by $T_{2,n-b-c+1}$ in the statement of Theorem 2.2 and $\rho_{n-b}\rho_{n-b-1} \cdots \rho_t$ by $T_{t-c+1,n-b-c+1}$ in the assertion of Theorem 2.4. Let us also recall that a prime q divides $T_{2,n-b-c+1}$, respectively $T_{t-c+1,n-b-c+1}$, if and only if $q \leq (n-b-c+3)/2$.

REMARK 2.7. It is of course possible to formulate the results of this section for an *unreduced* homology theory $E_*(-)$ by using the isomorphism $E_n(X) \cong \tilde{E}_n(X) \oplus \pi_n E$ for all n.

3. Upper and lower bounds for the exponent of extraordinary homology groups

In order to show a first application of the homomorphisms given by Theorems 2.2 and 2.4, let us investigate the *exponent* of the groups $\tilde{E}_n(X)$, where $\tilde{E}_*(-)$ is the reduced homology theory associated with a bounded below spectrum E and X a CW-complex. Our argument produces the following comparison of the exponent of the extraordinary homology groups of X with the exponent of the ordinary homology groups of X. (We shall write $\exp(G)$ for the exponent of a group G.)

THEOREM 3.1. *Let E be a bounded below spectrum (with $c \in \mathbb{Z}$ such that $\pi_i E = 0$ for $i < c$) and for $i \geq c$ let ρ_i denote the order of the k-invariant $k^{i+1}(E)$ in $H^{i+1}(E[i-1]; \pi_i E)$. Assume that X is a $(b-1)$-homologically-connected CW-complex with the property that its integral homology groups $H_i(X; \mathbb{Z})$ are of finite exponent for $i \geq b$, then for any integer $n \geq b + c + 1$*

(a) $\exp(\tilde{E}_n(X))$ *divides*

$$\rho_{n-b}\rho_{n-b-1}\cdots\rho_{c+1}\cdot\max\{\exp(H_{n-t}(X;\pi_t E))\mid c\leq t\leq n-b\},$$

(b) $\exp(\tilde{E}_n(X))$ *is a positive multiple of*

$$\frac{\exp(H_{n-t}(X;\pi_t E))}{\rho_{n-b}\rho_{n-b-1}\cdots\rho_t}\quad \textit{for all integers } t \textit{ such that } c\leq t\leq n-b.$$

Proof. Since

$$\exp\left(\bigoplus_{t=c}^{n-b} H_{n-t}(X;\pi_t E)\right)=\max\{\exp(H_{n-t}(X;\pi_t E))\mid c\leq t\leq n-b\},$$

we get that

$$(\max\{\exp(H_{n-t}(X;\pi_t E))\mid c\leq t\leq n-b\}\cdot\rho_{n-b}\rho_{n-b-1}\cdots\rho_{c+1})\tilde{E}_n(X)=0$$

because of Corollary 2.3(a). On the other hand, we may deduce from the same argument and Corollary 2.5(a) that the integer $\rho_{n-b}\rho_{n-b-1}\cdots\rho_t\cdot\exp(\tilde{E}_n(X))$ is a positive multiple of $\exp(H_{n-t}(X;\pi_t E))$ for all integers t with $c \leq t \leq n-b$.

REMARK 3.2. Because of Remarks 1.6 and 2.6, the approximations of $\exp(\tilde{E}_n(X))$ given by the previous theorem may be done with the *universal bounds* (i.e., integers which are independent of E and of X) $T_{2,n-b-c+1}$ instead of $\rho_{n-b}\rho_{n-b-1}\cdots\rho_{c+1}$ in assertion (a), and $T_{t-c+1,n-b-c+1}$ instead of $\rho_{n-b}\rho_{n-b-1}\cdots\rho_t$ in assertion (b).

4. Brown–Peterson homology

In this section, we consider our results for a specific example, the *Brown–Peterson homology* associated with the spectrum BP for a given prime number p (recall that $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$, where the degree of v_i is $2(p^i - 1)$). Let us start with the study of the k-invariants of BP; for this purpose, we first need the following general result.

LEMMA 4.1. *Let E be a spectrum, n an integer and $h_n : \pi_n E \to H_n(E; \mathbb{Z})$ the Hurewicz homomorphism. Assume furthermore that the k-invariant $k^{n+1}(E)$ is of order ρ_n in the group $H^{n+1}(E[n-1]; \pi_n E)$. Then there exists a homomorphism $v_n : H_n(E; \mathbb{Z}) \to \pi_n E$ such that the composition $v_n h_n : \pi_n E \to \pi_n E$ is multiplication by ρ_n.*

Proof. According to Theorem 1.1, the composition

$$f_n : E \xrightarrow{\alpha_n} E[n] \xrightarrow{\zeta_n} E[n-1] \vee \Sigma^n H(\pi_n E) \xrightarrow{\text{proj}} \Sigma^n H(\pi_n E),$$

where proj denotes the projection on the second factor, induces multiplication by ρ_n on $\pi_n E$. This map f_n produces the following commutative diagram:

$$\begin{array}{ccc} \pi_n E & \xrightarrow{(f_n)_*} & \pi_n \Sigma^n H(\pi_n E) \\ \downarrow{\scriptstyle h_n} & & \cong \downarrow {\scriptstyle \text{Hurewicz}} \\ H_n(E; \mathbb{Z}) & \xrightarrow{(f_n)_\#} & H_n(\Sigma^n H(\pi_n E); \mathbb{Z}). \end{array}$$

It is clear that $H_n(\Sigma^n H(\pi_n E); \mathbb{Z}) \cong \pi_n E$ and that the composition $(f_n)_\# h_n$ is multiplication by ρ_n: thus, we obtain the conclusion of the lemma by defining $v_n := (f_n)_\#$.

PROPOSITION 4.2. *The Postnikov k-invariants of BP, $k^{i+1}(BP) \in H^{i+1}(BP[i-1]; BP_i)$, satisfy:*

(a) $k^{i+1}(BP) = 0$ *if* $i \not\equiv 0$ *mod* $2(p-1)$,

(b) $k^{2m(p-1)+1}(BP)$ *is of order* p^m *for all* $m \geq 1$.

Proof. The first assertion is trivial since $BP_i = 0$ if $i \not\equiv 0 \bmod 2(p-1)$. Let us show the second. It is known that there is a non-trivial element $y \in H_{2(p-1)}(BP; \mathbb{Z})$ such that $h_{2(p-1)}(v_1) = py$, where $h_{2(p-1)}$ is the Hurewicz homomorphism $BP_{2(p-1)} \to H_{2(p-1)}(BP; \mathbb{Z})$. (See [R], p. 71, Theorem 3.1.5, for the similar statement for the spectrum MU: the Hurewicz homomorphism $h_* : MU_* =$

$\mathbb{Z}[x_1, x_2, \ldots] \to H_*(MU; \mathbb{Z}) = \mathbb{Z}[b_1, b_2, \ldots]$ fulfills $h_*(x_{(p^i-1)}) = -pb_{(p^i-1)}$ for $i \geq 1$ and $h_*(x_j) = -b_j$ otherwise, where the degree of x_j is $2j$ and the degree of b_j is $2j$, $j \geq 1$; because the usual map from MU to BP sends $x_{(p-1)}$ onto v_1, the above result on $h_{2(p-1)}(v_1)$ follows from the injectivity of $h_* : BP_* \to H_*(BP; \mathbb{Z})$ [AL].) Since BP is a ring spectrum, the Hurewicz homomorphism $h_* : BP_* \to H_*(BP; \mathbb{Z})$ is a ring homomorphism with respect to the Pontryagin ring structure, and we then may conclude that

$$h_{2m(p-1)}(v_1^m) = p^m y^m \qquad \text{for } m \geq 1.$$

On the other hand, if $\rho_{2m(p-1)}$ denotes the order of the k-invariant $k^{2m(p-1)+1}(BP)$, the previous lemma implies that $v_{2m(p-1)}h_{2m(p-1)}(v_1^m) = \rho_{2m(p-1)}v_1^m$ and consequently that

$$\rho_{2m(p-1)}v_1^m = p^m v_{2m(p-1)}(y^m) \in BP_{2m(p-1)}.$$

This shows that p^m divides $\rho_{2m(p-1)}$.

In order to prove that $\rho_{2m(p-1)}$ is exactly p^m, it is then sufficient to verify that $p^m k^{2m(p-1)+1}(BP) = 0$ in $H^{2m(p-1)+1}(BP[2m(p-1)-1]; BP_{2m(p-1)})$. The spectrum $BP[2m(p-1)-1]$ has non-trivial homotopy groups (which are direct sums of copies of $\mathbb{Z}_{(p)}$) only in dimensions $0, 2(p-1), 4(p-1), \ldots, 2(m-1)(p-1)$; notice in particular that $BP[2m(p-1)-1] = BP[2(m-1)(p-1)]$. Therefore, we can consider the cofibrations of spectra

$$BP[2s(p-1)] \longrightarrow BP[2(s-1)(p-1)] \longrightarrow \Sigma^{2s(p-1)+1}H(BP_{2s(p-1)})$$

and the corresponding long exact homology sequences (with integral coefficients)

$$\begin{aligned} \cdots \longrightarrow H_{2m(p-1)+2}\Sigma^{2s(p-1)+1}H(BP_{2s(p-1)}) &\longrightarrow H_{2m(p-1)+1}BP[2s(p-1)] \\ \longrightarrow H_{2m(p-1)+1}BP[2(s-1)(p-1)] &\longrightarrow \cdots \end{aligned}$$

for $s = 1, 2, \ldots, m-1$. According to [C], $H_{2m(p-1)+2}\Sigma^{2s(p-1)+1}H(BP_{2s(p-1)})$ is a direct sum of copies of $\mathbb{Z}/p$, as is $H_{2m(p-1)+1}BP[0]$. By induction, it is then clear that $p^m H_{2m(p-1)+1}BP[2(m-1)(p-1)] = 0$, and it follows from the same argument that $p^m H_{2m(p-1)}BP[2(m-1)(p-1)] = 0$. Finally, the universal coefficient theorem implies that the exponent of the cohomology group $H^{2m(p-1)+1}(BP[2(m-1)(p-1)]; BP_{2m(p-1)})$ divides p^m and thus that $p^m k^{2m(p-1)+1}(BP) = 0$ because $k^{2m(p-1)+1}(BP)$ belongs to this cohomology group.

(I would like to thank Yuly Rudyak for useful discussions concerning the first part of the proof of this proposition.)

This enables us to apply Theorems 2.2 and 2.4 to the case $E = BP$, $\rho_i = 0$ if $i \not\equiv 0 \bmod 2(p-1)$ and $\rho_{2m(p-1)} = p^m$ for $m \geq 1$.

THEOREM 4.3. *Let X by any $(b-1)$-homologically-connected CW-complex, n an integer $\geq b+1$, and let r denote the positive integer such that $2r(p-1) \leq n-b < 2(r+1)(p-1)$.*

(a) *There exist homomorphisms*

$$\Phi_n : \widetilde{BP}_n(X) \to \bigoplus_{t=0}^{r} H_{n-2t(p-1)}(X; BP_{2t(p-1)})$$

and

$$\Psi_n : \bigoplus_{t=0}^{r} H_{n-2t(p-1)}(X; BP_{2t(p-1)}) \to \widetilde{BP}_n(X)$$

such that the composition $\Psi_n \Phi_n$ is multiplication by $p^{r(r+1)/2}$ on $\widetilde{BP}_n(X)$.

(b) *For any integer t with $0 \leq t \leq r$ there exist homomorphisms*

$$\Theta_{n,t} : H_{n-2t(p-1)}(X; BP_{2t(p-1)}) \to \widetilde{BP}_n(X)$$

and

$$\Lambda_{n,t} : \widetilde{BP}_n(X) \to H_{n-2t(p-1)}(X; BP_{2t(p-1)})$$

such that the composition $\Lambda_{n,t}\Theta_{n,t}$ is multiplication by $p^{(r+t)(r-t+1)/2}$ on $H_{n-2t(p-1)}(X; BP_{2t(p-1)})$.

EXAMPLE 4.4. Let us take $p = 2$ and $X = BSL(\mathbb{Z})^+$, the space obtained by performing the plus construction on the classifying space of the infinite special linear group $SL(\mathbb{Z}) = \varinjlim_m SL_m(\mathbb{Z})$ (this is the 0-th space of the spectrum of the algebraic K-theory of $\mathbb{Z}$). It is known that $H_1(BSL(\mathbb{Z})^+; \mathbb{Z}) = 0$, $H_2(BSL(\mathbb{Z})^+; \mathbb{Z}) \cong \mathbb{Z}/2$ and $H_3(BSL(\mathbb{Z})^+; \mathbb{Z}_{(2)}) \cong \mathbb{Z}/8$ (see [Ar1]). Therefore, the first dimension where the assertion (b) of the theorem gives a non-trivial information is $n = 5$ ($b = 2, r = 1$): there are homomorphisms

$$\mathbb{Z}/8 \xrightarrow{\Theta_{5,1}} \widetilde{BP}_5(BSL(\mathbb{Z})^+) \xrightarrow{\Lambda_{5,1}} \mathbb{Z}/8$$

such that the composition $\Lambda_{5,1}\Theta_{5,1}$ is multiplication by 2; this implies that the group $\widetilde{BP}_5(BSL(\mathbb{Z})^+)$ contains a cyclic subgroup of order 4.

Finally, let us describe our approximation of the exponent of the Brown–Peterson homology groups of a CW-complex.

COROLLARY 4.5. *Let X be any $(b-1)$-homologically-connected CW-complex with the property that its homology groups $H_i(X, \mathbb{Z}_{(p)})$ are of finite exponent p^{e_i} for $i \geq b$, n an integer $\geq b+1$, and r the positive integer such that $2r(p-1) \leq n-b < 2(r+1)(p-1)$. Then* $\exp(\widetilde{BP}_n(X)) = p^{\varepsilon_n}$, *where*

$$\varepsilon_n \leq \max\{e_{n-2t(p-1)} \mid 0 \leq t \leq r\} + \frac{r(r+1)}{2}$$

and

$$\varepsilon_n \geq \max\left\{e_{n-2t(p-1)} - \frac{(r+t)(r-t+1)}{2} \,\middle|\, 0 \leq t \leq r\right\}.$$

Proof. This is a direct consequence of Theorem 3.1.

EXAMPLE 4.6. If X is the classifying space BG of a group G and b the greatest positive integer such that $\tilde{H}_i(G; \mathbb{Z}_{(p)}) = 0$ for $i < b$, then the previous corollary gives an estimate of the exponent of $\widetilde{BP}_n(BG)$ in terms of the exponent p^{e_i} of the ordinary homology groups $H_i(G; \mathbb{Z}_{(p)})$, $i \geq b$. For $r = 1$ and 2, we get the following approximations. If $2(p-1) \leq n-b < 4(p-1)$, then $\exp(\widetilde{BP}_n(BG)) = p^{\varepsilon_n}$, where

$$\max\{e_n, e_{n-2(p-1)}\} - 1 \leq \varepsilon_n \leq \max\{e_n, e_{n-2(p-1)}\} + 1.$$

If $4(p-1) \leq n-b < 6(p-1)$, then $\exp(\widetilde{BP}_n(BG)) = p^{\varepsilon_n}$, where

$$\max\{e_n - 3, e_{n-2(p-1)} - 3, e_{n-4(p-1)} - 2\}$$
$$\leq \varepsilon_n \leq \max\{e_n, e_{n-2(p-1)}, e_{n-4(p-1)}\} + 3.$$

Choose for instance $p = 2$ and for G the infinite special linear group $SL(\mathbb{F}_q)$ with coefficients in the field of q elements (q a prime number). The integral homology groups of $SL(\mathbb{F}_q)$ are known from [H]: $H_1(SL(\mathbb{F}_q); \mathbb{Z}) = H_2(SL(\mathbb{F}_q); \mathbb{Z}) = 0$, $H_3(SL(\mathbb{F}_q); \mathbb{Z}) \cong \mathbb{Z}/(q^2-1)$, $H_4(SL(\mathbb{F}_q); \mathbb{Z}) = 0$, $H_5(SL(\mathbb{F}_q); \mathbb{Z}) \cong \mathbb{Z}/(q^3-1)$, $H_6(SL(\mathbb{F}_q); \mathbb{Z}) = 0$, $H_7(SL(\mathbb{F}_q); \mathbb{Z}) \cong \mathbb{Z}/(q^4-1) \oplus \mathbb{Z}/(q^2-1), \ldots$ Assume for example that q is a prime $\equiv 5 \bmod 8$: then $q^2 - 1 = 8 \cdot$ (odd integer), $q^3 - 1 = 4 \cdot$ (odd integer) and $q^4 - 1 = 16 \cdot$ (odd integer). When one computes the small-dimensional Brown–Peterson homology groups of $BSL(\mathbb{F}_q)$ with the Atiyah–Hirzebruch spectral sequence $E^2_{s,t} \cong \tilde{H}_s(BSL(\mathbb{F}_q); BP_t) \Rightarrow \widetilde{BP}_{s+t}(BSL(\mathbb{F}_q))$, one checks easily that $\widetilde{BP}_n(BSL(\mathbb{F}_q)) = 0$ for $n = 1, 2, 4, 6$ and that $\widetilde{BP}_3(BSL(\mathbb{F}_q)) \cong \mathbb{Z}/8$.

The first difficult dimension is $n = 5$, because there are two non-trivial groups, $\mathbb{Z}/8$ and $\mathbb{Z}/4$, on the line $s + t = 5$ in the E^2-term of the Atiyah–Hirzebruch spectral sequence; since no differentials may modify these groups, the spectral sequence gives the short exact sequence

$$0 \longrightarrow \mathbb{Z}/8 \longrightarrow \widetilde{BP}_5(BSL(\mathbb{F}_q)) \longrightarrow \mathbb{Z}/4 \longrightarrow 0.$$

But our approximation, for $n = 5$, $b = 3$, asserts that $\exp(\widetilde{BP}_5(BSL(\mathbb{F}_q)))$ divides 16. We then may conclude that $\widetilde{BP}_5(BSL(\mathbb{F}_q)) \cong \mathbb{Z}/8 \oplus \mathbb{Z}/4$ or $\mathbb{Z}/16 \oplus \mathbb{Z}/2$ (but not $\mathbb{Z}/32$).

The next interesting dimension is $n = 7$, where the line $s + t = 7$ in the E^2-term of the Atiyah–Hirzebruch spectral sequence contains three non-trivial groups: $\mathbb{Z}/8$, $\mathbb{Z}/4$ and $\mathbb{Z}/16 \oplus \mathbb{Z}/8$. It follows from our method that $\exp(\widetilde{BP}_7(BSL(\mathbb{F}_q)))$ is a positive multiple of 2 and a divisor of 128.

5. Universally strange torsion

The objective of this section is to present a second application of our results. For simplicity, we only consider here connective spectra E (i.e., such that $\pi_i E = 0$ for $i < 0$). Let p be a prime number, X a CW-complex, n a positive integer, and suppose that the group $H_n(X; \mathbb{Z})$ has p-torsion. We are interested in the following problem: does $\tilde{E}_n(X)$ also contain p-torsion when $\tilde{E}_*(-)$ runs over the reduced homology theories corresponding to all connective spectra E satisfying the condition that $\pi_0 E$ is not p-divisible? Our argument provides a positive answer to this question if p is large enough in comparison with n. More precisely, we get:

THEOREM 5.1. *Let X be a $(b-1)$-homologically-connected CW-complex, p a prime number and n an integer $\geq b$ with the property that $H_n(X; \mathbb{Z})$ contains p-torsion. If $p \geq (n-b)/2 + 2$, then $\tilde{E}_n(X)$ also contains p-torsion, for any reduced homology theory $\tilde{E}_*(-)$ associated with a connective spectrum E such that $\pi_0 E$ is not p-divisible.*

Proof. Since $\pi_0 E$ is not divisible by p, the tensor product $H_n(X; \mathbb{Z}) \otimes \pi_0 E$, which is a direct summand of $H_n(X; \pi_0 E)$, has p-torsion. According to Theorem 2.4 and Remark 2.6, there exist, for any connective spectrum E, homomorphisms $\Theta_{n,0} : H_n(X; \pi_0 E) \to \tilde{E}_n(X)$ and $\Lambda_{n,0} : \tilde{E}_n(X) \to H_n(X; \pi_0 E)$ such that the composition $\Lambda_{n,0} \Theta_{n,0}$ is multiplication by $T_{1,n-b+1}$. But $T_{1,n-b+1}$ is only divisible by primes $\leq (n-b+3)/2$. Consequently, the hypothesis $p \geq (n-b)/2 + 2$ implies that the p-torsion of $H_n(X; \pi_0 E)$ survives in $\tilde{E}_n(X)$.

This theorem is a suitable tool for the detection of p-torsion in all connective homology theories applied to a given CW-complex X. This game is of special interest if X is the classifying space BG of a p-torsion-free group G.

In [We], a group G is called a *group with strange p-torsion* if it is p-torsion-free and $H_*(G; \mathbb{Z})$ has p-torsion. Let us extend this definition as follows.

DEFINITION 5.2. A group G has *universally strange p-torsion* if it is p-torsion-free and $\tilde{E}_*(BG)$ contains p-torsion for any reduced homology theory $\tilde{E}_*(-)$ associated with a connective spectrum E such that $\pi_0 E$ is not p-divisible.

Now, let us give examples of groups with universally strange torsion. For an odd prime number q and a positive integer m, let $\Gamma_{m,q}$ denote the congruence subgroup of level q in $SL_m(\mathbb{Z})$, i.e., the kernel of the surjective homomorphism $SL_m(\mathbb{Z}) \to SL_m(\mathbb{F}_q)$ induced by the reduction modulo q (where $\mathbb{F}_q$ is the field with q elements), and define $\Gamma_q := \varinjlim_m \Gamma_{m,q}$ using upper left inclusions $\Gamma_{m,q} \hookrightarrow \Gamma_{m+1,q}$. The groups Γ_q are torsion-free. We have shown in Theorem 2.5 of [Ar2] that the groups Γ_q have strange torsion which comes from the torsion discovered in the algebraic K-theory of $\mathbb{Z}$ in [S]:

THEOREM 5.3. *Let p be a properly irregular prime and j an even integer $<p$ such that p divides the numerator of B_j/j (where B_j is the j-th Bernoulli number: $B_2 = \frac{1}{6}$, $B_4 = \frac{1}{30}, \ldots$). Then there is p-torsion in $H_{2j-2}(\Gamma_q; \mathbb{Z})$ for all odd primes $q \neq p$.*

Because $j < p$, the hypothesis of Theorem 5.1 is fulfilled for the CW-complex $B\Gamma_q$ (with $b = 1$) and $n = 2j - 2$. We then may conclude that the groups Γ_q have universally strange p-torsion (in dimension $2j - 2$) for all primes $q \neq p$:

COROLLARY 5.4. *Let p be a properly irregular prime and j an even integer $<p$ such that p divides the numerator of B_j/j. Then there exists p-torsion in $\tilde{E}_{2j-2}(B\Gamma_q)$ for all odd primes $q \neq p$ and all reduced homology theories $\tilde{E}_*(-)$ corresponding to connective spectra E such that $\pi_0 E$ is not p-divisible.*

REMARK 5.5. We established in Section 1 of [Ar2] that certain congruence subgroups have torsion in their ordinary integral homology in infinitely many dimensions (and thus have *very strange torsion* according to the definition given in [We]). However, we don't know if there are groups with universally strange p-torsion in infinitely many dimensions, in other words with *universally very strange p-torsion*.

6. Relationships between extraordinary and ordinary cohomology theories

Now, we would like to do the same with *cohomology* theories.

LEMMA 6.1. *Let E be a spectrum, X a CW-complex, $\tilde{E}^*(X)$ the reduced E-cohomology of X, and ρ an integer. The power map $\chi^\rho : E \to E$ induces multiplication by $\rho : \tilde{E}^*(X) \to \tilde{E}^*(X)$.*

Proof. As in the proof of Lemma 2.1, we may assume that E is an Ω-spectrum. Then $\tilde{E}^l(X) = [X, E_l]$ and the homomorphism $(\chi^\rho)^* : [X, E_l] \to [X, E_l]$ is just given by $(\chi^\rho)^*(\alpha)(x) = (\alpha(x))^\rho$ for $\alpha \in [X, E_l]$ and $x \in X$ (using the H-space structure of $E_l \simeq \Omega E_{l+1}$): by definition of the group structure of $[X, E_l]$, this means that $(\chi^\rho)^*(\alpha) = \rho\alpha$.

The point is that we cannot just formulate the results of Section 2 for cohomology theories, because an E-cohomology group of X may not, in general, be computed with a Postnikov section of E (as it was the case in the proof of Theorem 2.2). In order to do it, we must suppose that the space X is of *finite dimension* d (and we assume again that X is $(b-1)$-homologically-connected, $d \geq b+1$), but the spectrum E *does not need to be bounded below* any more.

If n is a given integer there is an obvious map of spectra $\gamma_{b-n} : E(b-n) \to E$, where $E(b-n)$ is a spectrum satisfying $\pi_i E(b-n) = 0$ for $i < b-n$ and $(\gamma_{b-n})_*$: $\pi_i E(b-n) \xrightarrow{\cong} \pi_i E$ for $i \geq b-n$; let us write $E(b-n, d-n]$ for the Postnikov section $E(b-n)[d-n]$ of $E(b-n)$: $E(b-n, d-n]$ satisfies $\pi_i E(b-n, d-n] = 0$ if $i < b-n$ or $i > d-n$ and $\pi_i E(b-n, d-n] \cong \pi_i E$ if $b-n \leq i \leq d-n$ (cf. Section 4 of [D], [M], or [V] for the existence of $E(b-n, d-n]$). It turns out that $\tilde{E}^n(X) \cong \tilde{E}(b-n, d-n]^n(X)$, according to the cohomological version of Lemma 4.2 of [Ar3]. Therefore, $\tilde{E}^n(X)$ may be calculated with the spectrum $E(b-n, d-n]$, which has finitely many non-trivial homotopy groups (even if E is not bounded below) and which has consequently all its k-invariants of finite order. This enables us to deduce from Corollary 1.4 and Theorem 1.5, applied to the spectrum $E(b-n, d-n]$, the following assertions.

THEOREM 6.2. *Let E be any spectrum, X a $(b-1)$-homologically-connected CW-complex of finite dimension d, n an integer, and let ρ_i denote the order of the k-invariant $k^{i+1}(E(b-n))$ in $H^{i+1}(E(b-n, i-1]; \pi_i E)$ for $b-n \leq i \leq d-n$ (observe that $\rho_{b-n} = 1$).*

(a) *There exist homomorphisms*

$$\Phi^n : \tilde{E}^n(X) \to \bigoplus_{t=b-n}^{d-n} H^{n+t}(X; \pi_t E)$$

and

$$\Psi^n : \bigoplus_{t=b-n}^{d-n} H^{n+t}(X; \pi_t E) \to \tilde{E}^n(X)$$

with the property that the composition $\Psi^n \Phi^n : \tilde{E}^n(X) \to \tilde{E}^n(X)$ *is multiplication by* $\rho_{d-n}\rho_{d-n-1} \cdots \rho_{b-n+1}$.

(b) *For any integer* t *with* $b - n \le t \le d - n$, *there exist homomorphisms*

$$\Theta^{n,t} : H^{n+t}(X; \pi_t E) \to \tilde{E}^n(X)$$

and

$$\Lambda^{n,t} : \tilde{E}^n(X) \to H^{n+t}(X; \pi_t E)$$

such that the composition $\Lambda^{n,t}\Theta^{n,t} : H^{n+t}(X; \pi_t E) \to H^{n+t}(X; \pi_t E)$ *is multiplication by* $\rho_{d-n}\rho_{d-n-1} \cdots \rho_t$.

REMARK 6.3. According to Remark 1.6, it is always possible to replace the integer $\rho_{d-n}\rho_{d-n-1} \cdots \rho_{b-n+1}$ by $T_{2,d-b+1}$ and $\rho_{d-n}\rho_{d-n-1} \cdots \rho_t$ by $T_{t-b+n+1,d-b+1}$.

REMARK 6.4. Theorem 6.2 also determines an approximation of the exponent of $\tilde{E}^n(X)$ as explained for the case of homology in Theorem 3.1. The results may be of special interest if X is the classifying space BG of a discrete group G of finite cohomological dimension d (for instance a congruence subgroup of odd level in $SL_m(\mathbb{Z})$, $m \ge 2$).

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