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Homotopy classes of truncated projective resolutions

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Let X be a finite connected m-dimensional CW-complex whose universal cover $\tilde{X}$ is $(m-1)$-connected. Then the homotopy type of X can be recognised in the following sense from the homotopy type of the cellular chain complex $C(\tilde{X})$: If Y is another space like X then Y is homotopically equivalent to X if and only if there exists an isomorphism of $\pi_1(X)$ to $\pi_1(Y)$ under which $C(\tilde{X})$ and $C(\tilde{Y})$ are equivariantly homotopically equivalent (as augmented chain complexes). This result is essentially due to S. Mac Lane and J. H. C. Whitehead ([MW]; cf. also [D]); it transforms a topological problem into one about the integral representation theory of the fundamental groups. When these groups are assumed to be finite, the algebraic problem was studied with great effect by Wesley Browning [B1, 2, 3]. He showed how the chain homotopy classes of truncated projective $\mathbb{Z}G$-resolutions of $\mathbb{Z}$ for a given finite group G can be parametrised by the elements of a certain naturally occurring group within the K-theory of G.

Browning worked with pointed lattices, whose relevance in this context was first explored by J. S. Williams [W]. (A pointed lattice means here a pair (L, x) consisting of a $\mathbb{Z}G$-lattice L and an element x belonging to some specified $H^k(G, L)$.) Browning developed a general theory of pointed lattices that parallels ordinary lattice theory. This led to a plethora of new definitions and notations and is perhaps one reason why his work is generally regarded as difficult to absorb. He circulated his papers in 1979 but they have unfortunately never been published. However the material in them is now beginning to be widely used. An account of the results in the 2-dimensional case together with Browning's proofs has appeared recently [GL]. One of the main results in [B1] is a cancellation theorem, generalising one of Williams, discovered independently at about the same time by P. Linnell, who did eventually publish his result in 1985 [L1].

The point of the present paper is to give a new treatment of all the main theoretical results in Browning's three papers (but excluding the cancellation theorem which I use in the form given in [L1]). I avoid entirely the notion of pointed lattice and base my account solely on old concepts from elementary K-theory (but I explain the connexion with pointed lattices in §2.7). The basic object of study here is an Abelian group of which the ordinary genus class group is a homomorphic image. This basic object I christen *Browning's class group.*

It is defined for any $\mathbb{Z}G$-lattice M and any covariant additive functor Φ from $\mathbb{Z}G$-lattices to Abelian groups such that $\Phi(M)$ has finite exponent. There is a natural homomorphism of the Browning class group $\mathrm{Cl}\,(M; \Phi)$ onto the genus class group $\mathrm{Cl}\,(M)$ of M (§2.1). In particular, $\mathrm{Cl}\,(\mathbb{Z}G)$ is the familiar reduced projective class group and if M is the $(m+1)$-st kernel in a projective resolution of $\mathbb{Z}$, there is a natural homomorphism of $\mathrm{Cl}\,(M; H^{m+1}(G, -))$ into $\mathrm{Cl}\,(\mathbb{Z}G)$ (§2.6). The kernel of this homomorphism is the group that classifies the G-linked homotopy classes of the spaces considered by Mac Lane and Whitehead (§4.1). Our discussion in the final part of the paper (§4.2) seems to go beyond the point reached by Browning.

In Browning's third paper he applies his theory to 2-complexes with finite Abelian fundamental group. His results in this case are subsumed in recent work of Linnell who obtains complete information for n-complexes [L2]. I shall indicate how Linnell's results fit into the scheme proposed here (§4.3).

I warmly thank John Moody and Alfred Weiss for helpful conversations and Peter Linnell for an enlightening correspondence.

1. Discussion of the relevant K-theory

Everything in this section is well known and readily accessible, or easy to deduce from the published literature. A convenient reference is the book by Swan–Evans [SE]. We give detailed references to [SE] as we proceed.

1.1. We work with a finite group G and the category $\mathrm{Lat}\,(\mathbb{Z}G)$ of $\mathbb{Z}G$-lattices. If M is a $\mathbb{Z}G$-lattice and π is a finite set of rational primes, then M_π will denote $M \otimes \mathbb{Z}_\pi$ (where $\mathbb{Z}_\pi$ is the semi-localization of $\mathbb{Z}$ at π). If M and N are $\mathbb{Z}G$-lattices and $\phi \in \mathrm{Hom}_G(M_\pi, N_\pi)$, then $\phi = f/s$, where $f \in \mathrm{Hom}_G(M, N)$ and s is an integer prime to π. We shall later work with a given π-number e (an integer involving only primes in π). Then we may and *shall always* assume s in the expression for ϕ is chosen so that $s \equiv 1 (\mathrm{mod}\ e)$: Let $d = et$, where t is the product of all primes in π not in e and suppose $ss' + dd' = 1$. Then $f/s = fs'/ss'$. Note that if $\mathrm{Coker} f$ is finite of order prime to π then the same holds for $\mathrm{Coker} fs'$.

We write $P_0(M)$ (P for "projective") for the Grothendieck group on the full subcategory of $\mathrm{Lat}\,(\mathbb{Z}G)$ consisting of direct summands of direct sums of copies of M, constructed relative to split exact sequences; and analogously $P_0(M_\pi)$. (So $P_0(\mathbb{Z}G)$ is the Grothendieck group on projective $\mathbb{Z}G$ modules.) Further, $T_0(M, \pi)$ (T for "torsion") will denote the Grothendieck group on the category of all finite G-modules U that are images of direct sums of copies of M and have order $|U|$ prime to π, constructed relative to all exact sequences. We shall always abbreviate $\mathrm{Aut}_{\mathbb{Z}_\pi G}\,(M_\pi)$ as $\mathrm{Aut}\, M_\pi$.

Henceforth we assume throughout that $\pi \supseteq \pi(G)$, the set of prime divisors of the group order $|G|$. K_0-theory provides us with the following exact sequence of groups:

$$\operatorname{Aut} M_\pi \xrightarrow{\alpha} T_0(M, \pi) \xrightarrow{\beta} P_0(M) \xrightarrow{\gamma} P_0(M_\pi) \longrightarrow 0,$$

with maps defined thus:

$$\gamma : [L] \mapsto [L_\pi];$$

$$\beta : [T] \mapsto s[M] - [K],$$

where $0 \to K \to M^{(s)} \to T \to 0$;

$$\alpha : \phi \mapsto [\phi],$$

where, if $\phi = f/s$ with $f \in \operatorname{End}_{\mathbb{Z}G}(M)$, the integer s prime to π and, when e is involved, $s \equiv 1 (\operatorname{mod} e)$, then $[\phi] = [\operatorname{Coker} f] - [\operatorname{Coker} s]$ (here s is viewed as the endomorphism of M given by multiplication by s). Cf. [SE], Chapter 8, pp. 140–147.

We propose to call the kernel of γ the *class group* of M and write it $\operatorname{Cl}(M)$. (Note that $\operatorname{Cl}(\mathbb{Z}G)$ is then the usual (reduced projective) class group.) The subgroup $\operatorname{Cl}(M)$ is finite and consists of the *set* of all differences $[N] - [M]$ for all $N \vee M$ (N is the genus of M). (Cf. [SE], pp. 113–4.) If $L \vee M$, then there exists a $\mathbb{Z}_\pi G$-isomorphism $\rho : L_\pi \xrightarrow{\sim} M_\pi$ and $T_0(L, \pi) = T_0(M, \pi)$. This last equality is true if one merely assumes $\mathbb{Q}L \simeq \mathbb{Q}M$: for S is a finite simple π'-image of L if and only if $\mathbb{Q}L$ has a simple $\mathbb{Q}G$-summand $\mathbb{Q}D$ with D a lattice and S an image of D; whence the simple π'-images of L and M coincide. Moreover, it is easy to see that the following triangle is commutative:

$$\begin{array}{ccc} \operatorname{Aut} L_\pi & \xrightarrow{\alpha_L} & \\ \downarrow & & T_0(M, \pi) \\ \operatorname{Aut} M_\pi & \xrightarrow{\alpha_M} & \end{array}$$

where the vertical down map is $\phi \mapsto \rho^{-1}\phi\rho$. (If $\phi = f/s$, $\rho = g/t$, $\rho^{-1} = g'/t'$, then $0 = [\rho^{-1}\rho] = [\operatorname{Coker} g'g] - [\operatorname{Coker} t't]$ and so

$$\begin{aligned}(\rho^{-1}\phi\rho)\alpha_M &= [\operatorname{Coker} g'fg] - [\operatorname{Coker} t'st] \\ &= [\operatorname{Coker} f] - [\operatorname{Coker} s] \\ &= \phi\alpha_L.)\end{aligned}$$

Call $A(M, \pi)$ the common image of α_M and α_L.

A further notational point: We shall frequently write $[f]$ instead of $[\operatorname{Coker} f]$.

1.2. We need some generalities about $T_0(M, \pi)$. These are essentially Lemmas 3 and 4 in Browning's first paper [B1]. Cf. also [S], Lemma 4.1.

If $[U] \in T_0(M, \pi)$ (so U is a finite G-module of order prime to π and there exists a surjection $M^{(r)} \to U$), then every composition factor of U is an image of M: cf. [SE], p. 171. This shows that $T_0(M, \pi)$ is $\mathbb{Z}$-free on all $[S]$, where $[S]$ is a simple image of M of order prime to π. Moreover we claim there exists an image V of M such that $[V] = [U]$. This is proved by an induction on the composition length of U: If S is a simple submodule of U, then by induction we have $[V_1] = [U/S]$ where V_1 is an image of M, say $0 \to M_1 \to M \to V_1 \to 0$. Now S is an image of M (as observed at the beginning of this paragraph) and since $M_1 \vee M$, we have $0 \to M_2 \to M_1 \to S \to 0$, whence

$$[M/M_2] = [M/M_1] + [M_1/M_2] = [V_1] + [S] = [U].$$

We next claim that every element in $T_0(M, \pi)$ can be written as $[U] - [M/rM]$, where U is an image of M, r is prime to π and (if e is involved, $r \equiv 1 (\operatorname{mod} e)$). For take $x = [A] - [B]$ and, by the above, find V so that $[V] = [B]$ and $0 \to M_1 \to M \to V \to 0$. Let r be an integer prime to π, $r \equiv 1 (\operatorname{mod} e)$ and so that $rM \subseteq M_1$. Then $[V] \oplus [M_1/rM] = [M/rM]$ and $x = [A \oplus (M_1/rM)] - [M/rM]$. Again by the last paragraph we can find an image U of M such that $[U] = [A \oplus (M_1/rM)]$.

1.3. Let us look at the connexion between $T_0(M, \pi)$ and $T_0(\mathbb{Z}G, \pi)$. It is clear that the former is a subgroup of the latter. Since $T_0(M, \pi)$ has a $\mathbb{Z}$-basis that is part of a $\mathbb{Z}$-basis of $T_0(\mathbb{Z}G, \pi)$ (cf. §1.2), so $T_0(M, \pi)$ is a direct summand of $T_0(\mathbb{Z}G, \pi)$. Also note the related fact, which we use later, that $T_0(M, \pi)$ is a direct summand of $T_0(M, \pi(G))$: a complement is the $\mathbb{Z}$-submodule on all $[S]$ with S a simple image of M of characteristic p, where $p \in \pi - \pi(G)$.

Now consider

$$\begin{array}{ccccc} T_0(M, \pi) & \xrightarrow{\beta_M} & \mathrm{Cl}\,(M) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \iota} & & & & \\ T_0(\mathbb{Z}G, \pi) & \xrightarrow{\beta_{\mathbb{Z}G}} & \mathrm{Cl}\,(\mathbb{Z}G) & \longrightarrow & 0, \end{array}$$

where ι is inclusion. The image of $\iota\beta_{\mathbb{Z}G}$ was studied by Swan in [S], p. 198. We shall here denote the image of $\iota\beta_{\mathbb{Z}G}$ as $C(M, \pi)$. Swan wrote $C_{\mathbb{Q}M}$ for our $C(M, \pi(G))$. This notation is permissible since $C(M, \pi)$ depends only on (π and) the $\mathbb{Q}G$-module determined by M. The reason is that, as we have already remarked in §1.1, $\mathbb{Q}L \simeq \mathbb{Q}M$ implies $T_0(M, \pi) = T_0(L, \pi)$.

2. The Browning class group

2.1. Let Φ be an additive covariant functor Lat $(\mathbb{Z}G) \to \mathrm{Ab}$ such that $\Phi(M)$ has finite exponent e. *Henceforth we assume* π *contains all primes in* e.

If $\phi \in \mathrm{Hom}_G(M_\pi, N_\pi)$ and $\phi = f/s = f'/s'$, where s and s' are prime to π and $s \equiv s' \equiv 1(\mathrm{mod}\ e)$, then $s'f = sf'$. If s and s' are viewed as endomorphisms of M, then $\Phi(s) = \Phi(s')$ is the identity on $\Phi(M)$. Therefore $\Phi(f) = \Phi(f')$ and consequently we can denote $\Phi(f)$ unambiguously as $\Phi(\phi)$.

If $\phi \in \mathrm{Aut}\, M_\pi$, then $\phi \mapsto \Phi(\phi)$ is a group homomorphism $\Phi_M : \mathrm{Aut}\, M_\pi \to \mathrm{Aut}\, \Phi(M)$, whose kernel we write $\mathrm{Aut}\,(M_\pi; \Phi)$. Let $A(M; \Phi, \pi)$ denote the image of $\mathrm{Aut}\,(M_\pi; \Phi)$ under $\alpha : \mathrm{Aut}\, M_\pi \to T_0(M, \pi)$. (We shall frequently omit π or M from $A(M; \Phi, \pi)$ when these are understood.) We now have the exact sequence of Abelian groups

$$\mathrm{Aut}\, M_\pi/\mathrm{Aut}\,(M_\pi; \Phi) \longrightarrow T_0(M, \pi)/A(\Phi) \longrightarrow \mathrm{Cl}\,(M) \longrightarrow 0. \qquad (*)$$

We propose to call $T_0(M, \pi)/A(\Phi)$ the *Browning class group of* M *with respect to* Φ and shall denote this group by $\mathrm{Cl}\,(M; \Phi, \pi)$. Note that if $\Phi(M) = 0$ then $\mathrm{Cl}\,(M; \Phi, \pi) \simeq \mathrm{Cl}\,(M)$. (Of course, $\mathrm{Cl}\,(M)$ does not depend on π because of our permanent assumption that $\pi \supseteq \pi(G)$.) For example if $\Phi(\mathbb{Z}G) = 0$, then $\mathrm{Cl}\,(\mathbb{Z}G; \Phi, \pi)$ is (isomorphic to) the usual projective class group $\mathrm{Cl}\,(\mathbb{Z}G)$.

2.2. *If* $\Phi(M)$ *is finite, then so is* $\mathrm{Cl}\,(M; \Phi, \pi)$. For the finiteness of $\Phi(M)$ implies that the image of Φ_M is finite and since $\mathrm{Cl}\,(M)$ is finite anyway, therefore the exact sequence $(*)$ in §2.1 gives the required conclusion.

Let L be a lattice in the genus of M and $\rho : L_\pi \xrightarrow{\sim} M_\pi$. Then $\Phi(\rho)$ is an isomorphism $\Phi(L) \xrightarrow{\sim} \Phi(M)$ and we have the following commutative square:

$$\begin{array}{ccc} \mathrm{Aut}\, L_\pi & \xrightarrow{\Phi_L} & \mathrm{Aut}\, \Phi(L) \\ \downarrow & & \downarrow \\ \mathrm{Aut}\, M_\pi & \xrightarrow{\Phi_M} & \mathrm{Aut}\, \Phi(M) \end{array}$$

where the left vertical map is $\phi \mapsto \rho^{-1}\phi\rho$ and the right one is $\lambda \mapsto \Phi(\rho)^{-1}\lambda\Phi(\rho)$, for λ in $\mathrm{Aut}\Phi(L)$. It follows, using also the commutative triangle displayed in §1.1, that the α_L-image of the kernel of Φ_L coincides with the α_M-image of the kernel of Φ_M. Thus $A(M; \Phi, \pi) = A(L; \Phi, \pi)$, whence $\mathrm{Cl}\,(M; \Phi, \pi) = \mathrm{Cl}\,(L; \Phi, \pi)$.

2.3. If $N \vee M$, we can embed N in M with finite cokernel prime to π, say

$$0 \longrightarrow N \xrightarrow{f} M \longrightarrow U \longrightarrow 0.$$

Call this embedding (N, f). If (L, g) is another embedding, we define

$$(N, f) \sim (L, g)$$

to mean that there exists an isomorphism $h : N \tilde{\to} L$ so that $\Phi(hg) = \Phi(f)$. Clearly $\sim$ is an equivalence relation. Let $[N, f]$ be the equivalence class containing (N, f) and $\mathbf{E}(M; \Phi, \pi)$ (E for "embedding") the set of all equivalence classes.

THEOREM 1. *There exists a surjection* $\mathbf{E}(M; \Phi, \pi) \to \mathrm{Cl}\,(M; \Phi, \pi)$.

We associate to (N, f) the element $[\mathrm{Coker} f] + A(\Phi, \pi)$. The fact that this procedure gives a well-defined mapping on $\mathbf{E}$ is a consequence of the following

LEMMA. *If* $(N, f) \sim (L, g)$, *then* $[\mathrm{Coker} f] \equiv [\mathrm{Coker}\, g] \bmod A(\Phi)$.

Proof. Let $h : N \tilde{\to} L$ so that $\Phi(hg) = \Phi(f)$. If $(f/1)^{-1} = f'/s'$, then $f'f = s'$ and so $\Phi(f'f)$ is the identity, whence $[\mathrm{Coker} f'f] \in A(\Phi)$. Consequently

$$[\mathrm{Coker}\, hgf'f] = [\mathrm{Coker}\, g] + [\mathrm{Coker} f'f] \equiv [\mathrm{Coker}\, g] \bmod A(\Phi).$$

Now $\Phi(hgf') = \Phi(f)\Phi(f') = \text{identity}$, whence $[\mathrm{Coker}\, hgf'] \in A(\Phi)$ and so

$$[\mathrm{Coker}\, hgf'f] \equiv [\mathrm{Coker} f] \bmod A(\Phi).$$

The map of Theorem 1 is surjective because every element in $\mathrm{Cl}\,(M; \Phi, \pi)$ can be written in the form $[U] + A(\Phi)$ for a suitable image U of M (cf. §1.2). So Theorem 1 is established.

2.4. When M satisfies the Eichler condition, the surjection of Theorem 1 becomes a bijection. The proof of this depends on the following result.

THE BROWNING–LINNELL CANCELLATION THEOREM [B1], [L1]. *Let M satisfy the Eichler condition. Then there exists π (containing $\pi(G)$ as always) so that if we are given*

$$0 \longrightarrow N \xrightarrow{f} M \longrightarrow U \longrightarrow 0,$$
$$0 \longrightarrow M \xrightarrow{g} M \longrightarrow V \longrightarrow 0$$

with $[U] = [V]$ *in* $T_0(M, \pi)$ *and* $\Phi(g) = \text{identity}$, *then there exists* $h : N \tilde{\to} M$ *so that* $\Phi(h) = \Phi(f)$ *(in other words,* $(N, f) \sim (M, g)$*).*

This statement follows directly from Linnell's Lemma 3.3 [L1].

The existence of π is explained in [SE], Chapter 9; especially p. 196. Sometimes $\pi(G)$ will do. A case in point, and this will be important for us later, is the following: Given *any* lattice M, then $M \oplus M$ always satisfies the Eichler condition and here (i.e., for $M \oplus M$) the Cancellation Theorem holds with $\pi = \pi(G)$ ([SE], p. 173 and [GrL], p. 364).

THEOREM 2. *If M satisfies the Eichler condition and π is chosen suitably for this, then the map in Theorem* 1 *is bijective.*

Proof. Suppose $[N, f], [L, g]$ have the same image in $\mathrm{Cl}(M; \Phi, \pi)$. Let $U = \mathrm{Coker} f$, $V = \mathrm{Coker}\, g$. So $[U] \equiv [V] \bmod A(\Phi)$. If $(g/1)^{-1} = g'/r$, then

$$[\mathrm{Coker}\, gg'] = [V] + [\mathrm{Coker}\, g'] = [\mathrm{Coker}\, r] \in A(\Phi)$$

(because $\Phi(r) = \text{identity}$); and so

$$[\mathrm{Coker} fg'] = [U] + [\mathrm{Coker}\, g'] \equiv 0 \bmod A(\Phi).$$

Hence $[\mathrm{Coker} fg']$ is the image under α (cf. §1.1) of some $\phi \in \mathrm{Aut}(L_\pi; \Phi)$. If $\phi = \ell/s$,

$$[\mathrm{Coker} fg'] + [\mathrm{Coker}\, s] = [\mathrm{Coker}\, \ell]$$

and

$$0 \longrightarrow N \xrightarrow{fg's} L \longrightarrow D \longrightarrow 0$$

is exact, where $[D] = [\mathrm{Coker}\, \ell]$. Since $\Phi(\ell) = \text{identity}$, therefore by the Browning–Linnell Theorem there exists an isomorphism $h: N \xrightarrow{\sim} L$ so that $\Phi(h) = \Phi(fg's)$. Since $\Phi(s)$ and $\Phi(g'g)$ are both identities, so $\Phi(hg) = \Phi(f)$. Thus $[N, f] = [L, g]$.

2.5. It is clear that always $T_0(M, \pi) = T_0(M \oplus M, \pi)$. Also $A(M, \Phi) \subseteq A(M \oplus M, \pi)$: For if $[f/s] \in A(M, \Phi)$, then f is an endomorphism of M satisfying $\Phi(f) = \text{identity}$ and $[f] \in A(M, \Phi)$ because $s \equiv 1 (\mathrm{mod}\ e)$. If g is the endomorphism of $M \oplus M$ given by $\begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix}$, then $[g] = [f]$ and $\Phi(g) = \text{identity}$ on $\Phi(M \oplus M)$. Hence $[f] \in A(M \oplus M, \Phi)$.

PROPOSITION [B2, 6.5]. $A(M, \Phi) = A(M \oplus M, \Phi)$.

Proof. Let $[f] \in A(M \oplus M, \Phi)$. We need to prove $[f] \in A(M, \Phi)$. Now f is an endomorphism of $M \oplus M$ with $\Phi(f) = \text{identity}$. By [L1], Lemma 3.1, we can find endomorphisms $\beta_1, \beta_2, \beta_3$ with matrices

$$\begin{pmatrix} n_1 & \xi_1 \\ 0 & n_1 \end{pmatrix}, \quad \begin{pmatrix} n_2 & \xi_2 \\ 0 & n_2 \end{pmatrix}, \quad \begin{pmatrix} n_3 & 0 \\ \xi_3 & n_3 \end{pmatrix},$$

where $n_i \equiv 1 (\bmod e)$ for all i, so that $\beta_1 f \beta_2 \beta_3 = g$ has matrix $\begin{pmatrix} \gamma & 0 \\ 0 & \delta \end{pmatrix}$. Since $\Phi(g) = \Phi(\beta_1)\Phi(\beta_2)\Phi(\beta_3)$, so $\Phi(\gamma) = \Phi(\delta) = n_1 n_2 n_3 = \text{identity}$ on $\Phi(M)$ (cf. [L1], p. 454), whence $[g] = [\gamma] + [\delta] \in A(M, \Phi)$. But

$$[g] = [f] + [\beta_1] + [\beta_2] + [\beta_3] \equiv [f] \bmod A(M, \Phi),$$

because $[\beta_i] = 2[n_i] \in A(M, \Phi)$. Thus $[f] \in A(M, \Phi)$, as required.

COROLLARY. $\mathrm{Cl}\,(M; \Phi, \pi) = \mathrm{Cl}\,(M \oplus M; \Phi, \pi)$.

2.6. Suppose we are given a projective presentation

$$0 \longrightarrow M \longrightarrow Q \longrightarrow D \longrightarrow 0, \qquad (*)$$

of a module D. Take $\Phi = \mathrm{Ext}^1_G(D, -)$. To apply our theory to this Φ we must check that $\Phi(M)$ has finite exponent. This is clear when D is a lattice because then $\Phi(M) \simeq H^1(G, \mathrm{Hom}\,(D, M))$; it is also clear when D itself has finite exponent. For a general D, apply Φ to the short exact sequence $0 \to \mathrm{Tor}\,(D) \to D \to \bar{D} \to 0$ to obtain the required conclusion.

If f is an endormorphism of M with cokernel of order prime to π and $\Phi(f) = \text{identity}$, then $[f] \in A(\Phi)$. We claim that the image of $[f]$ in $T_0(\mathbb{Z}G, \pi)$ (under the inclusion $T_0(M, \pi) \to T_0(\mathbb{Z}G, \pi)$) lies in $A(\mathbb{Z}G, \pi)$: Since $\Phi(f)$ is the identity, the pushout to f of $(*)$ is equivalent (as an extension of modules) to $(*)$ and hence there exists a map $\check{f}: Q \to Q$ which restricts to the identity on D and to f on M. Consequently $[\check{f}] = [f]$ and $[\check{f}] \in A(Q, \pi) = A(\mathbb{Z}G^{(r)}, \pi)$ if $Q \vee \mathbb{Z}G^{(r)}$; moreover $A(\mathbb{Z}G^{(r)}, \pi) = A(\mathbb{Z}G, \pi)$ by the Proposition in §2.5. So we have established

PROPOSITION 1. *There exists a natural homomorphism*

$$\mathrm{Cl}\,(M; \mathrm{Ext}^1_G(D, -), \pi) \longrightarrow \mathrm{Cl}\,(\mathbb{Z}G).$$

The image of the map in this Proposition is Swan's group $C(M, \pi)$ (§1.3).

There is an important special case when the map of Proposition 1 is an isomorphism. Assume M is not core-equal: this means that $M = M' \oplus P$ for some non-zero projective module P. Here $T_0(M, \pi) = T_0(P, \pi) = T_0(\mathbb{Z}G, \pi)$ and we claim $A(M; \Phi, \pi) \supseteq A(\mathbb{Z}G, \pi)$ provided $\Phi(\mathbb{Z}G) = 0$: for if $\phi = f/s$ is an automorphism of $\mathbb{Z}_\pi G$, then ϕ extends to an automorphism of P_π (since $\mathbb{Z}_\pi G$ is a summand of P_π) and hence $\psi = (\mathrm{id}, \phi) : M' \oplus P \to M' \oplus P$ is an automorphism of M for which $[\psi] = [\phi]$ and $\Phi(\psi) = \Phi(\mathrm{id})$ because $\Phi(P) = 0$. (Notice that this argument works with a general Φ). We have $\mathrm{Ext}^1_G(D, \mathbb{Z}G) = 0$ if D is a lattice and so we have established

PROPOSITION 2. *If M is not core-equal and D is a $\mathbb{Z}G$-lattice, then*

$$\mathrm{Cl}\,(M; \mathrm{Ext}^1_G(D, -), \pi) \simeq \mathrm{Cl}\,(\mathbb{Z}G).$$

2.7. We conclude this section by discussing the connexion with the theory of pointed lattices. A pair (N, x) is a Φ-pointed $\mathbb{Z}G$-lattice if N is a $\mathbb{Z}G$-lattice and x is an element in $\Phi(N)$. It is obvious how to define morphisms of such objects. In particular, $(N, x) \simeq (L, y)$ means that there exists a $\mathbb{Z}G$-isomorphism $h : N \tilde{\to} L$ such that $x\Phi(h) = y$; moreover, (N, x) and (L, y) are in the same genus if there exists an isomorphism $\rho : N_\pi \tilde{\to} L_\pi$ so that $x\Phi(\rho) = y$. Write $[N, x]$ for the isomorphism class containing (N, x) and $\bigvee(M, z)$ for the set of all isomorphism classes in the genus of (M, z).

Recall (from §2.1) that $\mathrm{Aut}\, M_\pi$ acts on $\Phi(M)$; it also acts on $\mathbf{E}(M; \Phi, \pi)$: if $\rho \in \mathrm{Aut}\, M_\pi$, then $[N, f]\rho = [N, fg]$, where $\rho = g/r$. Let $St(z)$ be the stabilizer of z in $\mathrm{Aut}\, M_\pi$.

PROPOSITION. *There exists a bijection* $\mathbf{E}(M; \Phi, \pi)/St(z) \tilde{\to} \bigvee(M, z)$.

Proof. Given an embedding (N, f), let $x = z\Phi((f/1)^{-1})$, thus producing the pointed lattice (N, x). It is easy to check that $[N, f] \to [N, x]$ is a well defined map of $\mathbf{E}(M; \Phi, \pi)$ into $\bigvee(M, z)$. Clearly this is surjective.

Suppose (L, g) produces (L, y) isomorphic to (N, x). So there exists $h : N \tilde{\to} L$ such that $x\Phi(h) = y$. If $(f/1)^{-1} = f'/r$ and $\sigma = (f'hg)/r$, then

$$z\Phi(\sigma) = x\Phi(hg) = y\Phi(g) = z,$$

so that $\sigma \in St(z)$. Now $[L, g] = [N, ff'hg] = [N, f]\sigma$, whence (N, x) and (L, y) belong to the same $St(z)$-orbit.

Note that by definition $St(z)$ contains $\mathrm{Aut}\,(M_\pi; \Phi)$, the kernel of the action of $\mathrm{Aut}\,(M_\pi)$ on $\Phi(M)$. There are situations where $St(z) = \mathrm{Aut}\,(M_\pi; \Phi)$. We shall meet one such in the next section (§3.5).

It should also be observed that $\mathrm{Aut}\, M_\pi$ acts on $\mathrm{Cl}\,(M; \Phi, \pi)$ via the group homomorphism $\mathrm{Aut}\, M_\pi \to \mathrm{Cl}\,(M; \Phi, \pi)$ (cf. $(*)$ in §2.1) and the kernel of this action is $\mathrm{Aut}\,(M_\pi; \Phi)$. The surjection of Theorem 1 (§2.3) is equivariant.

3. Truncated resolutions

3.1. If A is a G-module (not necessarily a lattice) and (P, C) denotes the projective resolution

$$\begin{array}{ccccccccccc} \cdots \longrightarrow & P_i & \longrightarrow & P_{i-1} & \longrightarrow \cdots & P_0 & \longrightarrow & A & \longrightarrow & 0, \\ & & \searrow \quad \nearrow & & & & & & & \\ & & C_i & & & & & & & \end{array}$$

then ${}_m(P, C)$ or just ${}_mP$ will mean the truncated resolution

$$0 \longrightarrow C_{m+1} \longrightarrow P_m \longrightarrow \cdots \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

and $\chi_m(P) = \Sigma_{i=0}^{m}(-1)^{m-i}\,\text{rank } P_i$ its Euler characteristic. (We interpret A as C_0.) If ${}_mP$ and ${}_mP'$ are chain homotopically equivalent (as augmented complexes over A), we shall write ${}_mP \sim {}_mP'$ and denote the equivalence class containing ${}_mP$ as $[{}_mP]$.

Let $\mathbf{P}(A; m, \ell)$ denote the set of all $[{}_mP]$ with Euler characteristic ℓ. Suppose that ${}_mP \sim {}_mP'$ and let $\alpha : {}_mP \to {}_mP'$, $\beta : {}_mP' \to {}_mP$ be chain maps yielding the homotopy. It is easy to see, but will be important for us later, that α induces an isomorphism $C_{m+1} \to C'_{m+1}$ (with inverse β).

Pick and fix one particular truncated projective resolution ${}_m(Q, D)$ of A with Euler characteristic ℓ and set $D_{m+1} = M$. Following Dyer [D], p. 256, we call (Q, D) the *reference resolution*. If ${}_m(P, C)$ is a truncated projective resolution with the same Euler characteristic, then it is easy to see that $C_{m+1} \vee M$ (compare Q_π and P_π by Schanuel's Lemma and use the fact that $\mathbb{Z}_\pi G$-projectives are free; e.g. [G], 3.3).

Let Φ be the functor $\text{Ext}_G^{m+1}(A, -)$ and π be a given finite set of primes containing $\pi(G)$. The π-number e of §1.1 will continue to be the exponent of $\Phi(M)$ (cf. §2.1). When $m \geq 1$ or A is a lattice, then $|G|\Phi(M) = 0$ and so e is actually a $\pi(G)$-number. But when $m = 0$ and A is not a lattice, e could very well involve primes outside $\pi(G)$.

Given ${}_m(P, C)$ with Euler characteristic ℓ, there exists an embedding of C_{m+1} in M with finite cokernel prime to π. We claim that among these embeddings there is one class that determines a well defined map of $\mathbf{P}(A; m, \ell)$ into $\mathbf{E}(M; \Phi, \pi)$. The main result (Theorem 3, below) is that this map is a bijection.

3.2. We begin by proving that we really do obtain a map. Given (P, C), we consider chain maps $\phi : {}_m(P_\pi) \to {}_m(Q_\pi)$ over the identity on A_π such that ϕ restricts to an isomorphism on $(C_{m+1})_\pi$. If $\phi = f/r$ on $(C_{m+1})_\pi$, where (as usual) $f : C_{m+1} \to M$, r is prime to π and chosen so that $r \equiv 1(\text{mod } |G|)$, then since f is injective, $[C_{m+1}, f] \in \mathbf{E}(M; \Phi, \pi)$.

If ψ is another chain map like ϕ, both ϕ and ψ extend to dimensions $>m$ and give chain maps $P_\pi \to Q_\pi$ over the identity on A_π. Hence they are homotopic: there exists a chain map θ of degree $+1$ such that $\phi - \psi = \theta\partial + \partial\theta$. Then θ induces

$$(C_{m+1})_\pi \xrightarrow{\iota} (P_m)_\pi \xrightarrow{\theta_m} (Q_{m+1})_\pi \xrightarrow{\partial} M_\pi$$

where ι is inclusion and ∂ is surjective. Now $\Phi(\iota\theta_m\partial) = 0$ because Φ vanishes on projectives. Hence $\Phi(\phi) = \Phi(\psi)$ on $\Phi(C_{m+1})$ and so, if $\psi = g/s$, then $\Phi(f) = \Phi(g)$ which, by the definition of $\mathbf{E}(M; \Phi, \pi)$, gives $[C_{m+1}, f] = [C_{m+1}, g]$.

We next show that maps like ϕ do exist. Adjust the resolutions P_* and Q_* to produce new resolutions $\tilde{P}_*$ and $\tilde{Q}_*$ that differ from P_* and Q_* only in dimensions $\leq m$ and satisfy rank $\tilde{P}_i =$ rank $\tilde{Q}_i$ for $0 \leq i \leq m$. If, say, rank $Q_0 -$ rank $P_0 = |G|r > 0$, replace P_0 by $P_0 \oplus \mathbb{Z}G^{(r)}$ and P_1 by $P_1 \oplus \mathbb{Z}G^{(r)}$ (this leaves the Euler characteristic unchanged) and then $\tilde{P}_0 = P_0 \oplus \mathbb{Z}G^{(r)}$ has the same rank as $\tilde{Q}_0 = Q_0$. We repeat this procedure all the way up to dimension $m-1$. Then the resolutions have their m-dimensional terms of equal rank since the m-th partial Euler characteristics are the same. Hence there exists an isomorphism $\psi : {}_m(\tilde{P}_\pi) \tilde{\to} {}_m(\tilde{Q}_\pi)$ of complexes over A_π (cf. [G], 3.5). Let ϕ be the composite of the following maps (all are over the identity on A_π):

$${}_m(P_\pi) \xrightarrow{\iota} {}_m(\tilde{P}_\pi) \xrightarrow{\psi} {}_m(\tilde{Q}_\pi) \xrightarrow{\rho} {}_m(Q_\pi),$$

where ι is the natural inclusion and ρ the natural projection. Then ϕ restricts to an isomorphism on $(C_{m+1})_\pi$.

Finally, suppose ${}_mP'$ is homotopic to ${}_mP$ with chain maps ${}_mP \xrightarrow{\alpha} {}_mP' \xrightarrow{\beta} {}_mP$. We may construct, by the method explained above, allowable chain maps $\phi : {}_m(P_\pi) \to {}_m(Q_\pi)$ and $\phi' : {}_m(P'_\pi) \to {}_m(Q_\pi)$. Let $\phi = f/r$ and $\phi' = f'/r'$, so that ${}_mP$ yields the embedding (C_{m+1}, f) and ϕ' the embedding (C'_{m+1}, f'). Since α restricts to an isomorphism on C_{m+1} (cf. §3.1), $\alpha_\pi\phi'$ restricts to an isomorphism on $(C_{m+1})_\pi$ and hence yields the embedding $(C_{m+1}, \alpha f')$. Because $\alpha f'/r'$ and f/r are maps over the identity on A_π, they induce the same map on homology: $\Phi(\alpha f') = \Phi(f)$. As α restricts to an isomorphism on C_{m+1}, so $(C_{m+1}, f) \sim (C'_{m+1}, f')$. Hence $[{}_m(P, C)] \to [C_{m+1}, f]$ is a well defined map. Call it η.

3.3. The proof that η is a bijection involves a well known fact from homological algebra: If ${}_m(P, C)$ and ${}_m(P', C')$ have $C_{m+1} = C'_{m+1} = L$, say and determine the same element ξ in $\Phi(L)$, then they are homotopically equivalent. I have been unable to locate a satisfactory reference for this, so here is a proof.

Construct a map $f: {}_mP \to {}_mP'$ lifting id_A and take the pushout to f on L:

$$\begin{array}{ccccccc} 0 \longrightarrow & L & \longrightarrow & P_m & \longrightarrow & P_{m-1} & \longrightarrow \cdots \\ & {\scriptstyle f}\downarrow & & \downarrow & & \downarrow{\scriptstyle =} & \\ 0 \longrightarrow & L & \longrightarrow & X & \longrightarrow & P_{m-1} & \longrightarrow \cdots . \end{array}$$

The pushout also determines ξ. Since Φ is naturally equivalent to $\mathrm{Ext}^1_G(C_m, -)$, therefore the extensions

$$0 \longrightarrow L \longrightarrow P_m \longrightarrow C_m \longrightarrow 0,$$

$$0 \longrightarrow L \longrightarrow X \longrightarrow C_m \longrightarrow 0$$

are equivalent. Hence the pushout is isomorphic (and therefore certainly homotopically equivalent) to ${}_mP$. Moreover, the pushout provides a map $g: X \to P'_m$ giving the following commutative diagram:

$$\begin{array}{ccccccccccccc} 0 \longrightarrow & L & \longrightarrow & P_m & \longrightarrow & P_{m-1} & \longrightarrow \cdots \longrightarrow & P_0 & \longrightarrow & A & \longrightarrow 0 \\ & {\scriptstyle f}\downarrow & & \downarrow & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} & \\ 0 \longrightarrow & L & \longrightarrow & X & \longrightarrow & P_{m-1} & \longrightarrow \cdots \longrightarrow & P_0 & \longrightarrow & A & \longrightarrow 0 \\ & {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle =} & \\ 0 \longrightarrow & L & \longrightarrow & P'_m & \longrightarrow & P'_{m-1} & \longrightarrow \cdots \longrightarrow & P'_0 & \longrightarrow & A & \longrightarrow 0. \end{array} \qquad (*)$$

Cf. [HS], Chapter 4, §9. The complex homomorphism in $(*)$ taking the middle complex (the pushout) to ${}_mP'$ is an isomorphism on homology, whence these two truncated resolutions are homotopically equivalent (cf. [Sp], Chapter 4, §2). But we already know the middle complex is homotopically equivalent to ${}_mP$ and so we are done.

3.4. We are now ready to prove

THEOREM 3. *The map* $\eta : \mathbf{P}(A; m, \ell) \to \mathbf{E}(M; \mathrm{Ext}^{m+1}_G(A, -), \pi)$ *is a bijection.*

Note that the **E**-set involves π but the **P**-set is independent of π. We return to this point in Theorem 5. We continue to write $\Phi = \mathrm{Ext}^{m+1}_G(A, -)$.

Surjectivity of η. Given (C, f), let $f_\pi^{-1} = g/s$ and take the pushout to g:

$$\begin{array}{ccccccccccccc} 0 \longrightarrow & M & \longrightarrow & Q_m & \longrightarrow & Q_{m-1} & \longrightarrow \cdots \longrightarrow & Q_0 & \longrightarrow & A & \longrightarrow 0 \\ & {\scriptstyle g}\downarrow & & \downarrow & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} & \\ 0 \longrightarrow & C & \longrightarrow & X & \longrightarrow & Q_{m-1} & \longrightarrow \cdots \longrightarrow & Q_0 & \longrightarrow & A & \longrightarrow 0. \end{array}$$

As $fg = s$, $g_\pi^{-1} = f/s$ and $[C, f]$ is the image under η of the class of

$$0 \longrightarrow C \longrightarrow X \longrightarrow Q_{m-1} \longrightarrow \cdots \longrightarrow Q_0 \longrightarrow A \longrightarrow 0.$$

Injectivity of η. Suppose $[_mP], [_mP']$ have the same image: $[C_{m+1}, f] = [C'_{m+1}, f']$. Construct the pushout to h (where $h : C_{m+1} \tilde{\to} C'_{m+1}$ and $\Phi(hf') = \Phi(f)$):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_{m+1} & \longrightarrow & P_m & \longrightarrow & P_{m-1} & \longrightarrow & \cdots \\ & & {\scriptstyle h}\downarrow & & \downarrow & & \downarrow{\scriptstyle =} & & \\ 0 & \longrightarrow & C'_{m+1} & \longrightarrow & X & \longrightarrow & P_{m-1} & \longrightarrow & \cdots . \end{array}$$

and let the lower sequence yield $[C'_{m+1}, g]$. Since the above two truncated resolutions are isomorphic, $[C_{m+1}, f] = [C_{m+1}, hg]$ and $\Phi(hg) = \Phi(f) = \Phi(hf')$, whence $\Phi(g) = \Phi(f')$. Thus $[C'_{m+1}, f'] = [C'_{m+1}, g]$ and we are reduced to showing

$$0 \longrightarrow C'_{m+1} \longrightarrow X \longrightarrow P_{m-1} \longrightarrow \cdots$$

is homomorphic to

$$0 \longrightarrow C'_{m+1} \longrightarrow P'_m \longrightarrow P'_{m-1} \longrightarrow \cdots$$

These two truncated resolutions represent the same element in $\mathrm{Ext}_G^{m+1}(A, C'_{m+1})$ (because $\Phi(g) = \Phi(f')$) and so are homotopically equivalent (§3.3).

3.5 Consider again pointed lattices, as in §2.7. Let z be the image of the identity map on M under the surjection $\mathrm{End}_G(M) \to \mathrm{Ext}_G^{m+1}(A, M)$. We claim that here $St(z) = \mathrm{Aut}(M_\pi, \Phi)$. Suppose $\rho \in St(z)$ and let $[N, f]$ determine the pointed lattice class $[N, x]$, where $x\Phi(f) = z$. Then $[N, x]\rho$ also determines $[N, x]$ and therefore, by §3.3, the corresponding elements in $\mathbf{P}(A; m, \ell)$ are equal. Thus $[N, f] = [N, f]\rho$ and we conclude that the bijection of the Proposition in §2.7 is here a bijection $\mathbf{E}(M; \Phi, \pi) \tilde{\to} \bigvee(M, z)$.

3.6. Theorems 3 (§3.4) and 2 (§2.4) provide a parametrization of the homotopy classes $[_mP]$ by the elements of the Browning class group of M for the functor $\Phi = \mathrm{Ext}_G^{m+1}(A, -)$ and relative to π chosen suitably for the Eichler condition on M. The next point is that, for this particular functor, π can be taken to be the smallest possible set, viz. $\pi(G)$, provided only that e (the exponent of $\Phi(M)$) is a $\pi(G)$-number.

Let $\pi \supseteq (G)$ and consider the links established by Theorems 1 (§2.3) and 3:

$$\begin{array}{ccccc} \mathbf{E}(M;\Phi,\pi) & \xleftarrow[\eta_1]{\sim} & \mathbf{P}(A;m,\ell) & \xrightarrow[\eta_2]{\sim} & \mathbf{E}(M;\Phi,\pi(G)) \\ {\scriptstyle\varepsilon_1}\downarrow & & & & \downarrow{\scriptstyle\varepsilon_2} \\ \mathrm{Cl}\,(M;\Phi,\pi) & & & & \mathrm{Cl}\,(M;\Phi,\pi(G)). \end{array}$$

The two horizontal bijections come from Theorem 3 and the two vertical surjections from Theorem 1.

Now $T(M,\pi)$ is a subgroup (even a direct summand) of $T(M,\pi(G))$ (cf. §1.3) and clearly $A(\Phi,\pi) \subseteq A(\Phi,\pi(G))$. Hence there is a natural group homomorphism

$$\gamma : \mathrm{Cl}\,(M;\Phi,\pi) \longrightarrow \mathrm{Cl}\,(M;\Phi,\pi(G)).$$

It is easy to check that γ is induced by the above diagram in the sense that $x\gamma = e\eta_1^{-1}\eta_2\varepsilon_2$, where e is any element satisfying $e\varepsilon_1 = x$.

Consider the above diagram with M replaced by $M \oplus M$ (so that necessarily A is replaced by $A \oplus A$, Φ by $\Psi = \mathrm{Ext}_G^{m+1}(A \oplus A, -)$ and ℓ by 2ℓ). The Eichler condition on $M \oplus M$ and the fact that $\pi(G)$ is suitable for $M \oplus M$ (cf. §2.4) make ε_1 and ε_2 bijections (Theorem 2). Hence γ for $M \oplus M$ is an isomorphism. It follows that γ for M is an isomorphism, by the Corollary in §2.5 and the easily verified fact that $\mathrm{Cl}\,(M;\Phi,\pi) = \mathrm{Cl}\,(M;\Psi,\pi)$. So we have

THEOREM 4. *Assume the exponent of* $\mathrm{Ext}_G^{m+1}(A, M)$ *is a* $\pi(G)$*-number. Then for any* $\pi \supseteq \pi(G)$,

$$\mathrm{Cl}\,(M;\mathrm{Ext}_G^{m+1}(A,-),\pi) \xrightarrow{\sim} \mathrm{Cl}\,(M;\mathrm{Ext}_G^{m+1}(A,-),\pi(G)).$$

THEOREM 5. *If M satisfies the Eichler condition and the exponent hypothesis of Theorem* 4 *holds, then there is a bijection*

$$\mathbf{P}(A;m,\ell) \xrightarrow{\sim} \mathrm{Cl}\,(M;\mathrm{Ext}_G^{m+1}(A,-),\pi(G)).$$

This is immediate: the class group over $\pi(G)$ is isomorphic to the class group over π by Theorem 4 and $\mathrm{Cl}\,(M;\Phi,\pi)$ is bijective with $\mathbf{P}(A;m,\ell)$ by Theorems 2 and 3.

The exponent hypothesis in Theorems 4 and 5 is not a serious restriction: the exponent can only fail to be a $\pi(G)$-number if $m = 0$ and A has torsion group involving primes not in $\pi(G)$.

4. Free elements

In this section we assume the exponent of $\mathrm{Ext}_G^{m+1}(A, M)$ is a $\pi(G)$-number. This enables us to work exclusively with $\pi(G)$.

4.1. We continue the discussion begun in §1.3 and explore the link between $T_0(M) = T_0(M, \pi(G))$ and $\mathrm{Cl}(\mathbb{Z}G)$. First we claim there is a well defined map $\mathbf{P}(A; m, \ell) \to P_0(\mathbb{Z}G)$ given by ${}_mP \to \varepsilon_m[P]$, where the *Euler class* $\varepsilon_m[P]$ is defined by

$$\varepsilon_m[P] = \sum_{i=0}^{m} (-1)^{m-i}[P_i].$$

To see that $\varepsilon_m[-]$ is well defined amounts to verifying that if ${}_mP \sim {}_mP'$ (homotopy equivalent), then $\varepsilon_m[P] = \varepsilon_m[P']$. Let $f: {}_mP \to {}_mP'$ be a map giving a chain homotopy. We noted (cf. §3.1) that f induces an isomorphism: $C_{m+1} \xrightarrow{\sim} C'_{m+1}$. It follows that the mapping cone M of f is acyclic and so $0 = \varepsilon_m[M] = \varepsilon_m[P] - \varepsilon_m[P']$.

Next, starting with $[{}_mP]$, construct its image $[C_{m+1}, f]$ and then as in the proof of the surjectivity in §3.4, find

$$0 \longrightarrow C_{m+1} \longrightarrow X \longrightarrow Q_{m-1} \longrightarrow \cdots \longrightarrow Q_0 \longrightarrow A \longrightarrow 0$$

which is chain homotopy equivalent to ${}_mP$ (by Theorem 3, §3.4). Hence

$$\begin{aligned}\varepsilon_m[P] &= [X] - [Q_{m-1}] + \cdots \\ &= \varepsilon_m[Q] - [Q_m] + [X]\end{aligned}$$

and $[Q_m] - [X]$ is the image of $[\mathrm{Coker} f]$ under $T_0(M) \to \mathrm{Cl}(\mathbb{Z}G)$ (§1.3). It will be safe to denote this image also as $[\mathrm{Coker} f]$. It is an element of $C(M)$. Thus $\varepsilon_m[P] = \varepsilon_m[Q] + [f]$.

Thus we have shown that $[{}_mP] \to \varepsilon_m[P] - \varepsilon_m[Q]$ is the composite of

$$\begin{aligned}\mathbf{P}(A; m, \ell) &\longrightarrow \mathbf{E}(M; \mathrm{Ext}_G^{m+1}(A, -), \pi(G)) \quad \text{(the bijection of Theorem 3, §3.4)} \\ &\longrightarrow \mathrm{Cl}(M; \mathrm{Ext}_G^{m+1}(A, -), \pi(G)) \ \text{(the surjection of Theorem 1, §2.3)} \\ &\longrightarrow \mathrm{Cl}(\mathbb{Z}G)\end{aligned}$$

by Proposition 1 in §2.6 and using that $\mathrm{Ext}_G^{m+1}(A, -)$ is naturally equivalent to the functor $\mathrm{Ext}_G^1(D_m, -)$. The image is $C(M)$. This gives

PROPOSITION 1. *The map* $\mathbf{P}(A; m, \ell) \to \mathrm{Cl}(\mathbb{Z}G)$ *given by* $[_mP] \to \varepsilon_m[P] - \varepsilon_m[Q]$ *has image* $C(M)$.

Let us call a class $[_mP]$ *free* if it contains a truncated *free* resolution. If $\mathbb{Q}P_i \simeq \mathbb{Q}G^{(e_i)}$ for $0 \leq i \leq m$, then the Euler characteristic $\chi_m(P) = |G| e_m(P)$, where $e_m(P) = \Sigma_{i=0}^{m} (-1)^{m-i} e_i$. Hence if $_mP$ is free, $\varepsilon_m[P] = e_m(P)[\mathbb{Z}G]$. The converse also holds and we have

PROPOSITION 2. $[_mP]$ *is free if and only if* $\varepsilon_m[P] = e_m(P)[\mathbb{Z}G]$.

Proof. Let $\varepsilon_m[P] = e_m(P)[G]$. We replace P_0 and P_1 by $P_0'' = P_0 \oplus P_0'$ and $P_1 \oplus P_0'$, respectively, where P_0' is chosen to make P_0'' free; then we continue this process up to dimension $m - 1$. This does not change the Euler class and so the constructed truncated resolution has its m-dimensional term stably free. Adjusting dimensions m and $m - 1$ finally produces a free resolution which is homotopically equivalent to $_mP$.

Suppose our reference resolution $_mQ$ is free. Then by Proposition 2, the set of free elements in $\mathbf{P}(A; m, \ell)$ is the inverse image of 0 under the map of Proposition 1 above. Hence, using also Theorem 5 (§3.6) and Proposition 2 of §2.6 (for the last part of the following result), we obtain

THEOREM 6. *If M satisfies the Eichler condition and $_mQ$ is free, then the set of all the free elements in* $\mathbf{P}(A; m, \ell)$ *is bijective with the kernel of the homomorphism*

$$\mathrm{Cl}\,(M; \mathrm{Ext}_G^{m+1}(A, -), \pi(G)) \longrightarrow \mathrm{Cl}\,(\mathbb{Z}G).$$

In particular, if M is not core-equal and $m \geq 1$ or $m = 0$ and A is a $\mathbb{Z}G$-lattice, then $_m[Q]$ is the only free element.

When A is a lattice, then to say that M is not core-equal is equivalent to the condition $\ell > \chi_m(A)$, where $\chi_m(A)$ means the m-th partial projective Euler characteristic of A in the sense of [G], §2.

Theorem 6 has topological content. The free elements in $\mathbf{P}(\mathbb{Z}; m, \ell)$ parametrise the G-linked homotopy classes of (G, m)-complexes of Euler characteristic ℓ. We use the term (G, m)-complex to mean a finite connected m-dimensional CW-complex X for which $\pi_i(X) = 0$ for $i = 2, \ldots, m - 1$ and there is given a group isomorphism $\theta_X : \pi_1(X) \to G$. This is the slight variant of Dyer's usage [D] proposed in [B3]. The notion of G-linked homotopy classes (though not the term) also appears in [B3]. It is this: If X and Y are (G, m)-complexes then a homotopy equivalence $f: X \to Y$ is G-linked if $f_1\theta_Y = \theta_X$, where f_1 is the isomorphism on fundamental groups determined by f.

By the theorem of Mac Lane and Whitehead [MW] *there is a one-one map of the set of all G-linked homotopy classes of* (G, m)*-complexes of Euler characteristic* ℓ *into the set of free elements in* $\mathbf{P}(\mathbb{Z}; m, \ell)$. This map is known to be surjective (and therefore bijective) if $m \geq 3$ and also if $m = 2$ provided G is Abelian ([D]; [B3], [GL], [L2]); it may possibly be surjective always.

If the Euler characteristic ℓ is non-minimal, then M satisfies the Eichler condition except when m is odd, $\ell = |G|$, $\mathbb{Z}$ is periodic with $m+1$ a free period and $\mathbb{Z}G$ does not satisfy the Eichler condition. This may be seen as follows. If we let $e = \ell/|G|$, then the free truncated resolution ${}_mQ$ of $\mathbb{Z}$ gives

$$\mathbb{Q}M = e\mathbb{Q}G + (-1)^{m+1}\mathbb{Q} \tag{$*$}$$

(cf. [GrL], proof of (3.5)) and if $M = M' \oplus P$, where M' is core-equal and P is projective, then $\ell = \chi_m(\mathbb{Z}) + \dim \mathbb{Q}P$ (cf. [G], proof of (3.2)). When m is even, $(*)$ yields $\mathbb{Q}M \simeq (e-1)\mathbb{Q}G \oplus \mathbb{Q}\Delta G$, where ΔG denotes the augmentation ideal of $\mathbb{Z}G$ and $e - 1 \geq 1$ because ℓ is non-minimal. Hence $\mathbb{Q}G \oplus \mathbb{Q}\Delta G$ is a summand of $\mathbb{Q}M$. When m is odd, $(*)$ gives $\mathbb{Q}M \simeq e\mathbb{Q}G \oplus \mathbb{Q}$, whence M satisfies the Eichler condition when $e \geq 2$. Suppose $e = 1$ ($e \neq 0$ by non-minimality). This implies $P \vee \mathbb{Z}G$ and $M' \simeq \mathbb{Z}$, so that $M \vee \mathbb{Z} \oplus \mathbb{Z}G$. Hence M satisfies the Eichler condition if $\mathbb{Z}G$ does. Finally consider the standard example due to Swan: G is the generalised quaternion group of order 32 and P is a projective $\mathbb{Z}G$-module so that $P \oplus \mathbb{Z}G \simeq \mathbb{Z}G \oplus \mathbb{Z}G$ and where also $d_G(P/P^G) = 2$. The free periodic resolution

$$0 \to \mathbb{Z} \to \mathbb{Z}G \to \mathbb{Z}G^{(2)} \to \mathbb{Z}G^{(2)} \to \mathbb{Z}G \to \mathbb{Z} \to 0$$

yields

$$0 \to \mathbb{Z} \oplus P \to \mathbb{Z}G^{(2)} \to \mathbb{Z}G^{(2)} \to \mathbb{Z}G^{(2)} \to \mathbb{Z}G \to \mathbb{Z} \to 0 \tag{1}$$

and

$$0 \to \mathbb{Z} \oplus \mathbb{Z}G \to \mathbb{Z}G^{(2)} \to \mathbb{Z}G^{(2)} \to \mathbb{Z}G^{(2)} \to \mathbb{Z}G \to \mathbb{Z} \to 0. \tag{2}$$

These two truncated free resolutions are not homotopically equivalent because $\mathbb{Z} \oplus P \not\simeq \mathbb{Z} \oplus \mathbb{Z}G$ (since $d_G(P/P^G) = 2$ while $d_G(\mathbb{Z}G/\mathbb{Z}G^G) = 1$). We sum up in

PROPOSITION 3. *If* X, Y *are* (G, m)*-complexes of equal non-minimal Euler characteristic* ℓ, *then* X, Y *are G-linked homotopically equivalent except possibly when* m *is odd,* $\ell = |G|$, $m+1$ *is a free period of* $\mathbb{Z}$ *and* $\mathbb{Z}G$ *does not satisfy the Eichler condition.*

4.2. Let W denote the kernel in Theorem 6 and again set $\Phi = \operatorname{Ext}_G^{m+1}(A, -)$; also π shall here stand for $\pi(G)$. So

$$W = \frac{A(\mathbb{Z}G) \cap T_0(M)}{A(M, \Phi)},$$

where we recall that $A(\mathbb{Z}G)$ is the image of Aut $\mathbb{Z}_\pi G = \mathrm{GL}_1(\mathbb{Z}_\pi G)$ under $\alpha_{\mathbb{Z}G} =: \alpha_1$ (cf. §1.1). In what follows M need not satisfy the Eichler condition.

As a rule the numerator in the above formula for W is just $A(\mathbb{Z}G)$. This happens when M is $\mathbb{Z}G$-faithful, for then $T_0(M) = T_0(\mathbb{Z}G)$. This is true when A is non-periodic for all sufficiently large m [GrL]. It is also true when $A = \mathbb{Z}$ except when m is odd and the Euler characteristic $\ell = 0$ (whence $M = \mathbb{Z}$) or m is even and $\ell = |G|$. In any case (for any A) $T_0(M)$ is a direct summand of $T_0(\mathbb{Z}G)$ and so $A(\mathbb{Z}G)/A(\mathbb{Z}G) \cap T_0(M)$ is free Abelian, whence W *is the torsion group of* $A(\mathbb{Z}G)/A(\Phi)$ (cf. §2.2).

Elementary K_1-theory gives us the exact sequence

$$K_1(\mathbb{Z}G) \xrightarrow{\alpha''} K_1(\mathbb{Z}_\pi G) \xrightarrow{\alpha'} A(\mathbb{Z}G) \longrightarrow 0,$$

where α' arises from a decomposition of α_t (for any $t \geq 1$):

$$\begin{array}{ccc} \mathrm{GL}_t(\mathbb{Z}_\pi G) & \xrightarrow{\alpha_t} & T_0(\mathbb{Z}G) \\ {\scriptstyle\kappa_t}\searrow & & \nearrow{\scriptstyle\alpha'} \\ & K_1(\mathbb{Z}_\pi G) & \end{array} .$$

Hence Ker $\alpha' = (\text{Ker } \alpha_t)\kappa_t$ and if K denotes the inverse image of $A(\Phi)$ under α', then W *is isomorphic to the torsion subgroup of* $K_1(\mathbb{Z}_\pi G)/K$.

Recall that M arises at the tail end of ${}_mQ$:

$$0 \longrightarrow M \longrightarrow Q_m \xrightarrow{\partial} D_m \longrightarrow 0.$$

Defining

$$B = \langle \phi \in \operatorname{Aut}(Q_m)_\pi \mid \phi\partial_\pi = \partial_\pi \rangle$$

and assuming $(Q_m)_\pi \simeq \mathbb{Z}_\pi G^{(t)}$, then α_t and κ_t contain B in their domain.

THEOREM 7. *The kernel W of the homomorphism of Theorem 6 is isomorphic to the torsion subgroup of the cokernel of*

$$(\kappa_t, \alpha'') : B \times K_1(\mathbb{Z}G) \longrightarrow K_1(\mathbb{Z}_\pi G).$$

Proof. We already know that W is isomorphic to the torsion subgroup of $K_1(\mathbb{Z}_\pi G)$ modulo K (the inverse image of $A(\Phi)$ under α'). So it remains to show that (i) $B\alpha_t \subseteq A(\Phi)$ and (ii) if $\phi \in \mathrm{Aut}\,(Q_m)_\pi$ and $\phi\alpha_t = [\phi] \in A(\Phi)$, then $[\phi] = [\psi]$ for some $\psi \in B$.

Let $\phi = g/s \in B$. Then $g\partial = \partial s$ and $Mg \subseteq M$. The connecting homomorphism $\mathrm{Hom}_G(M, M) \to \mathrm{Ext}^1_G(D_m, M)$ maps multiplication by s on D to $\mathrm{Ext}^1(s, M)$, which is the identity on $\mathrm{Ext}^1(D, M)$. However, $\mathrm{Ext}^1(s, M) = \mathrm{Ext}^1(D, g\downarrow)$, where $g\downarrow$ denotes the restriction of g to M. Now

$$[\phi] = [g] - [Q/sQ] = [g\downarrow] + [D/sD] - [Q/sQ] = [g\downarrow] - [M/sM] = [g\downarrow/s]$$

and $[g\downarrow/s]$ belongs to $A(\Phi)$. Hence (i) is proved.

Next let $\phi = g/s$ satisfy $[\phi] = [f/r]$, where f is an endomorpism of M with f_π an automorphism and $\Phi(f)$ the identity. As in the proof of Proposition 1 in §2.6, f gives a map $\check{f}: Q_m \to Q_m$, and similarly r gives $\check{r}$. Then $[f/r] = [\check{f}] - [\check{r}]$ and $\check{f}, \check{r}$ are both in B. So $[\phi] = [\psi]$, where $\psi = \check{f}_\pi(\check{r}_\pi)^{-1}$.

4.3. To conclude, let us see how the present set-up relates to two situations where W has been calculated. We shall need the obvious fact that $\mathrm{Ext}^{m+1}_G(\mathbb{Z}, M) \simeq \mathbb{Z}/|G|\mathbb{Z}$.

EXAMPLE 1. $A = \mathbb{Z}$ and $\mathbb{Z}$ is periodic with $m + 1$ a free period. This case was studied by Dyer [D].

Since $\mathrm{Cl}(\mathbb{Z}) = 0$ we have the exact sequence

$$\mathrm{Aut}\,\mathbb{Z} \longrightarrow \mathrm{Aut}\,\mathbb{Z}_\pi \xrightarrow{\alpha} T_0(\mathbb{Z}) \longrightarrow 0;$$

and there is also the exact sequence (cf. §2.2)

$$1 \longrightarrow \mathrm{Aut}\,(\mathbb{Z}_\pi, \Phi) \longrightarrow \mathrm{Aut}\,\mathbb{Z}_\pi \longrightarrow \mathrm{GL}_1(\mathbb{Z}/|G|\mathbb{Z}) \longrightarrow 1.$$

Hence

$$\mathrm{Cl}(\mathbb{Z}; \Phi) \simeq \mathrm{GL}_1(\mathbb{Z}/|G|\mathbb{Z})/\{\pm 1\},$$

where the isomorphism makes an integer k prime to $|G|$ correspond to $[\mathbb{Z}/k\mathbb{Z}] + A(\Phi)$. This element maps to $[\mathbb{Z}G] - [\Delta G + k\mathbb{Z}]$ in $\mathrm{Cl}(\mathbb{Z}G)$. (ΔG again denotes the augmentation ideal of $\mathbb{Z}G$.) Thus W is isomorphic to the kernel of $\mathrm{GL}_1(\mathbb{Z}/|G|\mathbb{Z})/\{\pm 1\} \to C(\mathbb{Z}G)$ and it is easily checked that this map is exactly the same as in [D].

EXAMPLE 2. $A = \mathbb{Z}$ and G is Abelian. This is the case studied by Browning when $m = 2$ and by Linnell in general [L2].

We may restrict attention to the non-cyclic case and to $m > 0$. Then $\mathbb{Q}G$ is a summand of $\mathbb{Q}M$ ([L2], 5.3) and so M satisfies the Eichler condition and $T_0(M) = T_0(\mathbb{Z}G)$, whence $A(\mathbb{Z}G)/A(\Phi)$ is finite. Therefore by Theorem 7 (§4.2), W is the cokernel of $B \times K_1(\mathbb{Z}G) \to \mathrm{GL}_1(\mathbb{Z}_\pi G)$. This map is the determinant homomorphism on B and the image of $K_1(\mathbb{Z}G)$ is the image of $\mathrm{GL}_2(\mathbb{Z}G)$ also under the determinant homomorphism. Hence

$$W \simeq \mathrm{GL}_1(\mathbb{Z}_\pi G)/(\det B)\, \mathrm{GL}_1(\mathbb{Z}G).$$

Let $\varepsilon : \mathbb{Z}_\pi G \to \mathbb{Z}_\pi$ be the augmentation homomorphism and take the composite of ε and $\mathbb{Z}_\pi \to \mathbb{Z}/t\mathbb{Z}$, where t is the greatest common divisor of the invariant factors of G. Then we obtain a surjection $\mathrm{GL}_1(\mathbb{Z}_\pi G) \to \mathrm{GL}_1(\mathbb{Z}/t\mathbb{Z})$ and the key result in [L2] (6.1 and 6.2) is that this surjection has kernel $(\det B)$. Hence

$$W \simeq \mathrm{GL}_1(\mathbb{Z}/t\mathbb{Z})/\{\pm 1\}.$$

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