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Density of states in spectral geometry

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Introduction

Recent studies on spectal geometry threw a light on the relationships between a discontinuous action of a group and the spectrum of the Laplacian (or more generally the spectra of elliptic operators) on a non-compact Riemannian manifold. The first result in this direction is the observation by R. Brooks [B] that the bottom of the L^2-spectrum of the Laplacian is related to the *amenability* of discontinuous transformation groups (see also [KOS] and [S1]). The purpose of this paper is to investigate the *integrated density of states* of a *periodic* Schrödinger operator on a manifold with *compact quotient* from the same point of view.

The integrated density of states, which is the concept introduced first by physicists in quantum theory of solids, is a non-decreasing function $\varphi(\lambda)$ on the real line defined roughly as the number of possible energy levels in the interval $(-\infty, \lambda)$ divided by the volume of a sufficiently large domain. To justify this physical definition, we must impose a suitable boundary condition on eigenfunctions and specify the way how to blow up the domain filling the whole space. For the Schrödinger operator with a *periodic potential* on the Euclidean space, a classical observation (cf. [Sh]) says that $\varphi(\lambda)$ is well-defined as far as the domain blows up in a sufficiently regular way and does not depend on the choice of the boundary conditions; say, Dirichlet, Neumann, and periodic boundary conditions. It is also a fact that the spectrum of the Schrödinger operator on the whole space is characterized as the set of increasing points of $\varphi(\lambda)$. One of the results in this paper gives a partial generalization of those facts to the case of a Riemannian manifold with compact quotient. In the discussion, we shall see a prominent role of amenability of discontinuous groups acting on manifolds, together with the role of spectral distribution functions defined by means of the concept of the *von Neumann trace*. See [SN], [S3], and [KOS] for the general background for the spectral theory of periodic Schrödinger operators on a manifold, also [ES] which gives us the stimulus for writing this paper.

§1. Definitions and results

Let X be a complete, connected, noncompact Riemannian manifold of dimension n, and $\mathscr{D} = \{D_j\}_{j=1}^{\infty}$ be a family of bounded connected open sets in X with *piecewise smooth* boundaries satisfying

$$\bar{D}_j \subset D_{j+1}, \qquad \bigcup_{j=1}^{\infty} D_j = X.$$

Let q be a smooth real-valued function on X. Consider the Schrödinger operator $H_{D_j} = -\Delta_{D_j} + q$ on each D_j acting on $L^2(D_j)$ with Dirichlet boundary conditions. We denote by $\varphi_{D_j}(\lambda)$ the number of eigenvalues of H_{D_j} not exceeding λ, where each eigenvalue is repeated according to its multiplicity. We now define the function $\varphi_{\mathscr{D}}$ by the limit (if it exists)

$$\varphi_{\mathscr{D}}(\lambda) = \lim_{j\to\infty} \operatorname{vol}(D_j)^{-1}\varphi_{D_j}(\lambda),$$

and call $\varphi_{\mathscr{D}}$ the integrand density of states for the Schrödinger operator $H_X = -\Delta_X + q$ associated with the family $\mathscr{D}$. The qeustions with which we are concerned are: (1) Under what condition does the limit exist? (2) When it exists, is $\varphi_{\mathscr{D}}$ independent of the choice of the expanding family $\mathscr{D}$?

Given a manifold with *compact quotient*, we may introduce the integrated density of states associated with *periodic boudary conditions*. Here a complet Riemannian manifold X is said to have compact quotient if there is a discrete subgroup Γ in the isometry group of X acting discontinuously on X such that the quotient space $M = \Gamma\backslash X$ is compact. We assume that q is Γ-invariant so that q may be regarded as a function on M. Let

$$H_X = \int \lambda \, dE(\lambda)$$

denote the spectral resolution of H_X. We define the *spectral distribution function* Φ_Γ by

$$\Phi_\Gamma(\lambda) = \operatorname{Tr}_\Gamma E(\lambda),$$

where Tr_Γ is the standard *von Neumann trace* on the von Neumann algebra of Γ-equivariant bounded operators of $L^2(X)$ (see [At], [ES], [S3]).

From the definition, it follows easily that the quantity $\operatorname{vol}(\Gamma\backslash X)^{-1}\Phi_\Gamma(\lambda)$ depends only on the commensurability class of Γ; that is, if Γ_1 and Γ_2 are discontinuous transformation groups of X such that q is invariant under Γ_1 and Γ_2, and $\Gamma_1 \cap \Gamma_2$ is of finite index in both Γ_1 and Γ_2, then one has

$$\operatorname{vol}(\Gamma_1 \backslash X)^{-1} \Phi_{\Gamma_1} = \operatorname{vol}(\Gamma_2 \backslash X)^{-1} \Phi_{\Gamma_2}.$$

In the special case that $q \equiv 0$ and X is a homogeneous Riemannian manifold, the quantity $\operatorname{vol}(\Gamma \backslash X)^{-1} \Phi_\Gamma(\lambda)$ does not depend on Γ. For example, if $X = \mathbb{R}^2$, one has

$$\operatorname{vol}(\Gamma \backslash X)^{-1} \Phi_\Gamma(\lambda) = (4\pi)^{-1} \lambda, \qquad \lambda \geq 0,$$

and if $X = \mathbb{H}^2$, the hyperbolic 2-plane, one has

$$\operatorname{vol}(\Gamma \backslash X)^{-1} \Phi_\Gamma(\lambda) = (4\pi)^{-1} \int_0^{\lambda - 1/4} \tanh \pi \sqrt{\lambda}\, d\lambda, \qquad \lambda \geq 1/4.$$

To see that the function Φ_Γ may be regarded as the integrated density of states associated with periodic boundary value conditions, we suppose that Γ acts freely on X and has a family of normal subgroup $\{\Gamma_i\}_{i=1}^\infty$ such that Γ_i is of finite index in Γ, Γ_{i+1} is contained in Γ_i, and $\bigcap_{i=1}^\infty \Gamma_i = \{1\}$. We then have a tower of finite-fold covering maps of closed manifolds

$$\cdots \longrightarrow M_{i+1} \longrightarrow M_i = \Gamma_i \backslash X \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M.$$

Let $\Phi_{M_i}(\lambda)$ denote the number of eigenvalues of H_{M_i} on the closed manifold M_i not exceeding λ. In [SN], it was observed that

$$\operatorname{vol}(\Gamma \backslash X)^{-1} \Phi_\Gamma(\lambda) = \lim_{i \to \infty} \operatorname{vol}(M_i)^{-1} \Phi_{M_i}(\lambda)$$

at all points of continuity of Φ_Γ. It should be noted ([ES], [SN]) that the set of increasing points of Φ_Γ coincides with the spectrum of H_X.

It is natural to compare $\varphi_{\mathscr{D}}(\lambda)$ with $\Phi_\Gamma(\lambda)$. In the case that X is the Euclidean space $\mathbb{R}^n$ and $\mathscr{D}$ is a family of concentric balls, it is known that $\varphi_{\mathscr{D}}$ exists and coincides with $\operatorname{vol}(\Gamma \backslash X)^{-1} \Phi_\Gamma$. On the other hand, if $X = \mathbb{H}^n$, the n-dimensional hyperbolic space, and $\mathscr{D}$ is a family of concentric geodesic balls in $\mathbb{H}^n$, we observe that $\varphi_{\mathscr{D}}$ is not equal to $\operatorname{vol}(\Gamma \backslash X)^{-1} \Phi_\Gamma$ (see Section 3). This is due to different geometric features of geodesic balls in $\mathbb{R}^n$ and $\mathbb{H}^n$ which may be clarified if one looks at the ratio

$$\operatorname{vol}(\partial_h D_j)/\operatorname{vol}(D_j), \qquad h > 0,$$

where $\partial_h D$ denotes the "thick" boundary $\{x \in D; \operatorname{dist}(x, \partial D) \le h\}$. In fact, for $\mathbb{R}^n$, this goes to zero as $j \to \infty$ for every h, while, for $\mathbb{H}^n$, this goes to the positive number $1 - e^{-h(n-1)}$. In terms of discrete transformation groups, this corresponds to the fact that a group Γ acting discontinuously on $\mathbb{R}^n$ is *amenable*, and a group Γ acting on $\mathbb{H}^n$ is non-amenable. Indeed, we may prove the following general criterion of amenability.

PROPOSITION 1.1. *The transformation group Γ is amenable if and only if there exists an expanding family $\mathscr{D} = \{D_j\}$ of bounded domains with piecewise smooth boundaries satisfying the following property*:

$$\lim_{j \to \infty} \operatorname{vol}(\partial_h D_j)/\operatorname{vol}(D_j) = 0 \tag{P}$$

for every $h > 0$.

In light of this criterion, we now state the main theorem of this paper, a generalization of the classical result for $X = \mathbb{R}^n$.

THEOREM 1.1. *If an expanding family $\mathscr{D} = \{D_j\}$ satisfies the property* (P) *in the above proposition, then $\varphi_{\mathscr{D}}$ exists and equals* $\operatorname{vol}(\Gamma \backslash X)^{-1}\Phi_\Gamma$ *at all points of continuity of Φ_Γ.*

An immediate consequence of this theorem is that, if Γ is amenable, the integrated density of states $\varphi_{\mathscr{D}}$ does not depend on the expanding family $\mathscr{D}$ with the property (P). We also conclude that $\operatorname{vol}(\Gamma \backslash X)^{-1}\Phi_\Gamma$ does not depend on Γ, which is by no means trivial from the definition of Φ_Γ since X is not supposed to be homogeneous.

It is interesting to consider the density of states associated with *Neumann boundary conditions*. We conjecture that the same statements as in Theorem 1 hold. Sznitman [Sz2] shows that, for the hyperbolic space, the integrated density of states associated wity Dirichlet boundary conditions is different from that associated with Neumann boundary conditions.

§2. Families of expanding domains and limit relations for the heat kernels

Henceforth we assume that X is a Riemannian manifold with compact quotient $\Gamma \backslash X$. We choose a fundamental domain $\mathscr{F}$ for the action of Γ with compact closure. The distance function on X will be denoted by $d(x, y)$.

Let $k(t, x, y)$ denote the heat kernel function for the semi-group $\exp(-tH_X)$, and $k_D(t, x, y)$ the heat kernel function on a domain D associated with Dirichlet boundary conditions. We readily get

$$\int e^{-\lambda t}\, d\varphi_D(\lambda) = \int_D k_D(t, x, x)\, dx.$$

The following lemma on the spectral distribution function Φ_Γ is immediate from the definition of Γ-trace.

LEMMA 2.1. $\displaystyle\int e^{-\lambda t}\, d\Phi_\Gamma(\lambda) = \int_{\mathscr{F}} k(t, x, x)\, dx.$

The idea of proof of Theorem 1.1 is based on a uniform estimate of the difference between the diagonal of the heat kernel and that of the Dirichlet heat kernel.

LEMMA 2.2. *Given a positive T, we have positive constants C_1 and C_2 such that*

$$0 \le k(t, x, y) \le C_1 t^{-n/2} \exp(-C_2 d(x, y)^2/t) \tag{1}$$

for $t \in (0, T]$, *and*

$$0 \le k(t, x, y) - k_D(t, x, y) \le C_1 t^{-n/2} \exp(-C_2\, d(y, \partial D)^2/t) \tag{2}$$

for $0 < t \le \min(T, 2C_2\, d(y, \partial D)^2/n)$.

Proof. The first inequality (1) is due to [Do] (see also [BS]). The second inequality is a consequence of the maximum principle (see [C] and [D]).

PROPOSITION 2.1. *If the family $\mathscr{D}$ satisfies the property* (P) *then*

$$\lim_{j \to \infty} \operatorname{vol}(D_j)^{-1} \int_{D_j} (k(t, x, x) - k_{D_j}(t, x, x))\, dx = 0.$$

Proof. Let $t > 0$, and take constants C_1 and C_2 in (1) for $T = t$. We have

$$\begin{aligned} &\operatorname{vol}(D_j)^{-1} \int_{D_j} (k(t, x, x) - k_{D_j}(t, x, x))\, dx \\ &\quad = \operatorname{vol}(D_j)^{-1} \int_{\partial_h D_j} (k(t, x, x) - k_{D_j}(t, x, x))\, dx \\ &\qquad + \operatorname{vol}(D_j)^{-1} \int_{D_j \setminus \partial_h D_j} (k(t, x, x) - k_{D_j}(t, x, x))\, dx. \end{aligned}$$

In view of Lemma 2.2, (1), the first term is estimated from above by

$$C_1 t^{-n/2} \operatorname{vol}(\partial_h D_j)/\operatorname{vol}(D_j),$$

which tends to zero as $j \uparrow \infty$. Take h with $t \leq 2C_2 h^2/n$. Then, for $x \in D_j \backslash \partial_h D_j$, one has $t \leq 2C_2\, d(x, \partial D_j)^2/n$, so that, by Lemma 2.2, (2) the second term is estimated from above by

$$C_1 t^{-n/2} \exp(-C_2 h^2/t).$$

By letting h go to infinity, we get the assertion.

PROPOSITION 2.2. *If $\mathscr{D}$ satisfies the property* (P), *then one has, for every Γ-periodic continuous function f, that*

$$\lim_{j \to \infty} \operatorname{vol}(D_j)^{-1} \int_{D_j} f(x)\, dx = \operatorname{vol}(\mathscr{F})^{-1} \int_{\mathscr{F}} f(x)\, dx.$$

Proof. Put $E_j = \{\sigma \in \Gamma; (D_j \backslash \partial_h D_j) \cap \sigma\bar{\mathscr{F}} \neq \varnothing\}$, and

$$D'_j = \bigcup_{\sigma \in E_j} \sigma\bar{\mathscr{F}}.$$

It is clear that $(D_j \backslash \partial_h D_j) \subset D'_j$. We show that, if $h \geq \operatorname{diam}(\mathscr{F})$, then $D'_j \subset D_j$. Let $x \in (D_j \backslash \partial_h D_j) \cap \sigma\bar{\mathscr{F}}$. since $d(x, \partial D_j) \geq h$, we find $\partial D_j \cap B_h(x) = \varnothing$, where $B_h(x) = \{z \in X; d(x, z) \leq h\}$. From the connectedness of $B_h(x)$, it follows that $B_h(x) \subset D_j$. Since $h \geq \operatorname{diam}(\mathscr{F})$, we have $\sigma\bar{\mathscr{F}} \subset B_h(x) \subset D_j$.

We now find

$$\begin{aligned}
\operatorname{vol}(D_j)^{-1} \int_{D_j} f(x)\, dx &= \operatorname{vol}(D_j)^{-1} \int_{D'_j} f(x)\, dx + \operatorname{vol}(D_j)^{-1} \int_{D_j \backslash D'_j} f(x)\, dx \\
&= \frac{\operatorname{vol}(D'_j)}{\operatorname{vol}(D_j)} \frac{1}{\operatorname{vol}(D'_j)} \int_{D'_j} f(x)\, dx \\
&\quad + \frac{\operatorname{vol}(D_j \backslash D'_j)}{\operatorname{vol}(D_j)} \frac{1}{\operatorname{vol}(D_j \backslash D'_j)} \int_{D_j \backslash D'_j} f(x)\, dx.
\end{aligned}$$

Since $D_j \backslash D_j' \subset \partial_h D_j$, we have $\lim_{j\to\infty} \operatorname{vol}(D_j \backslash D_j')/\operatorname{vol}(D_j) = 0$ and $\lim_{j\to\infty} \operatorname{vol}(D_j')/\operatorname{vol}(D_j) = 1$. In view of the Γ-periodicity of f, we find

$$\begin{aligned}\lim_{j\to\infty} \operatorname{vol}(D_j)^{-1} \int_{D_j} f(x)\, d(x) &= \lim_{j\to\infty} \operatorname{vol}(D_j')^{-1} \int_{D_j'} f(x)\, dx \\ &= \operatorname{vol}(\mathscr{F})^{-1} \int_{\mathscr{F}} f(x)\, dx.\end{aligned}$$

We shall make use of the following genral lemma to complete the proof of Theorem 1.1.

LEMMA 2.3 (cf. [Sh]). *Let $\{\varphi_j(\lambda)\}_{j=1}^{\infty}$ be a sequence of non-decreasing functions with $\varphi_j(\lambda) = 0$ for $\lambda \le c$, where c is a constant not depending on j. Suppose that there exists a function $C(t)$, not depending on j such that*

$$\Phi_j(t) := \int e^{-\lambda t}\, d\varphi_j(t) \le C(t),$$

and

$$\lim_{j\to\infty} \Phi_j(t) = \int e^{-\lambda t}\, d\varphi(\lambda),$$

where φ is a non-decreasing function. Then $\lim_{j\to\infty} \varphi_j(\lambda) = \varphi(\lambda)$ at all points of continuity of $\varphi(\lambda)$.

We apply this lemma to

$$\varphi_j(\lambda) = \operatorname{vol}(D_j)^{-1} \varphi_{D_j}(\lambda),$$
$$\varphi(\lambda) = \operatorname{vol}(\Gamma \backslash X)^{-1} \Phi_\Gamma(\lambda).$$

Since the first eigenvalue of H_{D_j} is not less than $\min q(x)$, we observe that $\varphi_j(\lambda) = 0$ for $\lambda < \min q(x)$. We also find that

$$\begin{aligned}\Phi_j(\lambda) &= \operatorname{vol}(D_j)^{-1} \int_{D_j} k_{D_j}(t, x, x)\, dx \\ &\le \operatorname{vol}(D_j)^{-1} \int_{D_j} k(t, x, x)\, dx \\ &\le \sup_{x\in X} k(t, x, x) =: C(t),\end{aligned}$$

where we should note that the function $k(t, x, x)$ is Γ-periodic with respect to the variable x. By Proposition 2.1,

$$\lim_{j\to\infty} \Phi_j(t) = \lim_{j\to\infty} \operatorname{vol}(D_j)^{-1} \int_{D_j} k_{D_j}(t, x, x)\, dx$$

$$= \lim_{j\to\infty} \operatorname{vol}(D_j)^{-1} \int_{D_j} k(t, x, x)\, dx.$$

Using again Γ-periodicity of $k(t, x, x)$, together with Proposition 2.2, we have

$$\lim_{j\to\infty} \Phi_j(t) = \operatorname{vol}(\mathscr{F})^{-1} \int_{\mathscr{F}} k(t, x, x)\, dx$$

$$= \operatorname{vol}(\mathscr{F})^{-1} \int e^{-\lambda t}\, d\Phi_\Gamma(\lambda)$$

$$= \int e^{-\lambda t}\, d\varphi(\lambda)$$

as desired.

§3. Manifolds with amenable group actions

In this section, we shall prove Proposition 1.1 in a slightly strong form. For this, we recall the Følner's characterization of amenability. Let Γ be a finitely generated group with a fixed finite set A of generators.

PROPOSITION 3.1 (Følner [F] and [Ad]). *Γ is amenable if and only if, for every positive ε, there exists a non-empty finite set E such that*

$$|EA\backslash E| \leq \varepsilon |E|,$$

where $|E|$ denotes the cardinality of the set E.

We first assume that a manifold X with compact quotient $\Gamma\backslash X$ has a family $\{D_j\}$ satisfying the property (P). Fixing a fundamental domain $\mathscr{F}$, we put

$$A = \{a \in \Gamma : a\bar{\mathscr{F}} \cap \bar{\mathscr{F}} \neq \varnothing\}.$$

The finite set A generates Γ. Taking a number $h > 2 \cdot \operatorname{diam}(\mathscr{F})$, we set

$$E_j = \{\gamma \in \Gamma;\, \gamma\bar{\mathscr{F}} \cap (D_j \backslash \partial_h D_j) \neq \varnothing\}.$$

Let $\sigma = \gamma \cdot a \in E_j A$ $(\gamma \in E_j, a \in A)$. We shall prove that $\sigma\bar{\mathscr{F}} \subset D_j$. For this, let $z \in \gamma\bar{\mathscr{F}} \cap (D_j \backslash \partial_h D_j)$. We then have $B_h(z) \subset D_j$ as before. Since

$$\sigma\bar{\mathscr{F}} \cap \gamma\bar{\mathscr{F}} = \gamma(a\bar{\mathscr{F}} \cap \bar{\mathscr{F}}) \neq \varnothing,$$

there exists an element $y \in \sigma\bar{\mathscr{F}} \cap \gamma\bar{\mathscr{F}}$, and hence, for every $x \in \sigma\bar{\mathscr{F}}$, one has

$$d(x, z) \leq d(x, y) + d(y, z) \leq 2 \cdot \operatorname{diam}(\mathscr{F}) < h,$$

which implies that $\sigma\bar{\mathscr{F}} \subset B_h(z)$, and hence $\sigma\bar{\mathscr{F}} \subset D_j$.

We now observe

$$\begin{aligned} |E_j A \backslash E_j|/|E_j| &= \frac{|E_j A| \operatorname{vol}(\mathscr{F})}{|E_j| \operatorname{vol}(\mathscr{F})} - 1 \leq \frac{\operatorname{vol}(D_j)}{\operatorname{vol}(D_j \backslash \partial_h D_j)} - 1 \\ &= \frac{\operatorname{vol}(\partial_h D_j)}{\operatorname{vol}(D_j)} \left(1 - \frac{\operatorname{vol}(\partial_h D_j)}{\operatorname{vol}(D_j)}\right)^{-1}, \end{aligned}$$

which goes to zero as $j \uparrow \infty$. Hence Γ is amenable by Følner's criterion.

Next we suppose that Γ is amenable. Using a smooth triangulation of the orbifold $\Gamma \backslash X$, we may lift up n-simplices one by one to X to obtain a *connected polyhedral* fundamental domain $\mathscr{F}$. The finite set $A = \{\sigma \in \Gamma;\ \sigma\bar{\mathscr{F}} \cap \bar{\mathscr{F}} \neq \varnothing\}$ is symmetric and contains the unit element. We associate the *Cayley graph* $\mathscr{C}(\Gamma, A)$; the set of vertices being Γ and the set of edges being $\{(\gamma, \sigma) \in \Gamma \times \Gamma;\, \gamma^{-1}\sigma \in A\}$. We denote by d_A the distance function on Γ associated with the graph $\mathscr{C}(\Gamma, A)$. A subset E in Γ will be called connected if, for any two vertices in E, there exists a path in $\mathscr{C}(\Gamma, A)$ joining those vertices and consisting of vertices in E. By use of Theorem 4 in [Ad], there is a family $\{E_j\}_{j=1}^{\infty}$ of connected subsets of Γ such that

$$\bigcup_{j=1}^{\infty} E_j = \Gamma, \qquad E_j \subset E_j \cdot A \subset E_{j+1} \quad \text{and}$$

$$|E_j \cdot A^j \backslash E_j| \leq |E_j|/j|A|^j \qquad \text{for every } j.$$

We put $F_j = \bigcup_{\gamma \in E_j} \gamma\bar{\mathscr{F}}$ and $F_j' = \bigcup_{\gamma \in E_j \cdot A} \gamma\bar{F}$, which are connected by the choice of A and the connectedness of E_j. It should be noted that there exists a positive ε such that the ε-neighborhood of F_j is contained in F_j'. Thus we may make a uniform regularization D_j of F_j satisfying $F_j \subset D_j \subset \bar{D}_j \subset F'_j$ (see [B]). It is clear that $\bigcup_{j=1}^{\infty} D_j = X$ and $\bar{D}_j \subset D_{j+1}$. Our goal is to show that $\{D_j\}_{j=1}^{\infty}$ satisfies the property (P). Let $x_0 \in \mathscr{F}$. Since the map $f: \Gamma \to X$, $f(\gamma) = \gamma x_0$, is a rough isometry (Kanai [K]), we have

$$d_A(\gamma, \mu) \leq c_1 d(\gamma x_0, \mu x_0) + c_2$$

with suitable constants $c_1 > 0$ and $c_2 \geq 0$.

LEMMA 3.1. *If* $h \leq (j - c_2)/c_1 - 2 \cdot \operatorname{diam}(\mathscr{F})$, *then the thick boundary* $\partial_h D_j$ *is contained in the set*

$$\partial^j F'_j = \bigcup \{\mu\sigma\mathscr{F}; \sigma \in A, \mu \in E_j, \textit{and there is } \gamma \in A^j \textit{ with } \mu\gamma \notin E_j\}.$$

Proof. Suppose x is contained in $\partial_h D_j \cap \mu\bar{\mathscr{F}}$ for some $\mu \in E_j$. Since $F_j \subset D_j$ there is $y \in \overline{X \backslash F_j}$ with $d(x, y) \leq h$. Choose $\rho \notin E_j$ so that $y \in \rho\bar{\mathscr{F}}$. Then $d(\mu x_0, \rho x_0) \leq h + 2 \cdot \operatorname{diam}(\mathscr{F})$, hence $d_A(\mu, \rho) \leq j$ and $\partial_h D_j \cap F_j \subset \partial^j F_j$, where

$$\partial^j F_j = \bigcup \{\mu\mathscr{F} \mid \mu \in E_j \text{ and there is } \gamma \in A^j \text{ with } \mu\gamma \notin E_j\}.$$

If $\gamma \in E_j A \backslash E_j$, it is clear that $\gamma\mathscr{F} \subset \partial^j F'_j$ (since $A \subset A^j$), therefore

$$\partial_h D_j \subset (\partial_h D_j \cap F_j) \cup (F'_j \backslash F_j) \subset \partial^j F'_j.$$

We now show that the family $\{D_j\}_{j=1}^{\infty}$ satisfies the property (P). By the definition of $\partial^j F_j$ and $\partial^j F'_j$ we have

$$\begin{aligned}
\operatorname{vol}(\partial^j F'_j) &\leq |A| \cdot \operatorname{vol}(\partial^j F_j) \\
&= |A| \cdot \operatorname{vol}(\mathscr{F}) \cdot |\{\mu \in E_j \mid \text{there is } \gamma \in A^j \text{ with } \mu\gamma \notin E_j\}| \\
&\leq |A| \cdot \operatorname{vol}(\mathscr{F}) \sum_{\gamma \in A^j} |E_j \backslash E_j \gamma^{-1}| \\
&= |A| \cdot \operatorname{vol}(\mathscr{F}) \sum_{\gamma \in A^j} |E_j \gamma \backslash E_j| \\
&\leq |A| \cdot \operatorname{vol}(\mathscr{F}) \cdot |A^j| \cdot |E_j A^j \backslash E_j| \\
&\leq \operatorname{vol}(\mathscr{F}) \cdot |E_j|/j \\
&= \operatorname{vol}(F_j)/j.
\end{aligned}$$

Therefore we get, for every $h > 0$, that

$$\mathrm{vol}\,(\partial_h D_j)/\mathrm{vol}\,(D_j) \leq \mathrm{vol}\,(\partial^j F_j')/\mathrm{vol}\,(F_j) \leq 1/j \to 0.$$

Summarizing up, we obtain

PROPOSITION 3.2. *If Γ is amenable, then there exists an expanding family $\mathscr{D} = \{D_j\}$ of bounded open domains with smooth boundaries satisfying the following conditions*:

(1) *$\mathscr{D}$ has the property* (P),

(2) *the boundary ∂D_j has a uniformly bounded second fundamental form h_j. More precisely, there exists positive constant c not depending on j with $-cg \leq h_j \leq cg$, where g denotes the Riemannian metric on X.*

A group of subexponential growth is amenable (see [B]). In this case, we may construct a family $\mathscr{D} = \{D_j\}$ satisfying the conditions in the above proposition by using the following property on concentric geodesic balls.

LEMMA 3.2. *Suppose that Γ is of subexponential growth. For an arbitrary point x in X, there is a sequences of positive numbers $\{R_j\}_{j=1}^{\infty}$ such that*

(1) $R_j \uparrow \infty$,

(2) $\lim_{j\to\infty} \mathrm{vol}\,(B_{R_j}(x))/\mathrm{vol}\,(B_{R_j - h}(x)) = 1$ *for every $h > 0$.*

(cf. [Ad]).

§4. Hyperoblic spaces

We now consider the density of states associated with the Laplacian on the hyperbolic space $X = \mathbb{H}^n$. The manifold $\mathbb{H}^n$ is a typical example of a manifold with a non-amenable discontinuous transformation group.

THEOREM 4.1. *Let $\mathscr{D} = \{D_j\}$ be a family of concentric geodesic balls in $\mathbb{H}^n$. Then one has*

$$\mathrm{vol}\,(\mathscr{F})^{-1} \int e^{-\lambda t}\, d\Phi_\Gamma(\lambda) > \lim_{j\to\infty} \sup \mathrm{vol}\,(D_j)^{-1} \int e^{-\lambda t}\, d\varphi_{D_j}(\lambda).$$

In particular, $\mathrm{vol}\,(\Gamma\backslash X)^{-1}\Phi_\Gamma \neq \varphi_{\mathscr{D}}$.

Proof. Since $\mathbb{H}^n$ is a homogeneous Riemannian manifold, $k(t, x, x)$ does not depend on the variable x, so that we write

$$k(t) = k(t, x, x).$$

We then find

$$\begin{aligned}
&\operatorname{vol}(\mathscr{F})^{-1} \int e^{-\lambda t}\, d\Phi_\Gamma(\lambda) - \operatorname{vol}(D_j)^{-1} \int_{D_j} k_{D_j}(t, x, x)\, dx \\
&\quad = k(t) - \operatorname{vol}(D_j)^{-1} \int_{D_j} k_{D_j}(t, x, x)\, dx \\
&\quad = \operatorname{vol}(D_j)^{-1} \int_{D_j} (k(t) - k_{D_j}(t, x, x))\, dx \\
&\quad \geq \operatorname{vol}(D_j)^{-1} \int_{\partial_h D_j} (k(t) - k_{D_j}(t, x, x))\, dx,
\end{aligned}$$

where we have used the fact that $k_D(t, x, y) \leq k(t)$.

To complete the proof, we need the following lemma.

LEMMA 4.1. *For a fixed $t > 0$, there exists a positive h such that*

$$k_D(t, x, x) \leq k(t)/2$$

for every geodesic ball D and every $x \in \partial_h D$.

Proof. Choose a unit speed geodesic $C : \mathbb{R} \to X$, and consider the horoball $H = \bigcup_{\tau > 0} B_\tau(c(\tau))$. Let $k_H(t, x, y)$ denote the Dirichlet heat kernel function for the horoball. Since $\lim_{x \to \partial H} k_H(t, x, x) = 0$, it follows that there exists a positive h such that, for a positive δ with $\operatorname{dist}(c(\delta), \partial H) = \delta \leq h$.

$$k_H(t, c(\delta), c(\delta)) \leq k(t)/2.$$

Let $x \in \partial_h D$. Since one can find an isometry f on $\mathbb{H}^n$ such that $f(D) = B_\tau(c(\tau))$, $\tau > 0$, and $f(x) = c(\delta)$ for some $\delta \leq h$. Hence we have, by the domain monotonicity of the Dirichlet heat kernel,

$$\begin{aligned}
k_D(t, x, x) &= k_{B_\tau(c(\tau))}(t, c(\delta), c(\delta)) \\
&\leq k_H(t, c(\delta), c(\delta)) \leq k(t)/2,
\end{aligned}$$

as desired.

Applying the above lemma, we get

$$\operatorname{vol}(\mathscr{F})^{-1}\int e^{-\lambda t}\,d\Phi_\Gamma(\lambda)-\operatorname{vol}(D_j)^{-1}\int_{D_j}k_{D_j}(t,x,x)\,dx$$
$$\geq\frac{k(t)}{2}\operatorname{vol}(\partial_h D_j)/\operatorname{vol}(D_j).$$

If r_j denotes the radius of D_j, one has $\operatorname{vol}(D_j)=e^{(n-1)r_j}$, so that the last term is written as

$$\frac{k(t)}{2}(1-e^{-(n-1)h})>0.$$

This completes the proof of Theorem 4.1.

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