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Coverings of 1-convex manifolds with 1-dimensional exceptional set

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To the memory of C. Banica

§0. Introduction

By a classical result of K. Stein [15] it is known that every covering $\tilde{X}$ of a Stein manifold X is itself Stein. The aim of this paper is to consider the case when X is a 1-convex manifold with 1-dimensional exceptional set and to study the convexity properties of $\tilde{X}$.

In [4] Grauert and Docquier have introduced several notions of convexity. For example a complex manifold Y is said to be p_3-convex if it can be exhausted by a sequence $\{Y_\nu\}_{\nu \in \mathbb{N}}$ of relatively compact strongly pseudoconvex domains. Our main result is Theorem 2 which says that every covering $\tilde{X}$ of a 1-convex manifold X with 1-dimensional exceptional set is p_3-convex.

We recall also [4] that a complex manifold Y is said to be p_1-convex if there exists a smooth plurisubharmonic exhaustion function $\varphi : Y \to \mathbb{R}$. Obviously every holomorphically convex manifold is p_1-convex. In §3 we exhibit an example of a strongly pseudoconvex surface X whose universal covering $\tilde{X}$ fails to be p_1-convex, in particular $\tilde{X}$ is not holomorphically convex. So, if we study the convexity properties of the coverings of 1-convex manifolds with 1-dimensional exceptional set, a natural condition is the p_3-convexity.

For embeddable 1-convex manifolds X (e.g. strongly pseudoconvex surfaces) Napier [11] has shown that their coverings $\tilde{X}$ have good meromorphic convexity properties: if $\{a_\nu\}_{\nu \in \mathbb{N}}$ is a discrete sequence of points in $\tilde{X}$ then there exists a meromorphic function f on $\tilde{X}$ which is holomorphic near $\{a_\nu\}_{\nu \in \mathbb{N}}$ and unbounded on $\{a_\nu\}_{\nu \in \mathbb{N}}$.

§1. Preliminaries

We assume all complex manifolds Hausdorff and countable at infinity. In [2] the following result is proved:

THEOREM 1. *Let X be a* 1*-convex manifold and $S \subset X$ its exceptional set. Then there is a strongly plurisubharmonic exhaustion function $\varphi : X \to [-\infty, \infty)$ such that $S = \{\varphi = -\infty\}$. Moreover φ can be chosen such that* $\exp \varphi$ *is smooth.*

Using the above result and a method of LeBarz [8] (see also ([13], p. 494)) we prove:

PROPOSITION 1. *Let X be a* 1*-convex manifold with exceptional set S and $p : \tilde{X} \to X$ any covering. Then there is a strongly plurisubharmonic function $\tilde{\varphi} : \tilde{X} \to [-\infty, \infty)$ such that $p^{-1}(S) = \{\tilde{\varphi} = -\infty\}$,* $\exp \tilde{\varphi}$ *is smooth and, for any open neighbourhood U of S, the restriction $\tilde{\varphi}|_{\tilde{X} \setminus p^{-1}(U)}$ is an exhaustion function on $\tilde{X} \setminus p^{-1}(U)$.*

Proof. We may assume that X and $\tilde{X}$ are connected. Let $\{U_i\}_{i \in \mathbb{N}}$ be a locally finite open covering of X such that $U_i \subset\subset X$ and each U_i is biholomorphic to a ball (so U_i is evenly covered). We get a decomposition $p^{-1}(U_i) = \bigcup_k W_{i,k}$ into disjoint open sets with $W_{i,k}$ biholomorphic to U_i via the projection map p. Let $\{\varphi_i\}_{i \in \mathbb{N}}$ be a partition of unity corresponding to $\{U_i\}_{i \in \mathbb{N}}$ and define $f : \tilde{X} \to \mathbb{R}$ as follows: fix some W_{i_0,k_0} and define $\lambda_{i,k}$ as the length of the shortest chain $W_{i_0,k_0}, W_{i_1,k_1}, \ldots, W_{i_s,k_s}$ such that $W_{i_\nu,k_\nu} \cap W_{i_{\nu+1},k_{\nu+1}} \neq \emptyset$ and $(i_s, k_s) = (i, k)$. If we set $f = \Sigma_{i,k} (\varphi_i \circ p)\lambda_{i,k}$ then one has ([13], p. 494):

(a) f is a smooth exhaustion function on $\tilde{X}$.

(b) The levi form $L(f)|_{p^{-1}(U_i)}$ is bounded from below.

Let $\varphi : X \to [-\infty, \infty)$ be a strongly plurisubharmonic exhaustion function having the properties stated in Theorem 1. By the conditions (a) and (b) we easily see that there exists a smooth convex strictly increasing function $\theta : \mathbb{R} \to \mathbb{R}$, $\theta(t) = At$ $(A > 0)$ near $-\infty$, $\lim_{t \to \infty} \theta(t) = \infty$ such that $\tilde{\varphi} = \theta \circ \varphi \circ p + f$ is strongly plurisubharmonic on $\tilde{X}$. From the definition of $\tilde{\varphi}$ it follows that it has all the required properties. Thus the proof of Proposition 1 is complete.

Another important ingredient for the proof of Theorem 2 is the following result due to Siu ([12], Corollary 1):

PROPOSITION 2. *Let A be a closed complex submanifold of a complex manifold Y. If A is Stein, then there exists a biholomorphic map from a neighbourhood W of A in Y onto an open neighbourhood of the zero cross section of the normal bundle of A in Y such that its restriction to A agrees with the canonical map from A onto the zero cross section. As a consequence, there is a holomorphic retract from W onto A.*

In fact we shall need this result in the case when A is a connected non-compact complex curve, which, by Behnke–Stein theorem, is Stein. The curve A will be obtained by removing finitely many points from the exceptional curve S of X.

§2. Proof of the main result

We begin by recalling the following definiton [4]: a complex manifold Y is said to be p_3-convex if there exists an increasing sequence $\{Y_\nu\}_{\nu \in \mathbb{N}}$ of relatively compact strongly pseudoconvex domains such that $Y = \bigcup_{\nu \in \mathbb{N}} Y_\nu$.

The aim of this paragraph is to prove the following:

THEOREM 2. *Let X be a 1-convex manifold with 1-dimensional exceptional set S and $p : \tilde{X} \to X$ any covering. Then $\tilde{X}$ is p_3-convex.*

The following lemma shows that it suffices to prove the above theorem for a suitable small neighbourhood of the exceptional set S.

LEMMA 1. *Let X be a 1-convex manifold, S its exceptional set and $p : \tilde{X} \to X$ any covering. Assume that there exists an open neighbourhood U of S in X such that $\tilde{U} = p^{-1}(U)$ is p_3-convex. Then $\tilde{X}$ is p_3-convex.*

Proof. Let $K \subset \tilde{X}$ be any compact subset of $\tilde{X}$. We prove the existence of a strongly pseudoconvex neighbourhood $D \subset\subset \tilde{X}$ of K. Let $\tilde{\varphi} : \tilde{X} \to [-\infty, \infty)$ be a strongly plurisubharmonic function on $\tilde{X}$ having the properties stated in Proposition 1 and let $\alpha > 0$ be such that $K \subset \{\tilde{\varphi} < \alpha\}$. Choose also strongly pseudoconvex neighbourhoods $U_1 \subset\subset U_2$ of S such that $\tilde{U}_1 = p^{-1}(U_1)$ and $\tilde{U}_2 = p^{-1}(U_2)$ are p_3-convex. Since $\tilde{\varphi}|_{\tilde{X} \setminus \tilde{U}_1}$ is an exhaustion function there is a compact set L containing K with $\{\tilde{\varphi} < \alpha\} \cap \complement L \subset \tilde{U}_1$ and because $\tilde{U}_2$ is p_3-convex there is a strongly pseudoconvex domain $M \subset\subset \tilde{X}$, $M \subset \tilde{U}_2$ such that $L \cap \tilde{U}_2 \subset M$. If we define D by

$$D = \begin{cases} \{\tilde{\varphi} < \alpha\} & \text{in } \complement \bar{\tilde{U}}_1 \\ \{\tilde{\varphi} < \alpha\} \cap M & \text{in } \tilde{U}_2 \end{cases}$$

then obviously D is a strongly pseudoconvex neighbourhood of K, so the proof of Lemma 1 is complete.

LEMMA 2. *Let W be a Stein manifold and $A \subset W$ a closed complex submanifold. Then there exists a smooth plurisubharmonic function $\varphi : W \to [0, \infty)$ such that $A = \{\varphi = 0\}$, φ is strongly plurisubharmonic on $W \setminus A$ and*

(α) if $U \subset W$ is any open subset, $\psi \in C^\infty(U)$ is plurisubharmonic and its restriction $\psi|_{A \cap U}$ is strongly plurisubharmonic, then for any $\varepsilon > 0$ the function $\psi + \varepsilon\varphi$ is strongly plurisubharmonic on U.

Proof. Let $\mathscr{I}$ be the ideal sheaf of A and let $g_1, \ldots, g_k \in \Gamma(W, \mathscr{I})$ be a set of generators of $\mathscr{I}$ on W. If we set $h = \Sigma_{i=1}^k |g_i|^2$ then h is plurisubharmonic, $h \geq 0$

and $A = \{h = 0\}$. Also if $z \in A$ then the Levi form $L(h)(t) > 0$ for any vector $t \in T_z W \backslash T_z A$. If $g_{k+1}, \ldots, g_m \in \Gamma(W, \mathscr{I})$ give an immersion at any point $z \in W \backslash A$ then $\varphi = \Sigma_{i=1}^m |g_i|^2$ has all the required properties.

Let us recall now some elementary results of algebraic topology which we shall need in the proof of Theorem 2.

If V is a complex manifold and $A \subset V$ is a closed analytic subset (or, more generally, a semi-analytic subset), by the triangulation theorem [9] and the results in [14], the following conditions are equivalent:

(a) the inclusion $A \hookrightarrow V$ is a weak homotopic equivalence,
(b) the inclusion $A \hookrightarrow V$ is a homotopic equivalence,
(c) A is a deformation retract of V,
(d) A is a strong deformation retract of V.

In order to lift some holomorphic retracts from the base space to the covering space we shall need also the following topological results (see for instance [6]):

PROPOSITION 2 (Covering homotopy theorem ([6], p. 18)). *Let* (E, e_0), (X, x_0) *be topological spaces with base points and* $p : (E, e_0) \to (X, x_0)$ *a covering map. Let* (Y, y_0) *be arbitrary and* $f : (Y, y_0) \to (X, x_0)$ *a map which has a lifting* $f' : (Y, y_0) \to (E, e_0)$. *Then every homotopy* $F : Y \times I \to X$ *with* $F(y, 0) = f(y)$ *for all* $y \in Y$ *can be lifted to a homotopy* $F' : Y \times I \to E$ *with* $F'(y, 0) = f'(y)$ *for all* $y \in Y$ (*here* I *denotes the interval* $[0, 1]$).

PROPOSITION 3 (Lifting criterion ([6], p. 22)). *Assume that the topological spaces* E, X, Y *are connected and locally pathwise connected. Consider the diagram*

$$\begin{tikzcd} & (E, e_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, dashed, "f'"] \arrow[r, "f"'] & (X, x_0) \end{tikzcd}$$

where p *is a covering map and* f *is arbitrary. Then there exists a lifting* f' *of* f $(p \circ f' = f)$ *iff* $f_* \pi_1(Y, y_0) \subset p_* \pi_1(E, e_0)$.

One application of these two results is described in the following:

REMARK 1. Let V be a complex manifold, $A \subset V$ a closed complex submanifold, $p : \tilde{V} \to V$ a covering map and $\tilde{A} = p^{-1}(A)$. If A is a deformation retract of V then $\tilde{A}$ is also a deformation retract of $\tilde{V}$.

This follows easily from the Covering homotopy theorem and the equivalence of the conditions (a), (b), (c), (d). Moreover every connected component $\tilde{A}_i$ of $\tilde{A}$ is contained in precisely one connected component $\tilde{V}_i$ of $\tilde{V}$ and every connected component of $\tilde{V}$ contains precisely one connected component of $\tilde{A}$. For every i the set $\tilde{A}_i$ is a deformation retract of $\tilde{V}_i$.

REMARK 2. Let V, $\tilde{V}$, A, $\tilde{A}$ be as above and assume that A is a deformation retract of V. Let $r : V \to A$ be any continuous retract (not necessarily a deformation retract). Then there is a continuous retract $\tilde{r} : \tilde{V} \to \tilde{A}$ such that the diagram

$$\begin{array}{ccc} \tilde{V} & \xrightarrow{\tilde{r}} & \tilde{A} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ V & \xrightarrow{r} & A \end{array} \tag{$*$}$$

is commutative. In particular, if r is a holomorphic retract then so is $\tilde{r}$.

This can easily be seen in the following way:

First we may assume that all the spaces V, $\tilde{V}$, A, $\tilde{A}$ are connected. From the Lifting criterion, one can conclude that an $\tilde{r}$ making the above diagram commutative exists, noting that the induced map $\pi_1(\tilde{A}) \to \pi_1(\tilde{V})$ is an isomorphism (by Remark 1). Indeed, let $i : A \hookrightarrow V$, $\tilde{i} : \tilde{A} \hookrightarrow \tilde{V}$ be the inclusions maps and consider the commutative diagram

$$\begin{array}{ccc} \tilde{V} & \xhookleftarrow{\tilde{i}} & \tilde{A} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ V & \xhookleftarrow{i} & A \end{array} \tag{$**$}$$

If $\alpha \in \pi_1(\tilde{V})$ then there is a unique $\beta \in \pi_1(\tilde{A})$ with $\tilde{i}_*(\beta) = \alpha$. It follows that $r_*(p_*(\alpha)) = r_*(p_*(\tilde{i}_*(\beta))) = r_*(i_*(p_*(\beta))) = p_*(\beta)$ where the last equality holds because r is a retract. Hence the conditions of the Lifting criterion are satisfied and the map $\tilde{r}$ making $(*)$ commutative exists. Restricting the diagram $(*)$ to A, the uniqueness theorem for liftings ([6], p. 17) shows that the restriction $\tilde{r}|_{\tilde{A}} = \mathrm{id}$ so $\tilde{r}$ is a retract. Clearly $\tilde{r}$ is holomorphic if r has this property.

LEMMA 3. *Let X be a* 1*-convex manifold with* 1*-dimensional exceptional set S and $p : \tilde{X} \to X$ any covering. Then there is an open neighbourhood U of S such that $\tilde{U} = p^{-1}(U)$ is p_3-convex.*

Proof. We may assume that $X, \tilde{X}, S$ are connected and that the covering $p : \tilde{X} \to X$ has infinite fibers. Let $M = \{s_1, \ldots, s_k\} \subset S$ be a finite set such that $A = S \setminus M$ is non-singular and Stein. Since A is a closed Stein submanifold of $X \setminus M$

it follows from Proposition 2 that there exists an open neighbourhood W of A in $X\backslash M$ and a biholomorphic map from W onto a neighbourhood of the zero cross section of the normal bundle $N_{A|X\backslash M}$ such that its restriction to A agrees with the canonical map from A onto the zero cross section. So we have an induced holomorphic retract $r : W \to A$ and we also may assume that W is Stein [12].

We set $\tilde{S} = p^{-1}(S)$, $\tilde{A} = p^{-1}(A)$ and let V be an open neighbourhood of A in $X\backslash M$, $\bar{V} \subset W$ (adherence with respect to $X\backslash M$), such that A is a deformation retract of V. If we denote $\tilde{V} = p^{-1}(V)$ and we consider r as a map $V \to A$ then r can be lifted (by Remark 2) to a holomorphic retract $\tilde{r} : \tilde{V} \to \tilde{A}$ such that the diagram

$$\begin{array}{ccc} \tilde{V} & \xrightarrow{\tilde{r}} & \tilde{A} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ V & \xrightarrow{r} & A \end{array} \tag{$*$}$$

is commutative. We may also assume that W and V are small enough such that if $K \subset\subset \tilde{A}$ then $\tilde{r}^{-1}(K) \subset\subset \tilde{X}$ (condition (C)). We choose balls $T_1, \ldots, T_k \subset\subset X$ centered at $s_1, \ldots, s_k$ with $\bar{T}_i \cap \bar{T}_j = \varnothing$ if $i \neq j$ such that some neighbourhoods of $\bar{T}_1, \ldots, \bar{T}_k$ are evenly covered and $T_1 \cap S, \ldots, T_k \cap S$ are connected. Let also $L_1 \subset\subset T_1, \ldots, L_k \subset\subset T_k$ be sufficiently small concentric balls such that:

(1) $x \in S\backslash T_i$ implies $r^{-1}(x) \cap \bar{L}_i = \varnothing$, $i = 1, \ldots, k$,

(2) $r^{-1}(L_i \cap A) \subset T_i$, $i = 1, \ldots, k$.

Clearly these conditions may easily be satisfied if W is chosen from the beginning small enough.

We now consider an exhaustion $\{D_\lambda\}_{\lambda \in \mathbb{N}}$ of $\tilde{S}$ by relatively compact domains with smooth boundary, $D_\lambda \subset\subset D_{\lambda+1}$, such that $\partial D_\lambda \cap \mathrm{Sing}\,(\tilde{S}) = \varnothing$ and ∂D_λ does not intersect $p^{-1}(S \cap \bar{T})$, where $T = T_1 \cup \cdots \cup T_k$. Since $\tilde{S}$ is 1-dimensional each D_λ is strongly pseudoconvex. We set $U_1 = V \cup L_1 \cup \cdots \cup L_k$ and $\tilde{U}_1 = p^{-1}(U_1)$. We first describe an exhaustion of $\tilde{U}_1$ by relatively compact domains in $\tilde{X}$ whose boundaries (relative to $\tilde{U}_1$) are pseudoconvex (not necessarily strongly pseudoconvex). Then, by a simple perturbation argument and Lemma 2, we may achieve that their boundaries (relative to $\tilde{U}_1$) become strongly pseudoconvex. Finally, replacing U_1 by a sufficiently small strongly pseudoconvex neighbourhood U of the exceptional set S, we get the desired exhaustion of $\tilde{U}$ by domains whose boundaries (relative to $\tilde{X}$) are strongly pseudoconvex.

In order to obtain the exhaustion of $\tilde{U}_1$ we define the open sets $\tilde{D}_\lambda \subset \tilde{U}_1$, $\lambda \in \mathbb{N}$, as follows:

Let $\{b_1, \ldots, b_m\}$, where $m = m(\lambda)$, be the subset of $p^{-1}(M)$ consisting of those points in $p^{-1}(M)$ contained in D_λ. Consider the decomposition into connected components $p^{-1}(L_i) = \bigcup_{j \in \mathbb{N}} L_i^j$, $i = 1, \ldots, k$ and denote by $B_1, \ldots, B_m$ those connected components L_i^j containing $b_1, \ldots, b_m$. We set

$\tilde{D}_\lambda = \tilde{r}^{-1}(D_\lambda \backslash \{b_1, \ldots, b_m\}) \cup B_1 \cup \cdots \cup B_m$. Clearly $\{\tilde{D}_\lambda\}_{\lambda \in \mathbb{N}}$ is an increasing sequence of open subsets of $\tilde{U}_1$ and $\bigcup_{\lambda \in \mathbb{N}} \tilde{D}_\lambda = \tilde{U}_1$. To see that $\tilde{D}_\lambda \subset\subset \tilde{X}$ it suffces to verify that $\tilde{r}^{-1}(D_\lambda \backslash \{b_1, \ldots, b_m\}) \subset\subset \tilde{X}$ and by the condition (C) it is enough to show that $\tilde{r}^{-1}(B_t \cap \tilde{S} \backslash b_t) \subset\subset \tilde{X}$, $t = 1, \ldots, m$. But this follows immediately from the commutativity of the diagram $(*)$ and the condition (2). Now we study the pseudoconvexity of the boundary of $\tilde{D}_\lambda$ (relative to $\tilde{U}_1$). First we remark that by the condition (1) and the commutativity of the diagram $(*)$ it follows that $\bar{B}_t \cap \tilde{V} \subset \tilde{r}^{-1}(D_\lambda \backslash \{b_1, \ldots, b_m\})$ and if E is another component L_i^j, different from $B_1, \ldots, B_m$, then $\bar{E} \cap \tilde{r}^{-1}(D_\lambda \backslash \{b_1, \ldots, b_m\}) = \varnothing$. Hence the boundary of $\tilde{D}_\lambda$ (relative to $\tilde{U}_1$) is precisely the boundary $\tilde{r}^{-1}(D_\lambda \backslash \{b_1, \ldots, b_m\})$ (relative to $\tilde{V}$). To describe this boundary we choose an open neighbourhood M_λ of ∂D_λ in $\tilde{S}$, $\bar{M}_\lambda \cap p^{-1}(S \cap \bar{T}) = \varnothing$, and a smooth strongly subharmonic function φ_λ defining D_λ in M_λ. Then $\varphi_\lambda \circ \tilde{r}$ is a plurisubharmonic defining function for $\tilde{D}_\lambda$ in $\tilde{U}_1$ so each $\tilde{D}_\lambda$ has a pseudoconvex boundary (relative to $\tilde{U}_1$). To obtain the desired exhaustion by strongly pseudoconvex domains we need Lemma 2. By this lemma there is a smooth plurisubharmonic function $\varphi : W \to [0, \infty)$ such that: $A = \{\varphi = 0\}$, φ is strongly plurisubharmonic on $W \backslash A$ and φ has the property (α). If we set $\tilde{\varphi} = \varphi \circ p|_{\tilde{V}}$ then for any $\varepsilon > 0$ the function $\varphi_\lambda \circ \tilde{r} + \varepsilon\tilde{\varphi}$ is strongly plurisubharmonic on $\tilde{r}^{-1}(M_\lambda)$ and its restriction to M_λ is φ_λ. The functions $\varphi_\lambda \circ \tilde{r} + \varepsilon_\lambda \tilde{\varphi}$ with $\varepsilon_\lambda > 0$ sufficiently small define in an obvious way the exhaustion of $\tilde{U}_1$ by domains $\tilde{D}'_\lambda$ having strongly pseudoconvex boundaries relative to $\tilde{U}_1$ (after replacing V if necessary by a smaller open subset V' with $A \subset V'$ and $\bar{V}' \subset V$, where the adherence is taken with respect to $X \backslash M$). If we choose now a strongly pseudoconvex neighbourhood U of S, $U \subset\subset U_1$, it follows then immediately, from our previous remarks, that $\tilde{U} = p^{-1}(U)$ is p_3-convex. The proof of Lemma 3 is complete.

Proof of Theorem 2. Theorem 2 is a direct consequence of Lemma 1 and Lemma 3.

If X is a 1-convex manifold of dimension 2 we shall call it a 1-convex surface (or strongly pseudoconvex surface). From Theorem 2 we get:

COROLLARY 1. *Let X be a* 1-*convex surface. Then every covering $\tilde{X}$ of X is p_3-convex.*

Let now $D \subset \mathbb{C}^n$ be the half-open unit polydisc, i.e.

$$D = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_1| \leq 1, |z_\nu| < 1, \nu = 2, \ldots, n\}$$

and set

$$\delta D = \{(z_1, \ldots, z_n) \in D \mid |z_1| = 1\}.$$

According to [4] a complex manifold Y is called p_6-convex if there is no biholomorphic map φ from an open neighbourhood of D onto an open subset of Y satisfying the following two conditions:

(a) $\varphi(\delta D) \subset\subset Y$,
(b) $\varphi(D) \not\subset \subset Y$.

In [4] it is proved that every p_3-convex manifold is p_6-convex. From Theorem 2 it follows:

COROLLARY 2. *Let X be a 1-convex manifold with 1-dimensional exceptional set. Then every covering $\tilde{X}$ of X is p_6-convex.*

We also obtain easily from Theorem 2:

COROLLARY 3. *Let X be a 1-convex manifold with 1-dimensional exceptional set and $\tilde{X}$ any covering of X. The $\tilde{X}$ is holomorphically convex iff $\tilde{X}$ is a proper modification of a Stein space at a discrete set.*

§3. Some counterexamples

In this paragraph we show that in general the covering spaces (even the universal covering spaces) of 1-convex surfaces are not holomorphically convex.

We recall [4] that a complex manifold Y is said to be p_1-convex if there exists a smooth plurisubharmonic exhaustion function $\varphi : Y \to \mathbb{R}$. Clearly any holomorphically convex manifold is p_1-convex. We shall exhibit an example of a 1-convex surface X whose universal covering $\tilde{X}$ fails to be p_1-convex, in particular X is not holomorphically convex. To construct X we shall need the following two results:

GRAUERT'S CRITERION [5]. *Let X be a 2-dimensional complex manifold and $S \subset X$ a 1-dimensional connected compact analytic subset with irreducible components $S_1, \ldots, S_m$. Then S is exceptional iff the intersection matrix $(S_i S_j)$ is negative definite.*

LEMMA 4. *Let X be a complex manifold and $S \subset X$ an exceptional set. Then there exists a strongly pseudoconvex neighbourhood V of S such that S is a deformation retract of V.*

Proof. Let $\varphi \geq 0$ be a real-analytic plurisubharmonic function in a neighbourhood of S such that $S = \{\varphi = 0\}$ and φ is strongly plurisubharmonic outside S. Such a function exists because S is exceptional. Since φ is real-analytic it follows from the "Curve selection lemma" [7] that for $\varepsilon_0 > 0$ small enough φ has no critical points in $\{\varphi < \varepsilon_0\}\backslash S$. So, for any $0 < \varepsilon < \varepsilon_0$, $\{\varphi \leq \varepsilon\}$ is a deformation retract of $V = \{\varphi < \varepsilon_0\}$. Choose an open neighbourhood V_1 of S, $V_1 \subset\subset \{\varphi < \varepsilon_0\}$ such that S is a deformation retract of V_1 and let $0 < \varepsilon_1 < \varepsilon_0$ be such that $\{\varphi \leq \varepsilon_1\} \subset V_1$. Because S is a deformation retract of V_1 and $\{\varphi \leq \varepsilon_1\}$ is a deformation retract of V it follows that the inclusion $S \hookrightarrow V$ is a weak homotopic equivalence. But (S, V) is a polyhedral pair so we deduce [14] that S is a deformation retract of V. The proof of Lemma 4 is complete.

We now begin constructing our example. First we make some remarks:

(i) Let X_1, X_2 be complex manifolds (Hausdorff), $U_1 \subset X_1$ and $U_2 \subset X_2$ open subsets and $\varphi : U_1 \to U_2$ a biholomorphic map. Let X be obtained by glueing X_1 and X_2 via the map φ. In general X is not Hausdorff but one can easily verify that the necessary and sufficient condition on X to be Hausdorff is the following: for every $x_1 \in \partial U_1$ (boundary relative to X_1) and every $x_2 \in \partial U_2$ (boundary relative to X_2) there exist open neighbourhoods $V_1 \subset X_1$ of x_1 and $V_2 \subset X_2$ of x_2 such that $\varphi(U_1 \cap V_1) \cap (U_2 \cap V_2) = \varnothing$.

(ii) We construct a complex manifold X of dimension 2 (complex surface) containing an exceptional curve S such that S has two irreducible components $S = S_1 \cup S_2$, $S_1 \cong \mathbb{P}^1$, $S_2 \cong \mathbb{P}^1$, the intersection $S_1 \cap S_2$ consists of two points and at these points S_1 and S_2 meet transversally. Let $S_1 = \mathbb{P}^1$ be the zero cross section of $\mathcal{O}(-3) = Y_1$ and $S_2 = \mathbb{P}^1$ the zero cross section of $\mathcal{O}(-3) = Y_2$. We glue suitable neighbourhoods of S_1 in Y_1 and of S_2 in Y_2 such that S_1 and S_2 meet transversally at two points. The precise construction is as follows: Let B^2 be the unit ball in $\mathbb{C}^2$ and $f: B^2 \to B^2$ the automorphism given by $f(z_1, z_2) = (z_2, z_1)$. Choose $p_1 \neq q_1$, $p_1 q_1 \in S_1$ and $p_2 \neq q_2$, $p_2, q_2 \in S_2$. Let $E_1 \subset\subset Y_1, F_1 \subset\subset Y_1, \bar{E}_1 \cap \bar{F}_1 = \varnothing$ be open neighbourhoods of p_1 and of q_1 respectively, such that there exist biholomorphic maps $\tau_1 : E_1 \to B^2$, $\psi_1 : F_1 \to B^2$. We also assume that, via the maps τ_1, ψ_1 the set S_1 corresponds to $z_1 = 0$ and the points p_1, q_1 to the origin $O \in \mathbb{C}^2$. Similarly we consider open neighbourhoods $E_2 \subset\subset Y_2, F_2 \subset\subset Y_2, \bar{E}_2 \cap \bar{F}_2 = \varnothing$ of p_2 and of q_2 respectively, and biholomorphic maps $\tau_2 : E_2 \to B^2$, $\psi_2 : F_2 \to B^2$ such that via these maps S_2 corresponds to $z_1 = 0$ and the points p_2, q_2 to the origin $O \in \mathbb{C}^2$. We have induced isomorphisms φ_1, φ_2 where $\varphi_1 : E_1 \to E_2$ is defined by $\varphi_1 = \tau_2^{-1} \circ f \circ \tau_1$ and $\varphi_2 : F_1 \to F_2$ by $\varphi_2 = \psi_2^{-1} \circ f \circ \psi_1$; so we get an isomorphism $\varphi = (\varphi_1, \varphi_2) : E_1 \cup F_1 \to E_2 \cup F_2$. Let V be an open neighbourhood of the zero cross section in $\mathcal{O}(-3)$ and set $X_1 = V \cup E_1 \cup F_1$, $X_2 = V \cup E_2 \cup F_2$, $U_1 = E_1 \cup F_1$, $U_2 = E_2 \cup F_2$. From the previous remark it follows that for sufficiently small V we can glue X_1 and

X_2 via the map φ and we get a complex manifold X containing a compact analytic curve S with two irreducible components $S = S_1 \cup S_2$, $S_1 \cong \mathbb{P}^1$, $S_2 \cong \mathbb{P}^1$, which meet transversally at two points. The intersection matrix is $\begin{pmatrix} -3 & 2 \\ 2 & -3 \end{pmatrix}$ which is negative definite. By Grauert's criterion S is exceptional and our construction is finished.

(iii) Let X be a 1-convex surface such that its exceptional curve S has two irreducible components $S = S_1 \cup S_2$, $S_1 \cong \mathbb{P}^1$, $S_2 \cong \mathbb{P}^1$ and the intersection $S_1 \cap S_2$ consists of two points where the intersection is transversal. Replacing X by a smaller strongly pseudoconvex neighbourhood of S we may assume (by Lemma 4) that S is a deformation retract of X. Let $p : \tilde{X} \to X$ be the universal covering of X. We assert that $\tilde{X}$ is not p_1-convex (hence it is not holomorphically convex). To see this it is enough to verify that $\tilde{S} = p^{-1}(S)$ is not p_1-convex. Since S is a deformation retract of X, it follows (from Remark 1 which clearly extends to singular A) that $\tilde{S}$ is a deformation retract of $\tilde{X}$, hence $\tilde{S}$ is the universal covering of S. But $\tilde{S}$ has a very simple description: its decompositoin into irreducible components can be written $\tilde{S} = \bigcup_{i \in \mathbb{Z}} \tilde{S}_i$, $\tilde{S}_i \cong \mathbb{P}^1$, $\tilde{S}_i$ meets $\tilde{S}_{i-1}$ in one point, $\tilde{S}_i$ meets $\tilde{S}_{i+1}$ in one point and $\tilde{S}_i$ does not meet any other component (so $\tilde{S}$ is an infinite necklace [10]; topologically $\tilde{S}$ is an infinite union of spheres each having one point in common with the next). Obviously $\tilde{S}$ is not p_1-convex because, by the maximum principle, any plurisubharmonic function on $\tilde{S}$ must be constant.

So we have shown:

THEOREM 3. *There exists a* 1-*convex surface* X *such that its universal covering* $\tilde{X}$ *is not holomorphically convex.*

REMARK 3. In our example the exceptional curve is not irreducible, so it is natural to ask if one can produce an example having the properties in Theorem 3 and with irreducible exceptional curve. Such an example can be obtained as follows: We glue suitable neighbourhoods of the zero cross sections in $\mathcal{O}(-1)$ and $\mathcal{O}(-5)$ exactly as before and we get a complex surface Z containing a compact analytic curve C with two irreducible components $C = C_1 \cup C_2$, $C_1 \cong \mathbb{P}^1$, $C_2 \cong \mathbb{P}^1$ such that the intersection $C_1 \cap C_2$ consists of two points where the intersection is transversal. The intersection matrix is $\begin{pmatrix} -1 & 2 \\ 2 & -5 \end{pmatrix}$ which is negative definite, hence by Grauert's criterion, C is exceptional. Because the autointersection $C_1^2 = -1$ it follows that C_1 is a curve of the first kind, so it can be contracted to a point in the complex (non-singular) surface X via the contraction map $h : Z \to X$. Then $S = h(C)$ is an irreducible exceptional curve (rational with one singular point). If we replace X by a smaller strongly pseudoconvex neighbourhood of S such that S is a deformation retract of this neighbourhood we get the desired example. The proof is exactly as in (iii) and so it is omitted.

REMARK 4. In ([11], Theorem 6.2) Napier has proved the following result: Let X be a 1-convex surface with exceptional set S and $p : \tilde{X} \to X$ any covering. Then $\tilde{X}$ is holomorphically convex iff $\tilde{S} = p^{-1}(S)$ is holomorphically convex. This last result explains why the exceptional curves in our previous examples are singular (for smooth S it follows that $\tilde{S}$ is holomorphically convex).

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