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Autor(en): **Panyushev, D.I.**

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Complexity and rank of double cones and tensor product decompositions

PANYUSHEV D. I.

(0.1) Let G be a reductive algebraic group defined over an algebraically closed field k of characteristic 0. Recently a rather significant progress has been achieved in the theory of algebraic transformation groups, which is related with the notions of *rank* and *complexity* of G-varieties, i.e. algebraic varieties, equipped with a regular action of G. The notion of complexity has been introduced in [6] (for homogeneous spaces of G) and in [9] (for arbitrary G-varieties). One of the reasons, which led to the appearance of this notion, was the thorough investigation of equivariant embeddings of homogeneous spaces of G, in particular, of spherical ones. The role of the rank of spherical homogeneous spaces has also been recognized. The general notion of rank of G-variety appeared independently in [3] and [8].

In [8] a method of calculation of the complexity $c(X)$ and the rank $r(X)$ of an irreducible G-variety X has been established. The idea is as follows. Let B be a Borel subgroup of G and U the unipotent radical of B. Let $k[X]$ and $k(X)$ denote the algebra of regular functions and the field of rational functions on X respectively. Then one has:

$$c(X) = \operatorname{tr\,deg} k(X)^B, \qquad c(X) + r(X) = \operatorname{tr\,deg} k(X)^U. \tag{1}$$

(Here $k(X)^A$ is the subfield of A-invariant functions for a subgroup A of G.) Thus, the definition uses only the solvable subgroups of G. But it has been shown in [8] that rank and complexity can be calculated in terms of the stabilizer of general position of the G-action on the "doubled" variety $X \times X^*$ (see (1.2)).

If X is affine, then rank and complexity give us an initial approach to the description of G-module structure of $k[X]$. But in the affine case a more useful object, than the rank can be defined. This is the *rank semigroup* which consists of the highest weights of all irreducible G-modules appearing in $k[X]$. Then $r(X)$ is equal to the rank of this semigroup.

(0.2) The purpose of this paper is to apply the general methods [8] to a special class of affine G-varieties. These varieties will be called *double cones*. Following [4] we define a double cone to be the product of two HV-varieties of G, where an HV-variety of G is the closure of the G-orbit of highest weight vectors in an irreducible G-module. Also, an HV-variety may be considered as an affine cone over a complete homogeneous space of G. The interest of the double cones is explained by the following observation, which is due to P. Littelmann [4]: the freeness of the algebra of U-invariants of a double cone provides a simple rule for decomposing of a series of tensor products.

A double cone is determined by two dominant weights λ, μ of G. Let $Z = Z(\lambda, \mu)$ be the corresponding double cone and $R(\lambda)$ be the irreducible G-module with highest weight λ. The algebra $k[Z]$ is bi-graded and the component of bi-degree (m, n) coincides with $R(m\lambda) \otimes R(n\mu)$. Therefore the knowledge of Poincare series of $k[Z]^U$ gives us the possibility to produce practical formulas for the decomposition of these tensor products.

A double cone Z is equipped with a natural action of the extended group $\tilde{G} = G \times (k^*)^2$. Let $\tilde{r}(Z)$, $\tilde{c}(Z)$ be the rank and the complexity of Z relative to the $\tilde{G}$-action. A simple but important observation connecting the complexity and the structure of $k[Z]^U$ is the following: If λ, μ are fundamental weights and $\tilde{c}(Z) = 0$, then $k[Z]^U$ is a polynomial algebra [4]. Moreover, [4] contains the classification of pairs of weights of this form together with the degrees and the weights of a homogeneous system of generators of $k[Z]^U$. A part of this results has been discovered independently by A. G. Elashvili.

(0.3) In this paper we shall give formulas for the complexity and the rank of a double cone, relative to both G- and $\tilde{G}$-actions and also a method of computing of them. This method, which is a development of the one from [8], reduces the problem under consideration to the determination of a *stabilizer of general position* of a representation of a reductive subgroup H of G. This representation is of the form $(H, V + V^*)$, where V is a H-module. Therefore, to determine a stabilizer of general position one can use the inductive procedure from [7].

A nice consequence of this theory is a fact, that c, r, $\tilde{c}$, $\tilde{r}$ coincide for $Z(\lambda, \mu)$ and $Z(\lambda, \mu^*)$, though the generators of the algebras of U-invariants and the formulas for the decompositions of tensor products change essentially. (The star denotes the passage to the highest weight of the dual G-module.) However, it is more useful to determine the *rank semigroup* $\Gamma(Z)$ of a double cone instead of the rank itself. The reason is that the description of $\Gamma(Z)$ allows us to restrict, a priori, the set of dominant weights, which may appear in the decomposition of $R(n\lambda) \otimes R(m\mu)$. We solve this problem by using a stabilizer of general position for $(H, V + V^*)$ and its *canonical embedding* in G.

(0.4) Applying our methods we also check the classification of the pairs of fundamental weights λ, μ with $\tilde{c}(Z(\lambda, \mu)) = 0$, given in [4]. Moreover, in chapter 3 we shall present the integers $c, r, \tilde{c}, \tilde{r}$ and the structure of the rank semigroup for *every* pair of fundamental weights of the classical simple algebraic groups (see Tables 1–3). For the exceptional simple algebraic groups we describe all the pairs of fundamental weights with $\tilde{c} \leq 2$ (Table 4). In a forthcoming paper we shall give a full description of $k[Z]^U$ for all pairs of fundamental weights with $\tilde{c} = 1$.

(0.5) Our basic reference for invariant theory is [11] and algebraic groups is [10]. We follow mainly the terminology and notatoins of them.

s.g.p. = stabilizer(s) of general position;

$[L, L]$ and L^0 are the commutator subgroup and the identity component respectively of an algebraic group L.

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1. Computing complexity and rank

(1.1) Throughout this paper (except 1.2) G denotes a connected and simply-connected simple algebraic group with a fixed maximal unipotent subgroup U and a fixed maximal torus $T \subset N_G(U) =: B$. Let $\mathscr{X}(T)$ be the character group of T and let $\mathscr{X}(T)_+$ be the semigroup of dominant weights relative to (B, T). $\mathscr{X}(T) \otimes \mathbf{Q}$ will be considered with a Weyl group invariant scalar product (,). A subgroup K of G is said to be regular, if $N_G(K)$ contains a maximal torus of G.

(1.2) Our results heavily rely upon the theory from [8]. Let us recall the necessary results in the affine case. Let G be an arbitrary reductive group and X be an irreducible affine G-variety. Put

$$\Gamma(X) = \{\lambda \in \mathscr{X}(T)_+ \mid k[X]^U_\lambda \neq 0\},$$

where $k[X]^U_\lambda = \{f \in k[X]^U \mid t * f = \lambda(t)f \text{ for any } t \in T\}$. The semigroup $\Gamma(X)$ is said to be the *rank semigroup* of X. Let $\theta \in \operatorname{Aut} G$ be an involution, such that $\theta(t) = t^{-1}$ for any $t \in T$. By X^* we shall denote a G-variety which is abstractly isomorphic to X, but provided with a twisted G-action. If $i : X \to X^*$ is the isomorphism, put $x^* = i(x)$. The twisted G-action is defined by $(g, x^*) \mapsto (\theta(g)x)^*$, $g \in G$, $x \in X$. Let us consider the diagonal action $(G, X \times X^*)$. It has been proved in [8, ch. 1] that there is a point $z = (x, x^*) \in X \times X^*$ such that:

(a) $U_* := U_x$ is a s.g.p. for (U, X);
(b) $B_* := B_x$ is a s.g.p. for (B, X);
(c) $S := G_z$ is a s.g.p. for $(G, X \times X^*)$;
(d) there is $t \in T$ such that $[Z_G(t), Z_G(t)] \subset S \subset Z_G(t)$;
(e) $U_* = U \cap S$, $B_* = B \cap S$.

It follows from (d), (e) that S is a reductive regular subgroup of G, U_* is a maximal unipotent subgroup of S and B_*^0 is a Borel subgroup of S^0. As a consequence of these statements the following important relations has been derived in [8, ch. 1]:

(α) If $\Sigma(G)$ is the root system of G relative T, then the root system of S^0 relative to $(S \cap T)^0$ is $\Sigma(S) = \Sigma(G) \cap \Gamma(X)^{\perp}$.

(β) $r(X) = \operatorname{rk} G - \operatorname{rk} S$,
$2c(X) + r(X) = 2 \dim X - \dim G + \dim S = \operatorname{tr} \deg k[X \times X^*]^G$;

(γ) $\Gamma(X) \subset \mathbf{Z}\Gamma(X) \cap \mathscr{X}(T)_+ = \{\omega \in \mathscr{X}(T)_+ \mid \omega|_{T \cap S} = 1\}$

A point $x \in X$, satisfying (a)–(e) is said to be a *canonical* one. If $x \in X$ is a canonical point and S is the stabilizer of (x, x^*), then the embedding $S \subset G$ is said to be the *canonical* one. This embedding is uniquely determined, if T and U are fixed. Clearly, these notions depend on the choice of $U \subset G$.

Warning. If $x \in X$ is a canonical point, then it is not true, that $x^* \in X^*$ is a canonical point relative to the same maximal unipotent subgroup. Actually, x^* is a canonical point relative $\theta(U)$. Therefore, it is important to distinguish X and X^*.

(1.3) Let $C(\lambda)$ ($\lambda \in \mathscr{X}(T)_+$) be the closure of the orbit of highest weight vectors in $R(\lambda)$. It is well-known that $k[C(\lambda)] = \oplus_{n=0}^{\infty} R(n\lambda^*)$ and $C(\lambda)$ is the affine cone over $G/P_\lambda \subset \mathbf{P}(R(\lambda))$, where P_λ is the parabolic subgroup of G, corresponding to λ. If λ, μ are two dominant weights, then the affine variety $Z(\lambda, \mu) = C(\lambda^*) \times C(\mu^*)$ is said to be a *double cone*. The variety $Z := Z(\lambda, \mu)$ is equipped with the natural action of the extended group $\tilde{G} = G \times (k^*)^2$.

By $c(Z), r(Z)$ we denote the complexity and the rank of Z relative to the G-action and by $\tilde{c}(Z), \tilde{r}(Z)$ – relative to the $\tilde{G}$-action. Since $G, \tilde{G}$ have the same maximal unipotent subgroup, (1) implies that

$$c(Z) + r(Z) = \tilde{c}(Z) + \tilde{r}(Z). \tag{2}$$

Since $k[Z(\lambda, \mu)] = k[C(\lambda^*)] \otimes k[C(\mu^*)] = \oplus_{n,m \ge 0} (R(n\lambda) \oplus R(m\mu))$, the rank semigroup of $Z(\lambda, \mu)$ coincides with the set

$$\Gamma(Z) = \{\omega \in \mathscr{X}(T)_+ \mid R(\omega) \subset R(n\lambda) \otimes R(m\mu) \text{ for some } n, m \ge 0\}. \tag{3}$$

Let L_λ be a Levi subgroup of P_λ, $L_\lambda \supset T$ and $L'_\lambda = \operatorname{Ker} \lambda \subset L_\lambda$. By $\tilde{S}_{\lambda\mu}$ we denote a s.g.p. for the natural action $(L_\lambda, G/L_\mu)$ and by $L_\lambda\backslash G/L_\mu$ we denote the (categorical) quotient [11]. Then $\dim L_\lambda\backslash G/L_\mu = \dim G + \dim \tilde{S}_{\lambda\mu} - \dim L_\lambda - \dim L_\mu$. Similarly, let $S_{\lambda\mu}$ be a s.g.p. for $(L'_\lambda, G/L'_\mu)$ and $L'_\lambda\backslash G/L'_\mu$ be the quotient.

(1.4) Now we formulate the main results on double cones.

THEOREM 1. *Let λ, μ be arbitrary dominant weights and $Z = Z(\lambda, \mu)$. Then $S_{\lambda\mu}$ is a s.g.p. for $(G, Z \times Z^*)$, $S_{\lambda\mu}$ is a reductive regular subgroup of G and*

$$2c(Z) + r(Z) = 2 + \dim L'_\lambda\backslash G/L'_\mu, \qquad r(Z) = \operatorname{rk} G - \operatorname{rk} S_{\lambda\mu}.$$

THEOREM 2. *The following relations are valid*:
(i) $2\tilde{c}(Z) + \tilde{r}(Z) = 2 + \dim L_\lambda\backslash G/L_\mu, \tilde{r}(Z) = \operatorname{rk} \tilde{G} - \operatorname{rk} \tilde{S}_{\lambda\mu}$;
(ii) $\tilde{c}(Z) = c(G/P_\lambda \times G/P_\mu), \tilde{r}(Z) = 2 + r(G/P_\lambda \times G/P_\mu)$.

COROLLARY 1. *The integers $c, r, \tilde{c}, \tilde{r}$ for $Z(\lambda, \mu)$ and $Z(\lambda, \mu^*)$ coincide. Moreover, the groups $S_{\lambda\mu}$, $S_{\lambda\mu^*}$ are conjugated in G.*

COROLLARY 2. *Let $\lambda = \mu$ or $\lambda = \mu^*$. Then*
(i) $c(Z) = c(G/L'_\mu) + 1, r(Z) = r(G/L'_\mu)$;
(ii) $\tilde{c}(Z) = c(G/L_\mu), \tilde{r}(Z) = r(G/L_\mu)$.

COROLLARY 3. *If $S_{\lambda\mu} \subset G$ is the canonical embedding, then* (i) $T \subset N_G(S_{\lambda\mu})$, (ii) *a subset of the simple roots of G form a system of simple roots for $S_{\lambda\mu}$, and* (iii) *if $\omega \in \Gamma(Z(\lambda, \mu))$, then $\omega|_{S_{\lambda\mu} \cap T} = 1$. In particular, if α is a root of $S_{\lambda\mu}$, then* $(\alpha, \omega) = 0$.

(1.5) To prove the theorems we need two simple lemmas.

LEMMA 1. *L'_λ is a s.g.p. for $(G, C(\lambda) \times C(\lambda^*))$.*

Proof. Let $v_\lambda \in R(\lambda)$ be a highest weight vector (relative to B) and let $v_{-\lambda} \in R(\lambda^*)$ be a lowest weight vector. Clearly, $x = (v_\lambda, v_{-\lambda}) \in C(\lambda) \times C(\lambda^*)$ and $G_x = L'_\lambda$. Since

$$\dim C(\lambda) = \dim C(\lambda^*) = \frac{\dim G - \dim L'_\lambda + 1}{2},$$

we have $\dim Gx = \dim(C(\lambda) \times C(\lambda^*)) - 1$. Thus, $\overline{G\langle x\rangle} = C(\lambda) \times C(\lambda^*)$ and G_x is a s.g.p. □

LEMMA 2. *Let G act on irreducible varieties X_1, X_2. If L_i is a s.g.p. for (G, X_i), then the s.g.p. for the diagonal action $(G, X_1 \times X_2)$ coincides with the s.g.p. for $(L_1, G/L_2)$ or (L_1, X_2).*

Proof. Obvious. □

(1.6) *Proof of the theorem* 1. We consider the G-action on $Z = C(\lambda^*) \times C(\mu^*)$. First we note, that $C(\lambda)^* \cong C(\lambda^*)$. Therefore $Z \times Z^* \cong C(\lambda^*) \times C(\mu^*) \times C(\lambda) \times C(\mu)$.

Let us apply (1.5) to $Y = Z \times Z^* \cong [C(\lambda) \times C(\lambda^*)] \times [C(\mu) \times C(\mu^*)]$. We can conclude that s.g.p. for (G, Y) coincides with s.g.p. for $(L'_\lambda, G/L'_\mu)$, i.e. with $S_{\lambda\mu} \subset G$. Thus, according to (1.2)(β) $S_{\lambda\mu}$ has the prescribed structure, $r(Z) = \operatorname{rk} G - \operatorname{rk} S_{\lambda\mu}$ and since

$$\dim Z = \frac{\dim G - \dim L'_\lambda + 1}{2} + \frac{\dim G - \dim L'_\mu + 1}{2},$$

we have

$$2c(Z) + r(Z) = 2 + \dim G - \dim L'_\lambda - \dim L'_\mu + \dim S_{\lambda\mu} = 2 + \dim L'_\lambda \backslash G/L'_\mu.$$

□

Proof of the theorem 2. We shall consider the action of $\tilde{G}$ on Z and the action of G on $PZ := G/P_{\lambda^*} \times G/P_{\mu^*}$. First we note, that $\tilde{B} = B \times (k^*)^2$ is a Borel subgroup of $\tilde{G}$ and PZ is the geometric quotient [11] for the action $((k^*)^2, Z)$. Therefore, $k(Z)^{\tilde{B}} \cong k(PZ)^B$, i.e.

$$\tilde{c}(Z) = c(PZ). \tag{4}$$

Further, the action of $(k^*)^2$ on Z is effective, hence

$$\operatorname{tr}\deg k(PZ)^U = \operatorname{tr}\deg k(Z)^{U \times (k^*)^2} = \operatorname{tr}\deg k(Z)^U - 2.$$

This equality, together with (1), (4), shows that $r(PZ) = \tilde{r}(Z) - 2$. That is, part (ii) of theorem 2 is proved. To prove (i) it is enough to show that

$$2c(PZ) + r(PZ) = \dim L_\lambda \backslash G/L_\mu, \qquad r(PZ) = \operatorname{rk} G - \operatorname{rk} \tilde{S}_{\lambda\mu}.$$

But this can be done in the same way as in (1.5). Consider the action $(G, PZ \times (PZ)^*)$. Taking into account that $(G/P_\lambda)^* \cong G/P_{\lambda^*}$ and that s.g.p. for $(G, G/P_\lambda \times G/P_{\lambda^*})$ is equal to L_λ, we get the assertion that s.g.p. for $(G, PZ \times (PZ)^*)$ is equal to s.g.p. for $(L_\lambda, G/L_\mu)$, i.e. coincides with $\tilde{S}_{\lambda\mu}$ and so on ... □

(1.7) *Proof of the corollaries* 1, 2, 3.

1. The replacement of μ by μ^* does not change $Z \times Z^*$ and $PZ \times (PZ)^*$, hence the subgroups $\tilde{S}_{\lambda\mu}$, $S_{\lambda\mu}$ do not change also. $\square$

2. Let $\lambda = \mu^*$ and $Z = Z(\mu, \mu^*)$. Then $\mathcal{O} = G/L'_\mu$ is a G-orbit of general position in Z (2.2), hence $r(\mathcal{O}) = r(Z)$. Since $\operatorname{codim}_Z \mathcal{O} = 1$ we have $c(Z) = c(\mathcal{O}) + 1$. The variety PZ has the open orbit $\tilde{\mathcal{O}} \cong G/L_\mu$. Hence, $c(\tilde{\mathcal{O}}) = c(PZ) = \tilde{c}(Z)$, $r(\tilde{\mathcal{O}}) = r(PZ) = \tilde{r}(Z) - 2$. $\square$

3. Since $S_{\lambda\mu}$ is a s.g.p. for $(G, Z \times Z^*)$, all assertions of the corollary are a direct consequence of the definition of a canonical point and of (1.2)(α),(γ). $\square$

(1.8) *Remarks.* 1. It may happen that $\Gamma(Z) \neq \mathbf{Z}\Gamma(Z) \cap \mathscr{X}(T)_+$, i.e. the canonical embedding $S_{\lambda\mu} \subset G$ does not determine $\Gamma(Z)$ completely.

2. The subgroups $S_{\lambda\mu}$ and $S_{\lambda\mu^*}$ are isomorphic and conjugated in G by corollary 1, nevertheless, they may have *different* canonical embeddings. For example, let $G = \mathrm{E}_6$ and $\mu = \lambda = \varphi_1$, $\mu^* = \varphi_5$ are the fundamental weights with numeration as in [10]. Then $S_{\lambda\mu} \cong S_{\lambda\mu^*} \cong \mathrm{A}_3$, but their canonical embeddings are described by the following pictures:

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(The black vertices indicate the simple roots of $S_{\lambda\mu}$ and $S_{\lambda\mu^*}$.)

(1.9) The previous results show that in order to compute the complexity and the rank semigroup of a double cone one has to find the canonical embedding of $S_{\lambda\mu}$ and $\tilde{S}_{\lambda\mu}$. The next result explains how this can be done.

THEOREM 3. *Let m_λ, m_μ be the orthogonal complements in* Lie G *to* Lie L_λ, Lie L_μ *respectively. Then*

(i) *$m_\lambda \cap m_\mu$ is a $L_\lambda \cap L_\mu$-module; moreover, if $V = (m_\lambda \cap m_\mu) \cap \operatorname{Lie} U$, then $m_\lambda \cap m_\mu \cong V \oplus V^*$;*

(ii) *$\tilde{S}_{\lambda\mu}$ is a s.g.p. for the linear action $(L_\lambda \cap L_\mu, m_\lambda \cap m_\mu)$;*

(iii) *$S_{\lambda\mu}$ is a s.g.p. for the linear action $(L'_\lambda \cap L'_\mu, m_\lambda \cap m_\mu)$;*

(iv) *The commutator subgroups of $\tilde{S}_{\lambda\mu}$, $S_{\lambda\mu}$ coincide and* $\dim \tilde{S}_{\lambda\mu} - \dim S_{\lambda\mu} = 2 + \tilde{c}(Z) - c(Z)$.

(v) *If $S_{\lambda\mu} \subset L'_\lambda \cap L'_\mu$ is the canonical embedding, corresponding to the action $(L'_\lambda \cap L'_\mu, V \oplus V^*)$, then the chain $S_{\lambda\mu} \subset L'_\lambda \cap L'_\mu \subset G$ gives us the canonical embedding, corresponding to the action $(G, Z \times Z^*)$.*

Proof. Part (i) is obvious.

(ii) To find a s.g.p. for $(L_\lambda, G/L_\mu)$, we consider the point $\tilde{x} = eL_\mu \in G/L_\mu$. Then $(L_\lambda)_{\tilde{x}} = L_\lambda \cap L_\mu$, hence, the orbit $L_\lambda \tilde{x}$ is closed, because $\operatorname{rk}(L_\lambda \cap L_\mu) = \operatorname{rk} L_\lambda$ and G/L_μ is affine. Therefore, a s.g.p. may be found from the slice-representation at $\tilde{x}$ [5]. Set $\ell_\lambda = \operatorname{Lie} L_\lambda$, $\ell_\mu = \operatorname{Lie} L_\mu$. Then $T_{\tilde{x}}(G/L_\mu) \cong \operatorname{Lie} G/\ell_\mu$, $T_{\tilde{x}}(L_\lambda \tilde{x}) \cong \ell_\lambda/\ell_\lambda \cap \ell_\mu$ as $L_\lambda \cap L_\mu$-modules. Therefore, the slice-module $N_{\tilde{x}}$ is of the form:

$$N_{\tilde{x}} \cong T_{\tilde{x}}(G/L_\mu)/T_{\tilde{x}}(L_\lambda \tilde{x}) \cong \operatorname{Lie} G/(\ell_\lambda + \ell_\mu) \cong (\ell_\lambda)^\perp \cap (\ell_\mu)^\perp = m_\lambda \cap m_\mu.$$

Thus, $\tilde{S}_{\lambda\mu}$ is a s.g.p. for $((L_\lambda)_{\tilde{x}}, N_{\tilde{x}}) = (L_\lambda \cap L_\mu, m_\lambda \cap m_\mu)$.

(ii) Similarly, let us consider the point $x = eL'_\mu \in G/L'_\mu$. Then $(L'_\lambda)_x = L'_\lambda \cap L'_\mu$. To prove, that $L'_\lambda x$ is closed we cannot proceed as in (ii), because $\operatorname{rk}(L'_\lambda)_x = \operatorname{rk} L'_\lambda - 1$, when $\dim \langle \lambda, \mu \rangle = 2$. But another way is not very difficult. By [7, ch. 1] G/L'_μ is equivariantly embedded in $R(\mu) \oplus R(\mu^*) = W$ as a *closed* G-orbit, $eL'_\mu \mapsto (v_\mu, v_{-\mu}) =: v$. Therefore, it is enough to prove that $L'_\lambda v$ is closed in W. But this easily follows from the Hilbert–Mumford criterium, because L'_λ is a regular subgroup of G and v has a very special form (cf. [7]). For the slice-module at x we have: $N_x \cong \operatorname{Lie} G/(\ell'_\lambda + \ell'_\mu) \cong m_\lambda \cap m_\mu + \Theta$, we have Θ is a trivial $L'_\lambda \cap L'_\mu$-module and $\dim \Theta = \dim \langle \lambda, \mu \rangle$.

(iv) The assertion about the commutator subgroups follows from the same assertion for $L_\lambda \cap L_\mu$ and $L'_\lambda \cap L'_\mu$. The dimension formula follows easily from theorem 1 and theorem 2(i).

(v) The inductive process of finding canonical points relative to a fixed maximal unipotent subgroup has been described in [7] (for linear representations) and in [8, ch. 1] (general case). We assume, that a fixed maximal unipotent subgroup of $L'_\lambda \cap L'_\mu$ is $L'_\lambda \cap L'_\mu \cap U$. In our situation the first two steps of this procedure transform consequently the action (G, Z) first into $(L'_\lambda, C(\mu^*))$ and then into $(L'_\lambda \cap L'_\mu, V)$. This means, that there is a $L'_\lambda \cap L'_\mu$-invariant closed subvariety in Z, which is isomorphic to V and if $v \in V$ is a canonical point for the action $(L'_\lambda \cap L'_\mu, V + V^*)$ then v is also a canonical point for the initial action. □

(1.10) *Remark.* The group $L'_\lambda \cap L'_\mu$ is connected and semisimple iff both λ and μ are fundamental weights [10]. In this case the Dynkin diagram of $L'_\lambda \cap L'_\mu$ is obtained from the Dynkin diagram of G by deleting the vertices corresponding to λ and μ.

2. Examples and applications

(2.1) In this chapter we shall show how to apply our results and present some classificational tables. We shall consider double cones only for pairs of fundamental weights. We enumerate them and the simple roots as in [10]. By $\tilde{\varphi}_i$ and α_i we denote the fundamental weights and the simple roots of a simple group G and by $\varphi_i, \varphi_i', \ldots$ – the fundamental weights of simple components of $L_{\tilde{\varphi}_i} \cap L_{\tilde{\varphi}_j}$; ε – is the exact one-dimensional representation of k^*. We shall write $Z(i,j), L_i, m_i, \ldots$ instead of $Z(\tilde{\varphi}_i, \tilde{\varphi}_j), L_{\tilde{\varphi}_i}, m_{\tilde{\varphi}_i}, \ldots$.

In this chapter we shall use the multiplicative notation for representations, i.e. we write $\varphi_i\varphi_j$ instead of $R(\varphi_i + \varphi_j)$ and 1 instead of $R(0)$, etc.

(2.2) EXAMPLE. $G = \mathrm{E}_8, Z = Z(1,1)$.

Here $(L_1', m_1) = (\mathrm{E}_7, 2\varphi_1 + 2)$. According to [1] we have Lie $S_{11} = \mathrm{D}_4$. Therefore $r = 4$, $2c + r = 12$ and $c = 4$. In particular, we find out that $\dim k[Z]^U = 8$.

Next, $(L_1, m_1) = (\mathrm{E}_7 \times k^*, \varphi_1 \otimes \varepsilon + \varphi_1 \otimes \varepsilon^{-1} + \varepsilon^2 + \varepsilon^{-2})$. Therefore Lie $\tilde{S}_{11} =$ Lie S_{11} and $\tilde{c} = c - 2 = 2$. According to the corollary 3(ii) the canonical embedding $S_{11} \subset E_8$ implies the embedding of the Dynkin diagrams. Here it can be done in a unique way. Hence, the canonical embedding is described by the following picture:

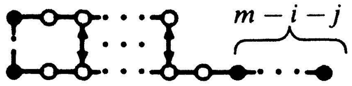
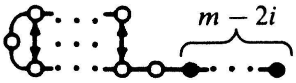
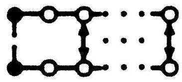
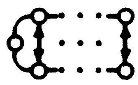
A/W

That is, $\alpha_4, \alpha_5, \alpha_6, \alpha_8$ is the system of simple roots of S_{11} and according to Corollary 3 $\Gamma(Z(1,1))$ is contained in $M = \langle \alpha_4, \alpha_5, \alpha_6, \alpha_8 \rangle^{\perp} = \langle \tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3, \tilde{\varphi}_7 \rangle$. This means, that for any n, m the tensor product decomposition $\tilde{\varphi}_1^n \otimes \tilde{\varphi}_1^m$ contains highest weights only from M.

(2.3) Now, we present the integers r, c, $\tilde{c}$ and Lie S_{ij} for any pair of fundamental weights for classical groups. Then $\tilde{r} = c + r - \tilde{c}$ and Lie $\tilde{S}_{ij} =$ Lie $S_{ij} + T_{2+\tilde{c}-c}$, where T_n denote the Lie algebra of an n-dimensional torus. In the column "Embedding" we indicate the canonical embedding of Lie S_{ij} in Lie G. The black vertices correspond to the simple roots of S_{ij} and the arrows show the embedding of the center of Lie S_{ij}. More exactly, each arrow gives us an element of a base of the center and the arrow, connecting the ith and jth vertices, presents the element $\alpha_i - \alpha_j$, lying in the center. Thus, the number of arrows is equal to the dimension of the center of S_{ij}. By the corollary 3 the obtained diagram indicates the structure of the rank semigroup. Namely, all weights from $\Gamma(Z(i,j))$ have zero labels on black vertices and equal labels on vertices, which are connected by an arrow. When Lie S_{ij} is semisimple, we can also indicate the simple roots of S_{ij} as a part of the set of simple roots of G, instead of a coloured diagram.

Since $\tilde{\varphi}_i, \tilde{\varphi}_j \in \Gamma(Z(i,j))$, we have the ith and jth vertices on the diagram of the canonical embedding are always white and without arrows. It is always assumed that $i \leq j$ and $m = \mathrm{rk}\, G$.

Table 1

N	Conditions	c	r	$\tilde{c}$	Lie S_{ij}	Embedding
1	$i<j$ $i+j\leq m$	0	$i+2$	0	$A_{j-i-1}\times A_{m-i-j}\times$ $\times T_{i-1}$	
2	$i=j$ $i+j\leq m$	1	$i+1$	0	$A_{m-2i}\times T_{i-1}$	
3	$i<j$ $i+j=m+1$	1	$i+1$	0	$A_{m-2i}\times T_{i-1}$	
4	$i=j=\frac{m+1}{2}$	2	i	0	T_{i-1}	

(2.4) $G=A_m$. We can assume here $i+j\leq m+1$. Actually, if $i+j>m+1$, then having replaced both weights $\tilde{\varphi}_i$, $\tilde{\varphi}_j$ on the dual ones ($\tilde{\varphi}_i^*=\tilde{\varphi}_{m+1-i}$), we get a double cone with $i+j\leq m+1$. Obviously, this procedure implies the same one on the rank semigroup and does not change its structure.

(2.5) $G=B_m, C_m$; $m\geq 2$. The representations $(L_i'\cap L_j', m_i\cap m_j)$ for these types of simple groups are distinct, nevertheless, it is found, a posteriori, that they have s.g.p. S_{ij}, $\tilde{S}_{ij}$ of the equal dimensions and ranks. Therefore, the integers c, r, $\tilde{c}$ for B_m and C_m coincide and we present the table for C_m only. To get the table for B_m one has to substitute always in the table 2 C_l on B_l.

Table 2

N	Conditions	c	r	$\tilde{c}$	Lie S_{ij}	Embedding
1	$i+j\leq m$ $1+2i\leq j$	$3\binom{i}{2}+1$	$3i+1$	$c-1$	$A_{j-2i-1}\times$ $\times C_{m-i-j}$	$\alpha_{i+1},\dots,\alpha_{j-i-1}$, $\alpha_{i+j+1},\dots,\alpha_m$
2	$i+j\leq m$ $i\leq j\leq 2i$	$2ij-\binom{i+1}{2}-$ $\binom{j+1}{2}+2$	$i+j$	$c-2$	C_{m-i-j}	$\alpha_{i+j+1},\dots,\alpha_m$
3	$i+j\geq m+1$ $1+2i\leq j$	$\binom{i}{2}+1+$ $(m-j)(2i+$ $j-1-m)$	$m+1+$ $2i-j$	$c-1$	A_{j-2i-1}	$\alpha_{i+1},\dots,\alpha_{j-i-1}$
4	$i+j\geq m+1$ $i\leq j\leq 2i$	$m^2-(m-i)^2-$ $(m-j)^2-$ $m-\binom{i}{2}-\binom{j}{2}+2$	m	$c-2$	0	

For the sake of completeness we indicate the slice-representation for C_m. Consideration of this representation allows us to determine Lie S_{ij} and to fill in the table.

$$(L_i' \cap L_j', m_i \cap m_j) = (A_{i-1} \times A_{j-i-1} \times C_{m-j}, V + V^*),$$

where $V = \varphi_1 \otimes \varphi_1' \otimes 1 + \varphi_1 \otimes 1 \otimes \varphi_1'' + \varphi_1^2 \otimes 1 \otimes 1$. We use the agreement that for

$$A_m \begin{cases} \varphi_1 = 0, & \text{if } m < 0 \\ \varphi_1 = 1, & \text{if } m = 0 \end{cases}, \qquad \text{and for } C_m \ \varphi_1 = 0, \quad \text{if } m = 0.$$

(2.6) $G = D_m, m \geq 4$.

In the table 3 we use the convention that $D_3 = A_3$, $D_2 = A_1 \times A_1$, $D_1 = T_1$. The slice-representation for D_m is of the form

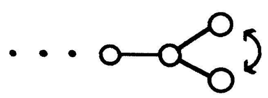
$$(L_i' \cap L_j', m_i \cap m_j) = (A_{i-1} \times A_{j-i-1} \times D_{m-j}, W + W^*),$$

and $W = \varphi_1 \otimes \varphi_1' \otimes 1 + \varphi_1 \otimes 1 \otimes \varphi_1'' + \varphi_2 \otimes 1 \otimes 1$.

In addition to the agreement from (3.5) we assume that for

$$A_m \begin{cases} \varphi_2 = 0, & \text{if } m < 1 \\ \varphi_2 = 1, & \text{if } m = 1 \end{cases}, \qquad D_m \begin{cases} \varphi_1 = \varepsilon + \varepsilon^{-1}, & \text{if } m = 1 \\ \varphi_1 = 0, & \text{if } m = 0 \end{cases}.$$

If $i + j = m - 1$, then Lie S_{ij} is not semisimple in n.1,2 in the table 3. In this case all vertices on the right-hand side of the Dynkin diagram are white and the last two vertices must be connected by an arrow:

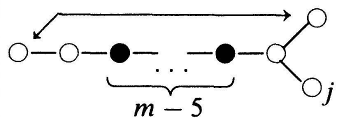
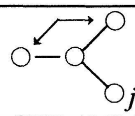


(2.7) Looking through these tables it is not difficult to pick out all pairs of fundamental weights with $\tilde{c}(Z(i,j)) = 0$. But we omit this simple consideration, because this result has already been obtained in [4]. Therefore we present here all pairs (i,j) with $\tilde{c} = 1$. This is a very short list:

- For B_m, C_m – $(2, m), m \geq 3$;
- For D_6 – $(4, 5), (4, 6)$.

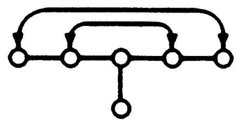
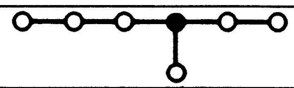
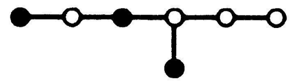
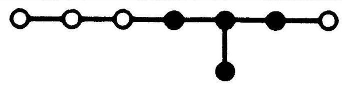
(2.8) For the exceptional simple groups we present the table of all pairs (i,j) with $\tilde{c} = 1, 2$. (The case $\tilde{c} = 0$ is described in [4].)

Table 3

N	Conditions		c	r	$\tilde{c}$	Lie S_{ij}	Embedding
1	$j \le m-2$	$i+j \le m$ $1+2i \le j$	$3\binom{i}{2}+1$	$3i+1$	$c-1$	$A_{j-2i-1} \times$ $\times D_{m-i-j}$	$\alpha_{i+1}, \ldots, \alpha_{j-i-1}$, $\alpha_{i+j+1}, \ldots, \alpha_m$
2		$i+j \le m$ $i \le j \le 2i$	$2ij+2-$ $\binom{i+1}{2}-\binom{j+1}{2}$	$i+j$	$c-2$	D_{m-i-j}	$\alpha_{i+j+1}, \ldots, \alpha_m$
3		$i+j \ge m+1$ $1+2i \le j$	$\binom{i-1}{2}+$ $(m-j)(2i+$ $j-m)$	$m+1+$ $2i-j$	$c-1$	A_{j-2i-1}	$\alpha_{i+1}, \ldots, \alpha_{j-i-1}$
4		$i+j \ge m+1$ $i \le j \le 2i$	$m^2-(m-i)^2-$ $(m-j)^2-$ $\binom{i+1}{2}-$ $\binom{j+1}{2}+2$	m	$c-2$	0	
5	$j = m-1$	$2i+1 \le m$ $i \ne 2$	$\binom{i-1}{2}$	$2i+1$	c, if $i=1$;	A_{j-2i-1}	$\alpha_{i+1}, \ldots, \alpha_{m-i-1}$
6	or	$2i+1 \le m$ $i=2$	1	4	$c-1$, if $i>1$	$A_{m-5} \times T_1$	
7	$j = m$	$i=2$ $m=4$	2	3	0	T_1	
8		$\frac{m}{2} \le i \le m-2$ $m \ge 5$	$m^2-(m-i)^2-$ $\binom{i+1}{2}-$ $\binom{m+1}{2}+2$	m	$c-2$	0	
9	$i=j=m-1$		$2, m=2l$	l	0	$(A_1)^l$	$\alpha_2, \alpha_4, \ldots, \alpha_{2l-2}, \alpha_j$
	or $i=j=m$		$1, m=2l+1$	$l+1$			$\alpha_2, \alpha_4, \ldots, \alpha_{2l}, \alpha_{2l+1}$
10	$i=m-1$		$1, m=2l$	$l+1$	0	$(A_1)^{l-1}$	$\alpha_1, \alpha_3, \ldots, \alpha_{2l-1}, \alpha_{2l}$
	$j=m$		$0, m=2l+1$			$(A_1)^l$	$\alpha_2, \alpha_4, \ldots, \alpha_{2l}, \alpha_{2l+1}$

This result is obtained with the aid of the case-by-case considerations. The E_8-case was elaborated in 2.2. Now we consider, for instance, the case E_7, $(1, 2)$. Here $L_1' = E_6$, $L_2' = D_5 \times A_1$ and $(L_1' \cap L_2', m_1 \cap m_2) = (D_5, \varphi_4 + \varphi_5 + 2\varphi_1)$. According to [1] Lie $S_{12} = A_1$, therefore $r = 6, c = 3$. For the determination of $\tilde{S}_{12}$ we have the representation $(L_1 \cap L_2, m_1 \cap m_2) = (D_5 \times (k^*)^2, \varphi_4 \otimes \varepsilon + \varphi_5 \otimes \varepsilon^{-1} + \varphi_1 \otimes \xi + \varphi_1 \otimes \xi^{-1})$. Hence, $\tilde{S}_{12} = A_1$ and $\tilde{c} = c - 2 = 1$.

Table 4

Group	weights	c	r	$\tilde{c}$	Lie S_{ij}	Embedding
E_6	(6, 6)	4	4	2	T_2	
E_7	(1, 2)	3	6	1	A_1	
	(6, 6)	4	4	2	$(A_1)^3$	
E_8	(1, 1)	4	4	2	D_4	
F_4	(1, 1)	4	4	2	0	
	(1, 4)	4	4	2	0	
	(4, 4)	4	4	2	0	
G_2	(1, 1)	4	2	2	0	
	(1, 2)	4	2	2	0	
	(2, 2)	4	2	2	0	

The relation Lie $S_{12} = A_1$ means that the Dynkin diagram of the canonical embedding has to contain a single black vertex. To determine the precise position of this vertex, we need a more careful analysis. There are two ways to do it.

1. If the black vertex has the number i, then by the corollary 3 $\tilde{\varphi}_i$ does not appear in $\Gamma(Z(1, 2))$. Therefore, by finding some first decompositions one has a chance to determine i. Actually, one may check, that $\tilde{\varphi}_1 \otimes \tilde{\varphi}_2 \supset \tilde{\varphi}_3 + \tilde{\varphi}_7 + \tilde{\varphi}_1 \tilde{\varphi}_6$ and $\tilde{\varphi}_1^2 \otimes \tilde{\varphi}_2 \supset \tilde{\varphi}_5$. Therefore, the only possibility is $i = 4$.

2. The more conceptual way is to apply the theorem 3(v). But this is a rather cumbersome procedure, beacuse we need to introduce notations for the simple roots of subgroups appearing on each step and we have to identify these roots with those of E_7. So we shall omit the details.

(2.9) In this chapter we have used Lie S_{ij} only. In a subsequent paper we shall show how the application of S_{ij} itself enables us to use the restriction theorem for U-invariants and to derive the explicit description of a series of non-free algebras of the form $k[Z(\lambda, \mu)]^U$.

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ul. akad. Anokhina
d.30, kor. 1, kv. 7
117602 Moscow, Russia

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