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Sharp borderline Sobolev inequalities on compact Riemannian manifolds

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1. Introduction and main results

In a 1971 paper [22], J. Moser proved the following theorem:

THEOREM 1.1 (Moser). *Let Ω be an open domain in $\mathbf{R}^n$, $n \geq 2$. There exists a constant C which depends only on n such that if u is smooth, has compact support contained in Ω and its gradient ∇u satisfies $\int_\Omega |\nabla u|^n \, dx \leq 1$, then*

$$\int_\Omega \exp\{\lambda(n)|u|^{n/(n-1)}\} \, dx \leq C|\Omega| \tag{1}$$

where $\lambda(n) = n\omega_{n-1}^{1/(n-1)}$ and ω_{n-1} is the surface measure of the unit sphere in $\mathbf{R}^n$. If $\lambda(n)$ is replaced by any $\lambda > \lambda(n)$, the integral on the left hand side of (1) is still finite, but can be made arbitrarily large by an appropriate choice of u.

And, with a modification of the same argument,

THEOREM 1.2 (Moser). *There exists an absolute constant c_0 such that if u is a smooth function on S^2 with $\int_{S^2} |\nabla u|^2 \, dS \leq 1$ and $\int_{S^2} u \, dS = 0$, then*

$$\int_{S^2} e^{4\pi u^2} \, dS \leq c_0. \tag{2}$$

The constant 4π is the best possible in the same sense as $\lambda(n)$ in Theorem 1.1.

Moser applied Theorem 1.2 in his work on the problem of prescribing the curvature on S^2 (see [23]).

Recall that Sobolev's theorems, see e.g. [29], assert existence of imbeddings $W_0^{1,p}(\Omega) \to L^q(\Omega)$ for $1 < p < n$ and $W_0^{1,p}(\Omega) \to C_0(\Omega)$ for $n < p$, where $1/q = 1/p - 1/n$. Thus, Theorem 1.1 represents a sharp way to fill in the gap at the critical exponent $p = n$. Theorem 1.2 plays the same role for the Sobolev Theorems on S^2.

In 1988 D. R. Adams proved a generalized version of Moser's Theorem 1.1, in which it is assumed control on the L^p norm of a higher order gradient of f.

THEOREM 1.3 (Adams). *Let Ω be a domain in $\mathbf{R}^n$ and m be a positive integer strictly smaller than n. There is a constant C, depending only on m and n, with the following property:*

If $u \in C^m(\mathbf{R}^n)$ has compact support contained in Ω and $\|\nabla^m u\|_{n/m} \leq 1$, then

$$\int_\Omega \exp\{\lambda(m,n)|u(x)|^{n/(n-m)}\}\,dx \leq C|\Omega| \tag{3}$$

where

$$\lambda(m,n) = \begin{cases} \dfrac{n}{\omega_{n-1}}\left[\dfrac{\pi^{n/2}2^m\Gamma\left(\dfrac{m+1}{2}\right)}{\Gamma\left(\dfrac{n-m+1}{2}\right)}\right]^{n/(n-m)} & \text{if } m \text{ is odd} \\[2ex] \dfrac{n}{\omega_{n-1}}\left[\dfrac{\pi^{n/2}2^m\Gamma(m/2)}{\Gamma\left(\dfrac{n-m}{2}\right)}\right]^{n/(n-m)} & \text{if } m \text{ is even} \end{cases} \tag{4}$$

and ∇^m denotes the iterated Laplacian $\Delta^{m/2}$ if m is even and $\nabla\Delta^{(m-1)/2}u$ if m is odd.

If $\lambda(m,n)$ is replaced by any larger number, the integral in (3) cannot be bounded uniformly by any constant.

Adams' approach to the problem is to express u as the Riesz Potential of its gradient of order m and then apply the following theorem

THEOREM 1.4 (Adams). *Let $1 < p < \infty$. There is a constant $c_0 = c_0(p,n)$ such that for all $f \in L^p(\mathbf{R}^n)$ with support contained in Ω,*

$$\int_\Omega \exp\left\{n\omega_{n-1}\left|\frac{I_\alpha f(x)}{\|f\|_p}\right|^{p'}\right\}dx \leq c_0|\Omega| \tag{5}$$

where $\alpha = n/p$, $1/p + 1/p' = 1$, and $I_\alpha f(x) = \int |x-y|^{\alpha-n}f(x)\,dy$. The constant n/ω_{n-1} cannot be replaced by any larger number without forcing c_0 to depend on f as well as on p and n.

To prove Theorem 1.4, Adams applied a result of R. O'Neil [24] about "convolution type" operators.

LEMMA 1.5 (O'Neil). *If $h = f * g$ then*

$$h^{**}(t) \le t f^{**}(t) g^{**}(t) + \int_t^\infty f^*(s) s^*(s)\, ds.$$

(By f^* we mean the usual non increasing rearrangement of $|f|$ and f^{**} is defined by $f^{**}(t) = 1/t \int_0^t f^*(s)\, ds$.)

Then, a change of variables reduces the problem to the following technical lemma.

LEMMA 1.6 (Adams). *Let $a(s, t)$ be a non negative measurable function defined on the set $(-\infty, \infty) \times [0, \infty)$ such that $a(s, t) \le 1$ a.e. when $0 < s < t$, and suppose that*

$$\sup_{t>0} \left(\int_{-\infty}^0 + \int_t^\infty a(s, t)^{p'}\, ds \right) = b^{p'} < \infty.$$

Then there is a constant $c_0 = c_0(p, b)$ such that, for all $\phi \ge 0$ satisfying $\int_{-\infty}^\infty \phi(s)^p\, ds \le 1$, the following inequality is true:

$$\int_0^\infty e^{-F(t)}\, dt \le c_0$$

where $F(t) = t - (\int_{-\infty}^\infty a(s, t)\phi(s)\, ds)^{p'}$.

There have been attempts to prove Moser–Adams type theorems for spheres of dimension $n \ge 3$ and for more general manifolds. For example, contributions have been made by Cherrier [14], [15], [16]. However, the arguments employed in the past failed to yield the largest coefficient $\lambda(m, n)$ in the exponential, even in the first order case $m = 1$.

In this paper we will show that the complete analogues of the Adams and Moser theorems are valid for every compact (smooth) Riemannian manifold M. In fact, the optimal λ's turn out to be the same for every such M as they are for domains in $\mathbf{R}^n$.

Our main result is the following

THEOREM 1.7. *Let M be a compact Riemannian manifold of dimension n and m a positive integer strictly smaller than n. There exists a constant $C = C(m, M)$ such* *that for all* $u \in C^n(M)$ *with* $\int_M u\, dV = 0$ *and* $\int_M |\nabla^m u|^{n/m}\, dV \le 1$, *the following uniform inequality holds*

$$\int_M \exp\{\lambda(m, n)|u(x)|^{n/(n-m)}\}\, dV(x) \le C \tag{6}$$

where the constant $\lambda(m, n)$ *is the one given in* (4) *and is sharp in the same sense.* ∇ *and* Δ *represent the gradient and the Laplace–Beltrami operator relative to the metric of* M, *so that, in a coordinate neighborhood,*

$$\nabla f = \sum_{i,j} (g^{ij}\partial_j f)\partial_i \quad \textit{and} \quad \Delta f = -\frac{1}{\sqrt{g}} \sum_{i,j} \partial_i (g^{ij}\sqrt{g}\, \partial_i f)$$

where g^{ij} *are the coefficients of* G^{-1}, $g = \det G$ *and* $G = (g_{ij})$ *is the metric tensor of* M.

We observe that, in the theorem above, as well as in Theorems 1.1, 1.2 and 1.3, the condition $u \in C^m(M)$ can be replaced by the slightly weaker condition $u \in W^{m,n/m}(M)$, where, as customary, $W^{m,n/m}(M)$ is the Sobolev space obtained by completion of $C^m(M)$ with respect to the norm $\|f\|_{m,n/m} = (\int_M \Sigma_{k=0}^m |\nabla^k f|^{n/m})^{m/n}$ or any of its equivalents. The proof is straightforward.

Theorem 1.7 was originally stated and proved for m even. While generalizing the proof to the case of m odd, the author was informed by Professor S-Y. A. Chang that, at about the same time, she, T. Branson and P. C. Yang had independently proved the special case corresponding to $n = 4$ and $m = 2$ by similar methods (see [9]).

Our outline for proving Theorem 1.7 is the same as that of Adams for his Theorem 1.3. Firstly, we formulate an appropriately modified version of O'Neil's lemma and an extended version of Lemma 1.6. These are Lemmas 3.1 and 3.2 in Section 3. They enable us to prove Theorem 1.9, which is an analogue on M of Adams' sharp fractional integral result Theorem 1.4, and is perhaps of independent interest. Theorem 1.7 follows from Theorem 1.9, a convolution type representation of u in terms of its gradients, and a precise estimate of the kernels given in Theorem 1.8.

The representation formulas we need are in terms of the Green's function G of M.

Let M be a compact Riemannian manifold of dimension n having volume V. The Green's function $G(P, Q)$ of M is a function which, as a distribution on M, satisfies the equation

$$\Delta_Q G(P, Q) = \delta_P(Q) - \frac{1}{V}$$

where Δ_Q is the distributional Laplacian of M with respect to the variable Q, δ_P is the Dirac measure at P and V is the volume of M.

It can be proved (see [3]) that the Green's function of M exists, can be normalized so that $\int_M G(P, Q)\, dV(Q) = 0$ for every $P \in M$, and enjoys the standard properties of Green's functions.

In particular, the following formula

$$u(P) = \int_M G(P, Q)\, \Delta u(Q)\, dV(Q) = \int_M \nabla_Q G(P, Q) \cdot \nabla u(Q)\, dV(Q)$$

is valid for u regular on M and satisfying $\int_M u\, dV = 0$.

Now define functions G_m for m even and vector fields G_m for m odd by

$$G_{2k}(P, Q) = \int_M G_{2(k-1)}(R, Q) G(P, R)\, dV(R)$$

for $k \geq 2$, with $G_2(P, Q) = G(P, Q)$, and

$$G_{2k-1}(P, Q) = \nabla_Q G_{2k}(P, Q)$$

for $k \geq 1$. Observe that

$$\Delta_Q G_{2k}(P, Q) = G_{2k-2}(P, Q) = \Delta_P G_{2k}(P, Q)$$

and, by Fubini's theorem,

$$G_{2k}(P, Q) = G_{2k}(Q, P) \quad \text{and} \quad \int_M G_{2k}(P, Q)\, dV(Q) = 0.$$

If u is a smooth function on M and $\int_M u\, dV = 0$, then by Green's formulas we have

$$u(P) = \int_M G_{2k}(P, Q)\, \Delta^k u(Q)\, dV(Q) \tag{7}$$

and

$$u(P) = \int_M G_{2k-1}(P, Q) \cdot \nabla^{2k-1} u(Q) \, dV(Q). \tag{8}$$

These are the representation formulas. We need the estimates of the kernels $G_m(P, Q)$ given by the following theorem

THEOREM 1.8. *There are constants $B_1, \ldots, B_{n-1}$ depending only on M, such that for every $P, Q \in M$ and $1 \le m \le n-1$*

$$|G_m(P, Q)| \le c_m(n) d(P, Q)^{m-n}(1 + B_m d(P, Q)^{1/2})$$

where

$$c_m(n) = \begin{cases} \dfrac{\Gamma\left(\frac{n-m}{2}\right)}{2^{m-1}\left(\frac{m}{2}-1\right)!\,\Gamma\left(\frac{n}{2}\right)\omega_{n-1}} & \text{if } m \text{ is even} \\[2ex] \dfrac{\Gamma\left(\frac{n-m+1}{2}\right)}{2^{m-1}\left(\frac{m-1}{2}\right)!\,\Gamma\left(\frac{n}{2}\right)\omega_{n-1}} & \text{if } m \text{ is odd.} \end{cases}$$

The cases $m = 1$ and $m = 2$ of this theorem follow from Aubin's basic analysis of the Green's function of a compact Riemannian manifold (see Theorem 2.5 and [2]). The cases $m > 2$ are deduced from the previous ones by using Rauch's comparison theorem (see Theorem 2.3 and [13]).

The above estimates reduce the proof of Theorem 1.7 to the proof of the following result which perhaps is of independent interest.

THEOREM 1.9. *Let M be a n-dimensional compact Riemannian manifold, $n \ge 1$. Let T be the operator defined by*

$$Tf(P) = \int_M K(P, Q) f(Q) \, dV(Q)$$

where

$$K(P, Q) = d(P, Q)^{\alpha - n}(1 + ad(P, Q)^{\beta}),$$

a is a non negative constant, $\beta > 0$, $0 < \alpha < n$ and d is the Riemannian distance. Then there exists a constant $c = c(\alpha, \beta, a, M)$ such that, for every function $f \in L^{n/\alpha}(M)$ satisfying $\int_M |f|^{n/\alpha}\, dV \le 1$, the following inequality holds

$$\int_M \exp\left\{\frac{n}{\omega_{n-1}} |Tf(P)|^{n/(n-\alpha)}\right\} dV(P) \le c. \tag{9}$$

The number n/ω_{n-1} is the largest possible constant for which the integral on the left hand side of inequality (9) *is bounded by a constant independent of f.*

Clearly Theorem 1.8 and Theorem 1.9 imply Theorem 1.7.

We shall prove Theorem 1.8 in Section 2 and Theorem 1.9 in Section 3. In Section 4 we prove analogue exponential integrability for fractional integrals of periodic functions and mention a few open problems.

2. The estimates for G_m

The crucial point in the proof of Theorem 1.8 is to reduce a certain integral on M involving Riemannian distances to an analogous integral on $\mathbf{R}^n$ involving ordinary Euclidean distances.

It is well known from potential theory and harmonic analysis, that, in $\mathbf{R}^n$, the following formula holds:

$$\int_{\mathbf{R}^n} |x - y|^{\alpha - n} |y|^{\beta - n}\, dy = \frac{\gamma(\alpha)\gamma(\beta)}{\gamma(\alpha + \beta)} |x|^{\alpha + \beta - n} \tag{10}$$

where α, $\beta > 0$, $\alpha + \beta < n$ and

$$\gamma(\alpha) = \pi^{n/2} 2^{\alpha} \frac{\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)}. \tag{11}$$

See [26], page 118.

To prove Theorem 1.8, we need a version of this formula valid on compact manifolds M.

Recall that $d(P, Q)$ denotes the Riemannian distance on the compact manifold M.

LEMMA 2.1. *Suppose that* $\alpha, \beta > 0$ *and that* $\alpha + \beta < n$. *Then*

$$\int_M d(P, R)^{\alpha - n} d(R, Q)^{\beta - n} \, dV(R) = \frac{\gamma(\alpha)\gamma(\beta)}{\gamma(\alpha + \beta)} d(P, Q)^{\alpha + \beta - n}(1 + E(P, Q)) \tag{12}$$

where $|E(P, Q)| \leq B d(P, Q)^{1/2}$, *with* B *constant depending only on* α, β, *and* M.

To prove this lemma we need to recall some results from Riemannian geometry.

PROPOSITION 2.2 (Volume form). *Let* $B_P(t)$ *be the geodesic ball of center* P *and radius* t *in* M, *i.e. the set of points in* M *at a distance from the point* P *smaller than* t. *Then*:

$$\mathrm{Vol}\,(B_P(t)) = \frac{\omega_{n-1}}{n} t^n \left(1 - \frac{1}{6(n+2)} s(P) t^2 + o(t^2)\right)$$

where $s(P)$ *is the scalar curvature at* P.

Moreover, in normal geodesic coordinates around P, *the volume form of* M *is*

$$dV(Q) = t^{n-1}\left(1 - \frac{1}{3} r(x) t^2 + o(t^2)\right) dx\, dt$$

where $Q = \exp_P tx$, $x \in S^{n-1}$ *and* dx *is the standard surface measure of* S^{n-1}. $r(x)$ *is the Ricci curvature (viewed as a quadratic form) evaluated on the vector* x.

For references and proofs see [7] pages 15–16 and [19].

All we really need, however, is

$$dV(Q) = t^{n-1}(1 + O(t))\, dx\, dt \quad \text{and} \quad \mathrm{Vol}\,(B_P(t)) = \frac{\omega_{n-1}}{n} t^n (1 + O(t))$$

which, for M compact, follows at once from Proposition 2.2.

The next result is the basic tool which allows us to estimate the quantity $d(R, Q)$ in terms of Euclidean distances.

THEOREM 2.3 (Rauch Comparison Theorem). *Let* M, M_0 *be Riemannian manifolds with* $\dim M_0 \geq \dim M$, *and let* $P(P_0) \in M(M_0)$.

Assume that for all plane sections $\sigma(\sigma_0)$ *of* $M(M_0)$, *the sectional curvatures satisfy* $K(\sigma_0) \geq K(\sigma)$. *Let* r *be chosen such that the exponential map of* M *with center at* P, $\exp_P$ *restricted to the ball* $B(r)$ *with radius* r *and center at the origin in* $T_P M$ *is an embedding and the exponential map of* M_0 *with center at* P_0, $\exp_{P_0}$, *is not singular on the corresponding ball* $B(r)$ *in* $T_{P_0}M_0$. *Let* $I : T_P M \to T_{P_0}M_0$ *be a linear injection preserving inner products. Then for any curve* $c : [0, 1] \to \exp_P(B(r))$, *defining the corresponding curve* c_0 *in* M_0 *by*

$$c_0(t) = \exp_{P_0} \circ I \circ \exp_P^{-1}(c)(t)$$

we have $L[c] \geq L[c_0]$ *where by* $L[\gamma]$ *we denote the length of the curve* γ.

For references and proofs, see [13], pages 30–31.

As an immediate consequence of this theorem, we see that if

$$Q_0 = \exp_{P_0} \circ I \circ \exp_P^{-1}(Q) \quad \text{and} \quad R_0 = \exp_{P_0} \circ I \circ \exp_P^{-1}(R)$$

with Q, R points in the ball of radius r in M, Q_0, R_0 points in M_0, then the geodesic joining Q_0 to R_0 in M_0 is not shorter than the geodesic joining Q to R in M.

We also need the following easy lemma which will take care of most of the error terms we will be producing throughout the paper.

LEMMA 2.4. *If* $0 < \alpha, \beta < n$, *the integral*

$$I(P, Q) = \int_M d(P, R)^{\alpha - n}\, d(R, Q)^{\beta - n}\, dV(R)$$

is bounded if $\alpha + \beta > n$, $O(d(P, Q)^{\alpha + \beta - n})$ *if* $n > \alpha + \beta$ *and* $O(\log(1/d(P,Q)))$ *if* $n = \alpha + \beta$.

The proof uses Proposition 2.2 and is straightforward.

We are now ready to prove Lemma 2.1.

Proof of Lemma 2.1. Since M is compact, certainly its sectional curvature is bounded above by a positive constant k_1 and below by a negative constant k_2, i.e. at each point $P \in M$, for every plane section σ, we have

$$k_2 \le K(\sigma) \le k_1.$$

Consider the n-dimensional sphere S of curvature k_1 and the n-dimensional hyperbolic space H of curvature k_2. Let d_S and d_H be their respective Riemannian distances. Denote by exp, exp$'$, and exp$''$ the exponential maps of M, S, and H respectively.

Let δ be $1/2$ of the smallest of the radii of injectivity of M, S and H (since M and S are compact their radii of injectivity are strictly positive, while H's one is infinite). Then, if P, P', P'' are points respectively in M, S, H, the exponential map of each of our manifolds is a diffeomorphism from the ball with center at the origin of the tangent space at P (res. P', P'') and radius δ, to the ball of the same radius around the point P (res. P', P'') in the manifold.

Consider normal geodesic coordinates on $B_P(\delta)$, $B_{P'}(\delta)$ and $B_{P''}(\delta)$ in M, S and H respectively. It is a feature of these coordinates that the tangent space at the center is isometric to the standard n-dimensional Euclidean space. So, by choosing orthonormal basis in T_PM, $T_{P'}S$ and $T_{P''}H$, we can identify all these spaces with the standard $\mathbf{R}^n$.

If R and Q are two points in $B_P(\delta) \subset M$, consider their normal geodesic coordinates y and x (points in $T_PM \simeq \mathbf{R}^n$), uniquely determined by $R = \exp_P(y)$ and $Q = \exp_P(x)$. Construct the points $R' = \exp'_{P'}(y)$ and $Q' = \exp''_{P''}(x)$ in S, and their analogues $R'' = \exp''_{P''}(y)$ and $Q'' = \exp''_{P'}$ in H.

The Rauch Comparison Theorem 2.3 implies that

$$d_S(R', Q') \le d(R, Q) \le d_H(R'', Q''). \tag{13}$$

In other words, we have geodesic triangles PQR in M, $P'Q'R'$ in S and $P''Q''R''$ in H, which, by the properties of the exponential map, satisfy

$$d(P, Q) = d_S(P', Q') = d_H(P'', Q'') = |x|$$
$$d(P, R) = d_S(P', R') = d_H(P'', R'') = |y|$$

and

$$\theta = R\widehat{P}Q = R'\widehat{P'}Q' = R''\widehat{P''}Q''$$

where θ is the Euclidean angle between the vectors y and x. Then, the "third sides" of the triangles satisfy (13).

We want to estimate $d(R, Q)$ in terms of $|y - x|$, the Euclidean distance between their normal geodesic coordinates. Inequality (13) reduces the problem to an estimate of distances on a sphere or a hyperbolic space, where we can use trigonometry.

We use spherical trigonometry on the triangle $P'R'Q'$, obtaining

$$\cos \mu d_S(R', Q') = \cos \mu|x| \cos \mu|y| + \cos \theta \sin \mu|x| \sin \mu|y|$$

where μ is a scaling constant depending only on k_1.

Standard algebra and trigonometry show that the above equation implies

$$d_S(R', Q') = |x-y|(1 + O(|x|+|y|)^2). \tag{14}$$

In the same way, using hyperbolic trigonometry on the triangle $R''P''Q''$, we have

$$\cosh v d_H(R'', Q'') = \cosh v|y| \cosh v|x| - \cos \theta \sinh v|y| \sinh v|x|$$

where v is a scaling factor depending only on the curvature of H k_2.

Again, after some algebra, we get

$$d_H(R'', Q'') = |x-y|(1 + O(|x|+|y|)^2). \tag{15}$$

As a consequence of (13), (14) and (15) we have

$$d(R, Q) = |x-y|(1 + O(|x|+|y|)^2) \tag{16}$$

where the quantity $O(|x|+|y|)^2/(|x|+|y|)^2$ is bounded by a constant depending only on M.

We can now estimate

$$\int_M d(P, R)^{\alpha-n} d(R, Q)^{\beta-n}\, dV(Q)$$

and conclude the proof of Lemma 2.1.

Decompose the integral over M into $\int_{B_P(\delta)} + \int_{M - B_P(\delta)}$. Since $d(R, Q)^{\beta-n}$ is integrable on M, the second integral is bounded by a constant, depending only on M and β, times $\delta^{\alpha-n}$.

To estimate

$$\int_{B_P(\delta)} d(P, R)^{\alpha-n} d(R, Q)^{\beta-n}\, dV(Q)$$

we write it in normal geodesic coordinates around P, and using the notations introduced above, we obtain the following integral in $\mathbf{R}^n$

$$\int_{B(\delta)} |y|^{\alpha-n}\phi(x, y)^{\beta-n}(1 + O(|y|))\, dy$$

where $\phi(x, y)$ is the coordinate expression of $d(R, Q)$. By (16), we get:

$$\int_{B(\delta)} |y|^{\alpha-n}|x - y|^{\beta-n}(1 + O(|x| + |y|)^2)(1 + O(|y|))\, dy.$$

Now, using the $\mathbf{R}^n$ version of Lemma 2.4, the fact that

$$\int_{\mathbf{R}^n - B(\delta)} |y|^{\alpha-n}|x - y|^{\beta-n}\, dy$$

is a bounded function of $|x|$ with the bound depending only on α, β and the geometric quantity δ, by recalling that $|x| = d(P, Q)$ and using formula (10), we get equality (12). Therefore Lemma 2.1 is completely proved.

Now we are ready to prove Theorem 1.8.

The first step was essentially done by T. Aubin in a 1974 paper [2]. He proved the following series representation for the Green's function of M:

THEOREM 2.5 (Aubin). *If M is a compact Riemannian manifold of dimension $n > 2$, its Green's function is given by*:

$$G(P, Q) = H(P, Q) + \int_M H(P, R) \sum_{i=1}^{\infty} K_i(Q, R)\, dV(R) \tag{17}$$

where $H(P, Q) = [(n-2)\omega_{n-1}]^{-1} t(d(P, Q))^{2-n}$ and $t(s)$ is a C^∞ non decreasing function defined on $(0, \infty)$ such that $\lim_{s\to 0^+}(t(s)/s) = 1$ and t is constant for $s \geq \delta$, with δ smaller than the radius of injectivity of M. K_i is defined inductively:

$$K_i(P, Q) = \int_M K_{i-1}(P, R) K_1(R, Q)\, dV(R)$$

where $K_1(P, Q) = -\Delta_Q H(P, Q) - 1$.

If $f(Q) = \phi(d(P, Q))$ is a function of the distance from P alone, then it is well known that:

$$\Delta f(Q) = -\frac{d^2\phi}{dr^2} - \frac{d\phi}{dr}\left[\frac{\partial}{\partial r}\log\sqrt{\det(g_{ij})} + \frac{n-1}{r}\right]$$

where r is the Riemannian distance from P to Q, and the g_{ij} are the coefficients of the metric (see [6] for instance).

By Aubin's Theorem 1 in the same paper, there exists r_0 depending only on the geometry of M such that for every $P \in M$ in $B_P(r)$, the ball of radius r and center P, $r \le r_0$, we have

$$\left|\frac{\partial}{\partial r}\log\sqrt{\det(g_{ij})}\right| \le Cr$$

the constant C depending only on M and not on P and r.

The quantity $|(\partial/\partial r)\log\sqrt{\det(g_{ij})}|$ is also globally bounded on M.

Using these facts we can easily estimate

$$H(P, Q) = H(r) = \frac{1}{(n-2)\omega_{n-1}} t^{2-n}(r) = \frac{1}{(n-2)\omega_{n-1}} r^{2-n}(1 + O(r))$$

and

$$\begin{aligned} -\Delta H(P, Q) &= \frac{n-1}{\omega_{n-1}} t^{-n}(r) t'(r)\left[t'(r) - \frac{t(r)}{r}\right] \\ &\quad - \frac{t^{1-n}(r)}{\omega_{n-1}}\left[t''(r) + t'(r) + \frac{\partial}{\partial r}\log\sqrt{\det(g_{ij})}\right]. \end{aligned}$$

Since $t(r)$ is C^∞, for small r, we have $t(r) = t'(0)r + O(r^2)$ and $t'(r) = 1 + t''(0)r + O(r^2)$ and thus

$$t^{-n}(r)t'(r)\left[t'(r) - \frac{t(r)}{r}\right] = O(r^{1-n}).$$

Hence $-\Delta H(p, Q)$ and K_1 are $O(d(P, Q)^{1-n})$ and therefore, for all m, $K_m(P, Q)$ is at least $O(d(P, Q)^{1-n})$ (much better, in fact).

Since Aubin proved that the series $\Sigma_{i=\lambda}^{\infty} K_i(P, Q)$ is absolutely convergent and bounded for λ suitable (depending only on M), we can conclude that

$$G_2(P, Q) = G(P, Q) = \frac{1}{(n-2)\omega_{n-1}} d(P, Q)^{2-n}[1 + O(d(P, Q))]. \tag{18}$$

Moreover, since

$$\nabla_P G(P, Q) = \nabla_P H(P, Q) + \int_M \nabla_P H(P, R) \sum_{i=1}^{\infty} K_i(R, Q)\, dV(R)$$

and

$$|\nabla_P H(P, Q)| = |\nabla_Q H(P, Q)| = \frac{1}{\omega_{n-1}} |t^{1-n}(r) t'(r)| = \frac{1}{\omega_{n-1}} r^{1-n}(1 + O(r))$$

we have

$$|G_1(P, Q)| = |\nabla_Q G(P, Q)| \leq \frac{1}{\omega_{n-1}} d(P, Q)^{1-n}[1 + O(d(P, Q))]. \tag{19}$$

In the above discussion, we assumed $\dim M > 2$. If $\dim M = 2$, G can still be represented by the series

$$H(P, Q) + \int_M H(P, R) \sum_{i=1}^{\infty} K_i(R, Q)\, dV(R)$$

where, this time, $H(P, Q) = (1/2\pi) \log(1/t(d(P, Q)))$.

The same kind of estimates as before, yields

$$|\nabla_Q G(P, Q)| \leq \frac{1}{2\pi} d(P, Q)^{-1}[1 + O(d(P, Q))] \tag{20}$$

which is the only kernel estimate we need when $n = 2$.

From now on, $n = \dim M > 2$.

We shall now prove the "even" part of Theorem 1.8.

LEMMA 2.6. *If* $1 < 2k < n$

$$G_{2k}(P, Q) = \frac{\Gamma\left(\frac{n}{2} - k\right)}{2^{2k-1}(k-1)!\,\Gamma\left(\frac{n}{2}\right)\omega_{n-1}} d(P, Q)^{2k-n}(1 + O(d(P, Q)^{1/2})).$$

Proof. By definition, when $2 < 2k < n$,

$$G_{2k}(P, Q) = \int_M G(P, R) G_{2k-2}(R, Q)\, dV(R).$$

Since we already have the right estimate for G_2, we can proceed by induction and assume that

$$G_{2k-2}(P, Q) = c_{2k-2}(n) d(P, Q)^{2k-2-n}[1 + O(d(P, Q))]$$

so that

$$G_{2k}(P, Q) = c_{2k-2}(n) \frac{1}{(n-2)\omega_{n-1}} \int_M d(P, R)^{2-n} d(R, Q)^{2k-2-n}\, dV(R)$$
$$+ \int_M d(P, R)^{2-n} O(d(R, Q)^{2k-1-n})\, dV(R).$$

By Lemma 2.4, the last integral is no worse than $O(d(P, Q)^{2k+1/2-n})$. Lemma 2.1 now implies

$$G_{2k}(P, Q) = c_{2k-2}(n) \frac{1}{(n-2)\omega_{n-1}} \frac{\gamma(2)\gamma(2k-2)}{\gamma(2k)} d(P, Q)^{2k-n}(1 + O(d(P, Q)^{1/2})).$$

Recalling the definition of γ (see (11)), we find

$$c_{2k}(n) = \frac{c_{2k-2}(n)}{2(n-2k)(k-1)} \quad \text{and} \quad c_2(n) = \frac{1}{(n-2)\omega_{n-1}}. \tag{21}$$

By induction, this implies

$$c_{2k}(n) = \frac{1}{(k-1)!2^{k-1}(n-2k)(n-2k+2)\cdots(n-2)\omega_{n-1}}$$

$$= \frac{\Gamma\left(\frac{n}{2}-k\right)}{2^{2k-1}\Gamma\left(\frac{n}{2}\right)(k-1)!\omega_{n-1}}$$

and the lemma is proved.

Now we prove:

LEMMA 2.7. *If* $1 < 2k \leq n$ *then*

$$|G_{2k-1}(P, Q)| \leq \frac{\Gamma\left(\frac{n}{2}-k+1\right)}{2^{2k-2}(k-1)!\Gamma\left(\frac{n}{2}\right)\omega_{n-1}} d(P, Q)^{2k-1-n}(1 + O(d(P, Q)^{1/2})). \tag{22}$$

Proof. By definition, for $1 < 2k \leq n$

$$G_{2k}(P, Q) = \int_M G(P, R)G_{2k-2}(R, Q)\, dV(R). \tag{23}$$

Taking the gradient with respect to the variable P we get

$$\nabla_P G_{2k}(P, Q) = \int_M \nabla_P G(P, R)G_{2k-2}(R, Q)\, dV(R).$$

Now recall that

$$\nabla_P G(P, R) = \frac{1}{(n-2)\omega_{n-1}} \nabla_P(d(P, R)^{2-n}) + O(d(P, R)^{2-n}).$$

By estimate (19) and Lemma 2.4 we are reduced to estimate the following integral

$$\int_{B_P(\delta)} \nabla_R(d(P,R)^{2-n}) d(R,Q)^{2k-2-n}\, dV(R) \tag{24}$$

where $B_P(\delta)$ is the ball of radius δ around P in M.

As in the proof of Lemma 2.1, we rewrite our integral using normal geodesic coordinates around P. We already have the expression of the volume form and of $d(R,Q)$ in these coordinates. All we need is the expression of $\nabla_P(d(P,R)^{2-n}) = (2-n)d(P,R)^{1-n}\nabla_P d(P,R)$.

$\nabla_P d(P,R)$ is nothing else than the unit vector tangent, in P, to the geodesic from R to P. Let $y = (y_1, \dots, y_n) = |y|\sigma$ with $\sigma \in S^{n-1}$ and $R = \exp y$. Then $\nabla_P d(P,R) = -\sigma = -y/|y|$ and $d(P,R)^{2-n} = |y|^{2-n} = (y_1^2 + \cdots + y_n^2)^{(2-n)/2}$ so that the i-th component of its gradient is

$$(\nabla_P d(P,R)^{2-n})_i = \sum_{j=i}^{n} (n-2)(y_1^2 + \cdots + y_n^2)^{(1-n)/2} \frac{y_j}{\sqrt{y_1^2 + \cdots + y_n^2}} g^{ij}$$

where the g^{ij} are the coefficients of $(g_{ij})^{-1}$, the inverse of the matrix of the coefficients of the metric.

Since in a normal geodesic coordinates neighborhood of P, $g^{ij} = \delta^{ij} + O(r)$, ($r$ being the distance from P and δ^{ij} Kronecker's delta), the i-th component of integral (24), by (16) and Lemma 2.4 becomes

$$(n-2)\int_{B(\delta)} |y|^{1-n}\frac{y_i}{|y|}|x-y|^{2k-2-n}\, dy + O(|x|^{2k-n-1/2}) \tag{25}$$

which, within an error that we can estimate as $O(|x|^{2k-n-1/2})$, is nothing else than

$$(n-2)\int_{\mathbf{R}^n} y_i |y|^{-n}|x-y|^{2k-2-n}\, dy.$$

Since in the ball $B(\delta)$ around the point P, $g^{ij} = \delta^{ij} + O(r)$, this means that, within an error not worse than $O(|x|^{2k-n-1/2})$, the length of the vector $\nabla_P G_{2k}(P,Q)$ in the Riemannian metric of M is estimated by the Euclidean modulus of the vector

$$\int_{\mathbf{R}^n} \nabla(|y|^{2-n})|x-y|^{2k-2-n}\, dy = \nabla\left(\int_{\mathbf{R}^n} |y|^{2-n}|x-y|^{2k-2-n}\, dy\right).$$

Since

$$\int_{\mathbf{R}^n} |y|^{2-n}|x-y|^{2k-2-n}\,dy = \frac{\omega_{n-1}(n-2)}{2(n-2k)(k-1)}\,|x|^{2k-n}$$

its gradient has modulus equal to $\omega_{n-1}(n-2)/2(k-1)|x|^{2k-n-1}$. Hence, recalling that $|x| = d(P, Q)$,

$$|G_{2k-1}(P, Q)| = \left|\frac{c_{2k-2}}{(n-2)\omega_{n-1}}\,\nabla\left[\int_M d(P, R)^{2-n}\,d(R, Q)^{2k-2-n}\,dV(R)\right] + E\right|$$

$$\leq \frac{c_{2k-2}}{2(k-1)}\,d(P, Q)^{2k-1-n} + E$$

where E represents an error (not necessarily the same in different lines) not worse than $O(d(P, Q)^{2k-n-1/2})$. The value of c_{2k-2} is given by Lemma 2.6 so the proof of Lemma 2.7 (and therefore of Theorem 1.8), is complete.

3. Proof of Theorem 1.9

The proof of Theorem 1.9 follows closely Adams' original proof. The point is that, because of the very local nature of the problem, perturbations of higher order of the kernel in our integral operators are negligible.

The main tool is O'Neil's Lemma. We already mentioned the version of this Lemma used by Adams in his paper (see Lemma 1.5). O'Neil's Theorem 1.7 in [24] actually covers operations more general than convolutions $f * g$ on $\mathbf{R}^n$. However, it does not apply, as stated, to integral operators on general manifolds M of the form T discussed below. This is because, when M is non homogeneous, balls with the same radii but different centers can have different volumes. Consequently, the kernels $K(P, Q)$ differ too much from those of the form $K(P-Q)$ on $\mathbf{R}^n$ for the hypotheses of O'Neil's Theorem 1.7 to be satisfied. There is, though, an asymtotic homogeneity in the manifold case when the radii tend to zero which enables us to prove the following lemma, which serves as suitable substitute.

LEMMA 3.1. *Let M be a compact Riemannian manifold of dimension n, and define the operator T, acting on functions defined on M, by*

$$Tf(P) = \int_M K(P, Q) f(Q)\,dV(Q)$$

where

$$K(P, Q) = d(P, Q)^{\alpha - n}(1 + Cd(P, Q)^{\beta})$$

with $0 < \alpha < n$, $\beta > 0$ *and* C *a non negative constant. Then, for every* $t > 0$ *and for every function* $f \geq 0$ *on* M *we have*

$$(Tf)^{**}(t) \leq \frac{\omega_{n-1}}{\alpha}\left(\frac{nt}{\omega_{n-1}}\right)^{\alpha/n}(1 + Bt^{\beta/n})f^{**}(t)$$
$$+ \int_t^{\infty} f^*(s)\left(\frac{ns}{\omega_{n-1}}\right)^{(\alpha - n)/n}(1 + Bs^{\beta/n})\, ds$$

where the constant B *is independent of* f.

Of course in this result, only $t \leq \mathrm{Vol}\,(M)$ is of interest.

O'Neil's original argument can be adapted to our situation by using the volume estimate of Proposition 2.2. We omit the proof.

Remark. Observe that, with suitable (and natural) conventions, Lemma 3.1 is true in the one-dimensional case too. In that case the small balls are just intervals on the one dimensional manifold, their volumes are twice the length of their radius (so that the volume estimate of Proposition 2.2 is trivially true), the boundary of such a ball is a set of two points and its zero-dimensional measure is $\omega_0 = 2$. With these conventions, Theorem 1.9 also is true for $n = 1$.

Proof of Theorem 1.9. Let $t_1 = \mathrm{Vol}\,(M)$. Then, using $f^* \leq f^{**}$,

$$\int_M \exp\left\{\frac{n}{\omega_{n-1}}|Tf(P)|^{n/(n-\alpha)}\right\} dV(P) = \int_0^{t_1} \exp\left\{\frac{n}{\omega_{n-1}}|(Tf)^*(t)|^{n/(n-\alpha)}\right\} dt$$
$$\leq \int_0^{t_1} \exp\left\{\frac{n}{\omega_{n-1}}|(Tf)^{**}(t)|^{n/(n-\alpha)}\right\} dt.$$

By Lemma 3.1, for $0 \leq t \leq t_1$,

$$(Tf)^{**}(t) \leq Ct^{(\alpha - n)/n}\int_0^t f^*(s)\, ds + \left(\frac{\omega_{n-1}}{n}\right)^{(n-\alpha)/n}$$
$$\times \int_t^{t_1} f^*(s)s^{(\alpha - n)/n}(1 + Cs^{\beta/n})\, ds. \tag{26}$$

C denotes constants independent of f which can vary from line to line. Hence

$$\int_M \exp\left\{\frac{n}{\omega_{n-1}}|Tf|^{n/(n-\alpha)}\right\} dV \le \int_0^{t_1} \exp\left\{\left[Ct^{(\alpha-n)/n}\int_0^t f^*(s)\,ds + \left(\frac{\omega_{n-1}}{n}\right)^{(n-\alpha)/n}\right.\right.$$
$$\left.\left.\times \int_t^{t_1} f^*(s)s^{(\alpha-n)/n}(1+Cs^{\beta/n})\,ds\right]^{n/(n-\alpha)}\right\} dt.$$

Let $x = \log 1/s$, $y = \log 1/t$, $y_1 = \log 1/t_1$ and

$$\phi(x) = f^*(e^{-x})\, e^{-\alpha x/n}.$$

The right hand side of the estimate above is equal to

$$\int_{y_1}^{\infty} \exp\left\{\left[C\int_y^{\infty} e^{-[(n-\alpha)/n](x-y)}\phi(x)\,dx\right.\right.$$
$$\left.\left. + \int_{y_1}^{y} (1 + C(e^{-\beta x/n})\phi(x)\,dx\right]^{n/(n-\alpha)} - y\right\} dy. \tag{27}$$

Define

$$g(x, y) = \begin{cases} 1 + C\,e^{-(\beta/n)x} & \text{if } y_1 \le x \le y \\ C\,e^{-[(n-\alpha)/n](x-y)} & \text{if } y_1 < y < x < \infty \end{cases}$$

and define, for $y \in [y_1, \infty)$,

$$F(y) = y - \left[\int_{y_1}^{\infty} g(x, y)\phi(x)\,dx\right]^{n/(n-\alpha)}. \tag{28}$$

The integral (27) becomes $\int_{y_1}^{\infty} e^{-F(y)}\,dy$. Since

$$1 \ge \int_M |f|^{n/\alpha}\,dV = \int_0^{t_1} (f^*)^{n/\alpha}(s)\,ds = \int_{y_1}^{\infty} \phi(y)^{n/\alpha}\,dy$$

the proof of the theorem is reduced to the proof of the following lemma.

LEMMA 3.2. *Suppose that* $\phi : [y_1, \infty) \to \mathbf{R}^+$ *satisfies* $\int_y^{\infty} \phi(x)^{n/\alpha}\,dx \le 1$. *Let* g *and* F *be as above with* $c, \beta > 0$, $0 < \alpha < n$. *Then*

$$\int_{y_1}^{\infty} e^{-F(y)}\, dy \le C_1 < \infty \tag{29}$$

where C_1 *depends on* y_1, α, β, C, *but not on* ϕ.

This lemma differs from Adams' Lemma 1.6 only by the perturbation $C\,e^{-x(\beta/n)}$ for $x \le y$ in the kernel $g(x, y)$.

Unfortunately, this modified lemma does not seem to be deducible in any obvious way from the statement of the original one, so we must check that Adams' proof can be successfully adapted to this case.

Proof of Lemma 3.2. First of all observe that

$$\sup_{t \ge y_1} \left(\int_{y}^{\infty} g(x, y)^{n/(n-\alpha)}\, dx \right)^{(n-\alpha)/n} = b < \infty. \tag{30}$$

Now

$$\int_{y_1}^{\infty} e^{-F(y)}\, dy = \int_{-\infty}^{\infty} |E_\lambda|\, e^{-\lambda}\, d\lambda \tag{31}$$

where E_λ is the set $\{y \ge y_1 : F(y) \le \lambda\}$. By $|E_\lambda|$ we denote its Lebesgue measure.

We prove the lemma in two steps

LEMMA 3.3 (Step 1). *There exists a constant* $c \ge 0$ *independent of* ϕ *such that if* $E_\lambda \ne \emptyset$ *then* $\lambda \ge -c$, *i.e.* $\inf_{y_1 \le y < \infty} F(y) \ge -c > \infty$. *Furthermore, if* $y \in E_\lambda$ *then*

$$(d + y)^{\alpha/n} \left(\int_{y}^{\infty} \phi(x)^{n/\alpha}\, dx \right)^{\alpha/n} \le A|\lambda|^{\alpha/n} + B \tag{32}$$

where d, A *and* B *are suitable constants independent of* ϕ *and* λ.

LEMMA 3.4 (Step 2). *There exist constants* C *and* D *independent of* ϕ *and* λ, *such that for every* λ

$$|E_\lambda| \le C|\lambda| + D. \tag{33}$$

Lemmas 3.3 and 3.4 immediately imply Lemma 3.2 since

$$\int_{y_1}^{\infty} e^{-F(y)}\,dy = \int_{-\infty}^{\infty} |E_\lambda|\, e^{-\lambda}\,d\lambda \le \int_{-c}^{\infty} (C|\lambda| + D)\, e^{-\lambda}\,d\lambda$$

and the last integral is just a finite constant independent of ϕ.

Proof of Step 1. If $y \in E_\lambda$, then by definition and Holder inequality we have

$$\begin{aligned}
y - \lambda &= \left[\int_{y_1}^{\infty} g(x,y)\phi(x)\right]^{n/(n-\alpha)} \\
&= \left[\int_{y_1}^{y} \phi(x)(1 + C\,e^{-x\beta/n})\,dx \right.\\
&\qquad \left. + \int_{y}^{\infty} \phi(x) C\, e^{(y-x)(n-\alpha)/n}\,dx\right]^{n/(n-\alpha)} \\
&\le \left[\left(\int_{y_1}^{y} \phi(x)^{n/\alpha}\,dx\right)^{\alpha/n} \left(\int_{y_1}^{y} g(x,y)^{n/(n-\alpha)}\,dx\right)^{(n-\alpha)/n}\right.\\
&\qquad \left. + \left(\int_{y}^{\infty} \phi(x)^{n/\alpha}\,dx\right)^{\alpha/n} \left(\int_{y}^{\infty} C\,e^{y-x}\,dx\right)^{(n-\alpha)/n}\right]^{n/(n-\alpha)}.
\end{aligned}$$

Now

$$\begin{aligned}
\left(\int_{y_1}^{y} g(x,y)^{n/(n-\alpha)}\,dx\right)^{(n-\alpha)/n} &\le \left[\int_{y_1}^{y} (1 + C\,e^{-x\beta/n})\,dx\right]^{(n-\alpha)/n} \\
&\le [y - y_1 + C(e^{-y_1\beta/n} - e^{-y\beta/n})]^{(n-\alpha)/n} \\
&= (y + d)^{(n-\alpha)/n}
\end{aligned}$$

and

$$\left[\int_{y}^{\infty} C\,e^{y-x}\,dx\right]^{(n-\alpha)/n} \le b.$$

The constant C in the above lines is not necessarily the same in different lines, the only important thing being that it is independent of ϕ and λ.

By letting

$$L(y) = \left[\int_y^\infty \phi(x)^{n/\alpha}\, dx \right]^{\alpha/n}$$

and observing that $L(y) \le 1$ for every y, we get

$$y - \lambda \le [(1 - L(y)^{n/\alpha})^{\alpha/n}(y + d)^{(n-\alpha)/n} + bL(y)]^{n/(n-\alpha)}. \tag{34}$$

Adams gets this estimates too, so the conclusions of Step 1 lemma follow from the same argument as in his paper [1].

Proof of Lemma 3.4. Let R be a positive arbitrary real number and suppose that $E_\lambda \cap [R, \infty) \ne \emptyset$. Take $t_1, t_2 \in E_\lambda \cap [R, \infty)$, $t_1 < t_2$. Then:

$$\begin{aligned} t_2 - \lambda &\le \left[\int_{y_1}^{t_1} g(s, t_2)\phi(s)\, ds + \int_{t_1}^{t_2} g(s, t_2)\phi(s)\, ds + bL(t_1) \right]^{n/(n-\alpha)} \\ &\le \left[\left(\int_{y_1}^{t_1} g(s, t_2)^{n/(n-\alpha)}\, ds \right)^{(n-\alpha)/n} \left(\int_0^{t_1} \phi^{n/\alpha} \right)^{\alpha/n} \right. \\ &\quad \left. + \left(\int_{t_1}^{t_2} g(s, t_2)^{n/(n-\alpha)}\, ds \right)^{(n-\alpha)/n} L(t_1) + bL(t_1) \right]^{n/(n-\alpha)}. \end{aligned}$$

Now note that

$$\int_{y_1}^{t_1} g(s, t_2)^{n/(n-\alpha)}\, ds = \int_{y_1}^{t_1} g(s, t_1)^{n/(n-\alpha)}\, ds$$

which we already showed to be smaller than $(t_1 + d)$. Moreover

$$\int_{t_1}^{t_2} g(s, t_2)^{n/(n-\alpha)}\, ds = \int_{t_1}^{t_2} (1 + C e^{-s\beta/n})^{n/(n-\alpha)}\, ds \le t_2 - t_1 + d_1.$$

Since $\int_0^{t_1} \phi^{n/\alpha}\, ds \le 1$, we get

$$t_2 - \lambda \le \{(t_1 + d)^{(n-\alpha)/n} + [(t_2 - t_1 + d_1)^{(n-\alpha)/n} + b]L(t_1)\}^{n/(n-\alpha)}. \tag{35}$$

At this point everything is as in Adams' paper and thus the proof is complete.

We shall show that the coefficients in the exponents of Theorems 1.7 and 1.9 are the largest possible.

We start with Theorem 1.9 and we prove the following

PROPOSITION 3.5. *If the operator T is defined as in Theorem* 1.9, *then n/ω_{n-1} is the maximum value of the constant γ for which the integral*

$$\int_M \exp\{\gamma|Tf(P)|^{n/(n-\alpha)}\}\,dV(P) \tag{36}$$

is bounded by a constant c_M independent of f whenever $\int_M |f|^{n/\alpha}\,dV \le 1$.

Observe that the local nature of the problem makes possible a complete reduction to the $\mathbf{R}^n$ case. Essentially the same extremizing sequence of radial functions works for $\mathbf{R}^n$ and for any n-dimensional Riemannian manifold, even non compact. Thus on any Riemannian manifold we expect the best constant to be not larger than n/ω_{n-1}.

Proof of Proposition 3.5. Fix $P \in M$ and consider the ball $B_P(\delta)$ where δ is smaller than 1/2 of the radius of injectivity of M at P. We want to evaluate the operator T on the extremizing family of functions f_r defined by

$$f_r(Q) = \frac{1}{\omega_{n-1}}\left(\log\frac{\delta}{r}\right)^{-1} d(P,Q)^{-\alpha}\chi_r(Q) \tag{37}$$

where χ_r denotes the characteristic function of the set

$$\{Q \in M \text{ such that } r < d(P,Q) < \delta\}.$$

Computations similar to the ones in the analogous case in $\mathbf{R}^n$ (see [1]) lead to the following estimate

$$Tf_r(Q) = 1 + O\left(\log\frac{\delta}{r}\right)^{-1}. \tag{38}$$

Thus, for every fixed $\epsilon > 0$, there exists r_0 such that $r \le r_0$ implies that $Tf_r(Q) \ge 1 - \epsilon$ for every $Q \in B_P(r)$.

For the $L^{n/\alpha}$-norm of f_r, we have

$$\|f_r\|_{n/\alpha} = \left(\omega_{n-1}\log\frac{\delta}{r}\right)^{-1}\left\{\int_{B_P(\delta)-B_P(r)} d((P, Q)^{-\alpha(n/\alpha)}\,dV(Q)\right\}^{\alpha/n}$$

$$= \frac{\left[\omega_{n-1}\log\frac{\delta}{r} + O(1)\right]^{\alpha/n}}{\omega_{n-1}\log\frac{\delta}{r}}$$

$$= \left(\omega_{n-1}\log\frac{\delta}{r}\right)^{-(n-\alpha)/n}\left(1 + O\left(\log\frac{\delta}{r}\right)^{-\alpha/n}\right)$$

and thus the inequality

$$\int_M \exp\left\{\gamma\left|\frac{Tf_r(Q)}{\|f_r\|_{n/\alpha}}\right|^{n/(n-\alpha)}\right\} dV(Q) \le c_0 \tag{39}$$

implies that γ must be less than or equal to n/ω_{n-1}. Proposition 3.5 is thus proved.

We will now show that Theorem 1.7 is sharp.

PROPOSITION 3.6. *With the same hypothesis of Theorem* 1.7, *the largest constant* η *for which the exponential integral*

$$\int_M \exp\{\eta|f(P)|^{n/(n-m)}\}\,dV(P) \tag{40}$$

is bounded by a constant c_0 *independent of* f *is* $\lambda(m, n)$ *of formula* (4).

Proof. As customary we work in the ball $B_P(\delta)$ with δ small enough so that the exponential map is a diffeomorphism and the volume and surface estimates for small balls are fairly precise. To avoid unnecessary notational complications we suppose $\delta = 1$. Everything works, with proper obvious modifications, for arbitrary δ. The construction is essentially the same as in the $\mathbf{R}^n$ case, see [1], but it has to be adapted to the new situation (different ambient space and different conditions on f).

The case $m = 1$ is easy.

We consider the family of radial functions

$$\tilde{f}_r(Q) = \begin{cases} 1 & \text{if } d(P, Q) < r \\ \left(\log \frac{1}{r}\right)^{-1} \log \dfrac{1}{d(P, Q)} & \text{if } r \le d(P, Q) \le 1 \\ 0 & \text{if } d(P, Q) > 1. \end{cases}$$

These functions belong to $W^{1n}(M)$. We normalize them in order to have zero mean, and define

$$f_r(Q) = \tilde{f}_r(Q) - \frac{1}{\operatorname{Vol}(M)} \int_M \tilde{f}_r(R)\, dV(R).$$

Now

$$\frac{1}{\operatorname{Vol}(M)} \int_M \tilde{f}_r(R)\, dV(R) = O\left(\log \frac{1}{r}\right)^{-1}$$

and thus

$$f_r(Q) = \begin{cases} \text{constant} = 1 - O\left(\log \frac{1}{r}\right)^{-1} & \text{if } 0 \le d(P, Q) < r \\ \left(\log \frac{1}{r}\right)^{-1} [-\log d(P, Q) + O(1)] & \text{if } r \le d(P, Q) \le 1 \\ \text{constant} = O\left(\log \frac{1}{r}\right)^{-1} & \text{if } d(P, Q) \ge 1. \end{cases}$$

The modulus of the gradient of these functions is easily calculated, and

$$\|\nabla f_r\|_n = \left(\log \frac{1}{r}\right)^{-(n-1)/n} \omega_{n-1}^{1/n} \left(1 + O\left(\log \frac{1}{r}\right)^{-1}\right).$$

The exponential integral (40) has to be bounded by c_0 for all admissible functions. Evaluating it for $f_r / \|\nabla f_r\|_n$ we see that η cannot be larger than $n\omega_{n-1}^{1/(n-1)}$.

The case $m > 1$ requires some regularizations. Let Φ be a C^∞ function defined on $[0, 1]$ for which the following is true

$$\Phi(0) = \Phi'(0) = \cdots = \Phi^{(m-1)}(0) = 0, \quad \Phi(1) = \Phi'(1) = 1$$

and, if $m > 2$,

$$\Phi''(1) = \cdots = \Phi^{(m-1)}(1) = 0.$$

For $0 < \epsilon < 1/2$ we define

$$H(t) = \begin{cases} \epsilon\Phi\left(\dfrac{t}{\epsilon}\right) & \text{if } 0 \le t \le \epsilon \\ t & \text{if } \epsilon < t \le 1 - \epsilon \\ 1 - \epsilon\Phi\left(\dfrac{1-t}{\epsilon}\right) & \text{if } 1 - \epsilon < t \le 1 \\ 1 & \text{if } 1 < t. \end{cases}$$

Now define, for $0 < r < 1, 0 < t < 1$,

$$\psi(t) = H\left(\left(\log\frac{1}{r}\right)^{-1}\log\frac{1}{t}\right)$$

and note that the functions

$$f_r(Q) = \psi(d(P, Q))$$

are defined in the unit ball around P in M, m times differentiable there and identically equal to 1 on $B_P(r)$.

We need to compute $\int_M |\nabla^m f_r(Q)|^{n/m}\, dV(Q)$, and begin with

$$\Delta f_r(Q) = -\frac{d^2}{dt^2}\left(H\left(\left(\log\frac{1}{r}\right)^{-1}\log\frac{1}{t}\right)\right)\Bigg|_{t=d(P,Q)}$$
$$-\frac{d}{dt}\left(H\left(\left(\log\frac{1}{r}\right)^{-1}\log\frac{1}{r}\right)\right)\Bigg|_{t=d(P,Q)}\left(\frac{\partial}{\partial\rho}\log\sqrt{|g|} + \frac{n-1}{d(P,Q)}\right)$$

$$= -\left[H'\left(\left(\log\frac{1}{r}\right)^{-1}\log\frac{1}{t}\right)\left(\log\frac{1}{r}\right)^{-1}d(P,Q)^{-2}\right.$$

$$\left. -(n-1)\left(\log\frac{1}{r}\right)^{-1}d(P,Q)^{-2}H'\left(\left(\log\frac{1}{r}\right)^{-1}\log\frac{1}{t}\right)\right]$$

$$+O\left(\left(\log\frac{1}{r}\right)^{-1}d(P,Q)^{-1}\right)+O\left(\left(\log\frac{1}{r}\right)^{-2}d(P,Q)^{-2}\right)$$

$$=(n-2)\left(\log\frac{1}{r}\right)^{-1}d(P,Q)^{-2}H'\left(\left(\log\frac{1}{r}\right)^{-1}\log\frac{1}{d(P,Q)}\right)$$

$$+O\left(\left(\log\frac{1}{r}\right)^{-1}d(P,Q)^{-1}\right)+O\left(\left(\log\frac{1}{r}\right)^{-2}d(P,Q)^{-2}\right)$$

where $\partial/\partial\rho$ denotes the derivative in the radial direction from P and we used the fact that $\log\sqrt{|g|}$ is C^∞ and bounded in our ball around P. At this point it is not difficult to see by induction that

$$\Delta^k f_r(Q)=(n-2)\left(\log\frac{1}{r}\right)^{-1}(\Delta^{k-1}d(P,Q)^{-2})H'\left(\left(\log\frac{1}{r}\right)^{-1}\log\frac{1}{d(P,Q)}\right)$$

$$+O\left(\left(\log\frac{1}{r}\right)^{-2}d(P,Q)^{-2k}\right)+O\left(\left(\log\frac{1}{r}\right)^{-1}d(P,Q)^{-2k+1}\right)$$

and, by an iterative computation of $\Delta^{k-1}d(P,Q)^{-2}$ and $(\partial/\partial\rho)(\Delta^{k-1}d(P,Q)^{-2})$ we get

$$|\nabla^m f_r(Q)|=c(m,n)H'\left(\left(\log\frac{1}{r}\right)^{-1}\log\frac{1}{d(P,Q)}\right)\left(\log\frac{1}{r}\right)^{-1}d(P,Q)^{-m}$$

$$+O\left(\log\frac{1}{r}\right)^{-1}d(P,Q)^{1-m}+O\left(\log\frac{1}{r}\right)^{-2}d(P,Q)^{-m} \tag{41}$$

where

$$c(m,n)=\begin{cases}2^{(m-2)/2}\,\Gamma\left(\dfrac{m}{2}\right)(n-m)(n-m+2)\cdots(n-2) & \text{for } m \text{ even,}\\[2ex] 2^{(m-1)/2}\,\Gamma\left(\dfrac{m+1}{2}\right)(n-m+1)(n-m+3)\cdots(n-2) & \text{for } m \text{ odd.}\end{cases}$$

Observe, now, that

$$H'(t) = \begin{cases} \Phi'\left(\dfrac{t}{\epsilon}\right) & \text{if } 0 < t \le \epsilon \\ 1 & \text{if } \epsilon < t \le 1 - \epsilon \\ \Phi'\left(\dfrac{1-t}{\epsilon}\right) & \text{if } 1 - \epsilon < t \le 1 \\ 0 & \text{if } 1 < t. \end{cases}$$

Thus, with some computations, we find

$$\left\{\int_M |\nabla^m f_r(Q)|^{n/m}\, dV(Q)\right\}^{m/n}$$
$$\le c(m,n)\left(\log\frac{1}{r}\right)^{-1}\left\{\omega_{n-1}\log\frac{1}{r} + C\epsilon\log\frac{1}{r} + O\left(\log\frac{1}{r}\right)^{-1}\right\}^{m/n}$$
$$+ c(m,n)\omega_{n-1}^{m/n}(1+C\epsilon)^{m/n}\left(\log\frac{1}{r}\right)^{m/n} + O\left(\log\frac{1}{r}\right)^{-m/n}.$$

Now we need to estimate the average of f_r over M:

$$\left|\frac{1}{\mathrm{Vol}(M)}\int_M f_r(Q)\, dV(Q)\right|$$
$$= \left|\frac{\omega_{n-1}}{\mathrm{Vol}(M)}\int_0^1 H\left(\left(\log\frac{1}{r}\right)^{-1}\log\frac{1}{s}\right)s^{n-1}(1+O(s))\, ds\right|.$$

Since $H(t) \le Ct$ (with $C = \|\Phi'\|_\infty$ for instance), we have

$$\left|\frac{1}{\mathrm{Vol}(M)}\int_M f_r(Q)\, dV(Q)\right| \le C\left(\log\frac{1}{r}\right)^{-1}\int_0^1 \log\frac{1}{s}\, s^{n-1}(1+O(s))\, ds$$
$$= C'\left(\log\frac{1}{r}\right)^{-1}.$$

Thus, the average of f_r over M is $O(\log 1/r)^{-1}$ and it is not large enough to influence the final result. In particular on $B_P(r)$ we have

$$f_r(Q) - \frac{1}{\mathrm{Vol}\,(M)} \int_m f_r\, dV = 1 - O(\log 1/r)^{-1}.$$

We define our extremizing family of functions $\tilde{f}_r$, as

$$\tilde{f}_r(Q) = \frac{f_r - \dfrac{1}{\mathrm{Vol}\,(M)} \displaystyle\int_M f_r\, dV}{\|\nabla^m f_r\|_{n/m}}.$$

The functions $\tilde{f}_r$ satisfy the hypothesis of Proposition 3.6, and if the exponential inequality is to hold, the coefficient η must satisfy the following condition

$$\omega_{n-1} r^n(1 + O(r)) \exp \frac{\eta\left(1 + O\left(\log \frac{1}{r}\right)^{-1}\right)}{\|\nabla^m f_r\|_{n/m}^{n/(n-m)}} \le c_0 \tag{42}$$

where, as usual, c_0 doesn't depend on r. Inequality (42) and the above computation of $\|\nabla^m f_r\|_{n/m}$ imply

$$\eta \le n\omega_{n-1}^{m/(n-m)}\, c(m, n)^{n/(n-m)}$$

which, remembering the expression of $c(m, n)$, says that $\eta \le \lambda(m, n)$. Proposition 3.6 is completely proved.

4. Some further results and remarks

The techniques described in the previous sections are quite general and can be successfully adapted to other situations in which exponential integrability is involved.

We start by examining the case of Weyl fractional integration for periodic functions. Following the presentation in Zygmund's classical work [30], Chapter XII, let f be an integrable 2π periodic function. Suppose its integral over the interval $(0, 2\pi)$ is zero so that the constant term of its Fourier series $S[f]$ is zero. This condition guarantees that the integral f_1 of f is a 2π periodic function itself and, upon choosing the constant of integration in such a way that $\int_0^{2\pi} f_1 = 0$, we can construct a periodic f_2 such that $f_2' = f_1$ and so on.

In terms of Fourier series, if $S[f] = \Sigma\, c_n\, e^{inx}$ with $c_0 = 0$ then $S[f_1] = \Sigma\, (c_n/in)\, e^{inx}$ and in general, for k positive integer

$$S[f_k] = \sum \frac{c_n}{(in)^k} e^{inx}. \tag{43}$$

The formula (43) can be taken as a definition of f_k and extended to any positive real value of k, provided that we define

$$\gamma_n^{(k)} = (in)^{-k} = |n|^{-k} \exp\left(-\frac{1}{2}\pi ik \operatorname{sign} n\right)$$

for $n \neq 0$ and $\gamma_0^{(k)} = 0$.

For every positive real number α and every integrable 2π periodic function with zero integral mean over $(0, 2\pi)$ and Fourier series $\Sigma\, c_n e^{int}$, we define the Weyl fractional integral of order α as that 2π periodic function f_α whose Fourier series is

$$\sum c_n \gamma_n^{(\alpha)} e^{int}.$$

It can be shown (see [30]) that f_α can be obtained from f by convolution with a periodic function $\Psi_\alpha(t)$ whose Fourier series is $\Sigma\, \gamma_n^{(\alpha)} e^{int}$.

If we define the operator I_α by $I_\alpha f = f_\alpha$, it turns out that, for every positive α and β,

$$I_{\alpha+\beta} = I_\alpha I_\beta$$

and since I_α for α positive integer is just an iteration of ordinary integrations, the case $0 < \alpha < 1$ is the most interesting one to study.

We recall, again from Zygmund's Chapter XII, that

PROPOSITION 4.1. *If* $1 < r < s < \infty$ *and* $\alpha = 1/r - 1/s$, *then* $f \in L^r$ *implies* $f_\alpha \in L^s$ *and*

$$\left(\int_0^{2\pi} |f_\alpha|^s\right)^{1/s} = \|f_\alpha\|_s \leq A_r s^{s'/r'} \|f\|_r \tag{44}$$

where s' *and* r' *are the conjugate exponents of* s *and* r *respectively.*

From this result, it is not difficult to deduce that if $f \in L^r$, $r > 1$, and $\int_0^{2\pi} f = 0$ then $\mathrm{I}_{1/r} f$ is integrable in every power. Moreover there are positive constants λ and Λ such that if $\|f\|_r \leq 1$, then

$$\int_0^{2\pi} \exp [\lambda |I_{1/r}f(x)|^{r'}]\, dx \le A \tag{45}$$

(see Zygmund [30], vol 2, page 142 and page 158). Our aim is to find the largest constant λ for which (45) is true.

The first step in this program is the following estimate of the kernel $\Psi_\alpha(t)$ (see Zygmund [30], vol 1, pages 69–70).

PROPOSITION 4.2. *With an error uniformly $O(1)$, the 2π periodic function $\Psi_\alpha(x)$ can be represented on the interval $-\pi < x < \pi$ as*

$$\Psi_\alpha(x) = \begin{cases} \dfrac{2\pi}{\Gamma(\alpha)} x^{\alpha-1} & \text{for } 0 < x < \pi \\ 0 & \text{for } -\pi < x < 0. \end{cases}$$

This kernel is not exactly of the same type as the one we studied in the previous sections, but it is close enough to allow us to modify suitably the argument and obtain the result we are aiming for.

THEOREM 4.3. *If f is a 2π periodic function with zero integral mean on $(0, 2\pi)$ and $\int_0^{2\pi} |f(x)|^p\, dx \le 1$ $(1 < p < \infty)$ then, letting $\alpha = 1/p$, $I_\alpha f$ satisfies the exponential inequality*

$$\int_0^{2\pi} \exp \left\{ \left(\frac{\Gamma(\alpha)}{2\pi} \right)^{1/(1-\alpha)} |I_\alpha f(x)|^{p/(p-1)} \right\} dx \le A \tag{46}$$

where A is a constant independent of f. For no coefficient larger than $(\Gamma(\alpha)/2\pi)^{1/(1-\alpha)}$ can inequality (46) be true with any A independent of f.

Proof. Let

$$K(x) = \chi_{(0,\pi)}(x) x^{\alpha-1} + B(x)$$

for $x \in (-\pi, \pi)$, where $B \in L^\infty$. Extend K to be a 2π periodic function. Define the operator

$$T_\alpha f(x) = \int_{-\pi}^{\pi} K(x-t) f(t)\, dt.$$

For integrable periodic functions f with $\int_{-\pi}^{\pi} f\, dx = 0$, Proposition 4.2 says that $(\Gamma(\alpha)/2\pi) I_\alpha$ is an operator of the form T_α.

Now, for every integrable 2π periodic f, by O'Neil's lemma (in its simpler form, Lemma 1.5) we have

$$(T_\alpha f)^{**}(t) \leq tf^{**}(t)K^{**}(t) + \int_t^\infty f^*(s)K^*(s)\, ds.$$

Choose a constant $A > 0$ such that, on $(-\pi, \pi)$

$$|K(x)| \leq \chi_{[0,\pi)}(x)x^{\alpha-1} + A = K_\alpha(x).$$

Thus

$$K^*(t) \leq K_\alpha^*(t) = \chi_{[0,\pi)}(t)t^{\alpha-1} + A$$

and

$$K^{**}(t) \leq K_\alpha^{**}(t) = \frac{1}{\alpha}(t^{\alpha-1} + \alpha A) + \frac{\pi^\alpha}{\alpha t}\chi_{(\pi,2\pi)}(t)$$

for $0 \leq t < 2\pi$. Therefore, for $0 \leq t < 2\pi$

$$\begin{aligned}(T_\alpha f)^{**}(t) &\leq 2^{\alpha-1}\left[\frac{2}{\alpha}\left(\frac{t}{2}\right)^\alpha \chi_{[0,\pi)}(t) + Bt\right]f^{**}(t)\\ &\quad + 2^{\alpha-1}\int_t^\infty \left[\chi_{[0,\pi)}(s)\left(\frac{s}{2}\right)^{\alpha-1} + B\right]f^*(s)\, ds. \qquad (47)\end{aligned}$$

We compare this estimate with (26) and recall that for $n = 1$ the value of ω_{n-1} is 2. Then we see that the only important difference lies in the coefficient $2^{\alpha-1}$ which can be factored out of the right hand side of (47).

If we assume $\|f\|_p \leq 1$ and carry through, step by step, the analysis done in proving Theorem 1.9, we get an exponential inequality in which the coefficient 1/2 given by the one dimensional case of Theorem 1.9 is replaced by 1:

$$\int_{-\pi}^{\pi} \exp\,[|T_\alpha f(x)|^{1/(1-\alpha)}]\, dx \leq A$$

with A independent of f.

To show that 1 is the sharp coefficient consider the following family of 2π periodic functions defined on $(-\pi, \pi)$ as

$$f_r(x) = \chi_{(-\pi,-r]}(x)|x|^{-\alpha}\left(\log\frac{1}{r}\right)^{-1}$$

with $0 < r < \pi$.

Since Weyl fractional integration is defined for functions with integral mean zero, we consider

$$\tilde{f}_r(x) = f_r(x) - \frac{1}{2\pi}\int_{-\pi}^{\pi} f_r(y)\,dy.$$

Observe that

$$\tilde{f}_r(x) = f_r(x) + O\left(\log\frac{1}{r}\right)^{-1}.$$

Now, for $0 < x \leq r$, r small, we see, after come computations, that

$$T_\alpha \tilde{f}_r(x) = 1 + O\left(\log\frac{1}{r}\right)^{-1}.$$

In particular, for every $\epsilon > 0$ there exists r_ϵ such that for $r \in (0, r_\epsilon)$, $T_\alpha \tilde{f}_r(x) \geq 1 - \epsilon$. Now

$$\|\tilde{f}_r\|_p = \left(\log\frac{1}{r}\right)^{\alpha-1}\left[1 + O\left(\log\frac{1}{r}\right)^{-\alpha}\right].$$

Thus, if

$$\int_{-\pi}^{\pi} \exp\left[\lambda\left|\frac{T_\alpha \tilde{f}_r(x)}{\|f\|_p}\right|^{1/(1-\alpha)}\right] dx \leq A$$

we must have

$$r\exp\left[\lambda\log\frac{1}{r}\left(1 + O\left(\log\frac{1}{r}\right)^{-\alpha}\right)^{1/(1-\alpha)}\right] \leq A$$

which implies $\lambda \leq 1$. We have shown that 1 is the best coefficient for the exponential integrability of $T_\alpha f$. Since the Weyl fractional integral operator I_α has the form $(2\pi/\Gamma(\alpha))T_\alpha$ we see that $(\Gamma(\alpha)/2\pi)^{1/(1-\alpha)}$ is the sharp constant for it, and the Theorem is proved.

There is also a notion of fractional integration for periodic functions in higher dimension. The definition, given by Wainger in [28], is suggested by fractional integration in $\mathbf{R}^n$ (the classical Riesz potentials).

Following Wainger [28], let f be in $L^1(\mathbf{T}^n)$ where $\mathbf{T}^n$ is the n-dimensional torus, that is $[0, 1]^n$ with the usual topological identifications.

Let c_k, $k \in Z^n$ be the Fourier coefficients of f. Define $I_\alpha f$, $0 < \alpha < n$ as the unique function defined on $\mathbf{T}^n$ with Fourier coefficients d_k given by $d_O = 0$ (O = zero vector of Z^n) and by $d_k = c_k(2\pi|k|)^{-\alpha}$ if $O \neq k = (k_1, \ldots, k_n)$. $|k| = (\Sigma\, k_j^2)^{1/2}$.

Wainger proved the following result

THEOREM 4.4 (Wainger). *For every $f \in L^1(\mathbf{T})^n$, $I_\alpha f$ is well defined and is given by*

$$I_\alpha f(x) = \int_{\mathbf{T}^n} g_\alpha(x - y) f(y)\, dy$$

where g_α is periodic of period 1 *in all its n variables and*

$$g_\alpha(x) = c(n, \alpha)|x|^{\alpha - n} + E_\alpha(x)$$

where $E_\alpha(x)$ is bounded (actually C^∞), and

$$\frac{1}{c(n, \alpha)} = \frac{\pi^{n/2} 2^\alpha \Gamma\left(\dfrac{\alpha}{2}\right)}{\Gamma\left(\dfrac{n - \alpha}{2}\right)}.$$

Exponential integrability for $I_\alpha f$ is an immediate consequence of Theorem 1.9. We can therefore state the following:

PROPOSITION 4.5. *There is a constant $C_{\alpha,n}$ such that if $f \in L^{n/\alpha}(\mathbf{T}^n)$ and satisfies $\|f\|_{n/\alpha} \leq 1$ then*

$$\int_{\mathbf{T}^n} \exp\{\gamma_\alpha(n)|I_\alpha f(x)|^{n/(n-\alpha)}\}\, dx \leq C_{\alpha,n}$$

where the constant $\gamma_\alpha(n)$, *given by*

$$\gamma_\alpha(n) = \frac{n}{\omega_{n-1}} \left[\frac{\pi^{n/2} 2^\alpha \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)} \right]^{n/(n-\alpha)},$$

is sharp.

Observe that $\gamma_\alpha(n)$, for α an even integer smaller than n, is the same critical exponent as in Theorem 1.7, as it should be, considering the meaning of fractional integration and the fact that the sharp exponential coefficient is the same in most situations (in particular for compact manifolds).

To conclude, we indicate two open problems. Firstly, we observe that for general non compact manifolds only very partial results about exponential integrability are known, see [14] and [15]. The argument in Section 3, shows that the coefficient λ cannot be larger than $\lambda(m, n)$, but we do not have yet complete results for exponential integrability over domains on non compact manifolds analogous to Moser and Adams' theorems for domains in $\mathbf{R}^n$.

The second problem concerns the possibility of higher dimensional extensions of a recent result of A. Chang and C. Yang (see [12]).

THEOREM 4.6 (Chang and Yang). *Suppose D is a piecewise C^2, bounded, finitely connected domain in the plane with a finite number of vertices. let θ_D be the minimum interior angle at the vertices of D. There exists a constant c_D such that for all $u \in C^1(\bar{D})$ with*

$$\int_D |\nabla u|^2\, dx \le 1 \qquad \int_D u\, dx = 0$$

we have

$$\int_D e^{2\theta_D u^2}\, dx \le c_D.$$

If we replace $2\theta_D$ with any positive β, the integral is still finite, but if $\beta > 2\theta_D$ it can be made arbitrarily large by appropriate choice of u.

Chang and Yang's proof of this theorem depends on an isoperimetric inequality and represents a different approach to the problem of exponential integrability.

The presence of the boundary of D seems to pose difficult problems to an approach based on the techniques of the previous chapters. It might be possible, however, to find (after some sort of symmetrization) a suitable and workable representation formula which could allow the extension of Chang and Yang's theorem to $\mathbf{R}^n$ with $n > 2$.

We should expect something like the following:

Let D be a domain in $\mathbf{R}^n$ with compact closure. Some regularity on ∂D has to be assumed. In particular, suppose that the limit

$$\theta(P) = \lim_{r \to 0} \frac{|\partial B_P(r) \cap D|}{r^{n-1}},$$

where the numerator denotes the $n-1$ surface measure of $\partial B_P(r) \cap D$, exists for every $P \in \partial D$. $\theta(P)$ can also be defined for $P \in D$ in which case $\theta(P) = \omega_{n-1}$. Assume $\theta(P)$ has a positive minimum θ_D on D.

Then there must be a constant C such that for all C^1 functions f satisfying

$$\int_D f(x)\, dx = 0 \quad \text{and} \quad \int_D |\nabla f(x)|\, dx \le 1$$

the following is true for $\alpha = n\theta_D^{1/(n-1)}$

$$\int_D \exp\left[\alpha |f(x)|^{n/(n-1)}\right] dx \le C. \tag{48}$$

If $\alpha > n\theta_D^{1/(n-1)}$, no constant C can bound the integral uniformly with respect to f.

It is easy to see why α cannot be larger than $n\theta_D^{1/(n-1)}$.

Let $P \in \partial D$ be a point for which $\theta(P) = \theta_D$. We can suppose P is the origin of $\mathbf{R}^n$. Consider, now, the functions $\tilde{f}_r$ such that:

$$\tilde{f}_r(Q) = \begin{cases} 1 & \text{if } |Q| \le r \\ \left(\log \dfrac{1}{r}\right)^{-1} \log \dfrac{1}{|Q|} & \text{if } r \le |Q| \le 1 \\ 0 & \text{if } |Q| > 1. \end{cases}$$

Then

$$f_r(Q) = \tilde{f}_r(Q) - \frac{1}{\operatorname{Vol}(D)} \int_D \tilde{f}_r(R)\, dR$$

is given by

$$f_r(Q) = \begin{cases} \text{constant} = 1 - O\left(\log \frac{1}{r}\right)^{-1} & \text{if } 0 \leq |Q| < r \\ \left(\log \frac{1}{r}\right)^{-1} \left[\log \frac{1}{|Q|} + O(1)\right] & \text{if } r \leq |Q| \leq 1 \\ \text{constant} = O\left(\log \frac{1}{r}\right)^{-1} & \text{if } |Q| \leq 1 \end{cases}$$

and

$$|\nabla f_r| = \begin{cases} \left(\log \frac{1}{r}\right)^{-1} \frac{1}{|Q|} & \text{if } r < |Q| < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Now, let $D_r = D \cap \{Q : r < |Q| < 1\}$. Fix ϵ small, in particular smaller than $\theta_D/2$, and let r_0 be so small that for $r < s < r_0$

$$\left| \int_{D_r \cap \partial B(s)} d\sigma - \theta_D \right| \leq \epsilon \tag{49}$$

so that

$$\begin{aligned} \int_D |\nabla f_r(x)|^n\, dx &= \left(\log \frac{1}{r}\right)^{-n} \int_{D_r} \frac{1}{s}\, d\sigma\, ds \\ &= \left(\log \frac{1}{r}\right)^{-n} \left\{ \int_r^{r_0} (\theta_D + E(s)) s^{-1}\, ds + \int_{D_{r_0}} s^{-1}\, d\sigma\, ds \right\} \end{aligned}$$

where, by (49), $|E(s)| \leq s$ for $r < s < r_0$. Therefore, for small r

$$\|\nabla f_r\|_n \leq (\theta + \epsilon)^{1/n}\left(\log\frac{1}{r}\right)^{-(n-1)/n}\left(1 + O\left(\log\frac{1}{r}\right)^{-1}\right).$$

So if (48) is true, for small r we must have

$$\frac{(\theta - \epsilon)r^n}{n}\exp\left\{\alpha\left[\frac{\left(1 + O\left(\log\frac{1}{r}\right)^{-1}\right)\left(\log\frac{1}{r}\right)^{-(n-1)/n}}{(\theta + \epsilon)^{1/n}}\right]^{n/(n-1)}\right\} \leq C$$

and therefore

$$\alpha \leq n(\theta + \epsilon)^{1/(n-1)}.$$

Since ϵ is arbitrary,

$$\alpha \leq n\theta^{1/(n-1)}.$$

What happens is that, since our functions need not be zero outside D, only part of the L^p norm of $|\nabla f|$ in a neighborhood of a boundary point P is controlled by our hypothesis on f. The rest (the part outside D) contributes to the values of f around P for free. Of course this circumstance cannot occur if we impose on f the condition of being compactly supported in D. In this case, as we know, the best integrability exponent is $n\omega_{n-1}^{1/(n-1)}$, and this is consistent with the equation $\theta(P) = \omega_{n-1}$ which holds when P is an interior point of D.

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