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Gluing Cohen–Macaulay modules with applications to quasihomogeneous complete intersections with isolated singularities

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Introduction

This paper deals with the problem of characterizing quasihomogeneous isolated singularities. The history begins in 1971 with the beautiful result of Saito [22]: an isolated complex hypersurface singularity with defining equation f is quasihomogeneous (i.e., after a change of coordinates f can be made into a quasihomogeneous polynomial) if and only if $f \in j(f)$, where $j(f)$ is the ideal generated by the partial derivatives of f (this ideal is also called the jacobian ideal of f).

In the subseqeunt years this result was extended to other fields and significantly generalized in papers by Scheja and Wiebe, see [24], [25] and [26]. Among other powerful results they showed that a complete intersection $(R, \mathfrak{m}, k)$ with isolated singularity is quasihomogeneous if and only if there exists a k-derivation δ of R which induces an isomorphism on the Zariski tangent space $\mathfrak{m}/\mathfrak{m}^2$. If $\dim R = 2$, then the assumption that R is a complete intersection can be discarded and the requirement on the derivation δ can be weakened: it suffices that δ induces a nonnilpotent transformation of $\mathfrak{m}/\mathfrak{m}^2$. A concise account of their work can be found in Platte's paper [21].

In 1985, Wahl [29] characterized quasihomogeneous Gorenstein surface singularities in terms of certain invariants associated with the resolution of singularities. There the aforementioned criterion of Scheja and Wiebe was used.

In 1984, Kunz and Waldi [15] characterized quasihomogeneous reduced Gorenstein algebroid curves over an algebraically closed field k of characteristic 0 by the condition that the cokernel R/J of the canonical homomorphism from the (universally finite) module of Kähler differentials to the module of regular differentials of R/k is Gorenstein. If R is a complete intersection then J is the Kähler different of R/k, i.e., the ideal generated by the maximal minors of the jacobian matrix.

In 1987 the second author noticed in his thesis [16] the relevance of maximal Cohen–Macaulay modules for the problem of quasihomogeneity. He conjectured that

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a two-dimensional complete normal analytic algebra R is quasihomogeneous if and only if the module $D_k(R)^{**}$ of Zariski differentials is isomorphic to the Cohen–Macaulay approximation of the maximal ideal of R. This conjecture was proved for certain cases in [17] and by Behnke in [5]. In [18] a similar conjecture was formulated for higher dimensions, and the easier implication was proved for hypersurfaces.

In this paper we further explore the properties of quasihomogeneous isolated singularities via module-theoretic techniques. Our main tool is what we call the gluing construction for Cohen–Macaulay modules which produces minimal Cohen–Macaulay approximations (and, if the ring is Gorenstein, hulls of finite injective dimension) for (nonmaximal) Cohen–Macaulay modules and their syzygy modules. As we recently learned, the gluing construction is a particular case of the so-called complete resolution of a module introduced by Buchweitz in his unpublished preprint [8]. The latter is constructed within the framework of complexes rather than modules. We should remark however that if the module under consideration is Cohen–Macaulay then, as the gluing construction shows, the approximations of the module and its syzygy modules can be obtained in a simple and direct way (i.e., without the pushout operations necessary in the general case, see [3]).

Using the gluing construction for the residue field k of a complete intersection R with isolated singularity we prove in Section 3 that, if R is quasihomogeneous, then some sufficiently high syzygy modules of k and the transpose $\operatorname{Tr} D_k(R)$ of the module of Kähler differentials coincide. This leads us to the following question: Let R be a complete Cohen–Macaulay analytic algebra over an algebraically closed field k of characteristic 0. Is it true that R is quasihomogeneous if and only if some sufficiently high syzygy modules of k and $\operatorname{Tr} D_k(R)$ are isomorphic?

We also show in Section 3 that the truncated symmetric (R/J)-algebra $G = (\bigoplus_{i=0}^{\infty} S_i(T^1))/(S_r(T^1))$ is Gorenstein, where J is the Kähler different of R/k, $T^1 \cong \operatorname{Tr} D_k(R)$ is the space of infinitesimal deformations of the quasihomogeneous complete intersection R with isolated singularity, and $r = \dim R$. In the hypersurface case this follows from Zariski's result on derivations of isolated singularities, Kunz' result stating that almost complete intersections are never Gorenstein, and the aforementioned result of Saito (see [17]).

We also give a new proof of the just mentioned result of Kunz based on the simplest form of the gluing construction and the nontrivial result of Eisenbud about the behaviour of the Eisenbud operators of a complete intersection.

1. The gluing construction for Cohen–Macaulay modules

We begin by recalling some basic facts about Cohen–Macaulay approximations and the dual construction of hulls of finite injective dimension. For a detailed account see [3]. Let $(R, \mathfrak{m})$ be a local Cohen–Macaulay ring with canonical module ω_R, and let N be a finitely generated R-module. A short exact sequence $0 \to Y_N \xrightarrow{f} X_N \to N \to 0$ is called a *Cohen–Macaulay approximation of* N if X_N is a maximal Cohen–Macaulay module and Y_N is a module of finite injective dimension. (Sometimes, by abuse of language, X_N is also called a Cohen–Macaulay approximation of N.) It is called *minimal* if X_N and Y_N do not have a common (under f) direct ω_R-summand. Minimal approximations always exist and are, in the obvious sense, uniquely defined.

Dually, a short exact sequence $0 \to N \to Y^N \xrightarrow{g} X^N \to 0$ is called a *hull of finite injective dimension for* N if X^N is a maximal Cohen–Macaulay module and Y^N is a module of finite injective dimension. It is called *minimal* if Y^N and X^N have no common (under g) direct ω_R-summands. Minimal hulls always exist and are again uniquely defined.

In this section we shall describe a construct which yields minimal Cohen–Macaulay approximations for a (non-maximal) Cohen–Macaulay module and its syzygy modules.

From now on the ring R will be assumed to be local Cohen–Macaulay with maximal ideal $\mathfrak{m}$ and canonical module ω_R.

Let M be a Cohen–Macaulay R-module of codepth n. The local dual $\operatorname{Ext}^n(M, \omega_R)$ of M will be denoted $M^\vee$. Let $F_\bullet$ be a minimal free resolution of M, and $G_\bullet$ a minimal free resolution of $M^\vee$. We define a complex $L_\bullet$ by setting $L_i = G_{n-i}^\vee$ and $d_i^L = (d_{n+1-i}^G)^\vee$.

Since $\operatorname{Ext}^i(M^\vee, \omega_R)$ vanishes for $i \neq n$ and is isomorphic to M for $i = n$, one has $H_i(L_\bullet) = 0$ for $i \neq 0$ and $H_0(L_\bullet) \cong M$. Thus the complexes $F_\bullet$ and $L_\bullet$ are quasi-isomorphic. Let $\alpha_\bullet : F_\bullet \to L_\bullet$ be a quasi-isomorphism, and let $C_\bullet = \operatorname{Con}(\alpha_\bullet)$ be its mapping cone. We obtain an exact sequence of complexes

$$0 \longrightarrow L_\bullet \longrightarrow C_\bullet \longrightarrow F_\bullet[-1] \longrightarrow 0.$$

Since $\alpha_\bullet$ is a quasi-isomorphism, this sequence implies that $C_\bullet$ is exact.

Truncating the exact sequence of complexes and observing that $\operatorname{Coker} d_i^F = \Omega^{i-1}(M)$, one obtains for all $i > 0$ exact sequences

$$0 \longrightarrow \operatorname{Coker} d_{i+1}^L \longrightarrow \operatorname{Coker} d_{i+1}^C \longrightarrow \Omega^{i-1}(M) \longrightarrow 0. \tag{1}$$

Here we set $\Omega^0(M) = M$.

Similarly one attaches to $\alpha_\bullet$ a complex $Z_\bullet = \operatorname{Cyl}(\alpha_\bullet)$, called the cylinder of $\alpha_\bullet$ (see [6], §2 no. 6) and an exact sequence of complexes

$$0 \longrightarrow F_\bullet \longrightarrow Z_\bullet \longrightarrow C_\bullet \longrightarrow 0,$$

which yields for all $i > 0$ exact sequences

$$0 \longrightarrow \Omega^{i-1}(M) \longrightarrow \operatorname{Coker} d_i^Z \longrightarrow \operatorname{Coker} d_i^C \longrightarrow 0. \tag{2}$$

PROPOSITION 1.1. *For $i > 0$, (1) is a minimal Cohen–Macaulay approximation of $\Omega^{i-1}(M)$, and if R is Gorenstein, (2) a hull of finite injective dimension for $\Omega^{i-1}(M)$.*

Proof. Since $C_{\bullet}$ is exact, all its syzygy modules are maximal Cohen–Macaulay modules, and so is $\operatorname{Coker} d_{i+1}^C$. Next observe that for all $i > 0$ we have an exact sequence

$$0 \longrightarrow L_n \longrightarrow \cdots \longrightarrow L_{i+1} \longrightarrow L_i \longrightarrow \operatorname{Coker} d_{i+1}^L \longrightarrow 0.$$

Since each L_j is a finite direct sum of copies of ω_R, and since ω_R has finite injective dimension, we conclude that $\operatorname{Coker} d_{i+1}^L$ has finite injective dimension. This proves that (1) is a Cohen–Macaulay approximation of $\Omega^{i-1}(M)$.

Suppose the approximation is not minimal, then $\operatorname{Coker} d_{i+1}^L \cong N \oplus \omega_R$, and we obtain a commutative diagram

$$\begin{tikzcd} L_i \arrow[r] \arrow[dr, "\rho"'] & \operatorname{Coker} d_{i+1}^L \arrow[d, "\pi"] \\ & \omega_R, \end{tikzcd}$$

where π is the projection onto the second summand and where ρ is an epimorphism. Since $L_i \cong \omega_R^{n_i}$ and since $\operatorname{End}(\omega_R) \cong R$, ρ can be described by a column vector $(a_1, \ldots, a_{n_i})^t$, $a_j \in R$. Since ρ is surjective we have $R = \Sigma\, Ra_j$. Dualizing into ω_R we obtain the exact sequence

$$0 \longrightarrow (\operatorname{Coker} d_{i+1}^L)^{\vee} \longrightarrow L_i^{\vee} \xrightarrow{(d_{i+1}^L)^{\vee}} L_{i+1}^{\vee}$$

which is isomorphic to

$$0 \longrightarrow \operatorname{Ker} d_{n-i}^G \longrightarrow G_{n-i} \xrightarrow{d_{n-i}^G} G_{n-i-1},$$

where $\operatorname{Ker} d_{n-i}^G = N^{\vee} \oplus R$, and where R is mapped to a generator of G_{n-i} since $\Sigma\, a_j R = R$.

If $i < n$, this contradicts the minimality of the resolution $G_{\bullet}$. For $i = n$, the exact sequence (1) is

$$0 \longrightarrow G_0^{\vee} \longrightarrow \operatorname{Coker} d_{n+1}^C \longrightarrow \Omega^{n-1}(M) \longrightarrow 0,$$

and as we assume it is not minimal there is a common ω_R-summand of $G_0^\vee$ and $\operatorname{Coker} d_{n+1}^C$ which we may cancel. Thus we obtain an exact sequence

$$0 \longrightarrow \omega_R^{n_0 - 1} \longrightarrow X \longrightarrow \Omega^{n-1}(M) \longrightarrow 0,$$

where n_0 is the rank of G_0, that is, the minimal number of generators of $M^\vee$. Again dualizing with respect to ω_R gives the exact sequence

$$X^\vee \longrightarrow R^{n_0 - 1} \longrightarrow \operatorname{Ext}^1(\Omega^{n-1}(M), \omega_r) \longrightarrow \operatorname{Ext}^1(X, \omega_R).$$

This is a contradiction since $\operatorname{Ext}^1(X, \omega_R) = 0$ and $\operatorname{Ext}^1(\Omega^{n-1}(M), \omega_R) \cong M^\vee$.

The statement concerning the cylinder of $\alpha_.$ and the hull is proved similarly. Here however we have to require that R be Gorenstein since otherwise $\operatorname{Coker} d_i^C$ will not be of finite injective dimension. $\square$

2. First applications of the gluing construction

In this section we draw a few quite straightforward consequences from the existence of the gluing construction for Cohen–Macaulay modules. The reader who is only interested in applications to quasihomogeneous complete intersections may skip this section.

Let $(R, \mathfrak{m}, k)$ be a local Cohen–Macaulay ring with canonical module ω_R, and let M be an R-module. Recall that $\delta^i(M)$, as defined by Auslander, is the rank of the largest free direct summand in the minimal Cohen–Macaulay approximation of the i-th syzygy module $\Omega^i(M)$. As was remarked in the lecture notes of Auslander [2], $\delta^0(M) = 0$ if and only if M is a homomorphic image of a maximal Cohen–Macaulay module without free summands. Correspondingly, we let $\gamma^i(M)$ be the rank of the largest ω_R-summand in the minimal Cohen–Macaulay approximation of $\Omega^i(M)$.

Clearly, if R is Gorenstein then $\delta^i(M) = \gamma^i(M)$ for all i. Moreover we have $\delta^i(M) = \gamma^i(M) = 0$ for $i > \operatorname{codepth} M$. Indeed, for these i the syzygy module $\Omega^i(M)$ is Cohen–Macaulay, so that $\delta^i(M)$ is simply the number of R-summands and $\gamma^i(M)$ the number of ω_R-summands of $\Omega^i(M)$. The conclusion follows since $\Omega^i(M)$ may have R- or ω_R-summands only for $i = \operatorname{codepth} M$. Thus, the alternating sums

$$\Delta(M) = \sum_{i \geq 0} (-1)^i \delta^i(M) \quad \text{and} \quad \Gamma(M) = \sum_{i \geq 0} (-1)^i \gamma^i(M)$$

are well-defined. The importance of the invariants $\delta^i(M)$ and $\Delta(M)$ have been shown in papers by Martsinkovsky [18], Ding [9] and Auslander–Ding–Solberg [4]. In particular, $\Delta(M)$ turned out to be an obstruction for weak lifting of modules defined over Gorenstein rings. If one wants to extend these results to non-Gorenstein rings, the invariant $\Gamma(M)$ comes into play.

PROPOSITION 2.1. *If M is a Cohen–Macaulay module of codepth n, then $\delta^i(M) = \gamma^{n-i}(M^\vee)$. In particular, if R is Gorenstein, then $\delta^i(M) = \delta^{n-i}(M^\vee)$.*

Proof. Let $\alpha_\bullet$ be a gluing map for M. Then $(\alpha_\bullet)^\vee$ is a gluing map for $M^\vee$. But, as is easily seen, $\operatorname{Con}(\alpha_\bullet)^\vee$ is identical, up to the sign of the differential, to $\operatorname{Con}(\alpha_\bullet^\vee)[-n]$ and the desired result follows from 1.1. □

The following simple example illustrates this result: let M be the Cohen–Macaulay module of codepth 1 defined by the short exact sequence

$$0 \longrightarrow \omega_R \longrightarrow R \xrightarrow{\varphi} M \longrightarrow 0$$

This sequence is both the beginning of a minimal projective resolution and a minimal Cohen–Macaulay approximation of M. Thus $\delta^0(M) = 1$, $\delta^1(M) = 0$, $\gamma^0(M) = 0$, $\gamma^1(M) = 1$. Dualizing this sequence into ω_R and using the fact that $\operatorname{Ann} M = \operatorname{Ann} M^\vee$ we conclude that $M^\vee \cong M$. Thus $\delta^0(M) = \gamma^1(M^\vee) = \gamma^1(M) = 1$ and $\delta^1(M) = \gamma^0(M^\vee) = \gamma^0(M) = 0$, as was expected.

COROLLARY 2.2. *Let M be a Cohen–Macaulay R-module of codepth n. Then*
(a) $\Delta(M) = (-1)^n \Gamma(M^\vee)$;
(b) *If R is Gorenstein, then $\Delta(M) = (-1)^n \Delta(M^\vee)$. In particular, if M is self-dual and n is odd, we have that $\Delta(M) = 0$. In any case, $\Delta(M)$ vanishes if and only if $\Delta(M^\vee)$ does.*

For a Gorenstein ring R the numbers $\delta^i(M)$ can be directly read off the gluing map:

LEMMA 2.3. *Suppose that R is Gorenstein and let $\alpha_\bullet$ be a gluing map for the R-module M. Then $\delta^i(M) = \operatorname{rank}_{R/\mathfrak{m}}(\alpha_i \otimes R/\mathfrak{m})$ for all $i \geq 0$.*

Proof. The assertion follows from the fact that $\operatorname{Con}(\alpha_\bullet)$ is an exact complex of free modules and the fact that unit entries in the differential of $\operatorname{Con}(\alpha_\bullet)$ may only come from $\alpha_\bullet$ itself. □

In the case of a hypersurface ring the δ^i are completely determined by the Betti numbers of M and $M^\vee$.

COROLLARY 2.4. *Let R be a hypersurface ring, M a Cohen–Macaulay module of codepth n, a_i the Betti numbers of M, b_i the Betti numbers of $M^\vee$, and set $\delta^i = \delta^i(M)$ for $i = 0, 1, \ldots$. Then $a_i + b_{n-i-1} - \delta^i - \delta^{i+1} = a_{n+1}$ and*

$$\delta^i(M) = (a_{i-1} - a_{i-2} + \cdots + (-1)^{i-1}a_0) + (b_{n-i} - b_{n-i+1} + \cdots + (-1)^i b_n) - \frac{1}{2}(1 + (-1)^i)a_{n+1}$$

for all $i = 0, \ldots, n$.

Proof. Since R is a hypersurface ring, the Betti numbers of M and $M^\vee$ stabilize at the $(n+1)$-th step: $a_{n+1} = a_{n+2} = \cdots$ and $b_{n+1} = b_{n+2} = \cdots$; see [10]. If $\alpha_\bullet$ is a gluing map for M then Con $(\alpha_\bullet)$ can be written as follows

$$\cdots \longrightarrow R^{a_{n+1}} \longrightarrow R^{a_n} \xrightarrow{t_n} R^{a_{n-1}} \oplus R^{b_0} \xrightarrow{t_{n-1}} R^{a_{n-2}} \oplus R^{b_1} \xrightarrow{t_{n-2}} \cdots$$
$$\xrightarrow{t_1} R^{a_0} \oplus R^{b_{n-2}} \xrightarrow{t_0} R^{b_n} \longrightarrow R^{b_{n+1}} \longrightarrow \cdots$$

By 2.3, rank $(t_i \otimes R/\mathfrak{m}) = $ rank $(\alpha_i \oplus R/\mathfrak{m}) = \delta^i(M)$. "Peeling off" the non-minimal part of this infinite exact complex we must obtain a periodic complex all of whose Betti numbers are equal to a_{n+1}. In particular, $a_{n+1} = b_{n+1}$, and

$$a_i + b_{n-i-1} - \delta^i - \delta^{i+1} = a_{n+1}$$

for $i = -1, 0, 1, \ldots, n$ where we set $a_{-1} = b_{-1} = \delta^{-1} = 0$. The assertions follow. □

REMARKS 2.5. (a) Since $\delta^i(M) \geq 0$ for all i, we have, in view of the 2.4, that $a_i + b_{n-i-1} \geq a_{n+1}$ for all i.

(b) Due to the fact that Con $(\alpha_\bullet)$, after cancellation of its nonminimal part, is periodic of period 2, we obtain the following isomorphisms

$$\Omega^{n+1}(M^\vee) \cong \begin{cases} \Omega^{n+2}(M)^\vee & \text{if } n \text{ is odd,} \\ \Omega^{n+1}(M)^\vee & \text{if } n \text{ is even.} \end{cases}$$

(c) Suppose M is self dual and n is even. Summing up the equations in 2.4 and taking into account that $a_i = b_i$ for all i and rank $\Omega^{n+1}(M) = a_n - a_{n-1} + \cdots$, we have that

$$\Delta(M) = (n+1) \text{ rank } \Omega^{n+1}(M) - a_{n+1}\binom{n+1}{2}.$$

As a last application we reprove and slightly generalize a result of Kunz [14]. The essential tool is a theorem of Eisenbud [10] concerning the nature of free resolutions over complete intersections. Our gluing construction comes into play only in the subsequent corollary, and actually could be avoided, but nevertheless let us do this proof.

PROPOSITION 2.6. *Let R be a complete intersection, that is, $R \cong S/I$ where S is a regular local ring and I is generated by a regular sequence $\mathbf{x} = x_1, \ldots, x_m$. Suppose further that M is an R-module with periodic minimal free R-resolution*

$$\cdots \longrightarrow R^n \xrightarrow{\alpha} R^n \xrightarrow{\beta} R^n \xrightarrow{\alpha} R^n \longrightarrow M \longrightarrow 0.$$

Let $\tilde{\alpha}$ be an $n \times n$ matrix with coefficients in S which, modulo I, gives α. Then $\mu(I^n, \det \tilde{\alpha}) \leq \mu(I^n)$. (Here, $\mu(J)$ denotes the minimal number of generators of an ideal J.)

Proof. Let $\tilde{\alpha}$ and $\tilde{\beta}$ be liftings of α and β to S. Since $\alpha\beta = 0$ it follows that

$$\tilde{\alpha}\tilde{\beta} = \sum_{i=1}^{m} x_i \tau_i$$

where the τ_i are certain $n \times n$ matrices with coefficients in S. Eisenbud's theorem [10, Theorem 3.1] tells us that at least one τ_i, say τ_1, is invertible, provided α and β represent high enough sysygies of M (which is satisfied in our case since the resolution is periodic). Modulo $x_2, \ldots, x_m$, the above matrix equation yields $\det \tilde{\alpha} \det \tilde{\beta} \equiv \det (x_1 \tau_1) \equiv x_1^n \det \tau_1$. Now our assertion follows easily from the fact that $\det \tau_1$ is a unit and x_1^n a minimal generator of I^n. □

COROLLARY 2.7 (Kunz). *An almost complete intersection is not Gorenstein.*

Proof. Let R be an almost complete intersection. Then R is Cohen–Macaulay, and can be written as $R = S/I$ where S is a regular local ring and I is minimally generated by $m + 1 = \operatorname{codim} R + 1$ elements $x_1, \ldots, x_{m+1}$. By standard general position arguments (if necessary extend the residue class field), we may assume that $x_1, \ldots, x_m$ form a regular sequence. Let $\bar{R}$ denote the complete intersection $R/(x_1, \ldots, x_m)$. Then R is a maximal Cohen–Macaulay $\bar{R}$-module, and, if we assume that R is a Gorenstein ring, R is a self-dual $\bar{R}$-module. Hence if we dualize the $\bar{R}$-resolution $\cdots \to \bar{R} \xrightarrow{a} \bar{R} \to R \to 0$, $a = x_{m+1} \bmod (x_1, \ldots, x_m)$, then we get the exact sequence $0 \to R \to \bar{R} \xrightarrow{a} \bar{R} \to \cdots$. Thus our gluing construction gives the exact sequence

$$\cdots \bar{R} \xrightarrow{\alpha} \bar{R} \longrightarrow \bar{R} \xrightarrow{\alpha} \bar{R} \longrightarrow \cdots.$$

It is now clear that the infinite complex must be periodic. But then, 2.6 implies that $\mu(x_1, \ldots, x_{m+1}) \le \mu(x_1, \ldots, x_m)$, a contradiction. □

3. The gluing construction for the residue field of a complete intersection

Suppose $(R, \mathfrak{m}, k)$ is a local Gorenstein ring of dimension r. Let $\cdots \to T_1 \to T_0 \to k \to 0$ be a minimal free R-resolution of k. Since R is Gorenstein, we have

$$\operatorname{Ext}^i_R(k, R) \cong \begin{cases} k & \text{if } i = r, \\ 0 & \text{if } i \neq r. \end{cases}$$

Thus the dual complex $0 \to T_0^* \to T_1^* \to \cdots \to T_r^* \to \cdots$ has homology only at T_r^* (namely $H_r(T_\bullet^*) \cong k$), and there exists gluing map $\nu_\bullet : T_\bullet \to T_\bullet^*[-r]$. Here we denote by M^* the R-dual of an R-module M. (Note that for all maximal Cohen–Macaulay R-modules, $M^* \cong M^\vee$ since R is Gorenstein.)

In this section we will explicitly describe a specific gluing map for k in case R is a complete intersection: so let $R \cong S/(h_1, \ldots, h_m)$ where S is a regular local ring whose maximal ideal is minimally generated by $x_1, \ldots, x_n$, and where $h_1, \ldots, h_m$ is a regular sequence.

There is a general procedure to construct a minimal free resolution (with algebra structure) of the residue class field. One starts with the Koszul complex $K_\bullet(\mathbf{x}; R)$ which, in case R is regular, provides already a resolution. Otherwise the first homology of the Koszul complex does not vanish, and one adjoints variables in degree 2 in order to kill the homology in degree 1. The new complex has non-vanishing homology at worst in degree 2. If so, one adjoins variables in degree 3, etc. This process leads to the co-called Tate resolution; see [27] for details. In his paper [27], Tate also shows that this process of adjoining variables terminates already in the second step if R is a complete intersection. As a consequence, the Tate resolution of the residue class field of a complete intersection may be viewed as the total complex of a certain double complex. We shall now describe this complex in a way which is appropriate for our purposes.

The maps in this complex are homotheties and their duals: let A be an arbitrary commutative ring, E a free A-module with basis $e_1, \ldots, e_n$ and $z_1, \ldots, z_n$ a sequence of elements of A. We set $\mathbf{z} = \Sigma_{i=0}^n z_i e_i$; then $\mathbf{z}$ may be viewed as degree 1 element in the (graded) exterior algebra $\bigwedge^\bullet E$. Multiplication μ_z by $\mathbf{z}$ makes this algebra into a complex

$$0 \to \overset{0}{\bigwedge} E \xrightarrow{\mu_z} \overset{1}{\bigwedge} E \xrightarrow{\mu_z} \overset{2}{\bigwedge} E \xrightarrow{\mu_z} \cdots$$

since $\mathbf{z} \wedge \mathbf{z} = 0$. Upon dualizing the above complex and using the natural isomorphisms $(\bigwedge^i E)^* \cong \bigwedge^i E^*$ we obtain the complex

$$\cdots \longrightarrow \overset{2}{\bigwedge} E^* \xrightarrow{\partial_z} \overset{1}{\bigwedge} E^* \xrightarrow{\partial_z} \overset{0}{\bigwedge} E^* \longrightarrow 0$$

with $\partial_z = \mu_z^*$, and this is exactly the Koszul complex on the sequence $z_1, \ldots, z_n$.

We return to our situation, and let F be a free R-module with basis $f_1, \ldots, f_m$ and G a free R-module with basis $g_1, \ldots, g_n$. The symmetric algebra of F will be denoted by $S_{\bullet}F$. Notice that $S_{\bullet}F$ is just a polynomial ring in m indeterminates over R, and we may view $G \otimes S_{\bullet}F$ a free module over $S_{\bullet}F$. Since $\bigwedge$ commutes with ring extensions, we have the natural isomorphism

$$\overset{\bullet}{\bigwedge} H \otimes S_{\bullet}F \cong \overset{\bullet}{\bigwedge} (H \otimes S_{\bullet}F) \tag{1}$$

for any R-module H.

Now we write $h_j = \Sigma_{i=1}^n h_{ij} x_i$ for $j = 1, \ldots, m$ with certain $h_{ij} \in S$, and define the elements $\mathbf{x} = \Sigma_{i=1}^n \bar{x}_i g_i \otimes 1 \in G \otimes S_0 F$ and $\mathbf{y} = \Sigma_{i=1}^n (g_i^* \otimes \Sigma_{j=1}^m \bar{h}_{ij} f_j) \in G^* \otimes S_1 F$, where overbar denotes the canonical surjection $S \to R$. We also introduce the R-linear map $\varphi : F^* \to G^*$ with

$$\varphi(f_j^*) = \sum_{i=0}^{n} \bar{h}_{ij} g_i^*$$

for $j = 1, \ldots, m$.

Then, in view of (1) (with $H = G$ or $H = G^*$) and the above considerations, we obtain complexes

$$\cdots \xrightarrow{\mu_x} \overset{i}{\bigwedge} G \otimes S_j F \xrightarrow{\mu_x} \overset{i+1}{\bigwedge} G \otimes S_j F \xrightarrow{\mu_x} \cdots$$

and

$$\cdots \xrightarrow{\partial_y} \overset{i+1}{\bigwedge} G \otimes S_j F \xrightarrow{\partial_y} \overset{i}{\bigwedge} G \otimes S_{j+1} F \xrightarrow{\partial_y} \cdots$$

where, in the explicit form,

$$\partial_y(g_{l_1} \wedge \cdots \wedge g_{l_i}) = \sum_{j=1}^{i} (-1)^{j+1} \varphi^*(g_{l_j}) g_{l_1} \wedge \cdots \wedge g_{\hat{l_j}} \wedge \cdots \wedge g_{l_i}.$$

These complexes fit together to build the rows and columns of a double complex $C_{\bullet\bullet}$ since the following diagrams anti-commute

$$\begin{array}{ccc}\bigwedge^i G\otimes S_jF & \xrightarrow{\partial_y} & \bigwedge^{i-1} G\otimes S_{j+1}F\\ {\scriptstyle\mu_x}\downarrow & & {\scriptstyle\mu_x}\downarrow\\ \bigwedge^{i+1} G\otimes S_jF & \xrightarrow{\partial_y} & \bigwedge^{i} G\otimes S_{j+1}F.\end{array}$$

Now the Tate resolution $T_{\bullet}$ of k is just the total complex of the double complex $T_{\bullet\bullet} = \mathrm{C}^*_{\bullet\bullet}$ with

$$T_{ij} = \left(\bigwedge^i G\otimes S_jF\right)^*$$

Following [7, pp. 17–18] we can now define a gluing map $v_{\bullet} : T_{\bullet} \to T^*_{\bullet}[-r]$. First we choose orientations (i.e. isomorphisms) $\gamma : \bigwedge^m F^* \to R$ of F^* and $\delta : \bigwedge^n G^* \to R$ of G^*, and define $v_i : \bigwedge^i G^* \to (\bigwedge^{r-i} G^*)^*$

$$(v_i(u))(v) = \varepsilon\delta(u \wedge v \wedge ((\bigwedge^m \varphi)(z)))$$

for all $u \in \bigwedge^i G^*$, $v \in \bigwedge^{r-i} G^*$, where $z = \gamma^{-1}(1)$, and $\varepsilon = 1$ when $i = 0, 3 \bmod 4$ and $\varepsilon = -1$ when $i \equiv 1, 2 \bmod 4$. The factor ε is only introduced to make the cone of $v_{\bullet}$ a double complex.

The map v_i gives rise to the homomorphism

$$\left(\bigwedge^i G\otimes S_0F\right)^* \cong \bigwedge^i G^* \xrightarrow{v_i} \left(\bigwedge^{r-i} G^*\right)^* \cong \bigwedge^{r-i} G\otimes S_0F.$$

We now extend this map by 0 outside of $(\bigwedge^i G\otimes S_0F)^*$ to the whole of T_i and call it, without the danger of confusion, again v_i. Thus we have the family of maps

$$v_{\bullet} : T_{\bullet} \to T^*_{\bullet}[-r]$$

The following diagram illustrates the situation for $r = 2$ (Henceforth the symbol $\bigwedge^i \otimes S_j$ stands for $\bigwedge^i G\otimes S_jF$):

$$\begin{array}{ccccccccc}
 & & 0 & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow{\scriptstyle\mu_x} & & \uparrow & & \\
0 & \longrightarrow & (\bigwedge^0\otimes S_0)^* & \xrightarrow{v_0} & \bigwedge^2\otimes S_0 & \xrightarrow{\partial_y} & \bigwedge^1\otimes S_1 & \longrightarrow & \cdots\\
\uparrow & & \uparrow{\scriptstyle\mu_x^*} & & \uparrow{\scriptstyle\mu_x} & & \uparrow{\scriptstyle\mu_x} & & \\
0 \longrightarrow (\bigwedge^0\otimes S_1)^* & \xrightarrow{\partial_y^*} & (\bigwedge^1\otimes S_0)^* & \xrightarrow{v_1} & \bigwedge^1\otimes S_0 & \xrightarrow{\partial_y} & \bigwedge^0\otimes S_1 & \longrightarrow & 0\\
\uparrow{\scriptstyle\mu_x^*} & & \uparrow{\scriptstyle\mu_x^*} & & \uparrow{\scriptstyle\mu_x} & & \uparrow & & \\
\cdots \longrightarrow (\bigwedge^1\otimes S_1)^* & \xrightarrow{\partial_y^*} & (\bigwedge^2\otimes S_0)^* & \xrightarrow{v_2} & \bigwedge^0\otimes S_0 & \xrightarrow{\partial_y} & 0 & & \\
\uparrow{\scriptstyle\mu_x^*} & & \uparrow{\scriptstyle\mu_x^*} & & \uparrow & & & & \\
\vdots & & \vdots & & 0 & & & &
\end{array}$$

THEOREM 3.1. *The just defined map*

$$\nu_\bullet : T_\bullet \longrightarrow T_\bullet^*[-r]$$

is a gluing map for k.

Proof. First we show that $\nu_\bullet$ is a chain map. Since the compositions of $\nu_\bullet$ with both ∂_y^* and ∂_y are zero (see [7], page 17), we only have to show that the following diagram

$$\begin{array}{ccc} \bigwedge^{i-1} G^* & \xrightarrow{\nu_{i-1}} & (\bigwedge^{r-i+1} G^*)^* \\ \uparrow{\scriptstyle \mu_x^*} & & \uparrow{\scriptstyle \mu_x^{**}} \\ \bigwedge^{i} G^* & \xrightarrow{\nu_i} & (\bigwedge^{r-i} G^*)^* \end{array}$$

anti-commutes for all $i = 1, \ldots, r$. To this end we choose arbitrary $u \in \bigwedge^i G^*$ and $v \in \bigwedge^{r-i+1} G^*$. Then $\nu_{i-1}(\mu_x^*(u))(v) = \varepsilon\delta(\mu_x^*(u) \wedge v \wedge ((\bigwedge^m \varphi)(z)))$. On the other hand, $[\mu_x^{**}(\nu_i(u))](v) = [\nu_i(u) \circ \mu_x^*](v) = \nu_i(u)(\mu_x^*(v)) = \varepsilon\delta(u \wedge \mu_x^*(v) \wedge ((\bigwedge^m \varphi)(z)))$. Since δ is an isomorphism, we therefore must show that $\mu_x^*(u) \wedge v \wedge ((\bigwedge^m \varphi)(z)) = (-1)^{i+1}(u \wedge \mu_x^*(v) \wedge ((\bigwedge^m \varphi(z)))$.

To do this we note that $u \wedge v \wedge ((\bigwedge^m \varphi(z)))$ belongs to $\bigwedge^{n+1} G^*$, and hence is zero. Therefore, since μ_x^* is a derivation, it suffices to show that $\mu_x^*((\bigwedge^m \varphi)(z)) = 0$. The element z is a scalar multiple of $f_1^* \wedge \cdots \wedge f_m^*$. We now have that

$$\mu_x^*((\bigwedge^m \varphi)(f_1^* \wedge \cdots \wedge f_m^*)) = \mu_x^*(\varphi(f_1^*) \wedge \cdots \wedge \varphi(f_m^*))$$

and, since $\varphi(f_j^*) = \Sigma_{i=1}^n \bar{h}_{ij} g_i^*$, the just computed element equals

$$\mu_x^*\left(\sum_{i=1}^n \bar{h}_{i1} g_i^* \wedge \cdots \wedge \sum_{i=1}^n \bar{h}_{im} g_i^*\right)$$
$$= \sum_{k=1}^m (-1)^{k-1}\left(\sum_{i=1}^n \bar{h}_{i1} g_i^* \wedge \cdots \wedge \mu_x^*\left(\sum_{i=1}^n \bar{h}_{ik} g_i^*\right) \wedge \cdots \wedge \sum_{i=1}^n \bar{h}_{im} g_i^*\right).$$

But

$$\mu_x^*\left(\sum_{i=1}^n \bar{h}_{ik} g_i^*\right)(1) = \sum_{i=1}^n \bar{h}_{ik} g_i^*(\mu_x(1)) = \left(\sum_{i=1}^n \bar{h}_{ik} g_i^*\right)\left(\sum_{l=1}^n \bar{x}_l g_l\right) = \sum_{i=1}^n \bar{h}_{ik} \bar{x}_i = 0.$$

Next we want to show that $\nu_0(1)$ generates the homology of the complex $T_\bullet^*[-r]$ (concentrated in degree 0). This means that $\nu_0(1)$ should not be in the image of the differential of $T_\bullet^*[-r]$, which, in turn, means that $\nu_0(1)$ is not in the image of the map $\mu_x : (\bigwedge^{r-1} G^*)^* \to (\bigwedge^r G^*)^*$. Thus it suffices to show that $\nu_0(1)$

generates the homology of the co-Koszul complex $K^{\cdot}(\mathbf{x}; R) = ((\bigwedge^{\cdot} G^*)^*, \mu_x) = (\bigwedge^{\cdot} G, \mu_x)$. Consider the corresponding Koszul complex $K_{\cdot}(\mathbf{x}, R)$. Its homology $H_{\cdot}(\mathbf{x}; R)$ is the exterior algebra of $H_1(\mathbf{x}; R)$ since R is a complete intersection; see [1] and [27]. The homology classes $[z_j]$ of the cycles z_j, where $z_j = \Sigma_{i=1}^n \bar{h}_{ij} g_i$, $j = 1, \ldots, m$, form a k-basis of $H_1(\mathbf{x}; R)$, and therefore $H_m(\mathbf{x}; R) \cong k([z_1] \wedge \cdots \wedge [z_m]) = k[z]$, where

$$z = z_1 \wedge \cdots \wedge z_m = \sum_{1 \le i_1 < \cdots < i_m \le n} \det A_{i_1, \ldots, i_m} g_{i_1} \wedge \cdots \wedge g_{i_m} = \sum_{\substack{I \subset \{1, \ldots, n\} \\ |I| = m}} (\det A_I) g_I.$$

In the last expression, for $I = \{i_1, \ldots, i_m\}$, $1 \le i_1 < \cdots < i_m \le n$, we set $G_I = g_{i_1} \wedge \cdots \wedge g_{i_m}$ and

$$A_I = A_{i_1, \ldots, i_m} = \begin{pmatrix} \bar{h}_{i_1 1} & \cdots & \bar{h}_{i_1 m} \\ \vdots & & \vdots \\ \bar{h}_{i_m 1} & \cdots & \bar{h}_{i_m m} \end{pmatrix}$$

Now we consider the isomorphism of complexes

$$\alpha_{\cdot} : K_{\cdot}(\mathbf{x}; R) \longrightarrow K^{\cdot}(\mathbf{x}; R)$$

which is defined as follows: for each $i = 0, \ldots, n$, the isomorphism $\alpha_i : \bigwedge^i G \to \operatorname{Hom}(\bigwedge^{n-i} G, \bigwedge^n G)$ sends $a \in \bigwedge^i G$ to the map $b \mapsto (-1)^i a \wedge b$, where $b \in \bigwedge^{n-i} G$.

The map $\alpha_{\cdot}$ gives rise to an isomorphism $\beta : H_m(\mathbf{x}; R) \cong H^r(\mathbf{x}; R)$. Thus $\beta([z])$ generates $H^r(\mathbf{x}; R)$. Computing this generator explicitly we have $\beta([z]) = [\Sigma_{I \subset \{1, \ldots, n\}, |I| = m} (-1)^{\sigma(I)} \det A_I g^*_{CI}]$, where CI denotes the complement of I in $\{1, \ldots, n\}$ and $\sigma(I)$ is given by the equation $g_I \wedge g_{CI} = (-1)^{\sigma(I)} g_1 \wedge \cdots \wedge g_n$. A straightforward computation for $v_{\cdot}$ shows that $v_0(1) = \beta([z])$. This finishes the proof of the theorem. □

4. Quasihomogeneous complete intersection with isolated singularity

In this section we shall produce a minimal resolution for the module of derivations of a quasihomogeneous complete intersection with isolated singularity. More precisely, we let $S = k[\![X_1, \ldots, X_n]\!]$, where k is a field of characteristic zero, and assign the variables X_i positive degrees: $\deg X_i = a_i$, $a_i \in \mathbb{N}$, $i = 1, \ldots, n$. Also let $h_1, \ldots, h_m$, where $m < n$, be homogeneous polynomials (with respect to the grading) of degree $b_i > 0$, $i = 1, \ldots, m$, respectively, and assume the h_i form a regular sequence. We then call $R = S/(h_1, \ldots, h_m) = k[\![x_1, \ldots, x_n]\!]$ a quasihomogeneous complete intersection. We will further assume that R has an isolated singularity, i.e., $R_{\mathfrak{p}}$ is regular for all prime ideals different from the maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n)$ of R. We set $r = \dim R$. Note that $r = n - m$.

By the Euler formula, we have that

$$b_j h_j = \sum_{i=1}^{n} a_i \frac{\partial h_j}{\partial X_i} X_i,$$

for $j = 1, \ldots, m$. Because the characteristic of k equals 0, the ideals $(h_1, \ldots, h_m)$ and $(b_1 h_1, \ldots, b_m h_m)$ are the same. Hence the elements $b_i h_i$, $i = 1, \ldots, m$, can be viewed as defining equations for R and therefore, as the Euler formula shows, we may take $\mathbf{x} = \Sigma_{i=1}^{n} a_i x_i g_i \otimes 1$ and $\mathbf{y} = \Sigma_{i=1}^{n} (g_i^* \otimes \Sigma_{j=1}^{m} (\partial h_j / \partial x_i) f_j)$ in the construction of the Tate resolution of the residue class field (see Section 3). Here we denote the image of $\partial h_j / \partial X_i$ in R by $\partial h_j / \partial x_i$. It is clear that the cokernel of the map $\varphi^* : G \to F$, $\varphi^*(g_i) = \Sigma_{j=1}^{m} (\partial h_j / \partial x_i) f_j$, as defined in Section 3, is just the transpose of the (universally finite) module $D_k(R)$ of Kähler differentials. Since we assume that R is a complete intersection, the latter module is isomorphic to the module of infinitesimal deformations $T^1 \cong \operatorname{Ext}_R^1 (D_k(R), R)$.

We now form the cone $\operatorname{Con}(v_\bullet)$ of $v_\bullet : T_\bullet \to T_\bullet^*[-r]$ as defined in Section 3. Note that, since $\operatorname{Con}(v_\bullet)$ has the structure of a double complex, there are two natural filtrations defined on $\operatorname{Con}(v_\bullet)$. We consider one of them – the horizontal filtration in regard to the diagram preceding 3.1. Thus we let L_i be the subcomplex of $\operatorname{Con}(v_\bullet)$ containing the modules $\bigwedge^s \otimes S_t$ with $s + t > i$, and the modules $(\bigwedge^s \otimes S_t)^*$ with $s + t < r - i$. We then have the chain of subcomplexes

$$\cdots \subset L_2 \subset L_1 \subset L_0 \subset L_{-1} \subset \cdots \subset \operatorname{Con}(v_\bullet)$$

whose successive quotients L_i / L_{i+1} are isomorphic to the rows of the underlying double complex of $\operatorname{Con}(v_\bullet)$.

For all $i \in \mathbb{Z}$ we consider the quotient complexes $K_i = \operatorname{Con}(v_\bullet)/L_i$. As $\operatorname{Con}(v_\bullet)$ is self-dual, it is easily seen that for all i, up to a shift, the complex L_i and the dualized complex $(K_{r-1-i})^*$ are isomorphic, a fact that will be used later.

The complex K_i begins with

$$\cdots \longrightarrow \bigwedge^{1} \otimes S_{i-1} \xrightarrow{\partial_y} \bigwedge^{0} \otimes S_i \longrightarrow 0$$

if $i > 0$, and with

$$\cdots \longrightarrow \left(\bigwedge^{r} \otimes S_0\right)^* \xrightarrow{v_0} \bigwedge^{0} \otimes S_0 \longrightarrow 0$$

if $i = 0$. In the first case, $\operatorname{Coker} \partial_y$ is isomorphic to the i-th symmetric power $S_i(T^1)$

of T^1 which, as we remarked already, is isomorphic to the cokernel of $\varphi : G^* \to F^* = \partial_y : \bigwedge^1 \otimes S_0 \to \bigwedge^0 \otimes S_1$. In the second case, Coker $v_0 \cong R/J$ where J is the Kähler different, i.e. the ideal of $m \times m$-minors of the Jacobian matrix $(\partial h_j/\partial x_i)$. As T^1 is annihilated by J we may view T^1 an R/J-module, and therefore set $S_0(T^1) = R/J$.

THEOREM 4.1. *For all $i = 0, \ldots, r$, the complexes K_i are acyclic. In particular, for those i the complex K_i is a minimal free resolution of $S_i(T^1)$.*

Proof. The filtration $\{L_j\}$ on Con $(v_\bullet)$ induces a filtration F_j on K_i with $F_j = L_j \cap K_i$ and the following properties:

(i) $F_j = 0$ for $j \leq i$.

(ii) $F_j/F_{j+1} \cong L_j/L_{j+1}$ is a row in the double complex Con $(v_\bullet)$.

(iii) For all l and j large enough, the module of l-chains P_l of K_i is a submodule of F_j.

Since R is an isolated singularity the map φ in the construction of Con $(v_\bullet)$ is split surjective on the punctured spectrum of R, and this implies that all rows of Con $(v_\bullet)$ are exact on the punctured spectrum of R. That this is the case for the rows containing the maps v_i follows from [7, Proposition 2.7], and for the other rows this is immediate since these are the homogeneous parts of a Koszul complex or its dual. It now follows that all rows have finite length homology, and thus (ii) and (iii) imply that this is true for the homology of K_i as well.

To simplify notation we denote the complex K_i by $(P_\bullet, \alpha_\bullet)$. Consider the beginning of K_i

$$0 \longrightarrow B \longrightarrow P_{r-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow 0. \tag{1}$$

where $B = \operatorname{Im} \alpha_{r-1}$. Since, for the i in the specified range, K_i coincides with Con $(v_\bullet)$ from P_r on, it follows that the homology of the complex

$$\cdots \longrightarrow P_{r+1} \xrightarrow{\alpha_r} P_r \longrightarrow B \longrightarrow 0$$

may be nonzero only at P_r. Since Coker α_r is an infinite syzygy module of Con $(v_\bullet)$ this cokernel is a maximal Cohen–Macaulay module. It is clear that we have the following short exact sequence

$$0 \longrightarrow H_r(P_\bullet) \longrightarrow \operatorname{Coker} \alpha_r \longrightarrow B \longrightarrow 0.$$

As we remarked already, $H_r(P_\bullet)$ has finite length, and therefore, as a submodule of a maximal Cohen–Macaulay module of positive dimension, is zero. We now conclude that B is maximal Cohen–Macaulay.

To finish the proof it suffices to show that (1) is acyclic. But all modules in this complex are maximal Cohen–Macaulay and its homology is of finite length. Thus the acyclicity follows from [20]. □

As a first application of the theorem we generalize one direction of a result in [16].

COROLLARY 4.2. *For all i and j we have that $\delta^j(S_i(T^1)) = 0$.*

Proof. All the modules $S_j(T^1)$ as well as their syzygy modules are homomorphic images of certain syzygy modules of $\operatorname{Con}(v_\bullet)$, which are all maximal Cohen–Macaulay modules without free summands. According to the comment at the beginning of Section 2, this proves the assertion. □

COROLLARY 4.3. *K_1 truncated at degree 2 is a minimal free resolution of the module $D_k(R)^*$ of derivations of R over k. In particular, the module of derivations is minimally generated by the $\binom{n}{r-1}$ elements in the image of v_{r-1}, the so-called trivial derivations, and one extra generator, the Euler derivation.*

Proof. We write down the beginning of the complex K_1:

$$\cdots \longrightarrow \left(\bigwedge^{r-1} \otimes S_0\right)^* \otimes \bigwedge^0 \otimes S_0 \xrightarrow{\psi} \bigwedge^1 \otimes S_0 \longrightarrow \bigwedge^0 \otimes S_0 \longrightarrow 0.$$

The complex K_1 is a minimal free resolution of T^1. Therefore, $\tau_2 K_1$ resolves the second syzygy module $\operatorname{Im}\psi$ of T^1 which is just $D_k(R)^*$. The image of ψ restricted to the first summand of $(\bigwedge^{r-1} \otimes S_0)^* \oplus \bigwedge^0 \otimes S_0$ gives the trivial derivations, the restriction to the second summand the Euler derivation. □

REMARK 4.4. The fact that the trivial derivations and the Euler derivation generate $D_k(R)^*$ has been observed by Kunz and Waldi [15] in dimension one, even when R is a quasihomogeneous Gorenstein ring, and by Kersken [13] in all dimensions. Corollary 4.3 shows that those are minimal generators.

DEFINITION 4.5. We call two modules M an N *syzygetically equivalent* if there exist natural numbers i and j such that $\Omega^i(M) \cong \Omega^j(N)$.

Thus we can paraphrase Corollary 4.3 by saying that if R is a quashihomogeneous complete intersection with isolated singularity over a field k of characteristic 0, then the residue field of R is syzygetically equivalent to T^1. We see that the integers $i = 2r - 3$ and $j = r$ given in Corollary 4.3 are the smallest possible with respect to the property in Definition 4.5. This does not however mean that for some particular rings they cannot be improved; for example, if R is a hypersurface of odd dimension ≥ 3 then, as was shown in [18, Prop 2.2], $\Omega^r(k) \cong \Omega^r(T^1)$. In particular, Conjecture 2.1 from [18] that gave a smaller number i for the residue field is not true. But taking now 4.5 into account we can reintroduce that conjecture in the following modified form:

Let R be a complete local analytic algebra of dimension ≥ 1 with an isolated singularity over a field k of characteristic zero. Is it true that R is quasihomogeneous if and only if the residue field k of R is syzygetically equivalent to the transpose $\operatorname{Tr} D_k(R)$ of the module of Kähler differentials of R over k?

Note that $T^1 \cong \operatorname{Tr} D_k(R)$ only when R is a complete intersection. The results known so far indicate however that if R is not a complete intersection, T^1 should be replaced by $\operatorname{Tr} D_k(R)$. See the papers [19] and [11] for the state of this problem in dimension 2.

By the definition of the complexes K_i and L_i, we have an exact sequence

$$0 \longrightarrow L_i \longrightarrow \operatorname{Con}(v_\bullet) \longrightarrow K_i \longrightarrow 0. \tag{1}$$

But, as we mentioned already, $L_i \cong (K_{r-1-i})^*$, so that for all i the long exact sequence derived from (1) gives us the isomorphisms

$$H_0(K_i) \cong H_1((K_{r-1-i})^*[-r-1]) \cong H_{-r}((K_{r-1-i})^*).$$

Notice that

$$H_{-r}((K_{r-1-i})^*) \cong \operatorname{Ext}^r(H_0(K_{r-1-i}), R) = H_0(K_{r-1-i})^\vee.$$

Combining these observations with 4.1 we get

COROLLARY 4.6. $(S_i(T^1))^\vee \cong S_{r-1-i}(T^1)$ *for all* $i = 0, \ldots, r-1$.

Note that 4.6 in particular implies that $S_{r-1}(T^1)$ is the canonical module of R/J where J, as before, is the Kähler different (also called the jacobian ideal of R).

If $r = 2$ and $i = 1$ we deduce from 4.6 that $T^1 \cong (R/J)^\vee$, and when $r = 3$ and $i = 2$ we have that T^1 is self-dual. Thus 4.6 may be viewed as an extension, although in the more restrictive case of complete intersections, to higher dimensions and arbitrary fields of characteristic zero of the results of J. Wahl [28, Th. 2.2, Th. 2.3]. If $r = 1$, i.e., R is a complete intersection curve, we have that $(R/J)^\vee \cong R/J$, which means that R/J is a Gorenstein ring. Thus we recover (the easier part) of a theorem of Kunz and Waldi [15]. They actually prove that R is quasihomogeneous if and only if R/J is Gorenstein.

Quasihomogeneity in higher dimensions is also reflected by the Gorenstein property of a certain algebra. Indeed, the next statement follows immediately from 4.6.

COROLLARY 4.7. *The truncate symmetric algebra* $G = (\bigoplus_{i=0}^{\infty} S_i(T^1))/(S_r(T^1))$ *is Gorenstein.*

Note that this statement once again contains the aforementioned result of Kunz and Waldi (when $r = 1$), and when R is a hypersurface ring the algebra G is simply $(R/J)[t]/(t^r)$ where J is the Kähler different. It is clear that in this case, G is Gorenstein if and only if R/J is Gorenstein. On the other hand, the second author noticed in his paper [18] that Saito's theorem [22] can be rephrased by saying that an isolated hypersurface singularity R is quasihomogeneous if and only if R/J is Gorenstein. Thus in view of these results there is some evidence that for an isolated complete intersection R the algebra G is Gorenstein if and only if R is quasihomogeneous.

We close this paper with a few observations concerning the module structure of the symmetric powers of T^1.

COROLLARY 4.8. *Let J be the Kähler different of R. Then for $i = 0, \dots, r-1$ the symmetric powers $S_i(T^1)$ are faithful R/J-modules of type $\binom{m+r-2-i}{r-1-i}$ if $r > 1$, and $\binom{m+1}{2}$ if $r = 1$.*

Proof. By 4.6, the $(r-1)$-th symmetric power $S_{r-1}(T^1)$ of T^1 is the canonical module of R/J, and hence is a faithful R/J-module. Then, clearly all the lower powers $S_i(T^1)$, $i = 0, \dots, r-1$, are faithful R/J-modules as well.

Next observe that the type of a module M is the minimal number of generators of its dual $M^\vee$. Thus the formula for the type follows for $r > 1$ from 4.6, and for $r = 1$ notice that $H_0(K_0)^\vee \cong H_0(K_{-1})$; see the arguments preceding 4.6. Since K_{-1} ends with $(\bigwedge^2 \otimes S_0)^*$ we obtain the desired result. □

Let $I \subset S$ denote the ideal of maximal minors of the matrix $Y = (\partial h_j/\partial X_i)$. Note that the extension ideal IR of I in R equals the Kähler different, and hence, since we assume that R is an isolated singularity, has height r. It follows that the ideal I in S has height at least $r = n - m$. On the other hand, the generic height of this determinantal ideal is at most $n - m + 1$. So that we have

$$n - m + 1 \geq \text{height}\, I \geq n - m.$$

We are grateful to Ulrich who told us the following argument showing that the upper bound is always attained: we let X be the column vector $(a_1X_1, \dots, a_nX_n)^t$.

Then, because of the Euler equations, $I_1(YX) = (h_1, \ldots, h_n)$, and thus, since R is an isolated singularity, height $(I_1(YX), I_m(Y)) = n$. (Here we denote as usual by $I_l(C)$ the ideal of all l-minors of a matrix C.)

Let T be the column vector $(T_1, \ldots, T_n)'$ in the new variables T_i, and set $J = (I_1(YT), I_m(Y))$. Then height $J \geq n$, and hence $\dim B \leq n$ where $B = S[T]/J$. Set $A = S/I_m(Y)$; then B may be interpreted as the symmetric algebra of the A-module M which is defined as the cokernel of the homomorphism $A^m \to A^n$ given by the matrix Y (modulo $I_m(Y)$). Assume the height of $I_m(Y)$ is $n - m$. Then there exists a minimal prime ideal $\mathfrak{p}$ of A with $\dim A/\mathfrak{p} = m$. By the Huneke–Rossi dimension formula [12] for symmetric algebras we have

$$n = \dim B \geq \dim A/\mathfrak{p} + \mu(M_\mathfrak{p}) = m + \mu(M_\mathfrak{p}).$$

But $\mu(M_\mathfrak{p}) > n - m$ since $I_m(Y) \subset \mathfrak{p}$.

By different arguments, communicated to us by Kunz and Waldi, this result can be shown even when the h_i are not quasihomogeneous.

PROPOSITION 4.9. *The symmetric powers* $S_i(T^1)$, $i = 0, \ldots, r-1$, *all have the same length.*

Proof. Let us denote by C_i the row of the double complex $\operatorname{Con}(v_\cdot)$ with last non-zero term $\bigwedge^0 \otimes S_i$ on the right. Then we get the exact sequence of complexes $0 \to C_i \to K_i \to K_{i-1} \to 0$. Since both complexes, K_{i-1} and K_i, are acyclic the corresponding long exact homology sequence yields for $i = 1, \ldots, r-1$ the isomorphisms

$$H_0(C_i) \cong S_i(T^1) \quad \text{and} \quad H_1(C_i) \cong S_{i-1}(T^1). \tag{1}$$

There are complexes D_i of free S-modules with $D_i \otimes R \cong C_i$, and these complexes are acyclic since height $I = n - m + 1$; see [7, Theorem 2.16]. It follows that $H_j(C_i) \cong \operatorname{Tor}_j^S(H_0(D_i), S) \cong H_j(h_1, \ldots, h_m; H_0(D_i))$, the Koszul homology of $H_0(D_i)$ with respect to the sequence $h_1, \ldots, h_m$. The last isomorphism is valid since the h_i form a regular S-sequence. Now we use the fact, due to Serre [23, Chapitre IV], that the Euler characteristic of the Koszul homology is non-negative. In our case this implies that $\ell(H_1(C_i)) \geq \ell(H_0(C_i))$ for all i. Therefore it follows from (1) that

$$\ell(S_0(T^1)) \leq \ell(S_1(T^1)) \leq \cdots \leq \ell(S_{r-1}(T^1)).$$

On the other hand, $\ell(S_{r-1}(T^1)) = \ell(S_0(T^1))$ since the module is the canonical module of $R/J = S_0(T^1)$. Thus all lengths under consideration must be the same. $\square$

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