Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 68 (1993)

Artikel: Groups with no infinite perfect subgroups and aspherical 2-complexes.
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DOI: https://doi.org/10.5169/seals-51773

Comment. Math. Helvetici 68 (1993) 333–339 0010-2571/93/020333-07$1.50 + 0.20/0

Groups with no infinite perfect subgroups and aspherical 2-complexes

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Abstract. The purpose of this paper is to generalize a theorem of J. F. Adams. He showed in [A] that if X is a subcomplex of an aspherical 2-complex and the fundamental group G of X has no non-trivial perfect subgroups, then X is aspherical. We weaken the hypothesis on G to "no infinite perfect subgroups."

1. Introduction

In [W], J. H. C. Whitehead, asked the following question: *Is a subcomplex of an aspherical 2-complex aspherical?*

A $[G, 2]$-*complex* X is a connected two-dimensional CW-complex with fundamental group $\pi_1 X \cong G$. If N is a subgroup of $\pi_1 X$, let X_N denote the covering of X corresponding to N. For any group G, let $H_i G$ denote the ith homology of G with coefficients in the integers $\mathbb{Z}$. A group G is said to be *perfect* if the abelianization $H_1 G$ of G is trivial; G is *superperfect* if $H_1 G = H_2 G = 0$.

A $[G, 2]$-complex X is *aspherical* iff its second homotopy group $\pi_2 X$ vanishes. If X is a $[G, 2]$-complex which is a subcomplex of an aspherical 2-complex, then J. F. Adams showed in [A] that X is aspherical provided G has no non-trivial perfect subgroups. In this note we show that X is aspherical provided G is finitely presented and has no *infinite* perfect subgroups.

The idea of the proof is to show that if X is a $[G, 2]$-complex and G is a finitely presented group which has a finite, non-trivial, normal, superperfect subgroup P such that $Q = G/P$ has cohomological dimension 1 or 2, then the Hurewicz homomorphism $\pi_2 X \to H_2 X_P$ is non-trivial.

2. Basic definitions

If X is a connected 2-complex and N is a subgroup of $\pi_1 X$ then X *is* N-*Cockcroft* if the Hurewicz homomorphism $\pi_2 X = \pi_2(X_N) \to H_2(X_N)$ is trivial. The N-Cockcroft property has been extensively studied in [Bo, BD, BDS, D, GH, H].

Let N be a subgroup of G. Then we say that G *is* N-*Cockcroft* if there is a $[G, 2]$-complex X and an isomorphism $\varphi : G \to \pi_1 X$ such that X is φN-Cockcroft.

The following is the main theorem of this paper.

2.1 THEOREM. *Let P be a non-trivial, finite, superperfect, normal subgroup of a finitely presented group G such that $Q = G/P$ has cohomological dimension 1 or 2. Then G is not P-Cockcroft.*

Note that the theorem is false if $Q = 1$. In this case, $G = P$ is finite and superperfect. Let G be the binary icosahedral group. In this case, G admits a presentation with 2 generators and 2 relators. The realization of this presentation as a $[G, 2]$-complex has $H_2X = 0 = H_1X$, so X is P-Cockcroft.

If G is a group, the *maximal perfect subgroup PG of G* is defined as the normal subgroup of G generated by all perfect subgroups; it is also the intersection of the (transfinite) derived series of G.

2.2 COROLLARY. *Let G be a finitely presented group with maximal perfect subgroup PG finite. Then any $[G, 2]$-complex X which is the subcomplex of an aspherical 2-complex is aspherical.*

Proof. If G is finite, the result is well known (see [BD]). Hence we will assume that Q is infinite. If the $[G, 2]$-complex X is a subcomplex of an aspherical 2-complex and X is not aspherical, then by the main theorem of [BDS], we see that there must exist a superperfect, normal, non-trivial subgroup P of G such that G is P-Cockcroft and the quotient Q has $\operatorname{cd} Q \leq 2$. The group Q is infinite, so the cohomological dimension of Q is 1 or 2. But the maximal perfect subgroup of G is finite, so P is infinite. The theorem then says that G cannot be P-Cockcroft. Thus X must be aspherical. $\square$

3. Two lemmas

In this section we will prove two lemmas preliminary to giving a proof of the theorem.

Let G be a group and let $\mathbf{C}$ be a projective $\mathbb{Z}G$-resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$. To each integer $i \geq 0$ we have an associated kernel $K_i = \ker \{\partial_i : C_i \to C_{i-1}\}$ $(C_{-1} = \mathbb{Z})$. For any $[G, 2]$-complex X, let $\tilde{X}$ be the universal covering of X. Then $C_*\tilde{X}$, the cellular chain complex of $\tilde{X}$, can be thought of as a partial resolution (of length two) of free left $\mathbb{Z}G$-modules. For any $[G, 2]$-complex X, the kernel $K_1 = \ker \{\partial_1 : C_1\tilde{X} \to C_0\tilde{X}\}$ is called the *relation module determined by X*.

For any left $\mathbb{Z}G$-module M, we let M^G denote the subgroup of elements fixed by the action of G; we let $M_G = \mathbb{Z} \otimes_{\mathbb{Z}G} M = M/IG \cdot M$ (IG is the augmentation ideal in $\mathbb{Z}G$) be M with the G-action killed.

3.1 LEMMA. *If P is a finite, normal subgroup of a group G and $Q = G/P$, then $H^i(G, \mathbb{Z}G) \cong H^i(G, \mathbb{Z}Q) \cong H^i(Q, \mathbb{Z}Q)$ for all $i > 0$. The first isomorphism is induced by $\mathbb{Z}G \to \mathbb{Z}Q$ and the second by $G \to Q$.*

Proof. Because P is finite, we have $H^j(P, \mathbb{Z}G) = 0$ for $j > 0$. By using the Lyndon–Hochschield–Serre spectral sequence, we see that $H^i(G, \mathbb{Z}G) \cong H^i(Q, \mathbb{Z}G^P)$ for $i > 0$. But clearly $\mathbb{Z}G^P \cong \bigoplus_{a\in Q} (\mathbb{Z}P)_a^P \cong \bigoplus_{a\in Q} (\mathbb{Z})_a \cong \mathbb{Z}Q$ as a $\mathbb{Z}Q$-module. □

3.2 LEMMA. *Let X be a $[G, 2]$-complex and suppose P is a superperfect, normal subgroup of $\pi_1 X$ such that the Hurewicz map from $\pi_2 X = \pi_2 X_P \to H_2 X_P$ is trivial (i.e., G is P-Cockcroft with respect to X). Let $K_1 = \ker\{\partial_1 : C_1\tilde{X} \to C_0\tilde{X}\}$ be the relation module determined by X, where $\tilde{X}$ is the universal covering of X. Then $\mathbb{Z} \otimes_{\mathbb{Z}P} K_1 = \ker\{\partial_1(X_P) : C_1(X_P) \to C_0(X_P)\} \cong \mathbb{Z} \otimes_{\mathbb{Z}P} C_2\tilde{X}$ is a relation module for $Q = (\pi_1 X)/P$. Furthermore, the surjection $G \to Q$ induces an isomorphism $H^2(G, \mathbb{Z} \otimes_{\mathbb{Z}P} K_1) \cong H^2(Q, \mathbb{Z} \otimes_{\mathbb{Z}P} K_1)$.*

Proof. Because P is a subgroup of $\pi_1 X$ we have $C_i X_P \cong \mathbb{Z} \otimes_{\mathbb{Z}P} C_i\tilde{X}$. That P is superperfect and G is P-Cockcroft with respect to X implies that

$$0 \to C_2 X_P \to C_1 X_P \to C_0 X_P \to \mathbb{Z} \to 0$$

is an exact sequence of free $\mathbb{Z}Q$-modules (a free resolution of the trivial module $\mathbb{Z}$). Tensoring the exact sequence (of $\mathbb{Z}G$-modules) $0 \to \pi_2 X \to C_2\tilde{X} \to K_1 \to 0$ with $\mathbb{Z} \otimes_{\mathbb{Z}P} -$ and using the fact that X is P-Cockcroft, we see that $\mathbb{Z} \otimes_{\mathbb{Z}P} K_1 \cong C_2 X_P$.

The isomorphism $H^2(G, \mathbb{Z} \otimes_{\mathbb{Z}P} K_1) \cong H^2(Q, \mathbb{Z} \otimes_{\mathbb{Z}P} K_1)$ follows from the LHS spectral sequence for the extension

$$1 \to P \to G \to Q \to 1$$

together with the facts that P is superperfect and that $\mathbb{Z} \otimes_{\mathbb{Z}P} K_1$ is a trivial $\mathbb{Z}P$-module. □

4. Proof of Theorem 2.1

From now on we assume that X is a $[G, 2]$-complex with fundamental group *equal* to G. We let P be a finite, superperfect, normal subgroup of G so that the Hurewicz map $\pi_2 X \to H_2 X_P$ is trivial. We let $Q = G/P$ have cohomological dimension 1 or 2 and K_1 be the relation module determined by X. The proof by contradiction is given in a series of steps as follows.

STEP 1 is devoted to the proof of the following claim. Let p be the order of the finite group P and consider the inclusion $K_1^P \to \mathbb{Z} \otimes_{\mathbb{Z}P} K_1 = K_{1P}$.

CLAIM. *If P is superperfect, then the image of K_1^P inside $\mathbb{Z} \otimes_{\mathbb{Z}P} K_1$ is $p \cdot \mathbb{Z} \otimes_{\mathbb{Z}P} K_1$.*

Proof of the Claim. Let $F_2 \to F_1 \to \mathbb{Z}P \to \mathbb{Z} \to 0$ be a partial resolution of $\mathbb{Z}$ over $\mathbb{Z}P$ by finitely generated free modules. Let L_1 denote the kernel of the map $\partial_1 : F_1 \to \mathbb{Z}P$. Then the following diagram commutes:

$$\begin{tikzcd}
F_2^P \arrow[rr, "\partial_1^P"] \arrow[dr] \arrow[ddd] & & F_1^P \arrow[ddd] \\
& L_1^P \arrow[ur] \arrow[d] & \\
& L_{1P} \arrow[dr] & \\
F_{2P} \arrow[ur] \arrow[rr, "\partial_{1P}"'] & & F_{1P}
\end{tikzcd}$$

The group P is finite implies that the vertical arrows are monomorphisms. The two outer vertical arrows are clearly multiplication by p because the modules are free. The group P is perfect implies that ∂_{1P} and ∂_1^P are epimorphisms and hence $L_{1P} = F_{1P}$ and $L_1^P = F_1^P$. Thus the interior vertical arrow has image which is multiplication by p. Now one uses Schanuel's lemma and a simple argument to show that the same is true of $K_1^P \to K_{1P}$. This completes the proof of the claim.

Hence the $\mathbb{Z}Q$-module $A = \mathbb{Z} \otimes_{\mathbb{Z}P} K_1 / K_1^P = \mathbb{Z} \otimes_{\mathbb{Z}P} K_1 / p \cdot \mathbb{Z} \otimes_P K_1$. If we write $\mathbb{Z} \otimes_{\mathbb{Z}P} K_1 \cong \mathbb{Z}Q^\alpha$ ($= \mathbb{Z} \otimes_{\mathbb{Z}P} C_2 \tilde{X}$; this follows from lemma 3.2), then $A \cong \mathbb{Z}_p Q^\alpha$.

STEP 2. The following diagram is commutative, with top and vertical sequences exact:

$$\begin{tikzcd}
& 0 \arrow[d] & & & \\
H^1(Q, A) \arrow[r] & H^2(Q, K_1^P) \arrow[r, "f'"] \arrow[d, "i"] & H^2(Q, \mathbb{Z} \otimes_{\mathbb{Z}P} K_1) \arrow[r] \arrow[d, "j"', "\cong"] & H^2(Q, A) \arrow[r] & H^3(Q, K_1^P) \arrow[d, equal] \\
& H^2(G, K_1) \arrow[r, "f"] \arrow[d, "h"] & H^2(G, \mathbb{Z} \otimes_{\mathbb{Z}P} K_1) \arrow[d] & & 0 \\
& \mathbb{Z}_p \cong H^2(P, K_1)^Q \arrow[r] \arrow[d] & H^2(P, \mathbb{Z} \otimes_{\mathbb{Z}P} K_1)^Q = 0 & & \\
& 0 & & &
\end{tikzcd} \tag{4.1}$$

The horizontal maps f and f' are induced by $K_1 \to \mathbb{Z} \otimes_{\mathbb{Z}P} K_1$ and $K_1^P \to \mathbb{Z} \otimes_{\mathbb{Z}P} K_1$, respectively. By using a dimension shifting argument one shows that $H^2(P, K_1) \cong \mathbb{Z}_p$ has trivial $\mathbb{Z}Q$-action. The fact that $p \cdot A = 0$ shows that $p \cdot H^2(Q, A) = 0$ also. The vertical sequences come from the LHS-spectral sequence. The left-most vertical sequence is exact, because $\operatorname{cd} Q \leq 2$ and $H^1(P, K_1) = 0$ (this is a consequence of the finiteness of P). The fact that $H^2(P, \mathbb{Z} \otimes_{\mathbb{Z}P} K_1) = 0$ follows because P is superperfect and $\mathbb{Z} \otimes_{\mathbb{Z}P} K_1$ is a trivial $\mathbb{Z}P$-module. We observe that the map f' is an isomorphism modulo torsion; that is to say, the kernel and the cokernel of f' are torsion groups. The group $H^3(Q, K_1^P) = 0$ because Q is two dimensional. By lemma 3.2, $\mathbb{Z} \otimes_{\mathbb{Z}P} K_1$ is a free $\mathbb{Z}Q$-module, so j is an isomorphism, by lemma 3.1.

STEP 3. Let M be any $\mathbb{Z}G$-module and $\rho(M) : M \to \mathbb{Z} \otimes_{\mathbb{Z}P} M$ be the natural surjection. We will show that $\rho(K_1) : K_1 \to \mathbb{Z} \otimes_{\mathbb{Z}P} K_1$ induces a split epimorphism

$$f : H^2(G, K_1) \to H^2(G, \mathbb{Z} \otimes_{\mathbb{Z}P} K_1).$$

We will show that there is a map $s : H^2(G, \mathbb{Z} \otimes_{\mathbb{Z}P} K_1) \to H^2(G, K_1)$ such that fs is an isomorphism.

Now $H^2(G, C_2\tilde{X}) \cong H^2(G, \mathbb{Z} \otimes_{\mathbb{Z}P} K_1)$, by lemma 3.1; the isomorphism is induced by $\rho(K_1)\,\partial_2$, where $\partial_2 : \mathbb{Z}G^\alpha = C_2\tilde{X} \to K_1$. This last follows because $\rho(K_1)\partial_2 = (1 \otimes \partial_2)\rho(C_2\tilde{X})$. The map $1 \otimes \partial_2$ is an isomorphism because G is P-Cockcroft and $\rho(C_2\tilde{X})$ induces an isomorphism on $H^2(G, -)$ by lemma 3.1. Thus the map ∂_2 induces a map $g : H^2(G, C_2\tilde{X}) \to H^2(G, K_1)$ whose composite gf is induced by the natural map $\mathbb{Z}G^\alpha \to \mathbb{Z}Q^\alpha$. Thus gf is an isomorphism, again by 3.1. Hence f is a split epimorphism and the map s can be chosen as $s = \partial_{2*}(\rho(K_1)\,\partial_2)_*^{-1}$.

STEP 4. We will show that, if $i : H^2(G, K_1^P) \to H^2(G, K_1)$ is the map in diagram 4.1, then $\operatorname{im} s = \operatorname{im} i$.

First we observe that, by definition, $\operatorname{im} s = \operatorname{im} \partial_{2*}$. Let $K_2 = \ker \partial_2$ and consider the long exact sequence arising from the short exact sequence $0 \to K_2 \to C_2\tilde{X} \to K_1 \to 0$;

$$\cdots \to H^2(G, C_2\tilde{X}) \xrightarrow{\partial_{2*}} H^2(G, K_1) \to H^3(G, K_2) \to H^3(G, C_2\tilde{X}) = 0.$$

The group $H^3(G, C_2\tilde{X}) = 0$ by 3.1 and the fact that $\operatorname{cd} Q \leq 2$ (3.2).

The commutativity of the diagram below (where we identify $H^3(P, K_2)$ with $H^2(P, K_1)$) shows that $\operatorname{im} i = \operatorname{im} \partial_{2*} = \operatorname{im} s$:

$$\begin{array}{ccccc} & & H^2(G, K_1^P) & & \\ & & \downarrow{\scriptstyle i} & & \\ H^2(G, C_2\tilde{X}) & \xrightarrow{\partial_{2*}} & H^2(G, K_1) & \longrightarrow & H^3(G, K_2) \longrightarrow 0 \\ & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \cong} \\ & & H^2(P, K_1)^Q & \xrightarrow[\cong]{} & H^3(P, K_2)^Q. \end{array} \tag{4.2}$$

STEP 5. We show that $\mathbb{Z}_p \otimes H^2(Q, \mathbb{Z}Q) = 0$.

The map fi (see 4.1) is an isomorphism because $\ker f \cap \operatorname{im} i = \ker f \cap \operatorname{im} s = 0$. This implies $f' = j^{-1}fi$ is an isomorphism. Thus, $H^2(Q, A) = 0$ and hence $\mathbb{Z}_p \otimes H^2(Q, \mathbb{Z}Q) = H^2(Q, \mathbb{Z}_pQ) = 0$.

STEP 6. The contradiction.

Case 1 (Q is free). The same proof above works (by simply reducing the dimension of the cohomology groups and the kernels by one in 4.1 and 4.2) to show that $\mathbb{Z}_p \otimes H^1(Q, \mathbb{Z}Q) = 0$. But this is impossible because $H^1(Q, \mathbb{Z}Q)$ is known to be free abelian and non-trivial [Sw, corollary 3.7]. Thus, G is P-Cockcroft and Q free leads to a contradiction.

Case 2: ($\operatorname{cd} Q = 2$). Because P is finite and $\operatorname{cd} Q = 2$ we have that $\mathbb{Z}_p \otimes H^2(Q, \mathbb{Z}Q) = 0$ by step 5.

Because G is finitely presented, so is Q. We observe that ([BE], theorem 5.2) Q is a free product of duality groups of dimension 1 or 2. Let R be one of the factors, and define $D = H^2(Q, \mathbb{Z}Q)$ and $E = H^2(R, \mathbb{Z}R)$. Let q be any prime divisor of p. The fact that $\mathbb{Z}_p \otimes D = 0$ implies that $\mathbb{Z}_q \otimes D = 0$. This in turn implies that $\mathbb{Z}_q \otimes E = 0$. If R is a duality group of dimension 2, we have, for any $\mathbb{Z}_qQ$-module M, $H^2(R, M) \cong \mathbb{Z} \otimes_{\mathbb{Z}R} (M \otimes D)$. But because M is a $\mathbb{Z}_q$-module, we have $M \otimes D \cong \mathbb{Z}_q \otimes M \otimes D = 0$. Hence, the cohomological dimension of $R \leq 1$ over the ring $\mathbb{Z}_q$. This, together with the fact that R is torsion-free, shows that $\operatorname{cd} R = 1$. Hence R is free and so Q is free. This brings us back to case 1. Hence no such group G can be P-Cockcroft. This finishes the proof of Theorem 2.1. □

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Received July 21, 1992

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