

**Zeitschrift:** Commentarii Mathematici Helvetici  
**Herausgeber:** Schweizerische Mathematische Gesellschaft  
**Band:** 68 (1993)

**Artikel:** Local fundamental groups of surface singularities in characteristic p.  
**Autor:** Cutkosky, S.D. / Srinivasan, Hema  
**DOI:** <https://doi.org/10.5169/seals-51772>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 02.08.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

## Local fundamental groups of surface singularities in characteristic $p$

STEVEN DALE CUTKOSKY\* AND HEMA SRINIVASAN\*

The local fundamental group of a normal singularity gives much information about the nature of the singularity. For instance, there is Mumford's theorem [M] that the local fundamental group of the germ of a normal complex analytic surface is zero if and only if the surface is smooth. This has been generalized by Flenner [F] to show that if  $(A, m)$  is a normal henselian equicharacteristic zero local ring of dimension two, with algebraically closed residue field, then the algebraic fundamental group  $\pi_1(\text{spec}(A) - m) = 0$  if and only if  $A$  is smooth. Artin has shown that Mumford's characterization of smooth surface germs is false in characteristic  $p$ . (c.f. [A3]) The simplest example is the rational double point  $k[[x, y, z]]/x^2 + y^2 + z^p$  which has trivial local fundamental group in characteristic  $p$ .

In Section 1 we generalize the results of Mumford [M] to characteristic  $p \geq 0$ . Suppose that  $(S, x)$  is a surface singularity of characteristic  $p \geq 0$ . We first demonstrate that if  $\pi_1(S - x)$  is finite, then the intersection diagram of a resolution of singularities of  $S$  is simply connected, with vertices of genus 0. When the intersection diagram of a resolution of singularities of  $S$  is of this form, we show that there is an expression for the generators and relations of the prime to  $p$  part of the local fundamental group of  $S$ , which is determined by the intersection matrix of the resolution of singularities of  $S$ . This is proved in Theorem 3.

**THEOREM 3.** *Let  $(A, m)$  be a complete normal local domain of dimension two over an algebraically closed field  $k$  of characteristic  $p \geq 0$ . Let  $\sigma : X \rightarrow \text{spec}(A)$  be a resolution of singularities, such that the reduced exceptional fiber has simple normal crossings, with irreducible exceptional curves  $E_1, \dots, E_n$ . Suppose that the intersection graph of the exceptional locus is simply connected, and that each  $E_i$  is a nonsingular rational curve. Let  $F_n$  be the free group on the symbols  $\alpha_1, \dots, \alpha_n$ . Then there exists a reindexing of the  $E_i$  such that*

$$\pi_1^{(p)}(\text{spec}(A) - m) \cong \pi_1^{(p)}\left(X - \sum_{i=1}^n E_i\right) \cong (F_n/N)^{(p)}$$

---

\* Both authors were partially supported by NSF.

where  $N$  is the normal subgroup of  $F(\alpha_1, \dots, \alpha_n)$  generated by the relations

$$\alpha_{j_1} \cdots \alpha_{j_{m(i)}} \alpha_i^{d_i} = 1,$$

$$[\alpha_i, \alpha_{j_1}] = 1, \dots, [\alpha_i, \alpha_{j_{m(i)}}] = 1,$$

for each  $1 \leq i \leq n$ , where  $E_{j_1}, \dots, E_{j_{m(i)}}$  with  $j_1 < \dots < j_{m(i)}$  are the  $m(i)$  curves which intersect  $E_i$  and  $d_i = (E_i)^2$ .

In Corollary 5 we give an arithmetic proof of the Theorem of Mumford and Flenner. To be precise, if  $(A, m)$  is a complete normal equicharacteristic zero local ring of dimension two, with algebraically closed residue field, then the algebraic fundamental group  $\pi_1(\operatorname{spec}(A) - m) = 0$  if and only if  $A$  is smooth.

In Section 3, we prove that for normal Brieskorn singularities, the triviality of the fundamental group is equivalent to the existence of a purely inseparable smooth cover. More precisely,

**THEOREM A.** *Let  $A = k[[x, y, z]]/x^a + y^b + z^c$  where  $k$  is an algebraically closed field of characteristic  $p \neq 2$  or  $3$ , and  $A$  is normal. Let  $S = \operatorname{spec}(A)$ , and  $m$  be the maximal ideal of  $A$ . Then the following are equivalent:*

- (i)  $\pi_1(\operatorname{spec}(A) - m) = 0$ .
- (ii)  $S$  has a purely inseparable smooth cover.

We prove this in Theorem 12. (ii)  $\Rightarrow$  (i) is always true (Lemma 2). Artin [A3] has proved that the conclusions of Theorem A are true for rational double points in characteristic bigger than two.

Our proof of Theorem 12 involves an analysis of the prime to  $p$  part of the local fundamental group. We use a group theoretic group, proved in Section 2 (Theorem 6).

M. Artin [A3] has asked if the following are equivalent for a surface singularity  $(S, x)$  of positive characteristic.

- (1)  $S$  has finite local fundamental group.
- (2)  $S$  has a smooth cover.

Artin has proved (2)  $\Rightarrow$  (1) in general, and proved (1)  $\Rightarrow$  (2) for rational double points in all characteristics.

Establishing that the conclusions of Theorem A hold for an arbitrary surface singularity would also answer Artin's question in the affirmative.

## 1. Local fundamental groups of surface singularities

$F(e_1, \dots, e_n)$  will denote the free group on  $e_1, \dots, e_n$ . If  $G$  is a group,  $p$  a prime,  $G^{(p)}$  will denote the pro-finite completion of  $G$  with respect to quotient groups of finite order prime to  $p$ .

**THEOREM 1.** *Suppose that  $(A, m)$  is a complete normal local domain of dimension two, with algebraically closed residue field  $k$ . Suppose that  $\pi_1^{(p)}(\text{spec}(A) - m)$  is a finite group. Then*

- (a) *The divisor class group of  $A$ ,  $CL(A)$ , is an extension of a finite group by a group with a composition series of factors isomorphic to  $k^+$ .*
- (b) *If  $f: X \rightarrow \text{spec}(A)$  is a resolution of singularities, such that the reduced exceptional fiber has simple normal crossings, then the irreducible exceptional curves are rational curves, and the intersection graph of the exceptional locus is a tree.*

*Proof.* We will first prove (a). Let  $f: X \rightarrow \text{spec}(A)$  be a resolution of singularities such that the reduced exceptional fiber has simple normal crossings. Let  $D$  be the reduced exceptional locus of  $f$ , and let  $D_i$  be the (nonsingular) irreducible components of  $D$ . There are exact sequences:

$$0 \rightarrow \text{Pic}^0(X) \rightarrow CL(A) \rightarrow G \rightarrow 0, \quad (1)$$

$$0 \rightarrow L \rightarrow \text{Pic}^0(X) \rightarrow \prod \text{Pic}^0(D_i) \rightarrow 0 \quad (2)$$

where  $L$  has a composition series with factors isomorphic to  $k^+$  and  $k^*$  and  $G$  is a finite group. (2) is derived in Section 1 of [A1], and (1) is Proposition 14.4 [L].

Suppose that  $\mathcal{L}$  is an element of order  $n$  in  $\text{Pic}^0(X)$ , such that  $p$  does not divide  $n$  if  $p > 0$ . Then there exists  $\sigma \in H^0(X, \mathcal{L}^{\otimes n})$  such that  $\sigma: \mathcal{O}_X \rightarrow \mathcal{L}^{\otimes n}$  is an isomorphism.  $\mathcal{A} = \bigoplus_{i=0}^{n-1} \mathcal{L}^{\otimes -i}$  has an  $\mathcal{O}_X$  algebra structure induced by identifying  $\mathcal{L}^{\otimes -n}$  with  $\mathcal{O}_X$  by  $\sigma$ .  $\text{spec}(f_* \mathcal{A})$  is a finite cover of  $\text{spec}(\mathcal{A})$  which restricts to be an irreducible, étale, kummer cover of  $\text{spec}(\mathcal{A}) - m$  of degree  $n$ .

Suppose that  $CL(A)$  is not as in (a). Then either some  $D_i$  has positive genus, so that  $\prod \text{Pic}^0(D_i)$  is a non-trivial abelian variety, or  $L$  has  $k^*$  as a term in a composition series. In either case, it can be shown that for each  $n > 0$  such that  $p$  does not divide  $n$ , we have an element  $\mathcal{L} \in \text{Pic}^0(X)$  of order  $n$ . We can then construct étale kummer covers of  $X$  of order  $n$ .  $\pi_1^{(p)}(\text{spec}(A) - m)$  is then infinite, which is a contradiction.

Let  $N = \sum (n_q - 1) - s + 1$ , where  $s$  is the number of irreducible components  $D_i$  of  $D$ , and  $n_q$  is the number of  $D_i$  containing the closed point  $q$ . The sum is over all closed points  $q$  of  $X$ . In the construction of the sequence (2), Artin [A1] shows



that the contribution of  $k^*$  to (2) is a term  $(k^*)^N$ . (a) is equivalent to  $N = 0$  and  $\text{Pic}^0(D_i) = 0$  for all  $i$ . Now  $\text{Pic}^0(D_i) = 0$  is equivalent to  $D_i$  being a rational curve. Further, if  $T$  is the intersection graph then

$$N = \sum (n_q - 1) - s + 1 = \text{number of edges} - \text{number of vertices} + 1 = 1 - \chi(T).$$

So  $N = 0$  if and only if  $T$  is a tree. This completes the proof.

The next Lemma gives one direction of the question (\*) raised in the introduction.

**LEMMA 2.** *Suppose that  $(A, m)$  is a complete, normal local domain with algebraically closed residue field  $k$ , and that  $A$  has a purely inseparable smooth cover. Then  $\pi_1(\text{spec}(A) - m) = 0$ .*

*Proof.* Let  $A \rightarrow B$  be the purely inseparable smooth cover, where  $(B, n)$  is a complete local ring. Since a purely inseparable morphism is radicial,  $\pi_1(\text{spec}(A) - m) = \pi_1(\text{spec}(B) - n)$  by IX 4.10 [S1]. But then,  $\pi_1(\text{spec}(B) - n) = \pi_1(\text{spec}(B)) = \pi_1(k) = 0$  by X 3.4, X 1.1 [S2].

We will introduce some notation which will be useful in the proof of Theorem 3. In Sections 3 and 4 of the book of Grothendieck and Murre on tame fundamental groups, [GM], it is shown that the notion of tame ramification over a divisor with simple normal crossings extends to formal schemes.

Let  $\mathcal{X}$  be a normal, connected formal scheme, with a divisor  $D$  on  $\mathcal{X}$  with simple normal crossings. Let  $\text{Rev}^D(\mathcal{X})$  be the category of formal  $\mathcal{X}$ -schemes which are tamely ramified over  $\mathcal{X}$  relative to  $D$ .  $\mathcal{D} \cdot \text{Rev}^D(\mathcal{X})$  is a Galois category by Proposition 4.2.2 of [GM], and hence has a fundamental group by Expose V of [S1].

**THEOREM 3.** *Let  $(A, m)$  be a complete normal local domain of dimension two over an algebraically closed field  $k$  of characteristic  $p \geq 0$ . Let  $\sigma : X \rightarrow \text{spec}(A)$  be a resolution of singularities, such that the reduced exceptional fiber has simple normal crossings, with irreducible exceptional curves  $E_1, \dots, E_n$ . Suppose that the intersection graph of the exceptional locus is simply connected, and that each  $E_i$  is a nonsingular rational curve. Let  $F_n$  be the free group on the symbols  $\alpha_1, \dots, \alpha_n$ . Then there exists a reindexing of the  $E_i$  such that*

$$\pi_1^{(p)}(\text{spec}(A) - m) \cong \pi_1^{(p)}\left(X - \sum_{i=1}^n E_i\right) \cong (F_n/N)^{(p)}$$

where  $N$  is the normal subgroup of  $F(\alpha_1, \dots, \alpha_n)$  generated by the relations

$$\alpha_{j_1} \cdots \alpha_{j_{m(i)}} \alpha_i^{d_i} = 1,$$

$$[\alpha_i, \alpha_{j_1}] = 1, \dots, [\alpha_i, \alpha_{j_{m(i)}}] = 1,$$

for each  $1 \leq i \leq n$ , where  $E_{j_1}, \dots, E_{j_{m(i)}}$  with  $j_1 < \dots < j_{m(i)}$  are the  $m(i)$  curves which intersect  $E_i$  and  $d_i = (E_i)^2$ .

The remainder of Section 1 will be devoted to the proof of Theorem 3. Without loss of generality, we may assume that  $n > 1$ . Set  $E = \sum_{i=1}^n E_i$ . Set  $p_{ij} = E_i \cap E_j$  whenever  $E_i$  and  $E_j$  intersect properly. Let  $\mathcal{S}$  be the formal completion of  $X$  along  $\sigma^{-1}(m)$ . Let  $\mathcal{S}_i$  be the formal completion of  $X$  along  $E_i$  for  $1 \leq i \leq n$ , and let  $\mathcal{S}_{ij}$  be the formal completion of  $X$  along  $p_{ij}$ .

Let  $\pi = \pi_1(\mathcal{S})^{(p)}$  be the prime to  $p$  part of a fundamental group  $\pi_1(\mathcal{S})$  for  $\text{Rev}^E(\mathcal{S})$ . Let  $\pi_i$  be the prime to  $p$  part of a fundamental group for  $\text{Rev}^E(\mathcal{S}_i)$ , and let  $\pi_{ij}$  be the prime to  $p$  part of a fundamental group for  $\text{Rev}^E(\mathcal{S}_{ij})$ . By Corollary 9.9 of [GM] we have

$$\pi \cong \pi_1^{(p)}(\text{spec}(A) - m). \quad (3)$$

Let  $\mu_r$  be the group of  $r$ -th roots of unity of  $k$ . Set

$$\mu' = \varprojlim_{p \nmid r} \mu_r.$$

Let  $w$  be a “generator” of  $\mu'$ . By Abhyankar’s Lemma, (c.f. XIII 5.3 [S1]), we have a canonical isomorphism  $\pi_{ij} \cong \mu' \oplus \mu'$ , which is the direct sum of limits of inertia groups of prime divisors ramified over  $E_i \cap \mathcal{S}_{ij}$  and  $E_j \cap \mathcal{S}_{ij}$ . The map  $\alpha_i \mapsto (w, 1)$ ,  $\alpha_j \mapsto (1, w)$  determines an isomorphism

$$\pi_{ij} \cong (F(\alpha_i, \alpha_j)/[\alpha_i, \alpha_j])^{(p)}.$$

Let  $E_{j_1}, \dots, E_{j_{m(i)}}$  be the exceptional curves of  $\sigma$  which intersect  $E_i$  properly. Suppose that

$$\lambda_i^{ijk} : \pi_{ijk} \rightarrow \pi_i$$

are paths. Then we will identify  $\alpha_{j_k}$  with  $\lambda_i^{ijk}(\alpha_{j_k})$  and  $\alpha_i$  with  $\lambda_i^{ijk}(\alpha_i)$  in  $\pi_i$ . We will verify in the proof of Lemma 4 below that this is well defined.

**LEMMA 4.** *Suppose that for some  $l$ , a path*

$$\lambda_i^{ijl} : \pi_{ijl} \rightarrow \pi_i$$

*is given, and that  $\tau$  is a permutation of  $[1, \dots, m(i)]$ . Then there exist paths*

$$\lambda_i^{ijk} : \pi_{ijk} \rightarrow \pi_i$$

such that

$$\pi_i = (F(\alpha_i, \alpha_{j_1}, \dots, \alpha_{j_{m(i)}})/N)^{(p)}$$

where  $N$  is the normal subgroup generated by the relations

$$\alpha_{j_{\tau(1)}} \alpha_{j_{\tau(2)}} \cdots \alpha_{j_{\tau(m(i))}} \alpha_i^{d_i} = [\alpha_i, \alpha_{j_1}] = \cdots = [\alpha_i, \alpha_{j_{m(i)}}] = 1.$$

*Proof.* Let  $\phi : \mathcal{X} \rightarrow \mathcal{S}_i \in \text{Rev}^E(\mathcal{S}_i)$  be connected and Galois. Then we have that  $\phi^{-1}(E_i)$  is irreducible, hence the inertia group of  $\phi^{-1}(E_i)_{\text{red}}$  is a normal subgroup of  $\text{Gal}(\mathcal{X}/\mathcal{S}_i)$ . This inertia group is naturally a quotient of  $\mu^t$ . Taking limits, we have a natural exact sequence (c.f. Corollary 5.1.11 [GM])

$$\mu^t \rightarrow \pi_i \rightarrow \pi_1^{(p)} \left( E_i - \sum p_{ij_k} \right) \rightarrow 1. \quad (4)$$

By our construction of  $\pi_{ij}$ , for any path  $\lambda_i^{y_k}$ ,  $\lambda_i^{y_k}(\alpha_i) = w \in \mu^t$ .

From the classical description of the fundamental groups of the  $m$ -times punctured projective line (c.f. Section 7 of [Ab] and Section 12 of [P]), paths  $\lambda_i^{y_k}$  can be chosen so that

$$\pi_1^{(p)} \left( E_i - \sum p_{ij_k} \right) = (F(\alpha_{j_1}, \dots, \alpha_{j_{m(i)}})/\alpha_{j_{\tau(1)}} \cdots \alpha_{j_{\tau(m(i))}})^{(p)}.$$

In particular,  $\pi_i$  is a quotient of  $F(\alpha_i, \alpha_{j_1}, \dots, \alpha_{j_{m(i)}})^{(p)}$ .

Let  $s$  be an integer between 1 and  $m(i)$ . Let  $r$  be an integer such that  $(r, p) = 1$ ,  $(r, d_i) = 1$  and  $r > -d_i$ . Since  $E_i$  can be contracted inside  $\mathcal{S}_i$  to a rational singularity, there exists  $f \in \Gamma(\mathcal{S}_i, \mathcal{O}_{\mathcal{S}_i})$  such that  $(f) = -d_i E_s + E_i$ . Let  $\phi : \mathcal{W}_r \rightarrow \mathcal{S}_i \in \text{Rev}^E(\mathcal{S}_i)$  be defined so that  $\phi_*(\mathcal{O}_{\mathcal{W}_r})$  is the normalization of  $\mathcal{O}_{\mathcal{S}_i}[t]/t^r - f$ . We can choose a surjection

$$\Lambda : \pi_i \rightarrow \text{Gal}(\mathcal{W}_r/\mathcal{S}_i).$$

$\phi$  is unramified over  $E_{j_k}$  if  $k \neq s$ . Hence  $\Lambda(\alpha_{j_k}) = 1$  if  $k \neq s$ . Consideration of the induced map

$$\pi_y \rightarrow \text{Gal}(\mathcal{W}_r/\mathcal{S}_i)$$

shows that

$$\text{Gal}(\mathcal{W}_r/\mathcal{S}_i) = (F(\alpha_i, \alpha_{j_s})/\alpha_i^r = \alpha_{j_s}^r = [\alpha_i, \alpha_{j_s}] = \alpha_{j_s} \alpha_i^{d_i} = 1). \quad (5)$$

By taking  $r$  arbitrarily large, we see from (5) that (4) is left exact. Hence

$$\pi_i = (F(\alpha_i, \alpha_{j_1}, \dots, \alpha_{j_{m(i)}}) / \alpha_{j_{\tau(1)}} \alpha_{j_{\tau(2)}} \cdots \alpha_{j_{\tau(m(i))}} \alpha_i^{e_i} = [\alpha_i, \alpha_{j_1}] = \cdots = [\alpha_i, \alpha_{i, j_{m(i)}}] = 1)^{(p)}$$

for some integer  $e_i$ . Now (5) shows that  $e_i = d_i$ .

Now we will return to the proof of Theorem 3. Since the intersection graph of  $E$  is a tree, it follows from Lemma 4 and induction that it is possible to choose paths

$$\lambda_i^y : \pi_{ij} \rightarrow \pi_i \quad \text{and} \quad \phi_i : \pi_i \rightarrow \pi$$

such that after a reordering of the  $E_i$ ,

$$\begin{array}{ccc} \pi_{ij} & \xrightarrow{\lambda_i''} & \pi_i \\ \lambda_j'' \downarrow & & \phi_i \downarrow \\ \pi_i & \xrightarrow{\phi_j} & \pi \end{array} \quad (6)$$

commutes, and

$$\pi_i = (F(\alpha_i, \alpha_{j_1}, \dots, \alpha_{j_{m(i)}}) / (\alpha_{j_1} \alpha_{j_2} \cdots \alpha_{j_{m(i)}} \alpha_i^{d_i} = [\alpha_i, \alpha_{j_1}] = \cdots = [\alpha_i, \alpha_{i, j_{m(i)}}] = 1)^{(p)},$$

where  $E_{j_1}, \dots, E_{j_m}$  with  $j_1 < \cdots < j_{m(i)}$  are the curves which intersect  $E_i$  properly. We can then identify  $\alpha_i$  with  $\phi_i(\alpha_i) = \phi_j(\alpha_i)$  in  $\pi$ .

The statement of Theorem 3 now follows from (3), (6), and the arithmetic analogue of Van Kampen's Theorem proved in Corollary 8.3.6 of [GM].

As a corollary, we get an arithmetic proof of Mumford and Flenner's Theorem.

**COROLLARY 5 (Mumford–Flenner).** *Suppose that  $(A, m)$  is a complete normal local domain of dimension two, with algebraically closed residue field  $k$  of characteristic zero. Then  $\pi_1(\text{spec}(A) - m) = 0$  if and only if  $A$  is smooth over  $k$ .*

*Proof.* By purity of Branch Locus (X.3.4 [S2] and X 1.1 [S2]),  $A$  smooth implies that  $\pi_1(\text{spec}(A) - m) = 0$ .

Suppose that  $\pi_1(\text{spec}(A) - m) = 0$ . Then by Theorems 1 and 3 we have an expression for  $\pi_1(\text{spec}(A) - m)$  in terms of generators and relations, depending on the intersection matrix of a resolution of singularities.  $\pi_1(\text{spec}(A) - m)$  is thus isomorphic to the profinite completion with respect to quotient groups of finite order of the group  $\pi(\Gamma)$  associated to the intersection diagram of a resolution of singularities defined in [F]. By Theorem 2.7 [F], this group is trivial if and only if  $A$  is smooth.

## 2. Existence of quotient groups of order prime to $p$

LEMMA 5. *Let  $s_1, \dots, s_t$  be integers, greater than one. For every prime number  $p > 3$  such that  $p$  does not divide  $s_i$  for  $i = 1, \dots, t$ , there exists a prime  $q > 3$  such that  $q \equiv 1 \pmod{s_i}$  for  $i = 1, \dots, t$ , but  $p$  does not divide  $q(q-1)(q+1)$ .*

*Proof.* Let  $a = \prod_{i=1}^t s_i$ . Since  $(a, p) = 1$ ,  $ma + np = 1$  for some integers  $m$  and  $n$ . There are indeed infinitely many primes in the set  $\{kap + (-np + 2) \mid k \in \mathbb{Z}\}$  because  $(ap, -np + 2) = 1$ . Choose a prime  $q > 3$  from this set

$$q \equiv -np + 2 \equiv -1 + 2 \equiv 1 \pmod{a}$$

and  $q \equiv 2 \pmod{p}$ . Thus  $q \equiv 1 \pmod{s_i}$  for  $i = 1, \dots, t$ , and  $p$  divides  $q - 2$ . Since both  $p$  and  $q$  are larger than 3,  $p$  does not divide  $q(q-1)(q+1)$ .

THEOREM 6. *Suppose that  $t \geq 3$ ,  $s_1, \dots, s_t$  are integers such that each  $s_i > 1$ , and  $p > 3$  is a prime such that  $p$  does not divide  $s_i$  for  $i = 1, \dots, t$ . Then*

$$F(e_1, \dots, e_t)/e_1^{s_1} = \dots = e_t^{s_t} = e_1 \cdots e_t = 1$$

*has a quotient of finite order prime to  $p$ .*

*Proof.* Let  $q > 3$  be a prime number such that  $q \equiv 1 \pmod{2s_i}$  for  $i = 1, \dots, t$ . Let  $F = F_q$  be the finite field with  $q$  elements. Since  $2s_i$  divides  $q - 1$ , we can pick an element  $x_i$  of  $F_q$  of order  $2s_i$ . Let

$$A_i = \begin{pmatrix} x_i & 0 \\ 0 & \frac{1}{x_i} \end{pmatrix}$$

for  $i = 1, \dots, t$ , so that the order of  $A_i$  is  $2s_i$  in  $SL(2, F)$ . Define

$$E_1 = \begin{pmatrix} 0 & -1 \\ 1 & x_1 + \frac{1}{x_1} \end{pmatrix},$$

$$E_2 = \begin{pmatrix} x_2 + \frac{1}{x_2} & x_3 \\ -\frac{1}{x_3} & 0 \end{pmatrix},$$

$$E_3 = \begin{pmatrix} x_3 & 0 \\ -x_2 - \frac{1}{x_2} + \frac{x_1}{x_3} + \frac{1}{x_1 x_3} & \frac{1}{x_3} \end{pmatrix}.$$

Define  $E_i = I$  for  $3 < i \leq t$ .  $\text{trace}(E_i) = x_i + 1/x_i = \text{trace}(A_i)$  for  $i = 1, 2, 3$ . Since  $s_i > 1$ ,  $A_i \neq \pm I$ . Hence  $E_i$  and  $A_i$  are conjugates in  $GL(2, F)$ . The order of  $E_i$  is thus  $2s_i$ .

For  $i = 1, \dots, t$ , define maps

$$\Phi_i : \mathbf{Z}_{s_i} \rightarrow SL(2, F)/\{\pm I\}$$

by  $\Phi_i(1) = E_i$ . We have

$$\prod_{i=1}^t \Phi_i(1) = E_1 E_2 E_3 I = I.$$

Let  $G = \mathbf{Z}_{s_1} * \dots * \mathbf{Z}_{s_t} / \prod_{i=1}^t e_i = 1$ . The  $\Phi_i$  define a unique map  $\Phi$  such that

$$\begin{array}{ccc} G & \xrightarrow{\Phi} & SL(2, F)/\{\pm I\} \\ \downarrow & \nearrow \Phi_i & \\ \mathbf{Z}_{s_i} & & \end{array}$$

commutes. Observe that

$$G = F(e_1, \dots, e_t)/e_1^{s_1} = \dots = e_t^{s_t} = e_1 e_2 \dots e_t = 1.$$

$\Phi$  is nontrivial since  $\Phi_1$ ,  $\Phi_2$  and  $\Phi_3$  are nontrivial. Thus  $G/\text{kernel}(\Phi)$  is a nontrivial quotient of  $G$  whose order  $|\Phi(G)|$  is a nontrivial factor of  $|SL(2, F)/\{\pm I\}|$ . So  $G$  has a nontrivial quotient of finite order dividing  $q(q-1)(q+1)/2$ . By Lemma 4, we can choose the prime  $q$  such that  $p$  does not divide  $q(q-1)(q+1)$ . Thus  $G$  has a finite nontrivial quotient of order prime to  $p$ .

### 3. Brieskorn singularities

In this section we will use the following notations. Suppose that  $k$  is an algebraically closed field of characteristic  $p > 3$ . Suppose that  $a_1, a_2, a_3$  are positive integers. Let

$$R(a_1, a_2, a_3) = k[[x_1, x_2, x_3]]/(x_1^{a_1} + x_2^{a_2} + x_3^{a_3}).$$

$R(a_1, a_2, a_3)$  is normal precisely when  $p$  divides at most one of the exponents  $a_1, a_2, a_3$ . Suppose that  $R(a_1, a_2, a_3)$  is normal. Let  $m$  be the maximal ideal of  $R(a_1, a_2, a_3)$ . Let  $S(a_1, a_2, a_3) = \text{spec}(R(a_1, a_2, a_3)) - m$ .

**PROPOSITION 7.** *Write  $a_i = p^{r_i} b_i$  where  $(b_i, p) = 1$ . Then*

$$\pi_1(S(a_1, a_2, a_3)) \cong \pi_1(S(b_1, b_2, b_3)).$$

*Proof.* Define

$$\phi : k[[x_1, x_2, x_3]]/(x_1^{b_1} + x_2^{b_2} + x_3^{b_3}) \rightarrow k[[y_1, y_2, y_3]]/(y_1^{a_1} + y_2^{a_2} + y_3^{a_3})$$

by  $x_1 \mapsto y_1^{p^{r_1}}, x_2 \mapsto y_2^{p^{r_2}}, x_3 \mapsto y_3^{p^{r_3}}$ .  $\phi$  is purely inseparable, hence radicial. The proposition follows from IX 4.10 [S1].

Resolutions of Brieskorn singularities are constructed in characteristic zero in [H–J] and [O–W]. the proofs easily extend to characteristic  $p$ .

**PROPOSITION 8.** *Suppose that  $p$  does not divide  $a_i$  for  $1 \leq i \leq 3$ . Then the intersection diagram of the minimal resolution of singularities of  $\text{spec}(R(a_1, a_2, a_3))$  can be described as follows: Let*

$$c = (a_1, a_2, a_3), \quad c_1 = \frac{(a_2, a_3)}{c}, \quad c_2 = \frac{(a_1, a_3)}{c}, \quad c_3 = \frac{(a_1, a_2)}{c},$$

$$\gamma_1 = \frac{a_1}{cc_2c_3}, \quad \gamma_2 = \frac{a_2}{cc_1c_3}, \quad \gamma_3 = \frac{a_3}{cc_1c_2}.$$

Let  $0 < r_1 < \gamma_1, 0 < r_2 < \gamma_2, 0 < r_3 < \gamma_3$  satisfy

$$c_1\gamma_2\gamma_3r_1 \equiv -1 \pmod{(\gamma_1)}, \quad c_2\gamma_1\gamma_3r_2 \equiv -1 \pmod{(\gamma_2)}, \quad c_3\gamma_1\gamma_2r_3 \equiv -1 \pmod{(\gamma_3)}$$

Let  $b_j^i$  for  $i = 1, 2, 3$  and  $1 \leq j \leq t_i$  denote the continued fraction expansions

$$\frac{\gamma_i}{r_i} = b_{t_i}^i - \frac{1}{b_{t_i-1}^i - \frac{1}{\cdots - \frac{1}{b_1^i}}}.$$

Let  $L_i$  be the linear graph with  $t_i + 1$  vertices and successive weights  $-b_1^i, \dots, -b_{t_i}^i, -b$ .

The intersection diagram of  $\text{spec}(R(a_1, a_2, a_3))$  is the star shaped graph obtained by identifying the vertex with weight  $-b$  of  $cc_1$  copies of  $L_1$ ,  $cc_2$  copies of  $L_2$ , and  $cc_3$  copies of  $L_3$  to a common point. The arms of the star in the  $cc_1$  copies of  $L_1$ ,  $cc_2$  copies of  $L_2$ , and  $cc_3$  copies of  $L_3$  which are glued together at the vertex of weight  $-b$ .

Each vertex in the intersection diagram corresponds to a smooth rational curve except for possibly the central vertex (with weight  $-b$ ), which corresponds to a smooth curve  $K$  of genus

$$g_K = \frac{1}{2}(2 + c^2 c_1 c_2 c_3 - cc_1 - cc_2 - cc_3).$$

**PROPOSITION 9.** Suppose that  $p$  does not divide  $a_i$  for  $i = 1, 2, 3$  and  $\pi_1(S(a_1, a_2, a_3)) = 0$ . Then  $g_K = 0$ , and one of the following cases must occur.

- (1)  $c = c_1 = c_2 = 1$  and  $c_3$  is arbitrary.
- (2)  $c = c_2 = c_3 = 1$  and  $c_1$  is arbitrary.
- (3)  $c = c_1 = c_3 = 1$  and  $c_2$  is arbitrary.
- (4)  $c = 2$  and  $c_1 = c_2 = c_3 = 1$ .

*Proof.*  $g_K = 0$  by Theorem 1. We will determine the positive integers  $c, c_1, c_2, c_3$  such that

$$2 + c(cc_1 c_2 c_3 - c_1 - c_2 - c_3) \leq 0.$$

Without loss of generality, we may assume that  $c_1 \leq c_2 \leq c_3$ . We immediately reduce to  $cc_1 c_2 c_3 - c_1 - c_2 - c_3 < 0$  which forces  $cc_1 c_2 < 3$ . The only solutions are  $c = 2, c_1 = c_2 = c_3 = 1$  and  $c = c_1 = 1, c_2 = c_3 = 2$  and  $c = c_1 = c_2 = 1, c_3$  arbitrary.

**PROPOSITION 10.** Suppose that  $p$  does not divide  $a_i$  for  $i = 1, 2, 3$ . Suppose that  $g_K = 0$ . Then

$$\begin{aligned} \pi_1^{(p)}(S(a_1, a_2, a_3)) \\ \cong (F(e; e_1^{1,1}, \dots, e_{t_1}^{1,1}, e_1^{2,1}, \dots, e_{t_1}^{cc_1,1}; e_1^{1,2}, \dots, e_{t_2}^{cc_2,2}; e_1^{1,3}, \dots, e_{t_3}^{cc_3,3})/N)^{(p)} \end{aligned}$$

where  $N$  is the normal subgroup generated by the relations

$$\begin{aligned} e_{t_1}^{1,1} \dots e_{t_1}^{cc_1,1} e_{t_2}^{1,2} \dots e_{t_2}^{cc_2,2} e_{t_3}^{1,3} \dots e_{t_3}^{cc_3,3} e^{-b} &= 1, \\ [e, e_{t_1}^{k,1}] &= [e, e_{t_2}^{k,2}] = [e, e_{t_3}^{k,3}] = 1, \end{aligned}$$



for  $1 \leq k \leq cc_i$  and the relations for  $1 \leq i \leq 3 \leq k \leq cc_i$

$$\begin{aligned} ee_{t_i-1}^{k,i}(e_{t_i}^{k,i})^{-b_{t_i}^i} &= 1, \\ e_2^{k,i}(e_1^{k,i})^{-b_1^i} &= 1, \\ e_{j-1}^{k,i}e_{j+1}^{k,i}(e_j^{k,i})^{-b_j^i} &= 1 \quad \text{for } 2 \leq j \leq t_i - 1, \\ [e_j^{k,i}, e_{j+1}^{k,i}] &= 1 \quad \text{for } 1 \leq j \leq t_i - 1. \end{aligned} \tag{i, k}$$

*Proof.* This is immediate for Theorems 1 and 3.

**PROPOSITION 11.** *Let assumptions be as in Proposition 10. Then*

$$\pi_1^{(p)}(S(a_1, a_2, a_3)) \cong (F(e; e_1^1, \dots, e_1^{cc_1}; e_2^1, \dots, e_2^{cc_2}; e_3^1, \dots, e_3^{cc_3})/M)^{(p)}$$

where  $M$  is the normal subgroup generated by the relations

$$\begin{aligned} e_1^1 \cdots e_1^{cc_1} e_2^1 \cdots e_2^{cc_2} e_3^1 \cdots e_3^{cc_3} e^{-b} &= 1, \\ e^{r_i}(e_i^k)^{-\gamma_i} &= 1 \quad \text{for } 1 \leq i \leq 3, 1 \leq k \leq cc_i, \\ [e, e_i^k] &= 1 \quad \text{for } 1 \leq i \leq 3, 1 \leq k \leq cc_i. \end{aligned}$$

*Proof.* The relations (i, k) determine relations

$$\begin{aligned} (e_{j+1}^{k,i})^{\alpha_j^i} (e_j^{k,i})^{-\alpha_{j+1}^i} &= 1, \quad 1 \leq j \leq t_i - 1, \\ e^{\alpha_{t_i}^i} (e_{t_i}^{k,i})^{-\alpha_{t_i+1}^i} &= 1 \end{aligned}$$

where  $\alpha_0^i = 0$ ,  $\alpha_j^i$  is determined by the recursion formula

$$\alpha_j^i = b_{j-1}^i \alpha_{j-1}^i - \alpha_{j-2}^i$$

for  $2 \leq j \leq t_i + 1$ . That is,

$$\frac{\alpha_j^i}{\alpha_{j-1}^i} = b_{j-1}^i - \frac{1}{\frac{\alpha_{j-1}^i}{\alpha_{j-2}^i}}.$$

So, we have

$$\frac{\alpha_{t_i+1}^i}{\alpha_{t_i}^i} = b_{t_i}^i - \frac{1}{b_{t_i-1}^i - \frac{1}{\cdots - \frac{1}{b_1^i}}} = \frac{\gamma_i}{r_i}$$

by Proposition 8. Since  $(\gamma_i, r_i) = 1$ , this gives  $e^{r_i}(e_{t_i}^{k,i})^{-\gamma_i} = 1$ . On the other hand, using the relations (i, k), one can eliminate the  $e_j^{k,i}$ , for  $1 \leq j \leq t_i - 1$ , since they can be written in terms of  $e_{t_i}^{k,i}$  and  $e$ . Set  $e_i^k = e_{t_i}^{k,i}$ . We then have the conclusions of Proposition 11.

**THEOREM 12.** *The following are equivalent.*

- (1)  $\pi_1(S(a_1, a_2, a_3)) = 0$ .
- (2)  $\text{Spec}(R(a_1, a_2, a_3))$  has a purely inseparable cover by a power series ring in  $k$ .
- (3) Some  $a_i$  is a power of  $p$ .

*Proof.* (3) implies (2) follows from the proof of Proposition 7. (2) implies (1) follows from Lemma 2. We must show that (1) implies (3).

We assume that  $b_i$  are such that  $\pi_1(S(b_1, b_2, b_3)) = 0$ , and prove that some  $b_i$  is a power of  $p$ . Let  $a_i$  be the positive integers such that  $(a_i, p) = 1$ , and  $b_i = a_i p^{\lambda_i}$ . Then  $\pi_1(S(a_1, a_2, a_3)) = 0$  by Proposition 7. By Proposition 9,  $g_K = 0$ . Proposition 11 shows that we have a surjection obtained by taking the quotient of  $\pi_1^{(p)}(S(a_1, a_2, a_3))$  by the normal subgroup generated by  $e$ .

$$\pi_1(S(a_1, a_2, a_3)) \rightarrow (F(e_1^1, \dots, e_1^{cc_1}; e_2^1, \dots, e_2^{cc_2}; e_3^1, \dots, e_3^{cc_3})/L)^{(p)} \quad (7)$$

where  $L$  is the normal subgroup generated by the relations

$$e_1^1 \cdots e_1^{cc_1} e_2^1 \cdots e_2^{cc_2} e_3^1 \cdots e_3^{cc_3} = 1,$$

$$(e_i^k)^{\gamma_i} = 1 \quad \text{for } 1 \leq i \leq 3, 1 \leq k \leq cc_i.$$

By Theorem 6, some  $\gamma_i = 1$ .

Suppose that one of the cases (1), (2), (3) of Proposition 9 occurs. After reindexing the  $a_i$ , we may assume that case (1) holds, so that  $c = c_1 = c_2 = 1$ , and  $c_3$  is arbitrary. Then  $(a_1, a_2, a_3) = (\gamma_1 c_3, \gamma_2 c_3, \gamma_3)$ . By (7), and Theorem 6,  $c_3 > 2$  implies that  $\gamma_3 = 1$  and  $a_3 = 1$ . If  $c_3 = 2$ , and  $\gamma_3 > 1$ , then  $\gamma_1 = \gamma_2 = 1$  which implies that the right hand side of (7) is  $\mathbf{Z}_{\gamma_3} \neq 0$ , a contradiction. If  $c_3 = 1$ , then some  $a_i = 1$ .

The remaining case of Proposition 9 is when  $c_1 = c_2 = c_3 = 1$  and  $c = 2$ , so that  $(a_1, a_2, a_3) = (2\gamma_1, 2\gamma_2, 2\gamma_3)$ . Suppose that some  $\gamma_i > 1$ . Since  $\pi_1(S(a_1, a_2, a_3))$  is trivial, we see by Theorem 6 and (7) that at most one  $\gamma_i$  is greater than 1. After reindexing the  $a_i$ , we may assume that  $\gamma_1 > 1$  and  $\gamma_2 = \gamma_3 = 1$ . The right hand side of (7) is then  $\mathbf{Z}_{\gamma_1}$ , a contradiction. Hence  $(a_1, a_2, a_3) = (2, 2, 2)$ .

In this case, the intersection graph of the minimal resolution of singularities of  $x_1^2 + x_2^2 + x_3^2 = 0$  is a single vertex, corresponding to a nonsingular rational curve, with weight  $-2$ . Hence  $\pi_1(S(a_1, a_2, a_3)) \cong \mathbf{Z}_2 \neq 0$ , so that this case cannot occur.

## REFERENCES

- [Ab] S. ABHYANKAR, *Coverings of algebraic curves*, Amer. J. Math. 79 (1957), 825–856.
- [A1] M. ARTIN, *Some numerical criteria for contractability of curves on algebraic surfaces*, Amer. J. Math. 84 (1962), 485–496.
- [A2] M. ARTIN, *Lifting of two-dimensional singularities to characteristic zero*, Amer. J. Math. 88 (1966), 747–762.
- [A3] M. ARTIN, *Coverings of the rational double points in characteristic  $p$* , in: W. Baily and T. Shioda (eds.), *Complex analysis and algebraic geometry*, Cambridge University press, Cambridge, 1977, 11–22.
- [F] H. FLENNER, *Reine lokale ringe der dimension zwei*, Math. Ann. 216 (1975), 253–263.
- [GM] A. GROTHENDIECK and J. MURRE, *The tame fundamental group of a formal neighborhood of a divisor with normal crossing on a scheme*, Lecture notes in Math. 208, Springer–Verlag, Berlin, Heidelberg, New York 1971.
- [H–J] F. HIRZEBRUCH and K. JANICH, *Involutions and singularities*, Algebraic Geometry (international colloquium, T.I.F.R., Bombay, 1968) 219–240, Oxford University Press, Bombay, 1969.
- [L] J. LIPMAN, *Rational singularities, with applications to algebraic surfaces and unique factorization*, Publ. Math. Inst. Hautes Etud. Sci. 36 (1969), 195–278.
- [M] D. MUMFORD, *The topology of normal singularities of an algebraic surface and a criterion for simplicity*, Publ. Math. Inst. Hautes Etud. Sci. 9 (1961), 5–22.
- [O–W] P. ORLICK and P. WAGREICH, *Isolated singularities of algebraic singularities with  $\mathbf{C}^*$  action*, Annals of Math. 93 (1971), 205–228.
- [P] H. POPP, *Fundamental gruppen algebraischer mannigfaltigkeiten*, Lecture notes in Math. 176, Springer–Verlag, Berlin, Heidelberg, New York, 1970.
- [S1] A. GROTHENDIECK, *Revêtements Etales et Groupe Fondamental (SGA1)*, Lecture notes in Math. 224, Springer–Verlag, Berlin, Heidelberg, New York, 1971.
- [S2] A. GROTHENDIECK, *Cohomologie locale des faisceaux cohérents et theorems locaux et globaux (SGA2)*, North-Holland, Amsterdam, 1968.

*Department of Mathematics*  
*University of Missouri*  
*Columbia, MO 65203*

Received July 7, 1992