Local fundamental groups of surface singularities in characteristic p.

Comment. Math. Helvetici 68 (1993) 319–332 0010-2571/93/020319-14\$1.50 + 0.20/0

Local fundamental groups of surface singularities in characteristic p

STEVEN DALE CUTKOSKY* AND HEMA SRINIVASAN*

The local fundamental group of a normal singularity gives much information about the nature of the singularity. For instance, there is Mumford's theorem [M] that the local fundamental group of the germ of a normal complex analytic surface is zero if and only if the surface is smooth. This has been generalized by Flenner [F] to show that if (A, m) is a normal henselian equicharacteristic zero local ring of dimension two, with algebraically closed residue field, then the algebraic fundamental group $\pi_1(\mathrm{spec}\,(A) - m) = 0$ if and only if A is smooth. Artin has shown that Mumford's characterization of smooth surface germs is false in characteristic p. (c.f. [A3]) The simplest example is the rational double point $k[[x, y, z]]/x^2 + y^2 + z^p$ which has trivial local fundamental group in characteristic p.

In Section 1 we generalize the results of Mumford [M] to characteristic $p \geq 0$. Suppose that (S, x) is a surface singularity of characteristic $p \geq 0$. We first demonstrate that if $\pi_1(S - x)$ is finite, then the intersection diagram of a resolution of singularities of S is simply connected, with vertices of genus 0. When the intersection diagram of a resolution of singularities of S is of this form, we show that there is an expression for the generators and relations of the prime to p part of the local fundamental group of S, which is determined by the intersection matrix of the resolution of singularities of S. This is proved in Theorem 3.

THEOREM 3. *Let (A, m) be a complete normal local domain of dimension two over an algebraically closed field k of characteristic $p \geq 0$. Let $\sigma : X \to \mathrm{spec}\,(A)$ be a resolution of singularities, such that the reduced exceptional fiber has simple normal crossings, with irreducible exceptional curves $E_1, \ldots, E_n$. Suppose that the intersection graph of the exceptional locus is simply connected, and that each E_i is a nonsingular rational curve. Let F_n be the free group on the symbols $\alpha_1, \ldots, \alpha_n$. Then there exists a reindexing of the E_i such that*

$$\pi_1^{(p)}(\mathrm{spec}\,(A) - m) \cong \pi_1^{(p)}\left(X - \sum_{i=1}^{n} E_i\right) \cong (F_n/N)^{(p)}$$

* Both authors were partially supported by NSF.

where N is the normal subgroup of $F(\alpha_1, \dots, \alpha_n)$ generated by the relations

$$\alpha_{j_1} \cdots \alpha_{j_{m(i)}} \alpha_i^{d_i} = 1,$$

$$[\alpha_i, \alpha_{j_1}] = 1, \dots, [\alpha_i, \alpha_{j_{m(i)}}] = 1,$$

for each $1 \le i \le n$, where $E_{j_1}, \dots, E_{j_{m(i)}}$ with $j_1 < \cdots < j_{m(i)}$ are the $m(i)$ curves which intersect E_i and $d_i = (E_i)^2$.

In Corollary 5 we give an arithmetic proof of the Theorem of Mumford and Flenner. To be precise, if (A, m) is a complete normal equicharacteristic zero local ring of dimension two, with algebraically closed residue field, then the algebraic fundamental group $\pi_1(\operatorname{spec}(A) - m) = 0$ if and only if A is smooth.

In Section 3, we prove that for normal Brieskorn singularities, the triviality of the fundamental group is equivalent to the existence of a purely inseparable smooth cover. More precisely,

THEOREM A. *Let $A = k[[x, y, z]]/x^a + y^b + z^c$ where k is an algebraically closed field of characteristic $p \neq 2$ or 3, and A is normal. Let $S = \operatorname{spec}(A)$, and m be the maximal ideal of A. Then the following are equivalent:*

(i) $\pi_1(\operatorname{spec}(A) - m) = 0$.

(ii) *S has a purely inseparable smooth cover.*

We prove this in Theorem 12. (ii) $\Rightarrow$ (i) is always true (Lemma 2). Artin [A3] has proved that the conclusions of Theorem A are true for rational double points in characteristic bigger than two.

Our proof of Theorem 12 involves an anlaysis of the prime to p part of the local fundamental group. We use a group theoretic group, proved in Section 2 (Theorem 6).

M. Artin [A3] has asked if the following are equivalent for a surface singularity (S, x) of positive characteristic.

(1) S has finite local fundamental group.

(2) S has a smooth cover.

Artin has proved (2) $\Rightarrow$ (1) in general, and proved (1) $\Rightarrow$ (2) for rational double points in all characteristics.

Establishing that the conclusions of Theorem A hold for an arbitrary surface singularity would also answer Artin's question in the affirmative.

1. Local fundamental groups of surface singularities

$F(e_1, \ldots, e_n)$ will denote the free group on $e_1, \ldots, e_n$. If G is a group, p a prime, $G^{(p)}$ will denote the pro-finite completion of G with respect to quotient groups of finite order prime to p.

THEOREM 1. *Suppose that (A, m) is a complete normal local domain of dimension two, with algebraically closed residue field k. Suppose that $\pi_1^{(p)}(\operatorname{spec}(A) - m)$ is a finite group. Then*

(a) *The divisor class group of A, $CL(A)$, is an extension of a finite group by a group with a composition series of factors isomorphic to k^+.*

(b) *If $f: X \to \operatorname{spec}(A)$ is a resolution of singularities, such that the reduced exceptional fiber has simple normal crossings, then the irreducible exceptional curves are rational curves, and the intersection graph of the exceptional locus is a tree.*

Proof. We will first prove (a). Let $f: X \to \operatorname{spec}(A)$ be a resolution of singularities such that the reduced exceptional fiber has simple normal crossings. Let D be the reduced exceptional locus of f, and let D_i be the (nonsingular) irreducible components of D. There are exact sequences:

$$0 \to \operatorname{Pic}^0(X) \to CL(A) \to G \to 0, \tag{1}$$

$$0 \to L \to \operatorname{Pic}^0(x) \to \prod \operatorname{Pic}^0(D_i) \to 0 \tag{2}$$

where L has a composition series with factors isomorphic to k^+ and k^* and G is a finite group. (2) is derived in Section 1 of [A1], and (1) is Proposition 14.4 [L].

Suppose that $\mathscr{L}$ is an element of order n in $\operatorname{Pic}^0(X)$, such that p does not divide n if $p > 0$. Then there exists $\sigma \in H^0(X, \mathscr{L}^{\otimes n})$ such that $\sigma: \mathcal{O}_X \to \mathscr{L}^{\otimes n}$ is an isomorphism. $\mathscr{A} = \bigoplus_{i=0}^{n-1} \mathscr{L}^{\otimes -i}$ has an $\mathcal{O}_X$ algebra structure induced by identifying $\mathscr{L}^{\otimes -n}$ with $\mathcal{O}_X$ by σ. $\operatorname{spec}(f_* \mathscr{A})$ is a finite cover of $\operatorname{spec}(\mathscr{A})$ which restricts to be an irreducible, étale, kummer cover of $\operatorname{spec}(\mathscr{A}) - m$ of degree n.

Suppose that $CL(A)$ is not as in (a). Then either some D_i has positive genus, so that $\Pi \operatorname{Pic}^0(D_i)$ is a non-trivial abelian variety, or L has k^* as a term in a composition series. In either case, it can be shown that for each $n > 0$ such that p does not divide n, we have an element $\mathscr{L} \in \operatorname{Pic}^0(X)$ of order n. We can then construct étale kummer covers of X of order n. $\pi_1^{(p)}(\operatorname{spec}(A) - m)$ is then infinite, which is a contradiction.

Let $N = \Sigma(n_q - 1) - s + 1$, where s is the number of irreducible components D_i of D, and n_q is the number of D_i containing the closed point q. The sum is over all closed points q of X. In the construction of the sequence (2), Artin [A1] shows that the contribution of k^* to (2) is a term $(k^*)^N$. (a) is equivalent to $N=0$ and $\mathrm{Pic}^0(D_i)=0$ for all i. Now $\mathrm{Pic}^0(D_i)=0$ is equivalent to D_i being a rational curve. Further, if T is the intersection graph then

$$N=\sum(n_q-1)-s+1=\text{number of edges}-\text{number of vertices}+1=1-\chi(T).$$

So $N=0$ if and only if T is a tree. This completes the proof.

The next Lemma gives one direction of the question (*) raised in the introduction.

LEMMA 2. *Suppose that (A,m) is a complete, normal local domain with algebraically closed residue field k, and that A has a purely inseparable smooth cover. Then $\pi_1(\mathrm{spec}\,(A)-m)=0$.*

Proof. Let $A\to B$ be the purely inseparable smooth cover, where (B,n) is a complete local ring. Since a purely inseparable morphism is radicial, $\pi_1(\mathrm{spec}\,(A)-m)=\pi_1(\mathrm{spec}\,(B)-n)$ by IX 4.10 [S1]. But then, $\pi_1(\mathrm{spec}\,(B)-n)=\pi_1(\mathrm{spec}\,(B))=\pi_1(k)=0$ by X 3.4, X 1.1 [S2].

We will introduce some notation which will be useful in the proof of Theorem 3. In Sections 3 and 4 of the book of Grothendieck and Murre on tame fundamental groups, [GM], it is shown that the notion of tame ramification over a divisor with simple normal crossings extends to formal schemes.

Let $\mathscr{X}$ be a normal, connected formal scheme, with a divisor D on $\mathscr{X}$ with simple normal crossings. Let $\mathrm{Rev}^D(\mathscr{X})$ be the category of formal $\mathscr{X}$-schemes which are tamely ramified over $\mathscr{X}$ relative to $\mathscr{D}\cdot\mathrm{Rev}^D(\mathscr{X})$ is a Galois category by Proposition 4.2.2 of [GM], and hence has a fundamental group by Expose V of [S1].

THEOREM 3. *Let (A,m) be a complete normal local domain of dimension two over an algebraically closed field k of characteristic $p\geq 0$. Let $\sigma: X\to\mathrm{spec}\,(A)$ be a resolution of singularities, such that the reduced exceptional fiber has simple normal crossings, with irreducible exceptional curves $E_1,\ldots,E_n$. Suppose that the intersection graph of the exceptional locus is simply connected, and that each E_i is a nonsingular rational curve. Let F_n be the free group on the symbols $\alpha_1,\ldots,\alpha_n$. Then there exists a reindexing of the E_i such that*

$$\pi_1^{(p)}(\mathrm{spec}\,(A)-m)\cong\pi_1^{(p)}\left(X-\sum_{i=1}^{n}E_i\right)\cong(F_n/N)^{(p)}$$

where N is the normal subgroup of $F(\alpha_1,\ldots,\alpha_n)$ generated by the relations

$$\alpha_{j_1}\cdots\alpha_{j_{m(i)}}\alpha_i^{d_i}=1,$$
$$[\alpha_i,\alpha_{j_1}]=1,\ldots,[\alpha_i,\alpha_{j_{m(i)}}]=1,$$

for each $1 \leq i \leq n$, *where* $E_{j_1}, \ldots, E_{j_{m(i)}}$ *with* $j_1 < \cdots < j_{m(i)}$ *are the* $m(i)$ *curves which intersect* E_i *and* $d_i = (E_i)^2$.

The remainder of Section 1 will be devoted to the proof of Theorem 3. Without loss of generality, we may assume that $n > 1$. Set $E = \Sigma_{i=1}^{n} E_i$. Set $p_{ij} = E_i \cap E_j$ whenever E_i and E_j intersect properly. Let $\mathscr{S}$ be the formal completion of X along $\sigma^{-1}(m)$. Let $\mathscr{S}_i$ be the formal completion of X along E_i for $1 \leq i \leq n$, and let $\mathscr{S}_{ij}$ be the formal completion of X along p_{ij}.

Let $\pi = \pi_1(\mathscr{S})^{(p)}$ be the prime to p part of a fundamental group $\pi_1(\mathscr{S})$ for $\mathrm{Rev}^E(\mathscr{S})$. Let π_i be the prime to p part of a fundamental group for $\mathrm{Rev}^E(\mathscr{S}_i)$, and let π_{ij} be the prime to p part of a fundamental group for $\mathrm{Rev}^E(\mathscr{S}_{ij})$. By Corollary 9.9 of [GM] we have

$$\pi \cong \pi_1^{(p)}(\mathrm{spec}\,(A) - m). \tag{3}$$

Let μ_r be the group of r-th roots of unity of k. Set

$$\mu' = \varprojlim_{p \nmid r} \mu_r.$$

Let w be a "generator" of μ'. By Abhyankar's Lemma, (c.f. XIII 5.3 [S1]), we have a canonical isomorphism $\pi_{ij} \cong \mu' \oplus \mu'$, which is the direct sum of limits of inertia groups of prime divisors ramified over $E_i \cap \mathscr{S}_{ij}$ and $E_j \cap \mathscr{S}_{ij}$. The map $\alpha_i \mapsto (w, 1)$, $\alpha_j \mapsto (1, w)$ determines an isomorphism

$$\pi_{ij} \cong (F(\alpha_i, \alpha_j)/[\alpha_i, \alpha_j]^{(p)}).$$

Let $E_{j_1}, \ldots, E_{j_{m(i)}}$ be the exceptional curves of σ which intersect E_i properly. Suppose that

$$\lambda_i^{ij_k} : \pi_{ij_k} \to \pi_i$$

are paths. Then we will identify α_{j_k} with $\lambda_i^{ij_k}(\alpha_{j_k})$ and α_i with $\lambda_i^{ij_k}(\alpha_i)$ in π_i. We will verify in the proof of Lemma 4 below that this is well defined.

LEMMA 4. *Suppose that for some* l, *a path*

$$\lambda_i^{ij_l} : \pi_{ij_l} \to \pi_i$$

is given, and that τ *is a permutation of* $[1, \ldots, m(i)]$. *Then there exist paths*

$$\lambda_i^{ij_k} : \pi_{ij_k} \to \pi_i$$

such that

$$\pi_i = (F(\alpha_i, \alpha_{j_1}, \ldots, \alpha_{j_{m(i)}})/N)^{(p)}$$

where N *is the normal subgroup generated by the relations*

$$\alpha_{j_{\tau(1)}}\alpha_{j_{\tau(2)}} \cdots \alpha_{j_{\tau(m(i))}}\alpha_i^{d_i} = [\alpha_i, \alpha_{j_1}] = \cdots = [\alpha_i, \alpha_{i,j_m(i)}] = 1.$$

Proof. Let $\phi : \mathscr{X} \to \mathscr{S}_i \in \mathrm{Rev}^E(\mathscr{S}_i)$ be connected and Galois. Then we have that $\phi^{-1}(E_i)$ is irreducible, hence the inertia group of $\phi^{-1}(E_i)_{\mathrm{red}}$ is a normal subgroup of $\mathrm{Gal}(\mathscr{X}/\mathscr{S}_i)$. This inertia group is naturally a quotient of μ^t. Taking limits, we have a natural exact sequence (c.f. Corollary 5.1.11 [GM])

$$\mu^t \to \pi_i \to \pi_1^{(p)}\left(E_i - \sum p_{ij_k}\right) \to 1. \tag{4}$$

By our construction of π_{ij}, for any path $\lambda_i^{ij_k}$, $\lambda_i^{ij_k}(\alpha_i) = w \in \mu^t$.

From the classical description of the fundamental groups of the m-times punctured projective line (c.f. Section 7 of [Ab] and Section 12 of [P]), paths $\lambda_i^{ij_k}$ can be chosen so that

$$\pi_1^{(p)}\left(E_i - \sum p_{ij_k}\right) = (F(\alpha_{j_1}, \ldots, \alpha_{j_{m(i)}})/\alpha_{j_{\tau(1)}} \cdots \alpha_{j_{\tau(m(i))}})^{(p)}.$$

In particular, π_i is a quotient of $F(\alpha_i, \alpha_{j_1}, \ldots, \alpha_{j_{m(i)}})^{(p)}$.

Let s be an integer between 1 and $m(i)$. Let r be an integer such that $(r, p) = 1$, $(r, d_i) = 1$ and $r > -d_i$. Since E_i can be contracted inside $\mathscr{S}_i$ to a rational singularity, there exists $f \in \Gamma(\mathscr{S}_i, \mathcal{O}_{\mathscr{S}_i})$ such that $(f) = -d_i E_s + E_i$. Let $\phi : \mathscr{W}_r \to \mathscr{S}_i \in \mathrm{Rev}^E(\mathscr{S}_i)$ be defined so that $\phi_*(\mathcal{O}_{\mathscr{W}_r})$ is the normalization of $\mathcal{O}_{\mathscr{S}_i}[t]/t^r - f$. We can choose a surjection

$$\Lambda : \pi_i \to \mathrm{Gal}(\mathscr{W}_r/\mathscr{S}_i).$$

ϕ is unramified over E_{j_k} if $k \neq s$. Hence $\Lambda(\alpha_{j_k}) = 1$ if $k \neq s$. Consideration of the induced map

$$\pi_{ij} \to \mathrm{Gal}(\mathscr{W}_r/\mathscr{S}_i)$$

shows that

$$\mathrm{Gal}(W_r/\mathscr{S}_i) = (F(\alpha_i, \alpha_{j_s})/\alpha_i^r = \alpha_{j_s}^r = [\alpha_i, \alpha_{j_s}] = \alpha_{j_s}\alpha_i^{d_i} = 1). \tag{5}$$

By taking r arbitrarily large, we see from (5) that (4) is left exact. Hence

$$\pi_i = (F(\alpha_i, \alpha_{j_1}, \ldots, \alpha_{j_{m(i)}})/\alpha_{j_{\tau(1)}}\alpha_{j_{\tau(2)}} \cdots \alpha_{j_{\tau(m(i))}}\alpha_i^{e_i} = [\alpha_i, \alpha_{j_1}] = \cdots = [\alpha_i, \alpha_{i,j_{m(i)}}] = 1)^{(p)}$$

for some integer e_i. Now (5) shows that $e_i = d_i$.

Now we will return to the proof of Theorem 3. Since the intersection graph of E is a tree, it follows from Lemma 4 and induction that it is possible to choose paths

$$\lambda_i^{ij} : \pi_{ij} \to \pi_i \quad \text{and} \quad \phi_i : \pi_i \to \pi$$

such that after a reordering of the E_i,

$$\begin{tikzcd} \pi_{ij} \arrow[r, "\lambda_i^{ij}"] \arrow[d, "\lambda_j^{ij}"'] & \pi_i \arrow[d, "\phi_i"] \\ \pi_j \arrow[r, "\phi_j"] & \pi \end{tikzcd} \tag{6}$$

commutes, and

$$\pi_i = (F(\alpha_i, \alpha_{j_1}, \ldots, \alpha_{j_{m(i)}})/(\alpha_{j_1}\alpha_{j_2} \cdots \alpha_{j_{m(i)}}\alpha_i^{d_i} = [\alpha_i, \alpha_{j_1}] = \cdots = [\alpha_i, \alpha_{i,j_{m(i)}}] = 1)^{(p)},$$

where $E_{j_1}, \ldots, E_{j_m}$ with $j_1 < \cdots < j_{m(i)}$ are the curves which intersect E_i properly. We can then identify α_i with $\phi_i(\alpha_i) = \phi_j(\alpha_i)$ in π.

The statement of Theorem 3 now follows from (3), (6), and the arithmetic analogue of Van Kampen's Theorem proved in Corollary 8.3.6 of [GM].

As a corollary, we get an arithmetic proof of Mumford and Flenner's Theorem.

COROLLARY 5 (Mumford–Flenner). *Suppose that (A, m) is a complete normal local domain of dimension two, with algebraically closed residue field k of characteristic zero. Then $\pi_1(\operatorname{spec}(A) - m) = 0$ if and only if A is smooth over k.*

Proof. By purity of Branch Locus (X.3.4 [S2] and X 1.1 [S2]), A smooth implies that $\pi_1(\operatorname{spec}(A) - m) = 0$.

Suppose that $\pi_1(\operatorname{spec}(A) - m) = 0$. Then by Theorems 1 and 3 we have an expression for $\pi_1(\operatorname{spec}(A) - m)$ in terms of generators and relations, depending on the intersection matrix of a resolution of singularities. $\pi_1(\operatorname{spec}(A) - m)$ is thus isomorphic to the profinite completion with respect to quotient groups of finite order of the group $\pi(\Gamma)$ associated to the intersection diagram of a resolution of singularities defined in [F]. By Theorem 2.7 [F], this group is trivial if and only if A is smooth.

2. Existence of quotient groups of order prime to p

LEMMA 5. *Let $s_1, \ldots, s_t$ be integers, greater than one. For every prime number $p > 3$ such that p does not divide s_i for $i = 1, \ldots, t$, there exists a prime $q > 3$ such that $q \equiv 1 \pmod{s_i}$ for $i = 1, \ldots, t$, but p does not divide $q(q-1)(q+1)$.*

Proof. Let $a = \Pi_{i=1}^{t} s_i$. Since $(a, p) = 1$, $ma + np = 1$ for some integers m and n. There are indeed infinitely many primes in the set $\{kap + (-np + 2) \mid k \in \mathbf{Z}\}$ because $(ap, -np + 2) = 1$. Choose a prime $q > 3$ from this set

$$q \equiv -np + 2 \equiv -1 + 2 \equiv 1 \bmod a$$

and $q \equiv 2 \bmod p$. Thus $q \equiv 1 \bmod s_i$, for $i = 1, \ldots, t$, and p divides $q - 2$. Since both p and q are larger than 3, p does not divide $q(q-1)(q+1)$.

THEOREM 6. *Suppose that $t \geq 3$, $s_1, \ldots, s_t$ are integers such that each $s_i > 1$, and $p > 3$ is a prime such that p does not divide s_i for $i = 1, \ldots, t$. Then*

$$F(e_1, \ldots, e_t)/e_1^{s_1} = \cdots = e_t^{s_t} = e_1 \cdots e_t = 1$$

has a quotient of finite order prime to p.

Proof. Let $q > 3$ be a prime number such that $q \equiv 1 \bmod 2s_i$ for $i = 1, \ldots, t$. Let $F = F_q$ be the finite field with q elements. Since $2s_i$ divides $q - 1$, we can pick an element x_i of F_q of order $2s_i$. Let

$$A_i = \begin{pmatrix} x_i & 0 \\ 0 & \dfrac{1}{x_i} \end{pmatrix}$$

for $i = 1, \ldots, t$, so that the order of A_i is $2s_i$ in $SL(2, F)$. Define

$$E_1 = \begin{pmatrix} 0 & -1 \\ 1 & x_1 + \dfrac{1}{x_1} \end{pmatrix},$$

$$E_2 = \begin{pmatrix} x_2 + \dfrac{1}{x_2} & x_3 \\ -\dfrac{1}{x_3} & 0 \end{pmatrix},$$

$$E_3 = \begin{pmatrix} x_3 & 0 \\ -x_2 - \dfrac{1}{x_2} + \dfrac{x_1}{x_3} + \dfrac{1}{x_1 x_3} & \dfrac{1}{x_3} \end{pmatrix}.$$

Define $E_i = I$ for $3 < i \leq t$. $\operatorname{trace}(E_i) = x_i + 1/x_i = \operatorname{trace}(A_i)$ for $i = 1, 2, 3$. Since $s_i > 1$, $A_i \neq \pm I$. Hence E_i and A_i are conjugates in $GL(2, F)$. The order of E_i is thus $2s_i$.

For $i = 1, \ldots, t$, define maps

$$\Phi_i : \mathbf{Z}_{s_t} \to SL(2, F)/\{\pm I\}$$

by $\Phi_i(1) = E_i$. We have

$$\prod_{i=1}^{t} \Phi_i(1) = E_1 E_2 E_3 I = I.$$

Let $G = \mathbf{Z}_{s_1} * \cdots * \mathbf{Z}_{s_t} / \Pi_{i=1}^{t} e_i = 1$. The Φ_i define a unique map Φ such that

$$\begin{tikzcd} G \arrow[r, "\Phi"] \arrow[d] & SL(2, F)/\{\pm I\} \\ \mathbf{Z}_{s_t} \arrow[ur, "\Phi_t"'] & \end{tikzcd}$$

commutes. Observe that

$$G = F(e_1, \ldots, e_t)/e_1^{s_1} = \cdots = e_t^{s_t} = e_1 e_2 \cdots e_t = 1.$$

Φ is nontrivial since Φ_1, Φ_2 and Φ_3 are nontrivial. Thus $G/\operatorname{kernel}(\Phi)$ is a nontrivial quotient of G whose order $|\Phi(G)|$ is a nontrivial factor of $|SL(2, F)/\{\pm I\}|$. So G has a nontrivial quotient of finite order dividing $q(q - q)(q + 1)/2$. By Lemma 4, we can choose the prime q such that p does not divide $q(q - 1)(q + 1)$. Thus G has a finite nontrivial quotient of order prime to p.

3. Brieskorn singularities

In this section we will use the following notations. Suppose that k is an algebraically closed field of characteristic $p > 3$. Suppose that a_1, a_2, a_3 are positive integers. Let

$$R(a_1, a_2, a_3) = k[[x_1, x_2, x_3]]/(x_1^{a_1} + x_2^{a_2} + x_3^{a_3}).$$

$R(a_1, a_2, a_3)$ is normal precisely when p divides at most one of the exponents a_1, a_2, a_3. Suppose that $R(a_1, a_2, a_3)$ is normal. Let m be the maximal ideal of $R(a_1, a_2, a_3)$. Let $S(a_1, a_2, a_3) = \operatorname{spec}(R(a_1, a_2, a_3)) - m$.

PROPOSITION 7. *Write* $a_i = p^{r_i} b_i$ *where* $(b_i, p) = 1$. *Then*

$$\pi_1(S(a_1, a_2, a_3)) \cong \pi_1(S(b_1, b_2, b_3)).$$

Proof. Define

$$\phi : k[[x_1, x_2, x_3]]/(x_1^{b_1} + x_2^{b_2} + x_3^{b_3}) \to k[[y_1, y_2, y_3]]/(y_1^{a_1} + y_2^{a_2} + y_3^{a_3})$$

by $x_1 \mapsto y_1^{p^{r_1}}$, $x_2 \mapsto y_2^{p^{r_2}}$, $x_3 \mapsto y_3^{p^{r_3}}$. ϕ is purely inseparable, hence radicial. The proposition follows from IX 4.10 [S1].

Resolutions of Brieskorn singularities are constructed in characteristic zero in [H–J] and [O–W]. the proofs easily extend to characteristic p.

PROPOSITION 8. *Suppose that p does not divide a_i for $1 \le i \le 3$. Then the intersection diagram of the minimal resolution of singularities of* $\operatorname{spec}(R(a_1, a_2, a_3))$ *can be described as follows: Let*

$$c = (a_1, a_2, a_3), \quad c_1 = \frac{(a_2, a_3)}{c}, \quad c_2 = \frac{(a_1, a_3)}{c}, \quad c_3 = \frac{(a_1, a_2)}{c},$$

$$\gamma_1 = \frac{a_1}{cc_2c_3}, \quad \gamma_2 = \frac{a_2}{cc_1c_3}, \quad \gamma_3 = \frac{a_3}{cc_1c_2}.$$

Let $0 < r_1 < \gamma_1$, $0 < r_2 < \gamma_2$, $0 < r_3 < \gamma_3$ *satisfy*

$$c_1\gamma_2\gamma_3r_1 \equiv -1 \bmod (\gamma_1). \quad c_2\gamma_1\gamma_3r_2 \equiv -1 \bmod (\gamma_2), \quad c_3\gamma_1\gamma_2r_3 \equiv -1 \bmod (\gamma_3)$$

Let b^i_j for $i = 1, 2, 3$ and $1 \le j \le t_i$ denote the continued fraction expansions

$$\frac{\gamma_i}{r_i} = b^i_{t_i} - \cfrac{1}{b^i_{t_i-1} - \cfrac{1}{\cdots - \cfrac{1}{b^i_1}}}.$$

Let L_i be the linear graph with $t_i + 1$ vertices and successive weights $-b^i_1, \ldots,$ $-b^i_{t_i}$, $-b$.

The intersection diagram of spec $(R(a_1, a_2, a_3))$ *is the star shaped graph obtained by identifying the vertex with weight* $-b$ *of* cc_1 *copies of* L_1, cc_2 *copies of* L_2, *and* cc_3 *copies of* L_3 *to a common point. The arms of the star in the* cc_1 *copies of* L_1, cc_2 *copies of* L_2, *and* cc_3 *copies of* L_3 *which are glued together at the vertex of weight* $-b$.

Each vertex in the intersection diagram corresponds to a smooth rational curve except for possibly the central vertex (with weight $-b$*), which corresponds to a smooth curve* K *of genus*

$$g_K = \frac{1}{2}(2 + c^2 c_1 c_2 c_3 - cc_1 - cc_2 - cc_3).$$

PROPOSITION 9. *Suppose that* p *does not divide* a_i *for* $i = 1, 2, 3$ *and* $\pi_1(S(a_1, a_2, a_3)) = 0$. *Then* $g_K = 0$, *and one of the following cases must occur.*

(1) $c = c_1 = c_2 = 1$ *and* c_3 *is arbitrary.*

(2) $c = c_2 = c_3 = 1$ *and* c_1 *is arbitrary.*

(3) $c = c_1 = c_3 = 1$ *and* c_2 *is arbitrary.*

(4) $c = 2$ *and* $c_1 = c_2 = c_3 = 1$.

Proof. $g_K = 0$ by Theorem 1. We will determine the positive integers c, c_1, c_2, c_3 such that

$$2 + c(cc_1c_2c_3 - c_1 - c_2 - c_3) \leq 0.$$

Without loss of generality, we may assume that $c_1 \leq c_2 \leq c_3$. We immediately reduce to $cc_1c_2c_3 - c_1 - c_2 - c_3 < 0$ which forces $cc_1c_2 < 3$. The only solutions are $c = 2$, $c_1 = c_2 = c_3 = 1$ and $c = c_1 = 1$, $c_2 = c_3 = 2$ and $c = c_1 = c_2 = 1$, c_3 arbitrary.

PROPOSITION 10. *Suppose that* p *does not divide* a_i *for* $i = 1, 2, 3$. *Suppose that* $g_K = 0$. *Then*

$$\pi_1^{(p)}(S(a_1, a_2, a_3))$$
$$\cong (F(e; e_1^{1,1}, \ldots, e_{t_1}^{1,1}, e_1^{2,1}, \ldots, e_{t_1}^{cc_1,1}; e_1^{1,2}, \ldots, e_{t_2}^{cc_2,2}; e_1^{1,3}, \ldots, e_{t_3}^{cc_3,3})/N)^{(p)}$$

where N *is the normal subgroup generated by the relations*

$$e_{t_1}^{1,1} \cdots e_{t_1}^{cc_1,1} e_{t_2}^{1,2} \cdots e_{t_2}^{cc_2,2} e_{t_3}^{1,3} \cdots e_{t_3}^{cc_3,3} e^{-b} = 1,$$
$$[e, e_{t_1}^{k,1}] = [e, e_{t_2}^{k,2}] = [e, e_{t_3}^{k,3}] = 1,$$

for $1 \le k \le cc_i$ *and the relations for* $1 \le i \le 3 \le k \le cc_i$

$$\begin{aligned}
&ee^{k,i}_{t_i-1}(e^{k,i}_{t_i})^{-b^i_{t_i}} = 1,\\
&e^{k,i}_2(e^{k,i}_1)^{-b^i_1} = 1,\\
&e^{k,i}_{j-1}e^{k,i}_{j+1}(e^{k,i}_j)^{-b^i_j} = 1 \qquad \text{for } 2 \le j \le t_i - 1,\\
&[e^{k,i}_j, e^{k,i}_{j+1}] = 1 \qquad \text{for } 1 \le j \le t_i - 1.
\end{aligned} \tag{i, k}$$

Proof. This is immediate for Theorems 1 and 3.

PROPOSITION 11. *Let assumptions be as in Proposition* 10. *Then*

$$\pi_1^{(p)}(S(a_1, a_2, a_3)) \cong (F(e; e_1^1, \dots, e_1^{cc_1}; e_2^1, \dots, e_2^{cc_2}; e_3^1, \dots, e_3^{cc_3})/M)^{(p)}$$

where M *is the normal subgroup generated by the relations*

$$\begin{aligned}
&e_1^1 \cdots e_1^{cc_1} e_2^1 \cdots e_2^{cc_2} e_3^1 \cdots e_3^{cc_3} e^{-b} = 1,\\
&e^{r_i}(e_i^k)^{-\gamma_i} = 1 \qquad \text{for } 1 \le i \le 3,\ 1 \le k \le cc_i,\\
&[e, e_i^k] = 1 \qquad \text{for } 1 \le i \le 3,\ 1 \le k \le cc_i.
\end{aligned}$$

Proof. The relations (i, k) determine relations

$$\begin{aligned}
&(e^{k,i}_{j+1})^{\alpha^i_j}(e^{k,i}_j)^{-\alpha^i_{j+1}} = 1, \qquad 1 \le j \le t_i - 1,\\
&e^{\alpha^i_{t_i}}(e^{k,i}_{t_i})^{-\alpha^i_{t_i+1}} = 1
\end{aligned}$$

where $\alpha^i_0 = 0$, α^i_j is determined by the recursion formula

$$\alpha^i_j = b^i_{j-1}\alpha^i_{j-1} - \alpha^i_{j-2}$$

for $2 \le j \le t_i + 1$. That is,

$$\frac{\alpha^i_j}{\alpha^i_{j-1}} = b^i_{j-1} - \cfrac{1}{\cfrac{\alpha^i_{j-1}}{\alpha^i_{j-2}}}.$$

So, we have

$$\frac{\alpha^i_{t_i+1}}{\alpha^i_{t_i}} = b^i_{t_i} - \cfrac{1}{b^i_{t_i-1} - \cfrac{1}{\cdots - \cfrac{1}{b^i_1}}} = \frac{\gamma_i}{r_i}$$

by Proposition 8. Since $(\gamma_i, r_i) = 1$, this gives $e^{r_i}(e_{t_i}^{k,i})^{-\gamma_i} = 1$. On the other hand, using the relations (i, k), one can eliminate the $e_j^{k,i}$, for $1 \leq j \leq t_i - 1$, since they can be written in terms of $e_{t_i}^{k,i}$ and e. Set $e_i^k = e_{t_i}^{k,i}$. We then have the conclusions of Proposition 11.

THEOREM 12. *The following are equivalent.*
(1) $\pi_1(S(a_1, a_2, a_3)) = 0$.
(2) Spec $(R(a_1, a_2, a_3))$ *has a purely inseparable cover by a power series ring in k.*
(3) *Some a_i is a power of p.*

Proof. (3) implies (2) follows from the proof of Proposition 7. (2) implies (1) follows from Lemma 2. We must show that (1) implies (3).

We assume that b_i are such that $\pi_1(S(b_1, b_2, b_3)) = 0$, and prove that some b_i is a power of p. Let a_i be the positive integers such that $(a_i, p) = 1$, and $b_i = a_i p^{\lambda_i}$. Then $\pi_1((S(a_1, a_2, a_3)) = 0$ by Proposition 7. By Proposition 9, $g_K = 0$. Proposition 11 shows that we have a surjection obtained by taking the quotient of $\pi_1^{(p)}(S(a_1, a_2, a_3))$ by the normal subgroup generated by e.

$$\pi_1(S(a_1, a_2, a_3)) \to (F(e_1^1, \dots, e_1^{cc_1}; e_2^1, \dots, e_2^{cc_2}; e_3^1, \dots, e_3^{cc_3})/L)^{(p)} \qquad (7)$$

where L is the normal subgroup generated by the relations

$$e_1^1 \cdots e_1^{cc_1} e_2^1 \cdots e_2^{cc_2} e_3^1 \cdots e_3^{cc_3} = 1,$$
$$(e_i^k)^{\gamma_i} = 1 \qquad \text{for } 1 \leq i \leq 3,\ 1 \leq k \leq cc_i.$$

By Theorem 6, some $\gamma_i = 1$.

Suppose that one of the cases (1), (2), (3) of Proposition 9 occurs. After reindexing the a_i, we may assume that case (1) holds, so that $c = c_1 = c_2 = 1$, and c_3 is arbitrary. Then $(a_1, a_2, a_3) = (\gamma_1 c_3, \gamma_2 c_3, \gamma_3)$. By (7), and Theorem 6, $c_3 > 2$ implies that $\gamma_3 = 1$ and $a_3 = 1$. If $c_3 = 2$, and $\gamma_3 > 1$, then $\gamma_1 = \gamma_2 = 1$ which implies that the right hand side of (7) is $\mathbf{Z}_{\gamma_3} \neq 0$, a contradiction. If $c_3 = 1$, then some $a_i = 1$.

The remaining case of Proposition 9 is when $c_1 = c_2 = c_3 = 1$ and $c = 2$, so that $(a_1, a_2, a_3) = (2\gamma_1, 2\gamma_2, 2\gamma_3)$. Suppose that some $\gamma_i > 1$. Since $\pi_1((S(a_1, a_2, a_3))$ is trivial, we see by Theorem 6 and (7) that at most one γ_i is greater than 1. After reindexing the a_i, we may assume that $\gamma_1 > 1$ and $\gamma_2 = \gamma_3 = 1$. The right hand side of (7) is then $\mathbf{Z}_{\gamma_1}$, a contradiction. Hence $(a_1, a_2, a_3) = (2, 2, 2)$.

In this case, the intersection graph of the minimal resolution of singularities of $x_1^2 + x_2^2 + x_3^2 = 0$ is a single vertex, corresponding to a nonsingular rational curve, with weight -2. Hence $\pi_1(S(a_1, a_2, a_3) \cong \mathbf{Z}_2 \neq 0$, so that this case cannot occur.

REFERENCES

[Ab] S. ABHYANKAR, *Coverings of algebraic curves*, Amer. J. Math. *79* (1957), 825–856.

[A1] M. ARTIN, *Some numerical criteria for contractability of curves on algebraic surfaces*, Amer. J. Math. *84* (1962), 485–496.

[A2] M. ARTIN, *Lifting of two-dimensional singularities to characteristic zero*, Amer. J. Math. *88* (1966), 747–762.

[A3] M. ARTIN, *Coverings of the rational double points in characteristic p*, in: W. Baily and T. Shioda (eds)., *Complex analysis and algebraic geometry*, Cambridge University press, Cambridge, 1977, 11–22.

[F] H. FLENNER, *Reine lokale ringe der dimension zwei*, Math. Ann. *216* (1975), 253–263.

[GM] A. GROTHENDIECK and J. MURRE, *The tame fundamental group of a formal neighborhood of a divisor with normal crossing on a scheme*, Lecture notes in Math. *208*, Springer–Verlag, Berlin, Heidelberg, New York 1971.

[H–J] F. HIRZEBRUCH and K. JANICH, *Involutions and singularities*, Algebraic Geometry (international colloquium, T.I.F.R., Bomaby, 1968) 219–240, Oxford University Press, Bombay, 1969.

[L] J. LIPMAN, *Rational singularities, with applications to algebraic surfaces and unique factorization.* Publ. Math. Inst. Hautes Etud. Sci. *36* (1969), 195–278.

[M] D. MUMFORD, *The topology of normal singularities of an algebraic surface and a criterion for simplicity.* Publ. Math. Inst. Hautes, Etud. Sci. *9* (1961), 5–22.

[O–W] P. ORLICK and P. WAGREICH, *Isolated singularities of algebraic singularities with* **C*** *action*, Annals of Math. *93* (1971), 205–228.

[P] H. POPP, *Fundamental gruppen algebraischer manigfaltigkeiten*, Lecture notes in Math. *176*, Springer–Verlag, Berlin, Heidelberg, New York, 1970.

[S1] A. GROTHENDIECK, *Revêtements Etales et Groupe Fondamental* (SGA1), Lecture notes in Math. *224*, Springer–Verlag, Berlin, Heidelberg, New York, 1971.

[S2] A. GROTHENDIECK, *Cohomologie locale des faisceaux cohérents et theorems locaux et globaux* (SGA2), North-Holland, Amsterdam, 1968.

Department of Mathematics
University of Missouri
Columbia, MO 65203

Received July 7, 1992