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Transcendental submanifolds of Rⁿ

S. AKBULUT AND H. KING

Dedicated to memory of Mario Raimondo

Abstract. In this paper we show how the restriction of the complex algebraic cycles to real part of a complex algebraic set is related to the real algebraic cycles of the real part. As a corollary we give examples of smooth submanifolds of a Euclidean space which can not be isotoped to real parts of complex nonsingular subvarieties in the corresponding projective space.

Algebraic numbers are dense in **R**. The problem of whether closed smooth submanifolds $M \subset \mathbf{R}^n$ can be approximated by nonsingular algebraic subsets could be viewed as a possible higher dimensional version of this property. By adapting a stronger version of the notion of nonsingularity (complexification is nonsingular) the results of this paper show that this is not the case. By identifying $\mathbf{R}^n \subset \mathbf{RP}^n$ we prove:

THEOREM. There are closed smooth submanifolds $M \subset \mathbb{R}^n$ which can not be isotoped to the real parts of nonsingular complex algebraic subvarieties of \mathbb{CP}^n . Furthermore, we can choose M to be nonsingular real algebraic subsets of \mathbb{R}^n .

Now a brief history: Seifert showed that if $M \subset \mathbb{R}^n$ has trivial normal bundle then it can be isotoped to a nonsingular component of an algebraic subset of \mathbb{R}^n ([S]). Nash proved that in general M can be isotoped to a nonsingular sheet of an algebraic subset of \mathbb{R}^n ; but the sheets might intersect each other ([N]). In [AK4] it was shown that M can be isotoped to a nonsingular component of an algebraic subset of \mathbb{R}^n , i.e. these sheets can be separated. Whether M can be isotoped to (not just to a component of) a nonsingular algebraic subset of \mathbb{R}^n still remains open.

We should emphasize that stably the answers of these problems are known. For example, Nash already proved that M can be isotoped to a nonsingular component of an algebraic set inside of a larger Euclidean space $\mathbb{R}^n \times \mathbb{R}^k$; and later Tognoli showed that in a larger Euclidean space the extra components of the algebraic set can be removed ([T]). In [AK4] and [AK5] it was shown that any $M \subset \mathbb{R}^n$ can be isotoped to a nonsingular algebraic subset of $\mathbb{R}^n \times \mathbb{R}$; more generally M can be isotoped to a nonsingular algebraic subset of \mathbb{R}^n if and only if M is cobordant through immersions to an algebraic subset of \mathbb{R}^n . We obtain the above results as a corollary to our main theorem which says that the restriction of complex algebraic cycles of the complexification of a nonsingular algebraic set consists of the cup products of the real algebraic cycles. Another corollary is that the Ponryagin classes of the tangent and the normal bundle of a nonsingular algebraic set in \mathbb{R}^n are represented by real algebraic subsets, a fact which was known only for the Steifel-Whitney classes.

1. Gysin homomorphism

Let $f: M^m \to N^n$ be a map. The Gysin homomorphism $f_+: H^*(M) \to H^{*+k}(N)$ is defined by the commutative diagram:

where k = n - m and the vertical maps are the Poincaré duality isomorphisms. Gysin homomorphism satisfies the following well known properties:

LEMMA 1. The Gysin homomorphism satisfies the following properties: (a) $f_+(f^*(u) - v) = u - f_+(v)$. (b) Given a commuting diagram:

$$\begin{array}{c} M \xrightarrow{j} N \\ \uparrow i & \uparrow j \\ K \xrightarrow{g} L \end{array}$$

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where i, j are imbeddings and g is transveral to L with $g^{-1}(L) = K$ then:

$$f^{\ast} \circ j_{+} = i_{+} \circ g^{\ast}.$$

LEMMA 2. Let $f: M \hookrightarrow N$ be an imbedding. Let u_M be the dual of the fundamental class of M in N, and $\chi(v_f)$ be the Euler class of the normal bundle of f then:

(a)
$$f_{+}(1) = u_{M}$$
,
(b) $f_{+}f^{*}(x) = x \smile u_{M}$,
(c) $f^{*}f_{+}(1) = f^{*}(u_{M}) = \chi(v_{f})$,
(d) $f_{+}(x) \smile f_{+}(y) = f_{+}(x \smile y) \smile f_{+}(1)$

These lemmas are standard facts of algebraic topology; we leave the proofs as an exercise to the reader.

2. Algebraic homology

Let V be a Zariski open real (or complex) algebraic set (defined over **R**), and $R = \mathbb{Z}_2$ (or $R = \mathbb{Z}$), then we can define algebraic homology groups $H^A_*(V; R)$ to be the subgroup of $H_*(V; R)$ generated by the compact real (or complex) algebraic subsets of V (cf. [AK1]). We define $H^*_A(V; R)$ to be the Poincaré duals of the groups $H^A_*(V; R)$ when defined. The resolution theorem of [H], implies that $H^A_*(V; R)$ is also the subgroup generated by the classes $g_*([S])$ where $g: S \to V$ is an entire rational function, S is a compact nonsingular real (or complex) algebraic set and [S] is the fundamental class of S. Therefore even when V is real, we can define $H^A_*(V; \mathbb{Z})$ to be the subgroup generated by $g_*([S])$ where $g: S \to V$ is an entire rational function from an oriented compact nonsingular real algebraic set and [S] is the fundamental class of S.

We call a real algebraic set V totally algebraic if $H_*(V; \mathbb{Z}_2) = H_*^A(V; \mathbb{Z}_2)$. It is known that not all nonsingular algebraic sets are totally algebraic. There are closed smooth manifolds which can not even be diffeomorphic to nonsingular totally algebraic sets ([BD]), even though every closed smooth manifold is homeomorphic to a totally algebraic set ([AK2]). Hence these algebraic homology groups are intimately related to the nonsingularity of the underlying algebraic set.

Recall from [BBK] that, for a compact nonsingular real algebraic set V, $H^*_{C-alg}(V; \mathbb{Z})$ is defined to be the subgroup of $H^*(V; \mathbb{Z})$ generated by the restriction of the classes of $H^*_A(V_C; \mathbb{Z})$ by the projective nonsingular complexification map $j: V \hookrightarrow V_C$ (this always exists). $H^*_{C-alg}(V; \mathbb{Z})$ is independent of the complexification V_C . Define $H^*_{C-alg}(V; \mathbb{Z}_2)$ to be the mod 2 reduction of $H^*_{C-alg}(V; \mathbb{Z})$.

The real algebraic cocycle groups $H^*_A(V, \mathbb{Z}_2)$ play useful role in real algebraic geometry. For example, they carry obstructions to isotoping submanifolds to algebraic subsets (see [AK1]). Likewise the groups $H^*_{C-alg}(V; \mathbb{Z}_2)$ also appear as obstructions to algebraic approximation problems (see [BK1]). Our main result describes the relation between these groups.

We need the next result in the proof of the main theorem. It is a special case of Fulton's theorem ([F]). In our notation $V_{\rm C}$ denotes a complex algebraic set defined over **R** with real part V. Square bracket such as [L] means the homology class induced by L, and D denotes the Poincaré duality homomorphism.

LEMMA 3. Let $\pi_{\mathbf{C}} : \tilde{V}_{\mathbf{C}} \to V_{\mathbf{C}}$ be a blowup of a compact nonsingular algebraic set along a nonsingular center $X_{\mathbf{C}} \subset V_{\mathbf{C}}$. Let $L_{\mathbf{C}}$ be an algebraic subset of $V_{\mathbf{C}}$ with $X_{\mathbf{C}} \subset L_{\mathbf{C}}$. Let $\tilde{L}_{\mathbf{C}} \subset \tilde{V}_{\mathbf{C}}$ be the strict transform and $\tilde{X}_{\mathbf{C}}$ be the exceptional locus $\pi_{\mathbf{C}}^{-1}(X_{\mathbf{C}})$. Then there is a proper algebraic subset $Z_{\mathbf{C}} \subset \tilde{X}_{\mathbf{C}}$ such that:

(a) $D^{-1}[\tilde{L}_{\mathbf{C}}] = \pi^* D^{-1}[L_{\mathbf{C}}] + D^{-1}[Z_{\mathbf{C}}].$

(b) $D^{-1}[\tilde{L}] = \pi^* D^{-1}[L] + D^{-1}[Z].$

Proof. The fact that $D^{-1}[\tilde{L}_C]$ and $\pi^*D^{-1}[L_C]$ differ by a cohomology class supported in \tilde{X}_C is standard (e.g. [AK1], Lemma 2.9.3). More specifically Theorem 6.7 of [F] gives an exact expression for the difference as an algebraic cohomology cycle.

3. Main results

Now for the rest of the paper let $V \subset \mathbb{RP}^n$ denote a compact nonsingular real algebraic set of dimension v, and $V_{\mathbb{C}} \subset \mathbb{CP}^n$ be a nonsingular projective complexification of V. Let $j: V \subseteq V_{\mathbb{C}}$ denote the inclusion.

Define $\overline{H}_{2k}^{A}(V_{\mathbb{C}}; \mathbb{Z})$ to be the subgroup of $H_{2k}^{A}(V_{\mathbb{C}}; \mathbb{Z})$ generated by irreducible complex algebraic subsets defined over \mathbb{R} with k-dimensional real parts. In other words it is generated by the complexification of k-dimensional real algebraic subsets of V in $V_{\mathbb{C}}$. As above, by the resolution theorem, $\overline{H}_{2k}^{A}(V_{\mathbb{C}}; \mathbb{Z})$ is generated by the classes $g_*([L_{\mathbb{C}}])$, where $L_{\mathbb{C}}$ is a compact irreducible nonsingular complex algebraic set defined over \mathbb{R} , i.e., it is the complexification of a k-dimensional real algebraic set L, and $g: L_{\mathbb{C}} \to V_{\mathbb{C}}$ is an entire rational function defined over \mathbb{R} , i.e., it is in the form $g = g_{\mathbb{C}}$. Define a subgroup of $H_{\mathbb{C}-alg}^{2k}(V; \mathbb{Z})$ by:

 $\bar{H}^{2k}_{\mathbf{C}-alg}(V;\mathbf{Z}) = j^* \bar{H}^{2k}_A(V_{\mathbf{C}};\mathbf{Z}).$

Let $\bar{H}_{C-alg}^{2k}(V; \mathbb{Z}_2)$ be its mod 2 reduction. The main theorem below implies that this last group is independent of the complexification V_C . Finally, define the following natural subgroup:

$$H^k_A(V; \mathbf{Z}_2)^2 = \{ \alpha^2 \mid \alpha \in H^k_A(V; \mathbf{Z}_2) \}$$

of $H_A^{2k}(V; \mathbb{Z}_2)$ (since cup product operation preserves algebraic cycles, [AK1])

THEOREM A. For all k the following hold: (a) $H_{\mathbf{C}-alg}^{2k}(V; R) \subset H_A^{2k}(V; R)$, where $R = \mathbf{Z}_2$ (or \mathbf{Z} when V is orientable). (b) $\bar{H}_{\mathbf{C}-alg}^{2k}(V; \mathbf{Z}_2) = H_A^k(V; \mathbf{Z}_2)^2$.

Proof. To prove (a) let $a \in H^{2k}_{C-alg}(V; R)$ be represented by the restriction of $\alpha \in H^{2k}_A(V_{\mathbb{C}}; R)$. Let $\beta \in H^A_{2v-2k}(V_{\mathbb{C}}; R)$ be the Poincaré dual of α in $V_{\mathbb{C}}$. Recall that the map j_1 induced by the restriction and the Poincaré duality maps:

is the homology intersection with the fundamental cycle [V], i.e., $j_!(\beta)$ is obtained by transversally intersecting V and a representative of β .

By definition β is represented by $g_*([S])$, where S is a compact nonsingular complex algebraic set and $g: S \to V_{\mathbb{C}}$ is an entire rational function. We can ϵ -isotop g to a smooth function $g_0: S \to V_{\mathbb{C}}$ which is transverse to $V \subset V_{\mathbb{C}}$. By [AK1] Proposition 2.8.8, we can find a nonsingular real algebraic set S' and a rational diffeomorphism $\pi: S' \to S$ and a rational map $F: S' \to V_{\mathbb{C}}$ such that $g_0 \circ \pi$ is ϵ -close to F (here we are viewing S and $V_{\mathbb{C}}$ as real algebraic sets by thinking C as \mathbb{R}^2). Hence F is transverse to V. If $T = F^{-1}(V)$ and $f: T \to V$ is the restriction of F, then $f_*[T]$ represents the Poincaré dual of a.

To see (b) let $a \in \overline{H}_{C-alg}^{2k}(V; \mathbb{Z}_2)$. Then *a* is the restriction of a class in $H_A^{2k}(V_C; \mathbb{Z}_2)$ whose Poincaré dual in $H_{2v-2k}^A(V_C; \mathbb{Z}_2)$ can be represented by $g_*[L_C]$, where $g: L_C \to V_C$ is an entire rational function from a nonsingular compact complex algebraic set which is the complexification of a v - k dimensional real algebraic set L.

We first prove (b) under the assumption that g is an inclusion $L_{\mathbf{C}} \subset V_{\mathbf{C}}$ of a nonsingular algebraic subset: We first ϵ -isotop g to a smooth function $g_0: L_{\mathbf{C}} \to V_{\mathbf{C}}$ which is transverse to $V \subset V_{\mathbf{C}}$. As before, by viewing $L_{\mathbf{C}}$ as a real algebraic set we can find a nonsingular real algebraic set L' and a rational diffeomorphism $\pi: L' \to L_{\mathbf{C}}$ and a rational map $F: L' \to V_{\mathbf{C}}$ such that $g_0 \circ \pi$ is ϵ -close to F. So F is transverse to V. If $T = F^{-1}(V)$ and $f: T \to V$ is the restriction of F, then $f_*[T]$ represents the Poincaré dual of a.

We claim that $f_*[T]$ is also the self intersection of the homology cycle $g_*[L]$ in V. In other words a is the cup product square of the dual of the map $g_*[L]$. To see this observe that the normal bundle of $V \subset V_{\mathbb{C}}$ is isomorphic to the tangent bundle of V (given by the multiplication by $\sqrt{-1}$). Hence the tubular neighborhoods of $V \subset V_{\mathbb{C}}$ and the diagonal $\Delta_V \subset V \times V$ are diffeomorphic. So for the purpose of computing $F^{-1}(V)$ we can assume that $F: L' \to V \times V$. Let $F = (F_1, F_2)$. Then

$$T = F^{-1}(V) = \{x \mid F_1(x) = F_2(x)\}.$$

But since $g \circ \pi$ is close to F and the maps F_1 and F_2 are generic this set is also the transverse self intersection of the homology cycle $g_*[L]$. We see this by looking carefully at the map F. First we look at the map g. We may identify a neighborhood of L in $L_{\mathbb{C}}$ with a neighborhood of the diagonal Δ_L in $L \times L$. Then in a neighborhood of Δ_L , the map g is given by $(x, y) \mapsto (g(x), g(y))$ (since locally g is an inclusion $\mathbb{C}^{v-k} \subset \mathbb{C}^{v}$). Thus our algebraic approximation F is given by $F(x, y) = (F_1(x, y), F_2(x, y))$ where F_1 approximates g(x) and F_2 approximates g(y). Consequently, $F^{-1}(\Delta_V)$ represents the Poincaré dual of the cup product of the Poincaré duals of $g_*(L)$ with itself.

Conversely if $a \in H_A^k(V; \mathbb{Z}_2)^2$, then $a = \alpha^2$ with $\alpha \in H_A^k(V; \mathbb{Z}_2)$. The dual of α is represented by a v - k dimensional real algebraic set $L \subset V$. Let $L_C \subset V_C$ be the complexification of L. Then by applying the above argument to the inclusion map $g: L_C \to V_C$ we see that the restriction of the dual of $g_*[L_C]$ to V is α^2 , i.e., $a \in \overline{H}_{C-alg}^{2k}(V; \mathbb{Z}_2)$.

Now in the general case, *a* is represented by restricting the dual of the fundamental class $[L_{\mathbf{C}}]$ of a possibly singular algebraic subset $L_{\mathbf{C}} \subset V_{\mathbf{C}}$. Let $\pi_{\mathbf{C}} : \tilde{V}_{\mathbf{C}} \to V_{\mathbf{C}}$ be a resolution of $V_{\mathbf{C}}$ turning $L_{\mathbf{C}}$ into a nonsingular subset $\tilde{L}_{\mathbf{C}} \subset \tilde{V}_{\mathbf{C}}$. In particular the restriction map $\pi : \tilde{V} \to V$ resolves L to \tilde{L} . Since $\pi_{\mathbf{C}}$ and π are degree one maps in \mathbf{Z} and \mathbf{Z}_2 coefficients respectively, the following commutes:

$$H_{2v-2k}(\tilde{V}_{\mathbf{C}}) \xleftarrow{D} H^{2k}(\tilde{V}_{\mathbf{C}}) \xrightarrow{j^{*}} H^{2k}(\tilde{V}) \xleftarrow{Sq^{k}} H^{k}(\tilde{V}) \xrightarrow{D} H_{v-k}(\tilde{V})$$

$$\downarrow^{\pi_{*}} \uparrow^{\pi^{*}} \uparrow^{\pi^{*}} \uparrow^{\pi^{*}} \uparrow^{\pi^{*}} \downarrow^{\pi_{*}}$$

$$H_{2v-2k}(V_{\mathbf{C}}) \xleftarrow{D} H^{2k}(V_{\mathbf{C}}) \xrightarrow{j^{*}} H^{2k}(V) \xleftarrow{Sq^{k}} H^{k}(V) \xrightarrow{D} H_{v-k}(V)$$

In this abbreviated diagram, the two left vertical maps are induced by $\pi_{\mathbf{C}}$, and the homologies of the complex algebraic sets are taken with Z coefficients and the real algebraic sets with \mathbf{Z}_2 coefficients. *D* denotes the Poincaré duality isomorphisms, and Sq^k is the Steenrod square, i.e. in our case $Sq^k(\theta) = \theta^2$. Also *j** denotes the composition map: \mathbf{Z}_2 reduction followed by the restriction. By the previous nonsingular case we have:

$$j^* D^{-1}[\tilde{\mathcal{L}}_{\mathbf{C}}] = Sq^k D^{-1}[\tilde{\mathcal{L}}] \tag{(*)}$$

We need to show that $j^*D^{-1}[L_C] = Sq^kD^{-1}[L]$; but since π_C is a composition of blowups along nonsingular centers, it suffices to prove the equality in the case where π_C is a single blowup along a nonsingular center $X_C \subset V_C$. We will prove this by induction on the dimension of V.

By substituting Lemma 3 to the identity (*) we see for some algebraic subset $Z_{\mathbf{C}}$ of the exceptional locus $\tilde{X}_{\mathbf{C}} = \pi_{\mathbf{C}}^{-1}(X_{\mathbf{C}})$ (so $Z \subset \tilde{X} = \pi^{-1}(X)$)

$$j^*\pi^*D^{-1}[L_{\mathbf{C}}] + j^*D^{-1}[Z_{\mathbf{C}}] = Sq^k\pi^*D^{-1}[L] + Sq^kD^{-1}[Z], \quad \text{hence}$$

$$\pi^*(j^*D^{-1}[L_{\mathbf{C}}] - Sq^kD^{-1}[L]) + j^*D^{-1}[Z_{\mathbf{C}}] - Sq^kD^{-1}[Z] = 0$$

Since π is degree one π^* is an injection, hence it suffices to prove $j^*D^{-1}[Z_C] = Sq^kD^{-1}[Z]$. To see this, consider the inclusions:

$$\begin{array}{cccc} \tilde{V}_{\mathbf{C}} & \stackrel{\prime}{\longleftarrow} & \tilde{V} \\ & \uparrow^{i} & \uparrow^{\tilde{i}} \\ \tilde{X}_{\mathbf{C}} & \stackrel{\tilde{j}}{\longleftarrow} & \tilde{X} \end{array}$$

By making *i* transversal to \tilde{V} and calling $i^{-1}(\tilde{V}) = Q$ we obtain the inclusions:

By the above discussion on the nonsingular case in fact $Q = \chi(v_i) = \tilde{X} \cap \tilde{X}$, i.e. Q is the transverse self intersection of \tilde{X} in \tilde{V} . Also $I = \tilde{i} \circ i_0$ and $J = \tilde{j} \circ i_0$ where $i_0 : Q \hookrightarrow \tilde{X}$ is the inclusion.



Define $\Phi: H^{2k-2}(\tilde{X}; \mathbb{Z}_2) \to H^{2k}(\tilde{V}; \mathbb{Z}_2)$ be the map $\Phi(x) = u_{\tilde{X}} \smile \tilde{i}_+(x)$. We claim that the following diagram commutes:

As above the homologies and cohomologies of the complex algebraic sets are taken with Z coefficients and the real algebraic sets with Z_2 coefficients, and j^* denotes the composition map: Z_2 reduction followed by the map induced by j.

Now given this, we can finish the proof as follows: Since $Z_{\rm C}$ lies in $X_{\rm C}$ we can write $[Z_{\rm C}] = i_*[Z_{\rm C}]$ and $[Z] = \tilde{i}_*[Z]$. Since dim $(\tilde{X}) = v - 1$ by induction $\tilde{j}^*D^{-1}[Z_{\rm C}] = Sq^{k-1}D^{-1}[Z]$. This with the commutativity of the diagram implies $j^*D^{-1}i_*[Z_{\rm C}] = Sq^kD^{-1}i_*[Z]$.

It remains to check the commutativity of the diagram. By Lemma 2(a), (d)

$$Sq^{k}\tilde{i}_{+}(x) = \tilde{i}_{+}(x) \smile \tilde{i}_{+}(x) = \tilde{i}_{+}(x^{2}) \smile \tilde{i}_{+}(1) = \Phi(x^{2}) = \Phi Sq^{k-1}(x).$$

By Lemma 1(b) and Lemma 2(b)

$$j^*i_+(x) = I_+J^*(x) = (\tilde{i} \circ i_0)_+ (\tilde{j} \circ i_0)^*(x) = \tilde{i}_+(i_0)_+ i_0^*(\tilde{j}^*(x)) = \tilde{i}_+(\tilde{j}^*(x) \smile u_Q).$$

Being over \mathbb{Z}_2 coefficients, in the last term we can commute the cup products, also by using Lemma 2(c), (a) and Lemma 1(a):

$$\tilde{i}_+(u_Q \smile \tilde{j}^*(x)) = \tilde{i}_+(\tilde{i}^*\tilde{i}_+(1) \smile \tilde{j}^*(x)) = \tilde{i}_+(\tilde{i}^*(u_{\tilde{X}}) \smile \tilde{j}^*(x)) = u_{\tilde{X}} \smile \tilde{i}_+(\tilde{j}^*(x)).$$

Hence we have shown $j^*i_+(x) = \Phi \tilde{j}^*(x)$.

Finally to start the induction observe that for algebraic sets V of dimension v - k + 1 any homology class [L] of dimension v - k has a nonsingular representative, so the proof in this case follows from the first part of the theorem. To see this observation, pick a codimension one closed smooth submanifold $S \subset V$ homologous to L. Then since the homology class $[S] = [L] \in H^A_{v-k}(V; \mathbb{Z}_2)$ is algebraic, the submanifold S can be isotoped to a nonsingular algebraic subset (e.g. [AK1] Theorem 2.8.2).

Remark. By defining $H_{2k}^{\mathbb{C}-alg}(V; \mathbb{Z}) = j_1 H_{2k}^A(V_{\mathbb{C}}; \mathbb{Z})$, even when V is nonorientable we can restate (a) in a slightly stronger form:

 $H_{2k}^{\mathbf{C}-alg}(V;\mathbf{Z}) \subset H_{2k}^{\mathcal{A}}(V;\mathbf{Z}).$

A useful corollary to the theorem is that we can estimate the number of complex algebraic cycles of a nonsingular complex algebraic set in terms of the real algebraic cycles of the real part:

COROLLARY 1. rank $H^{2k}_{\mathcal{A}}(V_{\mathbb{C}}; \mathbb{Z}) \ge \operatorname{rank} H^{k}_{\mathcal{A}}(V; \mathbb{Z}_{2})^{2}$.

It is well known that the duals of Steifel-Whitney classes of any compact nonsingular real algebraic set are represented by algebraic subsets. This is because the Grassmanian G(n, k) of unoriented k planes in \mathbb{R}^n is a nonsingular algebraic set in such a way that all the Steifel-Whitney classes are represented by algebraic subsets and the (tangent and normal) Gauss map $\alpha : V \to G(n, k)$ is entire rational (cf., [AK3], [AK1]). It is also well known that the Chern classes of a complex algebraic set are algebraic (cf., [F]). Since $p_k(V) = (-1)^k j^* c_{2k}(V_C)$ then Pontryagin classes are in $H^*_{C-alg}(V; Z)$.

COROLLARY 2. The duals of Pontryagin classes of V are represented by real algebraic subsets of V (in the unoriented case dual means the dual of mod 2 reduction with \mathbb{Z}_2 coefficient).

Recall that under the additional assumption: either $2k \le 2v - n$ or $V_{\mathbf{C}}$ is a complete intersection, for all 2k < v the group $H_{\mathbf{C}-alg}^{2k}(V_{\mathbf{C}}; \mathbf{Z})$ is equal to the image of the restriction homomorphism (see [BBK]):

 $H^{2k}(\mathbb{RP}^n;\mathbb{Z}) \to H^{2k}(V;\mathbb{Z}).$

COROLLARY 3. If $V \subset \mathbb{R}^n$ (here we are identifying $\mathbb{R}^n \subset \mathbb{R}\mathbb{P}^n$) and either $2k \leq 2v - n$ or $V_{\mathbb{C}}$ is a complete intersection, then no element $\alpha \in H^k(V; \mathbb{Z}_2)$ with 2k < v and $\alpha^2 \neq 0$ can be algebraic.

This corollary has the following amusing consequence:

THEOREM B. There exist closed smooth submanifolds $M \subset \mathbb{R}^n$ which can not be isotoped to the real parts of any nonsingular complex algebraic subvarieties of \mathbb{CP}^n .

Proof. Pick $M^m \subset \mathbb{R}^n$ with n = 2m - s, and $c_k \in H^k(M; \mathbb{Z}_2)$ such that:

(i) $k \le s/2$.

(ii) $c_k^2 \neq 0$.

(iii) c_k is either a Steifel-Whitney class or a \mathbb{Z}_2 reduction of a Ponryagin class of the tangent or normal bundle.

We claim that M can not be isotopic to the real part V of a nonsingular complex algebraic set in **CP**ⁿ. Otherwise, by Corollary 3 the class c_k could not be algebraic; on the other hand by Corollary 2 and the preceding discussion c_k would have to be algebraic. Contradiction.

It remains to find examples of M satisfying the above properties. Real or Complex projective spaces could be imbedded in this way ([J]). For example $\mathbf{RP}^{10} \subset \mathbf{R}^{18}$ ([Ha]), in which case we take k = 1 and c_1 the first Steifel-Whitney class $w_1(M)$. More generally, for any s there exists m such that there are imbeddings $\mathbf{RP}^m \subset \mathbf{R}^{2m-s}$ ([MM]). We claim that we can choose some of these M to be a nonsingular algebraic subset of \mathbf{R}^n . To see that first choose q so that $\mathbf{RP}^q \subset \mathbf{R}^{2q-5}$. If q is even we choose $M = \mathbf{RP}^q$ with k = 1 and $c_1 = w_1(M)$, otherwise we choose $M = \mathbf{RP}^{q-1} \subset \mathbf{RP}^q \subset \mathbf{R}^{2(q-1)-3}$. In any case, in our example we can assume that $M^m \subset \mathbf{R}^{2m-3}$. Hence by [AK4] we can isotop M to a nonsingular algebraic subset of \mathbf{R}^{2m-2} .

On the positive side we can prove the following:

COROLLARY 4. If $M \subset \mathbb{RP}^n$ is a topological complete intersection, that is if it is an intersection $\bigcap L_i$ of smooth codimension one submanifolds of \mathbb{RP}^n in general position, then M is isotopic to the real part V of a nonsingular complex complete intersection $V_{\mathbf{C}}$ in \mathbb{CP}^n . Furthermore, when $M \subset \mathbb{R}^n$ then for any $\alpha \in H^k_A(V; \mathbb{Z}_2)$ with 2k < v has the property $\alpha^2 = 0$.

Proof. We first isotop each L_i to a nonsingular hypersurface V_i in **RP**ⁿ (this is possible since the group $H_{n-1}(\mathbf{RP}^n; \mathbf{Z}_2)$ is algebraic, see [AK1]), then change the coefficients of the defining equations of each V_i a little so that the complex solutions become nonsingular and transverse to each other without affecting the isotopy type of $\bigcap V_i \approx M$. The last requirement follows from the above discussion.

As an application to this theorem we see that we can isotop \mathbb{RP}^3 a nonsingular real algebraic subset V of \mathbb{R}^5 such that $H^1_A(V; \mathbb{Z}_2) = 0$, a fact previously proven in [BBK]. According to [BK2] any closed smooth manifold M is diffeomorphic to a nonsingular algebraic set V with $H^2_{\mathbb{C}-alg}(V; \mathbb{Z}) = H^2(M; \mathbb{Z})$. According to [BD] if M is a smooth manifold approximating to a large finite skeleton of $K(\mathbb{Z}_2, 2)$, then for any nonsingular algebraic set V diffeomorphic to M we must have $H^2_A(V; \mathbb{Z}_2) = 0$. These two results together appear to contradict (a) of Theorem A. The reason they are consistent is that $H^2(K(\mathbb{Z}_2, 2); \mathbb{Z})$ and hence its \mathbb{Z}_2 reduction is zero.

Remark. Some results in this paper were announced in [A]. The reader should be warned that in [A] the distinction between the groups $\overline{H}_{C-alg}^{2k}(V; \mathbb{Z})$ and $H_{C-alg}^{2k}(V; \mathbb{Z})$ is intentionally suppressed. Also G. Mikhalkin independently observed a special case of (b) of Theorem A.

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