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## Transcendental submanifolds of $\mathbf{R}^n$

S. AKBULUT AND H. KING

*Dedicated to memory of Mario Raimondo*

*Abstract.* In this paper we show how the restriction of the complex algebraic cycles to real part of a complex algebraic set is related to the real algebraic cycles of the real part. As a corollary we give examples of smooth submanifolds of a Euclidean space which can not be isotoped to real parts of complex nonsingular subvarieties in the corresponding projective space.

Algebraic numbers are dense in  $\mathbf{R}$ . The problem of whether closed smooth submanifolds  $M \subset \mathbf{R}^n$  can be approximated by nonsingular algebraic subsets could be viewed as a possible higher dimensional version of this property. By adapting a stronger version of the notion of nonsingularity (complexification is nonsingular) the results of this paper show that this is not the case. By identifying  $\mathbf{R}^n \subset \mathbf{RP}^n$  we prove:

**THEOREM.** *There are closed smooth submanifolds  $M \subset \mathbf{R}^n$  which can not be isotoped to the real parts of nonsingular complex algebraic subvarieties of  $\mathbf{CP}^n$ . Furthermore, we can choose  $M$  to be nonsingular real algebraic subsets of  $\mathbf{R}^n$ .*

Now a brief history: Seifert showed that if  $M \subset \mathbf{R}^n$  has trivial normal bundle then it can be isotoped to a nonsingular component of an algebraic subset of  $\mathbf{R}^n$  ([S]). Nash proved that in general  $M$  can be isotoped to a nonsingular sheet of an algebraic subset of  $\mathbf{R}^n$ ; but the sheets might intersect each other ([N]). In [AK4] it was shown that  $M$  can be isotoped to a nonsingular component of an algebraic subset of  $\mathbf{R}^n$ , i.e. these sheets can be separated. Whether  $M$  can be isotoped to (not just to a component of) a nonsingular algebraic subset of  $\mathbf{R}^n$  still remains open.

We should emphasize that stably the answers of these problems are known. For example, Nash already proved that  $M$  can be isotoped to a nonsingular component of an algebraic set inside of a larger Euclidean space  $\mathbf{R}^n \times \mathbf{R}^k$ ; and later Tognoli showed that in a larger Euclidean space the extra components of the algebraic set can be removed ([T]). In [AK4] and [AK5] it was shown that any  $M \subset \mathbf{R}^n$  can be isotoped to a nonsingular algebraic subset of  $\mathbf{R}^n \times \mathbf{R}$ ; more generally  $M$  can be isotoped to a nonsingular algebraic subset of  $\mathbf{R}^n$  if and only if  $M$  is cobordant through immersions to an algebraic subset of  $\mathbf{R}^n$ .

We obtain the above results as a corollary to our main theorem which says that the restriction of complex algebraic cycles of the complexification of a nonsingular algebraic set consists of the cup products of the real algebraic cycles. Another corollary is that the Ponryagin classes of the tangent and the normal bundle of a nonsingular algebraic set in  $\mathbf{R}^n$  are represented by real algebraic subsets, a fact which was known only for the Steifel–Whitney classes.

### 1. Gysin homomorphism

Let  $f : M^m \rightarrow N^n$  be a map. The Gysin homomorphism  $f_+ : H^*(M) \rightarrow H^{*+k}(N)$  is defined by the commutative diagram:

$$\begin{array}{ccc} H^*(M) & \xrightarrow{f_+} & H^{*+k}(N) \\ \downarrow \cong & & \downarrow \cong \\ H_{m-*}(M) & \xrightarrow{f_*} & H_{m-*}(N) \end{array}$$

where  $k = n - m$  and the vertical maps are the Poincaré duality isomorphisms. Gysin homomorphism satisfies the following well known properties:

LEMMA 1. *The Gysin homomorphism satisfies the following properties:*

- (a)  $f_+(f^*(u) \smile v) = u \smile f_+(v)$ .
- (b) *Given a commuting diagram:*

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \uparrow i & & \uparrow j \\ K & \xrightarrow{g} & L \end{array}$$

where  $i, j$  are imbeddings and  $g$  is transversal to  $L$  with  $g^{-1}(L) = K$  then:

$$f^* \circ j_+ = i_+ \circ g^*.$$

LEMMA 2. *Let  $f : M \hookrightarrow N$  be an imbedding. Let  $u_M$  be the dual of the fundamental class of  $M$  in  $N$ , and  $\chi(v_f)$  be the Euler class of the normal bundle of  $f$  then:*

- (a)  $f_+(1) = u_M$ ,
- (b)  $f_+f^*(x) = x \smile u_M$ ,
- (c)  $f^*f_+(1) = f^*(u_M) = \chi(v_f)$ ,
- (d)  $f_+(x) \smile f_+(y) = f_+(x \smile y) \smile f_+(1)$ .

These lemmas are standard facts of algebraic topology; we leave the proofs as an exercise to the reader.

## 2. Algebraic homology

Let  $V$  be a Zariski open real (or complex) algebraic set (defined over  $\mathbf{R}$ ), and  $R = \mathbf{Z}_2$  (or  $R = \mathbf{Z}$ ), then we can define algebraic homology groups  $H_*^A(V; R)$  to be the subgroup of  $H_*(V; R)$  generated by the compact real (or complex) algebraic subsets of  $V$  (cf. [AK1]). We define  $H_*^A(V; R)$  to be the Poincaré duals of the groups  $H_*^A(V; R)$  when defined. The resolution theorem of [H], implies that  $H_*^A(V; R)$  is also the subgroup generated by the classes  $g_*([S])$  where  $g : S \rightarrow V$  is an entire rational function,  $S$  is a compact nonsingular real (or complex) algebraic set and  $[S]$  is the fundamental class of  $S$ . Therefore even when  $V$  is real, we can define  $H_*^A(V; \mathbf{Z})$  to be the subgroup generated by  $g_*([S])$  where  $g : S \rightarrow V$  is an entire rational function from an oriented compact nonsingular real algebraic set and  $[S]$  is the fundamental class of  $S$ .

We call a real algebraic set  $V$  **totally algebraic** if  $H_*(V; \mathbf{Z}_2) = H_*^A(V; \mathbf{Z}_2)$ . It is known that not all nonsingular algebraic sets are totally algebraic. There are closed smooth manifolds which can not even be diffeomorphic to nonsingular totally algebraic sets ([BD]), even though every closed smooth manifold is homeomorphic to a totally algebraic set ([AK2]). Hence these algebraic homology groups are intimately related to the nonsingularity of the underlying algebraic set.

Recall from [BBK] that, for a compact nonsingular real algebraic set  $V$ ,  $H_{\mathbf{C}-alg}^*(V; \mathbf{Z})$  is defined to be the subgroup of  $H^*(V; \mathbf{Z})$  generated by the restriction of the classes of  $H_A^*(V_{\mathbf{C}}; \mathbf{Z})$  by the projective nonsingular complexification map  $j : V \hookrightarrow V_{\mathbf{C}}$  (this always exists).  $H_{\mathbf{C}-alg}^*(V; \mathbf{Z})$  is independent of the complexification  $V_{\mathbf{C}}$ . Define  $H_{\mathbf{C}-alg}^*(V; \mathbf{Z}_2)$  to be the mod 2 reduction of  $H_{\mathbf{C}-alg}^*(V; \mathbf{Z})$ .

The real algebraic cocycle groups  $H_A^*(V, \mathbf{Z}_2)$  play useful role in real algebraic geometry. For example, they carry obstructions to isotoping submanifolds to algebraic subsets (see [AK1]). Likewise the groups  $H_{\mathbf{C}-alg}^*(V; \mathbf{Z}_2)$  also appear as obstructions to algebraic approximation problems (see [BK1]). Our main result describes the relation between these groups.

We need the next result in the proof of the main theorem. It is a special case of Fulton’s theorem ([F]). In our notation  $V_{\mathbf{C}}$  denotes a complex algebraic set defined over  $\mathbf{R}$  with real part  $V$ . Square bracket such as  $[L]$  means the homology class induced by  $L$ , and  $D$  denotes the Poincaré duality homomorphism.

LEMMA 3. *Let  $\pi_{\mathbf{C}} : \tilde{V}_{\mathbf{C}} \rightarrow V_{\mathbf{C}}$  be a blowup of a compact nonsingular algebraic set along a nonsingular center  $X_{\mathbf{C}} \subset V_{\mathbf{C}}$ . Let  $L_{\mathbf{C}}$  be an algebraic subset of  $V_{\mathbf{C}}$  with  $X_{\mathbf{C}} \subset L_{\mathbf{C}}$ . Let  $\tilde{L}_{\mathbf{C}} \subset \tilde{V}_{\mathbf{C}}$  be the strict transform and  $\tilde{X}_{\mathbf{C}}$  be the exceptional locus  $\pi_{\mathbf{C}}^{-1}(X_{\mathbf{C}})$ . Then there is a proper algebraic subset  $Z_{\mathbf{C}} \subset \tilde{X}_{\mathbf{C}}$  such that:*

- (a)  $D^{-1}[\tilde{L}_{\mathbf{C}}] = \pi^*D^{-1}[L_{\mathbf{C}}] + D^{-1}[Z_{\mathbf{C}}]$ .
- (b)  $D^{-1}[\tilde{L}] = \pi^*D^{-1}[L] + D^{-1}[Z]$ .

*Proof.* The fact that  $D^{-1}[\tilde{L}_C]$  and  $\pi^*D^{-1}[L_C]$  differ by a cohomology class supported in  $\tilde{X}_C$  is standard (e.g. [AK1], Lemma 2.9.3). More specifically Theorem 6.7 of [F] gives an exact expression for the difference as an algebraic cohomology cycle.  $\square$

### 3. Main results

Now for the rest of the paper let  $V \subset \mathbf{RP}^n$  denote a compact nonsingular real algebraic set of dimension  $v$ , and  $V_C \subset \mathbf{CP}^n$  be a nonsingular projective complexification of  $V$ . Let  $j : V \hookrightarrow V_C$  denote the inclusion.

Define  $\bar{H}_{2k}^A(V_C; \mathbf{Z})$  to be the subgroup of  $H_{2k}^A(V_C; \mathbf{Z})$  generated by irreducible complex algebraic subsets defined over  $\mathbf{R}$  with  $k$ -dimensional real parts. In other words it is generated by the complexification of  $k$ -dimensional real algebraic subsets of  $V$  in  $V_C$ . As above, by the resolution theorem,  $\bar{H}_{2k}^A(V_C; \mathbf{Z})$  is generated by the classes  $g_*([L_C])$ , where  $L_C$  is a compact irreducible nonsingular complex algebraic set defined over  $\mathbf{R}$ , i.e., it is the complexification of a  $k$ -dimensional real algebraic set  $L$ , and  $g : L_C \rightarrow V_C$  is an entire rational function defined over  $\mathbf{R}$ , i.e., it is in the form  $g = g_C$ . Define a subgroup of  $H_{C-alg}^{2k}(V; \mathbf{Z})$  by:

$$\bar{H}_{C-alg}^{2k}(V; \mathbf{Z}) = j^* \bar{H}_A^{2k}(V_C; \mathbf{Z}).$$

Let  $\bar{H}_{C-alg}^{2k}(V; \mathbf{Z}_2)$  be its mod 2 reduction. The main theorem below implies that this last group is independent of the complexification  $V_C$ . Finally, define the following natural subgroup:

$$H_A^k(V; \mathbf{Z}_2)^2 = \{\alpha^2 \mid \alpha \in H_A^k(V; \mathbf{Z}_2)\}$$

of  $H_A^{2k}(V; \mathbf{Z}_2)$  (since cup product operation preserves algebraic cycles, [AK1])

**THEOREM A.** *For all  $k$  the following hold:*

- (a)  $H_{C-alg}^{2k}(V; R) \subset H_A^{2k}(V; R)$ , where  $R = \mathbf{Z}_2$  (or  $\mathbf{Z}$  when  $V$  is orientable).
- (b)  $\bar{H}_{C-alg}^{2k}(V; \mathbf{Z}_2) = H_A^k(V; \mathbf{Z}_2)^2$ .

*Proof.* To prove (a) let  $a \in H_{C-alg}^{2k}(V; R)$  be represented by the restriction of  $\alpha \in H_A^{2k}(V_C; R)$ . Let  $\beta \in H_{2v-2k}^A(V_C; R)$  be the Poincaré dual of  $\alpha$  in  $V_C$ . Recall that the map  $j_!$  induced by the restriction and the Poincaré duality maps:

$$\begin{array}{ccc} H^{2k}(V_C; R) & \xrightarrow{j^*} & H^{2k}(V; R) \\ \cong \downarrow & & \downarrow \cong \\ H_{2v-2k}(V_C; R) & \xrightarrow{j_!} & H_{v-2k}(V; R) \end{array}$$

is the homology intersection with the fundamental cycle  $[V]$ , i.e.,  $j_1(\beta)$  is obtained by transversally intersecting  $V$  and a representative of  $\beta$ .

By definition  $\beta$  is represented by  $g_*([S])$ , where  $S$  is a compact nonsingular complex algebraic set and  $g : S \rightarrow V_{\mathbb{C}}$  is an entire rational function. We can  $\epsilon$ -isotop  $g$  to a smooth function  $g_0 : S \rightarrow V_{\mathbb{C}}$  which is transverse to  $V \subset V_{\mathbb{C}}$ . By [AK1] Proposition 2.8.8, we can find a nonsingular real algebraic set  $S'$  and a rational diffeomorphism  $\pi : S' \rightarrow S$  and a rational map  $F : S' \rightarrow V_{\mathbb{C}}$  such that  $g_0 \circ \pi$  is  $\epsilon$ -close to  $F$  (here we are viewing  $S$  and  $V_{\mathbb{C}}$  as real algebraic sets by thinking  $\mathbb{C}$  as  $\mathbb{R}^2$ ). Hence  $F$  is transverse to  $V$ . If  $T = F^{-1}(V)$  and  $f : T \rightarrow V$  is the restriction of  $F$ , then  $f_*[T]$  represents the Poincaré dual of  $a$ .

To see (b) let  $a \in \bar{H}_{\mathbb{C}\text{-alg}}^{2k}(V; \mathbb{Z}_2)$ . Then  $a$  is the restriction of a class in  $H_A^{2k}(V_{\mathbb{C}}; \mathbb{Z}_2)$  whose Poincaré dual in  $H_{2v-2k}^A(V_{\mathbb{C}}; \mathbb{Z}_2)$  can be represented by  $g_*[L_{\mathbb{C}}]$ , where  $g : L_{\mathbb{C}} \rightarrow V_{\mathbb{C}}$  is an entire rational function from a nonsingular compact complex algebraic set which is the complexification of a  $v - k$  dimensional real algebraic set  $L$ .

We first prove (b) under the assumption that  $g$  is an inclusion  $L_{\mathbb{C}} \subset V_{\mathbb{C}}$  of a nonsingular algebraic subset: We first  $\epsilon$ -isotop  $g$  to a smooth function  $g_0 : L_{\mathbb{C}} \rightarrow V_{\mathbb{C}}$  which is transverse to  $V \subset V_{\mathbb{C}}$ . As before, by viewing  $L_{\mathbb{C}}$  as a real algebraic set we can find a nonsingular real algebraic set  $L'$  and a rational diffeomorphism  $\pi : L' \rightarrow L_{\mathbb{C}}$  and a rational map  $F : L' \rightarrow V_{\mathbb{C}}$  such that  $g_0 \circ \pi$  is  $\epsilon$ -close to  $F$ . So  $F$  is transverse to  $V$ . If  $T = F^{-1}(V)$  and  $f : T \rightarrow V$  is the restriction of  $F$ , then  $f_*[T]$  represents the Poincaré dual of  $a$ .

We claim that  $f_*[T]$  is also the self intersection of the homology cycle  $g_*[L]$  in  $V$ . In other words  $a$  is the cup product square of the dual of the map  $g_*[L]$ . To see this observe that the normal bundle of  $V \subset V_{\mathbb{C}}$  is isomorphic to the tangent bundle of  $V$  (given by the multiplication by  $\sqrt{-1}$ ). Hence the tubular neighborhoods of  $V \subset V_{\mathbb{C}}$  and the diagonal  $\Delta_V \subset V \times V$  are diffeomorphic. So for the purpose of computing  $F^{-1}(V)$  we can assume that  $F : L' \rightarrow V \times V$ . Let  $F = (F_1, F_2)$ . Then

$$T = F^{-1}(V) = \{x \mid F_1(x) = F_2(x)\}.$$

But since  $g \circ \pi$  is close to  $F$  and the maps  $F_1$  and  $F_2$  are generic this set is also the transverse self intersection of the homology cycle  $g_*[L]$ . We see this by looking carefully at the map  $F$ . First we look at the map  $g$ . We may identify a neighborhood of  $L$  in  $L_{\mathbb{C}}$  with a neighborhood of the diagonal  $\Delta_L$  in  $L \times L$ . Then in a neighborhood of  $\Delta_L$ , the map  $g$  is given by  $(x, y) \mapsto (g(x), g(y))$  (since locally  $g$  is an inclusion  $\mathbb{C}^{v-k} \subset \mathbb{C}^v$ ). Thus our algebraic approximation  $F$  is given by  $F(x, y) = (F_1(x, y), F_2(x, y))$  where  $F_1$  approximates  $g(x)$  and  $F_2$  approximates  $g(y)$ . Consequently,  $F^{-1}(\Delta_V)$  represents the Poincaré dual of the cup product of the Poincaré duals of  $g_*(L)$  with itself.

Conversely if  $a \in H_A^k(V; \mathbf{Z}_2)^2$ , then  $a = \alpha^2$  with  $\alpha \in H_A^k(V; \mathbf{Z}_2)$ . The dual of  $\alpha$  is represented by a  $v - k$  dimensional real algebraic set  $L \subset V$ . Let  $L_C \subset V_C$  be the complexification of  $L$ . Then by applying the above argument to the inclusion map  $g : L_C \rightarrow V_C$  we see that the restriction of the dual of  $g_*[L_C]$  to  $V$  is  $\alpha^2$ , i.e.,  $a \in \bar{H}_{C-alg}^{2k}(V; \mathbf{Z}_2)$ .

Now in the general case,  $a$  is represented by restricting the dual of the fundamental class  $[L_C]$  of a possibly singular algebraic subset  $L_C \subset V_C$ . Let  $\pi_C : \tilde{V}_C \rightarrow V_C$  be a resolution of  $V_C$  turning  $L_C$  into a nonsingular subset  $\tilde{L}_C \subset \tilde{V}_C$ . In particular the restriction map  $\pi : \tilde{V} \rightarrow V$  resolves  $L$  to  $\tilde{L}$ . Since  $\pi_C$  and  $\pi$  are degree one maps in  $\mathbf{Z}$  and  $\mathbf{Z}_2$  coefficients respectively, the following commutes:

$$\begin{array}{ccccccc}
 H_{2v-2k}(\tilde{V}_C) & \xleftarrow{D} & H^{2k}(\tilde{V}_C) & \xrightarrow{j^*} & H^{2k}(\tilde{V}) & \xleftarrow{Sq^k} & H^k(\tilde{V}) & \xrightarrow{D} & H_{v-k}(\tilde{V}) \\
 \downarrow \pi_* & & \uparrow \pi^* & & \uparrow \pi^* & & \uparrow \pi^* & & \downarrow \pi_* \\
 H_{2v-2k}(V_C) & \xleftarrow{D} & H^{2k}(V_C) & \xrightarrow{j^*} & H^{2k}(V) & \xleftarrow{Sq^k} & H^k(V) & \xrightarrow{D} & H_{v-k}(V)
 \end{array}$$

In this abbreviated diagram, the two left vertical maps are induced by  $\pi_C$ , and the homologies of the complex algebraic sets are taken with  $\mathbf{Z}$  coefficients and the real algebraic sets with  $\mathbf{Z}_2$  coefficients.  $D$  denotes the Poincaré duality isomorphisms, and  $Sq^k$  is the Steenrod square, i.e. in our case  $Sq^k(\theta) = \theta^2$ . Also  $j^*$  denotes the composition map:  $\mathbf{Z}_2$  reduction followed by the restriction. By the previous nonsingular case we have:

$$j^*D^{-1}[\tilde{L}_C] = Sq^kD^{-1}[\tilde{L}] \tag{*}$$

We need to show that  $j^*D^{-1}[L_C] = Sq^kD^{-1}[L]$ ; but since  $\pi_C$  is a composition of blowups along nonsingular centers, it suffices to prove the equality in the case where  $\pi_C$  is a single blowup along a nonsingular center  $X_C \subset V_C$ . We will prove this by induction on the dimension of  $V$ .

By substituting Lemma 3 to the identity (\*) we see for some algebraic subset  $Z_C$  of the exceptional locus  $\tilde{X}_C = \pi_C^{-1}(X_C)$  (so  $Z \subset \tilde{X} = \pi^{-1}(X)$ )

$$j^*\pi^*D^{-1}[L_C] + j^*D^{-1}[Z_C] = Sq^k\pi^*D^{-1}[L] + Sq^kD^{-1}[Z], \quad \text{hence}$$

$$\pi^*(j^*D^{-1}[L_C] - Sq^kD^{-1}[L]) + j^*D^{-1}[Z_C] - Sq^kD^{-1}[Z] = 0$$

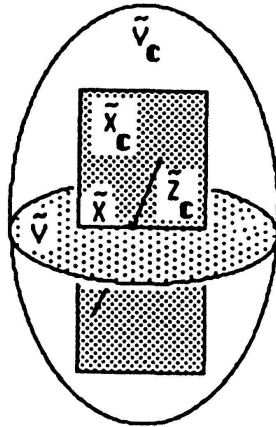
Since  $\pi$  is degree one  $\pi^*$  is an injection, hence it suffices to prove  $j^*D^{-1}[Z_C] = Sq^kD^{-1}[Z]$ . To see this, consider the inclusions:

$$\begin{array}{ccc}
 \tilde{V}_C & \xleftarrow{i} & \tilde{V} \\
 \uparrow i & & \uparrow \tilde{i} \\
 \tilde{X}_C & \xleftarrow{\tilde{j}} & \tilde{X}
 \end{array}$$

By making  $i$  transversal to  $\tilde{V}$  and calling  $i^{-1}(\tilde{V}) = Q$  we obtain the inclusions:

$$\begin{array}{ccc} \tilde{V}_C & \xleftarrow{j} & \tilde{V} \\ \uparrow i & & \uparrow I \\ \tilde{X}_C & \xleftarrow{J} & Q \end{array}$$

By the above discussion on the nonsingular case in fact  $Q = \chi(v_{\tilde{V}}) = \tilde{X} \cap \tilde{X}$ , i.e.  $Q$  is the transverse self intersection of  $\tilde{X}$  in  $\tilde{V}$ . Also  $I = \tilde{i} \circ i_0$  and  $J = \tilde{j} \circ i_0$  where  $i_0 : Q \hookrightarrow \tilde{X}$  is the inclusion.



Define  $\Phi : H^{2k-2}(\tilde{X}; \mathbf{Z}_2) \rightarrow H^{2k}(\tilde{V}; \mathbf{Z}_2)$  be the map  $\Phi(x) = u_{\tilde{X}} \smile \tilde{i}_+(x)$ . We claim that the following diagram commutes:

$$\begin{array}{ccccccc} H_{2v-2k}(\tilde{V}_C) & \xleftarrow{D} & H^{2k}(\tilde{V}_C) & \xrightarrow{j^*} & H^{2k}(\tilde{V}) & \xleftarrow{Sq^k} & H^k(\tilde{V}) & \xrightarrow{D} & H_{v-k}(\tilde{V}) \\ \uparrow i_* & & \uparrow i_+ & & \uparrow \Phi & & \uparrow \tilde{i}_+ & & \uparrow \tilde{i}_* \\ H_{2v-2k}(\tilde{X}_C) & \xleftarrow{D} & H^{2k-2}(\tilde{X}_C) & \xrightarrow{\tilde{j}^*} & H^{2k-2}(\tilde{X}) & \xleftarrow{Sq^{k-1}} & H^{k-1}(\tilde{X}) & \xrightarrow{D} & H_{v-k}(\tilde{X}) \end{array}$$

As above the homologies and cohomologies of the complex algebraic sets are taken with  $\mathbf{Z}$  coefficients and the real algebraic sets with  $\mathbf{Z}_2$  coefficients, and  $j^*$  denotes the composition map:  $\mathbf{Z}_2$  reduction followed by the map induced by  $j$ .

Now given this, we can finish the proof as follows: Since  $Z_C$  lies in  $X_C$  we can write  $[Z_C] = i_*[Z_C]$  and  $[Z] = \tilde{i}_*[Z]$ . Since  $\dim(\tilde{X}) = v - 1$  by induction  $\tilde{j}^*D^{-1}[Z_C] = Sq^{k-1}D^{-1}[Z]$ . This with the commutativity of the diagram implies  $\tilde{j}^*D^{-1}i_*[Z_C] = Sq^kD^{-1}i_*[Z]$ .

It remains to check the commutativity of the diagram. By Lemma 2(a), (d)

$$Sq^k\tilde{i}_+(x) = \tilde{i}_+(x) \smile \tilde{i}_+(x) = \tilde{i}_+(x^2) \smile \tilde{i}_+(1) = \Phi(x^2) = \Phi Sq^{k-1}(x).$$

By Lemma 1(b) and Lemma 2(b)

$$j^*i_+(x) = I_+ J^*(x) = (\tilde{i} \circ i_0)_+ (\tilde{j} \circ i_0)^*(x) = \tilde{i}_+ (i_0)_+ i_0^* (\tilde{j}^*(x)) = \tilde{i}_+ (\tilde{j}^*(x) \smile u_Q).$$

Being over  $\mathbf{Z}_2$  coefficients, in the last term we can commute the cup products, also by using Lemma 2(c), (a) and Lemma 1(a):

$$\tilde{i}_+ (u_Q \smile \tilde{j}^*(x)) = \tilde{i}_+ (\tilde{i}^* \tilde{i}_+ (1) \smile \tilde{j}^*(x)) = \tilde{i}_+ (\tilde{i}^*(u_{\tilde{X}}) \smile \tilde{j}^*(x)) = u_{\tilde{X}} \smile \tilde{i}_+ (\tilde{j}^*(x)).$$

Hence we have shown  $j^*i_+(x) = \Phi \tilde{j}^*(x)$ .

Finally to start the induction observe that for algebraic sets  $V$  of dimension  $v - k + 1$  any homology class  $[L]$  of dimension  $v - k$  has a nonsingular representative, so the proof in this case follows from the first part of the theorem. To see this observation, pick a codimension one closed smooth submanifold  $S \subset V$  homologous to  $L$ . Then since the homology class  $[S] = [L] \in H_{v-k}^A(V; \mathbf{Z}_2)$  is algebraic, the submanifold  $S$  can be isotoped to a nonsingular algebraic subset (e.g. [AK1] Theorem 2.8.2).  $\square$

*Remark.* By defining  $H_{2k}^{\mathbf{C}-alg}(V; \mathbf{Z}) = j_! H_{2k}^A(V_{\mathbf{C}}; \mathbf{Z})$ , even when  $V$  is nonorientable we can restate (a) in a slightly stronger form:

$$H_{2k}^{\mathbf{C}-alg}(V; \mathbf{Z}) \subset H_{2k}^A(V; \mathbf{Z}).$$

A useful corollary to the theorem is that we can estimate the number of complex algebraic cycles of a nonsingular complex algebraic set in terms of the real algebraic cycles of the real part:

COROLLARY 1.  $\text{rank } H_{\mathbf{A}}^{2k}(V_{\mathbf{C}}; \mathbf{Z}) \geq \text{rank } H_{\mathbf{A}}^k(V; \mathbf{Z}_2)^2.$

It is well known that the duals of Steifel–Whitney classes of any compact nonsingular real algebraic set are represented by algebraic subsets. This is because the Grassmanian  $G(n, k)$  of unoriented  $k$  planes in  $\mathbf{R}^n$  is a nonsingular algebraic set in such a way that all the Steifel–Whitney classes are represented by algebraic subsets and the (tangent and normal) Gauss map  $\alpha : V \rightarrow G(n, k)$  is entire rational (cf., [AK3], [AK1]). It is also well known that the Chern classes of a complex algebraic set are algebraic (cf., [F]). Since  $p_k(V) = (-1)^k j^* c_{2k}(V_{\mathbf{C}})$  then Pontryagin classes are in  $H_{\mathbf{C}-alg}^*(V; \mathbf{Z})$ .

COROLLARY 2. *The duals of Pontryagin classes of  $V$  are represented by real algebraic subsets of  $V$  (in the unoriented case dual means the dual of mod 2 reduction with  $\mathbf{Z}_2$  coefficient).*

Recall that under the additional assumption: either  $2k \leq 2v - n$  or  $V_{\mathbf{C}}$  is a complete intersection, for all  $2k < v$  the group  $H_{\mathbf{C}-alg}^{2k}(V_{\mathbf{C}}; \mathbf{Z})$  is equal to the image of the restriction homomorphism (see [BBK]):

$$H^{2k}(\mathbf{RP}^n; \mathbf{Z}) \rightarrow H^{2k}(V; \mathbf{Z}).$$

**COROLLARY 3.** *If  $V \subset \mathbf{R}^n$  (here we are identifying  $\mathbf{R}^n \subset \mathbf{RP}^n$ ) and either  $2k \leq 2v - n$  or  $V_{\mathbf{C}}$  is a complete intersection, then no element  $\alpha \in H^k(V; \mathbf{Z}_2)$  with  $2k < v$  and  $\alpha^2 \neq 0$  can be algebraic.*

This corollary has the following amusing consequence:

**THEOREM B.** *There exist closed smooth submanifolds  $M \subset \mathbf{R}^n$  which can not be isotoped to the real parts of any nonsingular complex algebraic subvarieties of  $\mathbf{CP}^n$ .*

*Proof.* Pick  $M^m \subset \mathbf{R}^n$  with  $n = 2m - s$ , and  $c_k \in H^k(M; \mathbf{Z}_2)$  such that:

- (i)  $k \leq s/2$ .
- (ii)  $c_k^2 \neq 0$ .
- (iii)  $c_k$  is either a Steifel–Whitney class or a  $\mathbf{Z}_2$  reduction of a Ponryagin class of the tangent or normal bundle.

We claim that  $M$  can not be isotopic to the real part  $V$  of a nonsingular complex algebraic set in  $\mathbf{CP}^n$ . Otherwise, by Corollary 3 the class  $c_k$  could not be algebraic; on the other hand by Corollary 2 and the preceding discussion  $c_k$  would have to be algebraic. Contradiction.

It remains to find examples of  $M$  satisfying the above properties. Real or Complex projective spaces could be imbedded in this way ([J]). For example  $\mathbf{RP}^{10} \subset \mathbf{R}^{18}$  ([Ha]), in which case we take  $k = 1$  and  $c_1$  the first Steifel–Whitney class  $w_1(M)$ . More generally, for any  $s$  there exists  $m$  such that there are imbeddings  $\mathbf{RP}^m \subset \mathbf{R}^{2m-s}$  ([MM]). We claim that we can choose some of these  $M$  to be a nonsingular algebraic subset of  $\mathbf{R}^n$ . To see that first choose  $q$  so that  $\mathbf{RP}^q \subset \mathbf{R}^{2q-5}$ . If  $q$  is even we choose  $M = \mathbf{RP}^q$  with  $k = 1$  and  $c_1 = w_1(M)$ , otherwise we choose  $M = \mathbf{RP}^{q-1} \subset \mathbf{RP}^q \subset \mathbf{R}^{2(q-1)-3}$ . In any case, in our example we can assume that  $M^m \subset \mathbf{R}^{2m-3}$ . Hence by [AK4] we can isotop  $M$  to a nonsingular algebraic subset of  $\mathbf{R}^{2m-2}$ .  $\square$

On the positive side we can prove the following:

**COROLLARY 4.** *If  $M \subset \mathbf{RP}^n$  is a topological complete intersection, that is if it is an intersection  $\cap L_i$  of smooth codimension one submanifolds of  $\mathbf{RP}^n$  in general position, then  $M$  is isotopic to the real part  $V$  of a nonsingular complex complete*

intersection  $V_C$  in  $\mathbf{CP}^n$ . Furthermore, when  $M \subset \mathbf{R}^n$  then for any  $\alpha \in H_A^k(V; \mathbf{Z}_2)$  with  $2k < v$  has the property  $\alpha^2 = 0$ .

*Proof.* We first isotop each  $L_i$  to a nonsingular hypersurface  $V_i$  in  $\mathbf{RP}^n$  (this is possible since the group  $H_{n-1}(\mathbf{RP}^n; \mathbf{Z}_2)$  is algebraic, see [AK1]), then change the coefficients of the defining equations of each  $V_i$  a little so that the complex solutions become nonsingular and transverse to each other without affecting the isotopy type of  $\bigcap V_i \approx M$ . The last requirement follows from the above discussion.  $\square$

As an application to this theorem we see that we can isotop  $\mathbf{RP}^3$  a nonsingular real algebraic subset  $V$  of  $\mathbf{R}^5$  such that  $H_A^1(V; \mathbf{Z}_2) = 0$ , a fact previously proven in [BBK]. According to [BK2] any closed smooth manifold  $M$  is diffeomorphic to a nonsingular algebraic set  $V$  with  $H_{C-alg}^2(V; \mathbf{Z}) = H^2(M; \mathbf{Z})$ . According to [BD] if  $M$  is a smooth manifold approximating to a large finite skeleton of  $K(\mathbf{Z}_2, 2)$ , then for any nonsingular algebraic set  $V$  diffeomorphic to  $M$  we must have  $H_A^2(V; \mathbf{Z}_2) = 0$ . These two results together appear to contradict (a) of Theorem A. The reason they are consistent is that  $H^2(K(\mathbf{Z}_2, 2); \mathbf{Z})$  and hence its  $\mathbf{Z}_2$  reduction is zero.

*Remark.* Some results in this paper were announced in [A]. The reader should be warned that in [A] the distinction between the groups  $\bar{H}_{C-alg}^{2k}(V; \mathbf{Z})$  and  $H_{C-alg}^{2k}(V; \mathbf{Z})$  is intentionally suppressed. Also G. Mikhalkin independently observed a special case of (b) of Theorem A.

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