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## Quadratic differentials with prescribed singularities and pseudo-Anosov diffeomorphisms

HOWARD MASUR AND JOHN SMILLIE

A quadratic differential on a Riemann surface  $M$  determines certain “topological” data: the genus of  $M$ ; the orders of zeros and poles; and the orientability of the horizontal foliation. In this note we determine which collections of data can be realized by quadratic differentials with finite area. A pseudo-Anosov diffeomorphism of  $M$  also determines certain topological data: the genus of the  $M$ ; the types of the singularities and the orientability of the stable foliation. As a corollary to our result on quadratic differentials we determine which topological data can be realized by pseudo-Anosov diffeomorphisms on oriented surfaces.

Let  $X$  be a closed Riemann surface of genus  $g$  with a system of holomorphic coordinate charts  $\{U_\nu, h_\nu\}$ . This means that  $\{U_\nu\}$  is a covering of  $X$  by open sets;  $h_\nu$  is a homeomorphism of  $U_\nu$  to an open set in the complex plane and  $h_\mu \circ h_\nu^{-1}$  is conformal whenever defined. Let  $q$  be a positive integer. A meromorphic  $q$ -differential  $\phi$  on  $X$  is a set of meromorphic function elements  $\phi_\nu$  in the local parameters  $z_\nu = h_\nu(p)$  for which the transformation law

$$\phi_\nu(z_\nu) dz_\nu^q = \phi_\mu(z_\mu) dz_\mu^q, \quad dz_\mu = \frac{dz_\mu}{dz_\nu} dz_\nu,$$

holds whenever  $z_\nu$  and  $z_\mu$  are parameter values corresponding to the same point  $p$  of  $X$ . The differentials corresponding to  $q = 2$  are classically called quadratic differentials and those corresponding to  $q = 1$  are called abelian differentials. At any point  $p$  in  $X$  we may choose a parameter  $z$  so that  $p$  corresponds to  $z = 0$ . We say the differential  $\phi$  has a zero (resp. pole) of order  $k$  at  $p$  if the meromorphic function element  $\phi(z)$  has a zero (resp. pole) of order  $k$  at zero. The quadratic differential  $\phi$  has zeroes and poles at a finite number of points  $p_1 \dots p_n$ . If the area is finite then all poles are simple. We will assume that this is the case. Define  $k_i$  to be the order of the zero at  $p_i$ , if  $p_i$  is a zero, and set  $k_i = -1$ , if  $p_i$  is a simple pole. Thus each quadratic differential  $\phi$  determines certain data

$$(k_1, \dots, k_n; \epsilon),$$

where  $\epsilon = +1$  if  $\phi$  is the square of an abelian differential, and  $\epsilon = -1$  if it is not. In this situation we say that  $\phi$  realizes the singularity data  $k = (k_1, \dots, k_n; \epsilon)$ . Theorem 1 describes precisely which sets of singularity data are realized by quadratic differentials. Note that the order in which the  $k_i$ 's are listed is irrelevant for the realization problem.

There are two obvious necessary conditions on a set  $k$  of singularity data which must be satisfied in order for  $k$  to be realized. To describe the first condition we note that the data  $k = (k_1, \dots, k_n)$  determines the topological type of the surface. Define  $\sigma(k)$  by the formula

$$\sigma(k) = \sum_{i=1}^n k_i.$$

It is a classical fact that if  $k$  can be realized on a surface of genus  $g$  then  $\sigma(k) = 4(g-1)$ . In particular we see that a necessary condition for the realization of the  $n+1$ -tuple  $k$  is that  $\sigma(k) \equiv 0 \pmod{4}$  and  $\sigma(k) \geq -4$ .

If a quadratic differential  $q$  has a zero or pole of odd order at a point  $p$  then near  $p$  the quadratic differential cannot be written as a square of an abelian differential. Thus if data  $(k_1, \dots, k_n; \epsilon)$  can be realized by a quadratic differential then if any  $k_i$  is odd it must be the case that  $\epsilon = -1$ .

We will show that with four exceptions every  $n+1$ -tuple of integers, satisfying these two necessary conditions can be realized by a quadratic differential on a compact Riemann surface.

**THEOREM 2.** *Let  $k = (k_1, \dots, k_n; \epsilon)$  be an  $(n+1)$ -tuple where each  $k_i \in \{-1, 1, 2, \dots\}$  and  $\epsilon = \pm 1$ . Then there is a quadratic differential  $\phi$  on a closed surface realizing  $k$  if and only if:*

- (a)  $\sigma(k) \equiv 0 \pmod{4}$  and  $\sigma(k) \geq -4$ ;
- (b)  $\epsilon = -1$  if any  $k_i$  is odd and
- (c)  $(k_1, \dots, k_n; \epsilon) \neq (4; -1), (1, 3; -1), (-1, 1; -1)$  or  $(\quad; -1)$ .

The set of all quadratic differentials with fixed data  $(k_1, \dots, k_n; \epsilon)$  forms a moduli space. These moduli spaces, called “strata”, are studied in [V] and [MS]. Theorem 1 arose out of our interest in understanding these strata.

Geometric structures equivalent to quadratic differentials arise in contexts other than complex analysis. A quadratic differential on a surface  $M$  determines a pair of transverse foliations. The horizontal (resp. vertical) trajectories are arcs along which

$$q(z) dz^2 > 0 \quad (\text{resp. } q(z) dz^2 < 0).$$

The horizontal (resp. vertical) measured foliation consists of the horizontal (resp. vertical) trajectories together with the transverse invariant measure

$$|\operatorname{Im} q^{1/2} dz| \quad (\text{resp. } |\operatorname{Re} q^{1/2} dz|).$$

Conversely a pair of transverse measured foliations determine a quadratic differential. For more information see [HM] and [G].

Let  $f$  be a pseudo-Anosov homeomorphism of a surface of genus  $g$  with  $n$  punctures. According to Thurston [FLP]  $f$  determines stable and unstable transverse measured foliations. These foliations have common “ $p$ -pronged” singularities at a finite number of points  $x_i$ . Let  $p_i$  be the number of prongs of the singularity at  $x_i$ . Let  $\epsilon$  be  $+1$  or  $-1$  depending on whether the stable foliation is or is not orientable.

**THEOREM 2.** *There is a pseudo-Anosov homeomorphism on a surface of genus  $g$  with  $n$  punctures realizing data  $(p_1, \dots, p_j; \epsilon)$  if and only if:*

- (a)  $\sum_{i=1}^j (p_i - 2) = 4(g - 1)$ ;
- (b)  $\epsilon = -1$  if any  $p_i$  is odd;
- (c)  $(p_1, \dots, p_j; \epsilon) \neq (6; -1), (3, 5; -1), (1, 3; -1)$  or  $(; -1)$  and
- (d) *The number of indices  $i$  for which  $p_i = 1$  is less than or equal to  $n$ , the number of punctures.*

*Proof of Theorem 1.* A pair of transverse measured foliations with singularity data  $(p_1, \dots, p_j; \epsilon)$  determines a quadratic differential with singularity data  $(k_1, \dots, k_j; \epsilon)$  where  $k_i = p_i - 2$ . Moreover every quadratic differential can be constructed in this fashion from two transverse measured foliations. Thus the necessity of the conditions (a), (b) and (c) in Theorem 2 follows directly from Theorem 1. Condition (d) is a result of the convention that all 1 pronged singularities occur at punctures. The sufficiency of the conditions in Theorem 2 is a consequence of Theorem 1 and a deep theorem of Veech [V]. Veech's theorem implies (among other things) that if  $k$  can be realized by some quadratic differential  $\phi$  then there is a quadratic differential  $\phi'$  with the same data as  $\phi$  and where  $\phi'$  is constructed from a pseudo-Anosov diffeomorphism in the manner described above. Q.E.D.

There is a third equivalent way to describe the geometric structure induced by a quadratic differential. This geometric formulation will be useful in the constructive half of our proof of Theorem 1. Let  $M$  be a surface and let  $\Sigma$  be a finite subset of  $M$ . A Riemannian metric on  $M - \Sigma$  is *flat* if it has Gaussian curvature zero. We say a Riemannian metric on  $M - \Sigma$  has cone type singularities if, in a neighbor-



hood of a point of  $\Sigma$ , the metric can be written as

$$ds^2 = dr^2 + (cr d\theta)^2,$$

where  $c > 0$ . In this case we say the metric has singularity with cone angle  $2\pi c$ . Parallel translation defines a homomorphism from  $\pi_1(M - \Sigma) \rightarrow SO(2, \mathbf{R})$ . The image of this map is contained in  $\{\pm I\}$  if and only if  $M$  possesses a parallel line field. The image of this map is trivial if and only if  $M$  possesses a parallel vector field. A quadratic differential gives rise to a surface with a flat metric and cone type singularities that also possesses a parallel line field. Conversely every flat structure with cone type singularities and parallel line field produces a quadratic differential.

Our proof of Theorem 1 uses complex analysis to show the necessity of condition (c). It would also be interesting to understand geometrically why the structures listed in condition (c) cannot be realized. We will give a short discussion here.

The nonexistence of a structure of type  $(; -1)$  is a special (and particularly simple) case. A structure of type  $(; \epsilon)$  is a flat structure on the torus without singularities. It is a consequence of Bieberbach's classification of manifolds with flat metrics that every such structure is isometric to  $\mathbf{R}^2$  modulo a lattice. Thus every such structure possesses a parallel vector field, not just a parallel line field. In particular  $(; -1)$  cannot be realized.

We now consider the other sets of singularity data listed in condition (c). Consider the problem of realizing flat structures with parallel line fields and prescribed sets of singularities on surfaces with boundary. It is a consequence of our proof of Theorem 1 that *any* data other than  $(; -1)$  can be realized on a surface with nonempty boundary so that the boundary components are parallel geodesics. One approach to realizing data on a surface  $M$  without boundary is to cut  $M$  along a closed curve, construct a flat structure with parallel boundary components and then glue the parallel boundary components together. This method fails in general because it is not always possible to realize the flat structure with boundary components of the same length. In fact the data sets  $(4; -1)$ ,  $(1, 3; -1)$ , and  $(-1, 1; -1)$  described in part (c) of Theorem 1 are precisely those which cannot be realized on surfaces with two parallel boundary components of equal length.

There is a related geometric realization problem. One can consider the problem of realizing prescribed sets of singularities by a flat structure with cone type singularities which does not necessarily possess a parallel vector field. It still makes sense to talk about cone angles in this case but the quantity  $\epsilon$  is not defined. In this case a result of Troyanov, (see [T]), implies that the obvious necessary topological condition (condition (a) in Theorem 1) is in fact sufficient.

*Proof of Theorem 1* (necessity of condition (c)). We will first prove that there is no quadratic differential  $\phi$  on a closed surface realizing the topological data listed in (c). We are indebted to Irwin Kra for pointing out the proofs in the first two cases. If  $(4; -1)$  or  $(3, 1; -1)$  can be realized, they are realized on a surface of genus 2. Accordingly, let  $X$  be a Riemann surface of genus 2 which supports  $\phi$ . Every Riemann surface of genus 2 is hyperelliptic. This means that it can be represented as a two sheeted branched covering of the Riemann sphere. Let  $z$  be the covering map and let  $P_1, \dots, P_6$  be the branch points in  $X$ .

Assume  $z(P_i) \neq \infty$ . There is a map  $w$  of  $X$  to the Riemann sphere which satisfies

$$w^2 = \prod_{j=1}^6 (z - z(P_j)),$$

and there is an involution  $\tau : X \rightarrow X$  satisfying  $z \circ \tau = z$  and  $w \circ \tau = -w$  and fixing precisely the points  $P_i$ . By III.7.5 Corollary 1 of Farkas–Kra the abelian differentials  $\omega_1 = dz/w$  and  $\omega_2 = z dz/w$  form a basis for the vector space of abelian differentials on  $X$ . Clearly  $\tau^*(\omega_i) = -\omega_i$ . By III.7.5 Corollary 2, the quadratic differentials  $\omega_1^2$ ,  $\omega_2^2$ , and  $\omega_1\omega_2$  form a basis for the vector space of quadratic differentials on  $X$ . Thus every quadratic differential  $\phi$  satisfies  $\tau^*(\phi) = \phi$ . Suppose  $\phi$  now is a quadratic differential with a single zero of order 4 at a point  $p \in X$ . We show  $\phi$  must realize  $(4; +1)$  and not  $(4; -1)$ . Since  $\tau^*(\phi) = \phi$  we have  $\tau(p) = p$ . Therefore  $p$  is one of the branch points  $P_j$ . By Theorem III.7.2 of Farkas–Kra [FK] the points  $P_i$  are also the Weierstrass points of  $X$ . This means that for each  $P_i$  there is an abelian differential on  $X$  with a single zero of order 2 at  $P_i$ . Let  $\omega_p$  be the abelian differential with a zero of order 2 at  $p$ . The quotient  $\phi/\omega_p^2$  is therefore a meromorphic function on  $X$  without zeroes or poles and therefore is a constant function. Thus  $\phi = c\omega_p^2$  and  $\phi$  determines  $(4; +1)$ .

Next we show there is no quadratic differential  $\phi$  on  $X$  with a zero of order 3 at a point  $p_1 \in X$  and a zero of order 1 at a point  $p_2 \in X$ . If such a  $\phi$  exists, we again have  $\tau^*(\phi) = \phi$  and therefore  $\tau(p_i) = p_i$  and so the  $p_i$  are Weierstrass points. Then  $\phi/\omega_{p_1}^2$  is a meromorphic function with a simple pole at  $p_1$  and a simple zero at  $p_2$ . Thus  $\phi/\omega_{p_1}^2$  defines a degree 1 covering of  $X$  to the Riemann sphere, which is impossible.

Next suppose  $\phi$  is a quadratic differential with a simple zero and a simple pole on a torus  $X$ . There exists a quadratic differential  $\psi$  on  $X$  which defines the flat structure without singularities ( $\psi$  defines  $(; +1)$ ). Then  $\phi/\psi$  defines a degree 1 covering of  $X$  to the Riemann sphere, which is impossible. If  $\phi$  defines  $(; -1)$ , then  $\phi/\psi$  is a constant, which also is impossible.

*Sufficiency of the Conditions of Theorem in the case  $g = 0$ .* The realization of  $(n + 1)$ -tuples,  $k$ , with  $\sigma(k) = -4$  is particularly simple and we dispense with it here.

Data with  $\sigma(k) = -4$  are realized on the sphere. Note that in this case some  $k_i$  is equal to  $-1$  so by condition (b)  $\epsilon = -1$ . Suppose  $k = (k_1, \dots, k_n; -1)$ , where  $k_i = -1$  for  $i > m$ . Let  $z_1, \dots, z_n$  be distinct points on the Riemann sphere  $X$ . Then the quadratic differential  $\phi$  defined by

$$\phi dz^2 = \frac{(z - z_1)^{k_1} \dots (z - z_m)^{k_m}}{(z - z_{m+1}) \dots (z - z_n)} dz^2$$

realizes  $k$ .

Our construction in the general case,  $\sigma(k) \geq 0$ , is by means of a “cut and paste” argument which relies on the geometric formulation of the quadratic differential structure. In the course of this proof it will be convenient to extend the definition of flat surfaces with cone points to include surfaces with boundaries and singularities on the boundaries. For our purposes it suffices to consider surfaces with the property that all boundary curves are unions of parallel geodesic segments. We require that for each boundary point  $p$ , there is a chart mapping a neighborhood of  $p$  to a neighborhood of the origin in the upper half plane so that with respect to this chart the metric has the form

$$ds^2 = dr^2 + (cr d\theta)^2.$$

The cone angle at  $p$  is then  $c\pi$ . In our examples  $c$  is always a positive integer.

Now suppose  $e_1$  and  $e_2$  are boundary edges of the same length on flat surfaces. Let  $p_1$  and  $p_2$  be endpoints of  $e_1$  and  $e_2$ . If we glue  $e_1$  to  $e_2$  isometrically, identifying  $p_1$  and  $p_2$ , then the cone angle at the identified point is the *sum* of the boundary cone angles at  $p_1$  and  $p_2$ .

We will use the following terminology. Let  $M$  be a surface possibly with boundary. A *flat structure* on  $M$  is a flat Riemannian metric with cone type singularities and a specified horizontal line field. If  $M$  has a boundary then we assume in addition that the boundary curves are horizontal. If  $M$  does not have a boundary then a flat structure is equivalent to a quadratic differential. We will call a surface with a flat structure a *flat surface*.

Our strategy for constructing examples is to partition the singularity data into subsets which are as small as possible, realize these subsets on flat surfaces with parallel boundaries and then assemble these surfaces to give a flat structure on a connected surface without boundary. The first step is to break up the data  $k$  into minimal pieces.

NOTATION. If  $k^1 = (k_1^1, \dots, k_{n_1}^1; \epsilon^1)$  and  $k^2 = (k_1^2, \dots, k_{n_2}^2; \epsilon^2)$  and  $\epsilon^1 = \epsilon^2 = \epsilon$  then we write  $k = k^1 \oplus k^2$  for  $(k_1^1, \dots, k_{n_1}^1, k_1^2, \dots, k_{n_2}^2; \epsilon)$ .

Let  $k = (k_1, \dots, k_n; \epsilon)$  be an  $(n + 1)$ -tuple. Define a collection  $k$  to be *admissible* if

- (1)  $\sigma(k) \equiv 0 \pmod{4}$ .
- (2)  $\sigma(k) \geq 0$ .

We say  $k$  is minimal if it cannot be written as  $k = k^1 \oplus k^2$  where  $k^1$  and  $k^2$  are admissible. Every admissible  $k$  can be written as  $k = k^1 \oplus \dots \oplus k^m$  where each  $k^j$  is minimal.

The following theorem shows that essentially every minimal collection can be realized on a flat surface with two parallel boundary components.

**THEOREM 1'.** *Let  $k = (k_1, \dots, k_n; \epsilon)$  be a minimal collection. Assume  $k \neq ( ; -1)$ .*

- (a) *If  $\epsilon = -1$ ,  $k \neq (4; -1)$ ,  $k \neq (3, 1; -1)$ , and  $k \neq (1, -1; -1)$ , then for any two positive numbers  $l_1$  and  $l_2$ , there is a flat surface  $M$  realizing the data  $k$  with two horizontal boundary components of lengths  $l_1$  and  $l_2$ .*
- (b) *If  $k = (4; -1)$ ,  $(3, 1; -1)$ , or  $(1, -1; -1)$ , then for any two positive numbers  $l_1 > l_2$  there is a flat surface realizing the data  $k$  with two horizontal boundary components of lengths  $l_1$  and  $l_2$ .*
- (c) *If  $\epsilon = +1$ , then for any positive number  $l$  there is a flat surface realizing the data  $k$  with two horizontal boundary components of equal length  $l$ .*

**DEFINITION.** We call the  $k$ -tuples  $(4; -1)$ ,  $(3, 1; -1)$  and  $(1, -1; -1)$  *restricted*.

We now show that Theorem 1' implies Theorem 1.

*Completion of the Proof of Theorem 1.* Let  $k = (k_1, \dots, k_n, \epsilon)$  and suppose  $\sigma \geq 0$  and  $k$  satisfies the hypotheses of Theorem 1. Write  $k$  as  $k = k^1 \oplus \dots \oplus k^m$  as a sum of  $m$  minimal collections  $(k_1^j, \dots, k_n^j, \epsilon^j)$ . By convention  $\epsilon^j = \epsilon$ .

Suppose first that  $m = 1$  so  $k$  itself is minimal. Hypothesis (c) of Theorem 1 implies that  $k$  is unrestricted. If  $\epsilon = +1$  we apply conclusion (c) of Theorem 1' to find a surface with two boundary components realizing  $k$  such that the boundary lengths are equal. If  $\epsilon = -1$ , conclusion (a) of Theorem 1' says the boundary lengths are arbitrary and thus we may choose them to be equal. In either case we may glue the boundary components isometrically to realize  $k$  on a closed surface.

Now assume  $m > 1$ .

**CASE A.**  $\epsilon = +1$ .

Choose a flat structure  $X_i$  with two boundary components realizing each minimal collection so that the boundary lengths of all  $X_i$  are equal. This is possible

by Theorem 1' part (c). For each  $1 \leq j \leq m-1$  glue a boundary component of  $X_j$  isometrically to a boundary component of  $X_{j+1}$  and glue a component of  $X_m$  to a component of  $X_1$ .

CASE B.  $\epsilon = -1$  and at least one minimal collection is unrestricted.

We may assume the unrestricted collection is realized on  $X_m$ . If  $m = 2$  choose the arbitrary boundary length of  $X_2$  to be equal to the boundary lengths of  $X_1$  and glue isometrically. Suppose then that  $m \geq 3$ . Realize  $k^1$  on  $X_1$ . Realize  $k^2$  on  $X_2$  so that one boundary length of  $X_2$  is equal to a boundary length of  $X_1$ . Glue these boundaries together isometrically. Similarly we realize  $k^3$  on  $X_3$  and glue one boundary component of  $X_3$  to the unglued component of  $X_2$ . Continue in this fashion eventually gluing a boundary component of  $X_{m-1}$  to the unglued boundary component of  $X_{m-2}$ . Since  $X_m$  has arbitrary boundary lengths we may choose them so that one boundary length is the same as the length of the unglued boundary of  $X_{m-1}$  and the other is the same as the length of the unglued boundary of  $X_1$ . We may glue  $X_m$  to the connected sum of the  $X_i$  along these boundaries to form the closed surface.

CASE C.  $\epsilon = -1$  and all minimal collections are restricted.

Realize the minimal collections on flat surfaces  $X_i$ . For each  $X_i$  label the boundaries  $C_i^j$ ,  $j = 1, 2$  so that  $|C_i^1| < |C_i^2|$  where the notation  $|\gamma|$  denotes the length of  $\gamma$ . The hypothesis of theorem 1 excludes the case that the data consists of a single restricted collection so we may assume that  $m \geq 2$ . If  $m = 2$  we may choose  $X_1$  and  $X_2$  so that  $|C_1^j| = |C_2^j|$ ,  $j = 1, 2$  and glue  $C_1^j$  to  $C_2^j$ . Suppose then  $m \geq 3$ . For  $1 \leq i \leq m-2$  we may choose  $X_i$  so that  $|C_i^2| = |C_{i+1}^1|$  and then form the connected sum of  $X_i$  and  $X_{i+1}$  by gluing  $C_i^2$  isometrically to  $C_{i+1}^1$ . Now  $|C_{m-1}^2| > |C_1^1|$ . Now choose  $X_m$  so that  $|C_m^2| = |C_{m-1}^2|$  and  $|C_m^1| = |C_1^1|$ . Glue  $C_m^2$  isometrically to  $C_{m-1}^2$  and  $C_m^1$  isometrically to  $C_1^1$ . Q.E.D.

Before proving Theorem 1' we will establish some properties of minimal collections.

**PROPOSITION 1.1.** *For each minimal collection  $(k_1, \dots, k_n)$  either*

- (a)  $n \leq 4$  or
- (b) *exactly one*  $k_i > 0$ .

*Proof.* Write  $m_-$  for the number of indices  $i$  for which  $k_i = -1$ .

CASE I.  $m_- < 4$ .

We will show (a) holds. For  $m \leq n$  let  $\alpha_m = \sum_{i=1}^m k_i$ . If  $n > 4$ , then for some  $1 \leq j_1 < j_2 \leq 5$  we have  $\alpha_{j_1} \equiv \alpha_{j_2} \pmod{4}$  by the pigeon hole principle.

Thus  $\alpha_{j_2} - \alpha_{j_1} = \sum_{i=j_1+1}^{j_2} k_i \equiv 0 \pmod{4}$ . This implies that both subcollections  $k^1 = (k_{j_1+1}, \dots, k_{j_2})$  and  $k^2 = (k_1, \dots, k_{j_1}, k_{j_2+1}, \dots, k_n)$  satisfy  $\sigma(k^i) \equiv 0 \pmod{4}$ . The condition,  $m_- < 4$ , that there are fewer than 4 indices with  $k_i = -1$ , implies that  $\sigma(k^i) \geq -3$  for both  $i = 1, 2$ . Therefore since  $\sigma(k^i) \equiv 0 \pmod{4}$  we have  $\sigma(k^i) \geq 0$ . We have subdivided  $(k_1, \dots, k_n)$  into two admissible collections contrary to the hypothesis.

CASE II.  $m_- \geq 4$ .

We will show that (b) applies. Suppose to the contrary that  $k_1 > 0$  and  $k_2 > 0$ .

Consider first the situation when  $m_- > k_1$  or  $m_- > k_2$ . Assume for the sake of definiteness that  $m_- > k_1$ . Consider the subcollection  $k^1 = (k_1, -1, \dots, -1)$  where the number of  $-1$  entries in the subcollection is  $k_1$ . We have  $\sigma(k^1) = 0$ . Therefore  $k^1$  and the complementary subcollection are admissible and we have subdivided  $(k_1, \dots, k_n)$ , contrary to assumption.

Now consider the situation when  $m_- \leq k_1$  and  $m_- \leq k_2$ . Since  $k$  is minimal,  $k_i \neq 4$ . Thus  $k_1, k_2 \geq 5$ . Form the subcollection  $(k_1, -1, \dots, -1)$  where the number of  $-1$  entries  $m'_-$  satisfies  $0 \leq m'_- \leq 3$  and  $k_1 \equiv m'_- \pmod{4}$ . Then both this collection and the complementary collection are admissible and again we have contradicted the minimality of  $(k_1, \dots, k_n)$ . Q.E.D.

The remainder of this paper will be devoted to the proof of Theorem 1'. We construct three different classes of examples. The lowest level are the building blocks. These are denoted below by  $A(1)$ ,  $A(m)$ ,  $B_1$  and  $B_2$  and are constructed by making identifications on the boundary of a metric cylinder. These building blocks will contain singularities on their boundaries.

We call the intermediate level pieces the *basic examples*. These are constructed by gluing together building blocks. The basic examples do not contain singularities on their boundaries and are in fact examples of surfaces with boundary which realize minimal data.

In the final step of the construction we use two inductive procedures to build all examples realizing minimal data from the basic examples.

### Basic examples

Let  $C$  denote a metric cylinder, that is to say  $C$  is isometric to the product  $S^1 \times I$ . We choose the horizontal direction so that the boundaries are horizontal.  $C$  realizes the data  $(\ ; +1)$ .

*Building Block  $A(1)$ .* Let  $C$  be a metric cylinder and let  $C^0$  be one of its boundary components. Divide  $C^0$  into three segments and labelled  $a$ ,  $b$  and  $b^{-1}$  so

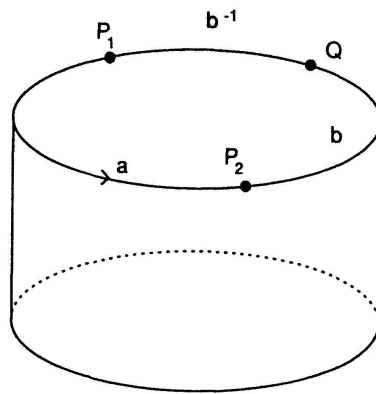


Figure 1

that  $|b| = |b^{-1}|$ . Let  $P_1$ ,  $Q$  and  $P_2$  be the vertices as shown in Figure 1. We construct a new surface,  $A(1)$ , by identifying  $b$  and  $b^{-1}$  by an orientation reversing isometry. Thus the endpoints of  $a$ ,  $P_1$  and  $P_2$ , are identified to become a single point  $P$  and  $a$  becomes a closed curve. The point  $P$  becomes a boundary singular point with cone angle  $2\pi$ . The point  $Q$  becomes an interior singular point in  $A(1)$  with cone angle  $\pi$ . In the language of quadratic differentials  $Q$  is a simple pole. In general we identify sides using the standard convention that a side labelled  $x$  and a side labelled  $x^{-1}$  are glued together by an orientation reversing isometry. The edge identifications force certain vertex identifications. We will identify two vertices only if their identification is forced in this way.  $A(1)$  has two boundary components  $A^0(1)$  and  $A^1(1)$  corresponding to the two components of  $C$ ;  $|A^0(1)| = |a|$  and  $|A^1(1)| = |a| + 2|b|$ . Note that by choosing the diameter of  $C$ ,  $|a|$  and  $|b|$  appropriately we can give the boundaries any assigned lengths with  $|A^0(1)| < |A^1(1)|$ .

*Building Block  $A(m)$ ,  $m > 1$ .* Let  $C$  be a metric cylinder and let  $C^0$  be one of its boundary components. Divide  $C^0$  into  $2m + 1$  segments and label them  $a, b_1, b_1^{-1}, b_2, b_2^{-1} \dots b_m b_m^{-1}$  (see Figure 2). Choose the lengths of segments so that

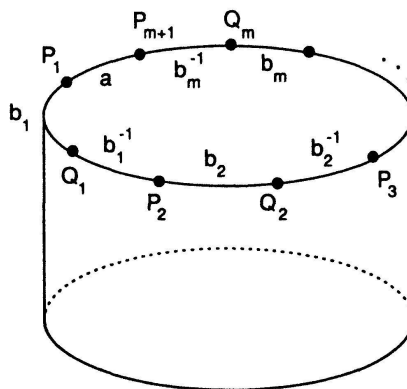


Figure 2

$|b_i| = |b_i^{-1}|$ . Identify sides using the standard convention. The points  $P_1 \dots P_m$  are identified to a single point  $P$ . Let  $A(m)$  be the resulting flat surface with boundary.  $A(m)$  has two boundary components; one,  $A^0(m)$ , corresponds to the edge  $a$  and the other  $A^1(1)$  corresponds to  $C^1$ . Then  $|A^0(m)| = |a|$  and  $|A^1(1)| = |a| + \sum 2|b_i|$ . Note that by choosing the lengths of segments appropriately we can give the boundaries any assigned lengths with  $|A^0(1)| < |A^1(1)|$ .  $A(m)$  has  $m$  singularities with cone angle  $\pi$  in its interior corresponding to  $Q_1 \dots Q_m$  and one singularity on the boundary  $A^0(m)$  with boundary cone angle  $(m+1)\pi$  at the point  $P$ .

We now construct the first set of basic examples.

*Basic Examples I.*  $k = (m, -1, \dots, -1; -1)$  with  $\sigma(k) = 0$ .

If  $m = 1$  then the data is restricted. Construct a surface  $M$  from  $C$  and  $A(1)$  by gluing one component of  $C$  to  $A^0(1)$ . We have that the length of  $A^1(1)$  is greater than the circumference of  $C$ , but otherwise the lengths are arbitrary.  $M$  has two singularities, one of cone angle  $\pi$  in the interior of  $A(1)$  and one of cone angle  $2\pi + \pi = 3\pi$  which arises from the identification of the boundary cone angle  $2\pi$  singularity on  $A^0(1)$  and a boundary cone angle  $\pi$  point on the boundary of  $C$ . Recall that boundary cone angles add under identification. The cone angle  $3\pi$  singularity is a zero of order 1.

If  $m > 1$  then we construct a surface  $M$  by gluing a copy of  $A(1)$  to a copy of  $A(m-1)$  so that the boundary singularities coincide. Using the fact that cone angles add we see that the singularity corresponding to the boundary singularity has cone angle  $(m+2)\pi$ . By choosing the boundary lengths of  $A(1)$  and  $A(m-1)$  appropriately we can realize any boundary lengths on  $M$ .

We now construct another building block,  $B_1$ , and use it to construct further basic examples.

*Building Block  $B_1$ .* Let  $C$  be a metric cylinder and let  $C^0$  be one of its boundary components. Divide  $C^0$  into 5 segments and label them  $aba^{-1}b^{-1}c$  (see Figure 3).

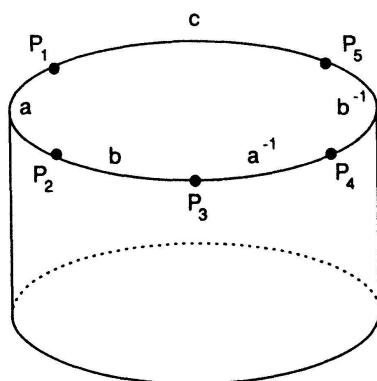


Figure 3



Choose the segments so that  $|a| = |a^{-1}|$  and  $|b| = |b^{-1}|$ . Identify sides using the standard convention. Let  $B_1$  be the resulting flat surface with boundary.  $B_1$  has two boundary components; one,  $B_1^0$ , corresponds to the edge  $c$  and the other,  $B_1^1$ , corresponds to  $C^1$ .  $|B_1^0| = |c|$  and  $|B_1^1| = 2|a| + 2|b| + |c|$ . Note that by choosing the lengths of segments appropriately we can give the boundaries any assigned lengths with  $|B_1^0| < |B_1^1|$ . All vertices are identified to a single point  $P$ .  $B_1$  has one singularity at  $P$  with cone angle  $5\pi$  on the boundary  $B_1^0$ .

We now construct the second group of basic examples.

*Basic Examples II.*  $k = (4; -1)$ ,  $(8; -1)$ ,  $(5, -1; -1)$  and  $(5, 3; -1)$ .

We first construct the example  $M$  that realizes the restricted data  $(4; -1)$ . Construct a surface  $M$  from  $B_1$  and  $C$  by gluing the boundary  $B_1^0$  to one boundary of  $C$ . The singularity of cone angle  $5\pi$  on  $B_1^0$  gives a singularity of cone angle  $6\pi$  in  $M$ . This is a zero of order 4. We can choose the boundary lengths of  $B_1$  and the circumference of  $C$  to realize any boundary lengths  $l_1 > l_2$ .

We construct  $(8; -1)$ . Construct a surface  $M$  from two copies of  $B_1$ , calling them  $B_1$  and  $B'_1$ . Glue boundaries  $B_1^0$  and  $B_1'^0$  so that the singularities coincide. The resulting singularity has cone angle  $10\pi$ . The boundary lengths can be chosen arbitrarily.

We construct a surface realizing  $(5, -1; -1)$ . Construct a surface  $M$  from a copy of  $B_1$  and a copy of  $A(1)$ . Glue the boundaries  $B_1^0$  and  $A^0(1)$  together so that the singularities coincide.

The resulting singularity has cone angle  $5\pi + 2\pi = 7\pi$ . There is one cone angle  $\pi$  singularity in the interior of  $M$  coming from  $A(1)$ . The boundary lengths can be chosen arbitrarily.

To construct  $(5, 3; -1)$  we need one additional building block.

*Building Block  $B_2$ .* Let  $C$  be a metric cylinder and let  $C^0$  be one of its boundary components. Divide  $C^0$  into 6 segments and label them  $efe^{-1}d_1f^{-1}d_0$  (see Figure 4). Choose the segments so that  $|e| = |e^{-1}|$  and  $|f| = |f^{-1}|$ . Identify sides using the standard convention. Let  $B_2$  be the resulting flat surface with boundary.  $B_2$  has two boundary components; one,  $B_2^0$ , corresponds to the edge  $d_1$  and  $d_0$  the other,  $B_2^1$ , corresponds to  $C^1$ . Then  $|B_2^0| = |d_1| + |d_0|$  and  $|B_2^1| = 2|e| + 2|f| + |d_1| + |d_0|$ . Note that by choosing the lengths of segments appropriately we can give the boundaries any assigned lengths with  $|B_2^0| < |B_2^1|$ ;  $B_2$  has a boundary singularity at  $P$  with cone angle  $2\pi$  and a boundary singularity at  $Q$  with cone angle  $4\pi$ .

We construct the basic example  $(5, 3; -1)$ . Construct a flat surface  $M$  by gluing a copy of  $B_1$  to a copy of  $B_2$  by gluing  $B_1^0$  to  $B_2^0$ .  $B_1^0$  contains a singularity with cone angle  $5\pi$ .  $B_2^0$  contains singularities with cone angles  $2\pi$  and  $4\pi$ . Choose the identification of the boundaries so that the  $5\pi$  singularity and the  $2\pi$  singularity  $P$  are

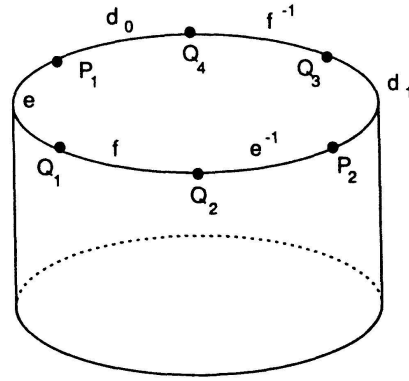


Figure 4

identified. The resulting singularity will have cone angle  $7\pi$ . The  $4\pi$  singularity  $Q$  will be identified with a nonsingular boundary point so the resulting singularity will have cone angle  $5\pi$ . The boundary lengths can be chosen arbitrarily.

We now construct the last set of basic examples.

*Basic Examples III.*  $(2, 2; -1)$ ,  $(3, 2, -1; -1)$ ,  $(3, 3, -1, -1; -1)$ ,  $(2, 1, 1; -1)$  and  $(1, 1, 1, 1; -1)$ .

The paradigm for these constructions is the construction of  $(2, 2; -1)$ . Take a metric cylinder  $C_1$  and identify two diametrically opposite points on one boundary component of  $C_1$ . This gives two boundary circles  $c_1$  and  $c_2$  of equal length joined at a point  $P$ . Note this space is a 2-complex but not a manifold with boundary. Take a second metric cylinder  $C_2$  with circumference  $|c_1| = |c_2|$  and glue one boundary component isometrically to  $c_1$ , the other to  $c_2$ . Then  $P$  becomes a singularity with cone angle  $4\pi$  which in the language of quadratic differentials is a zero of order 2. Now identify two not necessarily diametrically opposite points on the other boundary component of  $C_1$  giving two boundary circles  $d_1$  and  $d_2$  of arbitrary length joined together at a point  $Q$ . Glue one boundary component of a third cylinder  $C_3$  with circumference  $|d_1|$  isometrically to  $d_1$  and glue a boundary component of a fourth cylinder  $C_4$  with circumference  $|d_2|$  isometrically to  $d_2$ . The point  $Q$  is also a zero of order 2. This construction produces a surface with two boundary components corresponding to the unglued boundary components of  $C_3$  and  $C_4$  and they have arbitrary lengths  $|d_1|$  and  $|d_2|$  respectively. (See Figure 5.)

$(3, 2, -1; -1)$ : Replace the cylinder  $C_4$  in the previous construction with a copy of  $A(1)$ . Glue the  $2\pi$  singularity on the boundary of  $A(1)$  to  $Q$  on  $C_1$ . This makes  $Q$  a cone angle  $5\pi$  singularity and adds a singularity of cone angle  $\pi$  coming from the interior of  $A(1)$ .

$(3, 3, -1, -1; -1)$ : Replace the cylinder  $C_2$  in the previous construction with a copy of  $A(1)$ . (This necessitates identifying points on the boundary of  $C_1$  which are

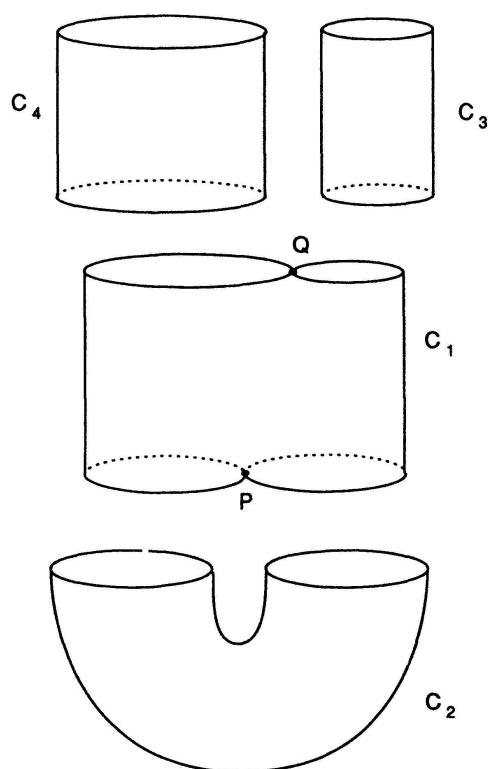


Figure 5

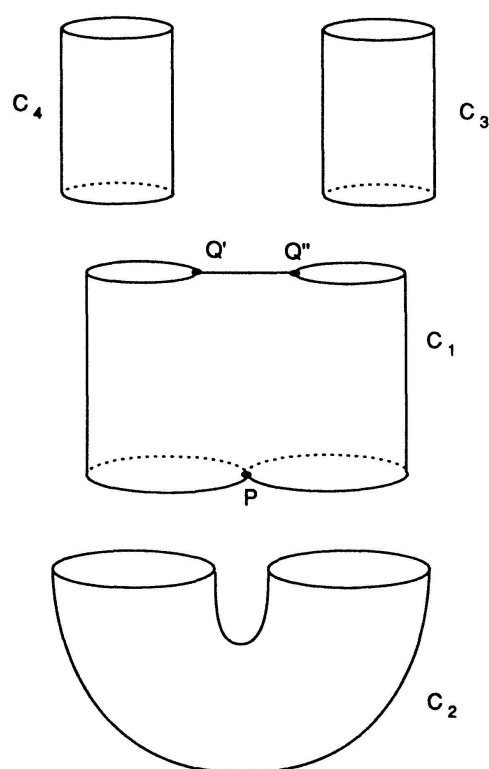


Figure 6

not diametrically opposite because the boundaries of  $A(1)$  must have different lengths.) In this way  $P$  becomes a cone angle  $5\pi$  singularity and a second cone angle  $\pi$  singularity is added.

$(2, 1, 1; -1)$ : We identify disjoint intervals of the same length on the “top” component of  $C_1$  by an orientation reversing isometry. (See Figure 6.) This gives two circles  $d_1$  and  $d_2$  based at distinct points  $Q'$  and  $Q''$  and an interval joining them. The lengths  $|d_1|$  and  $|d_2|$  of the circles are arbitrary. The points  $Q'$  and  $Q''$  are zeroes of order 1.

$(1, 1, 1, 1; -1)$ : We identify intervals on both boundary components of  $C_1$ . This produces four zeroes of order one:  $P'$ ,  $P''$ ,  $Q'$  and  $Q''$ . (See Figure 7.) As in the case of  $(2, 2; -1)$  we require that the two boundary circles  $c_1$  and  $c_2$  on one component have equal length so that they can be glued to boundary components of the same cylinder  $C_2$ .

The next two propositions allow us to build all minimal examples from the basic ones and allow us to prove Theorem 1'.

**DEFINITION.** A *saddle connection* is a geodesic segment joining two singularities which has no singularities in its interior.

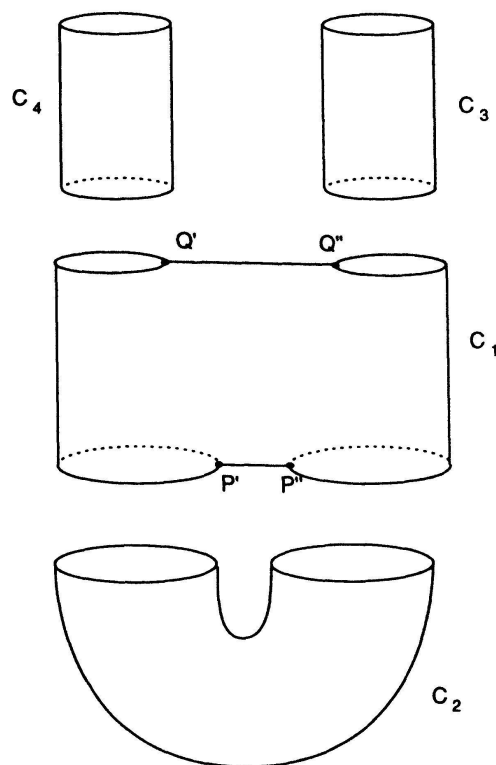


Figure 7

**PROPOSITION 1.2.** *Suppose  $M$  is a flat surface realizing  $k = (k_1, k_2, \dots, k_n; \epsilon)$ . Suppose there is a horizontal saddle connection joining the zeroes of order  $k_1$  and  $k_2$ . Then there is a new flat surface  $M'$  determining  $k' = (k_1 + 2, k_2 + 2, \dots, k_n; \epsilon)$ .*

*Proof.* Let  $\gamma$  be the saddle connection joining  $p_1$  and  $p_2$ , the zeroes of orders  $k_1$  and  $k_2$  and let  $d$  be its length. Take a flat torus  $T$ , points  $x_1, x_2 \in T$  and a horizontal geodesic segment  $\gamma'$  joining them of length  $d$ . Slit  $M$  and  $T$  along  $\gamma$  and  $\gamma'$ , and let  $M'$  be the connected sum of  $M$  and  $T$  formed by gluing  $M$  and  $T$  together along the slits with  $p_1$  glued to  $x_1$  and  $p_2$  glued to  $x_2$ . It is easy to see that  $2\pi$  is added to each of the cone angles at  $p_1$  and  $p_2$ . Thus 2 is added to the order of each zero.

Notice that if  $M$  has two boundary components which are horizontal geodesics without singularities then  $M'$  has the same property. Also note that  $M'$  has a horizontal saddle geodesic joining the two zeroes of order  $k_1 + 2$  and  $k_2 + 2$ . If  $M$  has a horizontal saddle connection which is a loop based at a zero, then there is horizontal saddle connection which is a loop based at the corresponding zero of  $M'$ .

**PROPOSITION 1.3.** *Suppose  $M$  is a flat structure realizing  $k = (k_1, \dots, k_n; \epsilon)$ . Suppose there is a closed horizontal saddle connection from the zero of order  $k_1$  to itself. Then there is a flat structure  $M'$  determining  $k' = (k_1 + 4, \dots, k_n; \epsilon)$ .*

*Proof.* Let  $p$  be the zero of order  $k_1$ . Let  $\gamma$  be the saddle connection based at  $p$  and suppose it has length  $d$ . Find a flat torus  $T$ , points  $x_1$  and  $x_2$  and a horizontal geodesic  $\gamma'$  with length  $2d$  joining them. Slit  $T$  along  $\gamma'$  and then identify  $x_1$  and  $x_2$ . The result is a flat torus  $T'$  with two boundary circles joined together at the point  $x_1 = x_2$ . Now slit  $M$  along  $\gamma$ . We form  $M'$  by gluing each side of the cut to one boundary circle of  $T'$ . We glue the point  $x_1 = x_2$  to the point on the cut corresponding to  $p$ .

Note that  $M'$  has a closed horizontal saddle connection from the zero of order  $k_1 + 4$  to itself. Suppose further that  $M$  has two horizontal boundary components, then  $M'$  has two horizontal boundary components with the same lengths as those of  $M$  and if there is closed horizontal saddle connection from a zero of  $M$  to itself, the corresponding zero of  $M'$  has such a saddle connection as well.

*Proof of Theorem 1'.*

CASE A.  $\epsilon = +1$ .

Since the  $k_i$  are even and  $k$  is minimal, it is easy to see that  $n = 1$  or  $n = 2$ . The proof is by induction on  $\sigma(k) = \sum_{i=1}^2 k_i$ . Assume  $k_1 \geq k_2$ . For the sake of applying Propositions 1.2 and 1.3 we make the following *induction hypotheses*:

- (1) if  $k_2 \neq 0$  there is a horizontal segment joining the two zeroes.
- (2) if  $k_1 > k_2$  there is a closed horizontal saddle connection from the zero of order  $k_1$  to itself.

The only case when  $\sigma(k) = 0$  is  $(; +1)$  which has already been constructed. Thus assume  $\sigma(k) \geq 4$ . To start the induction we construct the two examples for which  $\sigma = 4$ ; namely  $(2, 2; +1)$  and  $(4; +1)$ , and the example  $(6, 2; +1)$ . For the first, take a metric cylinder and choose a horizontal segment joining distinct points. Apply Proposition 1.2. Since the boundary lengths of the metric cylinder are equal, the same is true of  $(2, 2; +1)$ . To construct  $(4; +1)$ , choose a closed horizontal geodesic which is a waist curve of the metric cylinder. Apply Proposition 1.3. The induction hypotheses (1) and (2) are satisfied for both examples. The construction of  $(6, 2; +1)$  is somewhat special since it does not follow the general pattern. We construct this example from  $(4; +1)$  by joining the zero  $p$  of order 4 to a nonsingular point by a horizontal segment which is not a subsegment of the simple closed horizontal saddle connection from  $p$  to itself. Now apply Proposition 1.2. The remarks following the proof of Proposition 1.2 show that the induction hypotheses are satisfied for  $(6, 2; +1)$  as well. Now suppose inductively we have

constructed all examples  $k$  with  $\sigma(k) = 4m$ ,  $m \geq 1$ , and  $k$  satisfies  $\sigma(k) = 4m + 4$ . If  $k_1 \geq k_2 + 8$  let  $k' = (k_1 - 4, k_2; +1)$ . Then  $k_1 - 4 > k_2$ . The induction hypothesis (2) implies the existence of a horizontal saddle connection from the zero of order  $k_1 - 4$  to itself. Proposition 1.3 allows us to construct  $k$  from  $k'$ . The remarks following Proposition 1.3 show that  $k$  satisfies the induction hypotheses (1) and (2). Suppose  $k_2 \leq k_1 < k_2 + 8$ . Since  $k_1 + k_2 \geq 8$ ,  $k_2 > 0$ . The fact that  $k_1 + k_2 \equiv 0 \pmod{4}$  implies  $k = (6, 2; +1)$ , or  $k_2 > 2$ . The example  $(6, 2; +1)$  has already been constructed so assume  $k_2 > 2$ . Let  $k' = (k_1 - 2, k_2 - 2; +1)$ . Since  $k_2 - 2 > 0$ , the induction hypothesis (1) says there is a saddle connection joining the two zeroes. We may apply Proposition 1.2 to  $k'$  to construct  $k$ . Again the remarks following Proposition 1.2 show that  $k$  satisfies (1) and (2).

We now divide the general case of minimal  $k = (k_1, \dots, k_n; -1)$  with  $\sigma(k) \geq 0$  into two subcases based on the dichotomy of Proposition 1.1. We can assume  $k_1 > 0$ .

CASE B.  $\epsilon = -1$  and  $k_1$  is the only positive entry.

The proof is by induction on  $\sigma(k)$ . For the sake of applying Proposition 1.3 we make the following *Induction Hypothesis*: There is a closed horizontal loop based at the single zero of order  $k_1$ .

We construct the examples  $(4m; -1)$  by induction on  $m$ . We have constructed the examples for  $m = 1, 2$ . These examples indeed satisfy the induction hypothesis. Suppose we have constructed  $(4m; -1)$ . We may apply Proposition 1.3 to construct  $(4m + 4; -1)$ . By the remark following Proposition 1.3  $(4m + 4; -1)$  satisfies the induction hypothesis. Notice the unrestricted example  $(8; -1)$  cannot be constructed from the restricted example  $(4; -1)$  using Proposition 1.3 because we cannot then get arbitrary boundary lengths. That is why it was constructed separately.

Now we construct the examples  $k = (k_1, -1, \dots, -1; -1)$  by induction on  $k_1$ . We have constructed all examples with  $\sigma(k) = 0$  as well as  $(5, -1; -1)$ . For each the induction hypothesis holds. Suppose inductively we have constructed all examples for which  $k_1 \leq m$  where  $m \geq 5$ . Suppose  $k = (m + 1, \dots)$ . Let  $k' = (m - 3, -1, \dots)$  and  $m - 3 \geq 2$ . Then  $k_1 - 3 \geq 2$  so  $k'$  is unrestricted. We construct the unrestricted  $k$  from  $k'$  by applying Proposition 1.3. Notice again  $(5, -1; -1)$  cannot be constructed from  $(1, -1; -1)$  since the latter is restricted and we cannot get arbitrary boundary lengths. That is why we constructed  $(5, -1; -1)$  separately.

CASE C.  $\epsilon = -1$  and there is more than one positive  $k_i$ .

The proof is by induction on  $\sigma$ . By Proposition 1.1,  $n \leq 4$ . We can assume  $k_1$  is the largest positive entry. We make the following

*Induction Hypothesis*: If  $k$  is not the restricted example  $(3, 1; -1)$  then for each  $k_i$  satisfying  $k_i \geq \max(1, k_1 - 4)$  there is a closed horizontal saddle connection based at the zero of order  $k_i$ .

There are no minimal examples with  $\sigma = 0$ . If  $\sigma(k) = 4$ ,  $k_1 \leq 3$ . The possibilities are the basic examples constructed already in III:  $(2, 2; -1)$ ,  $(3, 2, -1; -1)$ ,  $(3, 3, -1; -1)$ ,  $(2, 1, 1; -1)$ ,  $(1, 1, 1, 1; -1)$  and the restricted example  $(3, 1; -1)$ . It is easy to check that the induction hypothesis is satisfied by the basic examples. The example  $(3, 1; -1)$  is constructed from  $(1, -1; -1)$  by applying Proposition 1.2 to the horizontal saddle connection joining the zero and the pole. Suppose inductively we have constructed all examples  $k'$  with  $\sigma(k') = 4m$  where  $m \geq 1$  and  $k = (k_1, k_2, \dots, k_n; -1)$  satisfies  $\sigma(k) = 4m + 4 \geq 8$ . We can assume  $k \neq (5, 3; -1)$  since it was already constructed.

Assume first that  $k_1 \leq 3$ . Since  $n \leq 4$  and  $\sigma(k) \geq 8$ , the possibilities are

$(3, 3, 2; -1)$ ,  $(3, 3, 3, -1; -1)$ , and  $(3, 3, 3, 3; -1)$ .

We construct  $(3, 3, 2; -1)$  and  $(3, 3, 3, -1; -1)$  from  $(1, 1, 2; -1)$  and  $(3, 1, 1; -1)$  respectively by applying Proposition 1.2 to the horizontal segment joining the two zeroes of order 1. Since for both  $(1, 1, 2; -1)$  and  $(3, 1, 1; -1)$  there is a horizontal loop from each zero to itself, the same is true for the examples built from them. Thus the induction hypothesis is still satisfied. To build  $(3, 3, 3, 3; -1)$  we first construct  $(3, 3, 1, 1; -1)$  from  $(1, 1, 1, 1; -1)$  by applying Proposition 1.2 to a segment joining two zeroes of order 1. Then  $(3, 3, 3, 3; -1)$  is constructed from  $(3, 3, 1, 1; -1)$  by again applying Proposition 1.2 to a segment joining the remaining two zeroes with order 1.

Now assume  $k_1 > 3$ . Then  $k_1 \geq 5$  by minimality. The case of  $(7, 1; -1)$  is special since it is an exception to the induction hypothesis. We recall the construction of the flat surface realizing  $(5, -1; -1)$  which has a horizontal saddle connection joining the zero and the pole. We apply Proposition 1.2 to construct  $(7, 1; -1)$ . For any other case let  $k' = (k_1 - 4, k_2, \dots, k_n; -1)$ . Since  $k$  is assumed not to be  $(5, 3; -1)$ ,  $k' \neq (1, 3; -1)$  so  $k'$  is unrestricted. Moreover  $k_1 - 4 \geq \max(1, k_i - 4)$ , for all  $i$ , and so by the induction hypothesis there is a closed horizontal saddle connection based at the zero of order  $k_1 - 4$ . Applying Proposition 1.3 to  $k'$ , constructs  $k$ . Again the induction hypothesis holds for  $k$ . This completes the proof of Theorem 1'.

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