

Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 68 (1993)

Artikel: Finiteness of local fundamental groups for quotients of affine varieties under reductive groups.
Autor: Kumar, Shrawan
DOI: <https://doi.org/10.5169/seals-51766>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 29.12.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Finiteness of local fundamental groups for quotients of affine varieties under reductive groups

SHRAWAN KUMAR

0. Introduction

Let us recall the following conjecture due to C. T. C. Wall:

(C_1) CONJECTURE [W; §1]. *Let G be a reductive linear algebraic group / \mathbb{C} acting linearly on an affine space \mathbb{C}^n . Assume that $\dim \mathbb{C}^n//G = 2$ (cf. §1). Then the variety $\mathbb{C}^n//G$ is biregular isomorphic with the variety \mathbb{C}^2/Γ , where Γ is some finite group acting linearly on \mathbb{C}^2 .*

In our attempt to prove the above conjecture, we (together with R. V. Gurjar) were led to the following question (or conjecture) vastly generalizing the above conjecture:

(C_2) CONJECTURE. *Let G be as above, and assume that G acts on an irreducible normal affine variety X over \mathbb{C} . If the local fundamental groups (cf. §1.2) of X at all the points of X are finite, then the same is true for the quotient variety $X//G$, provided $\dim X//G \geq 2$.*

Recently Gurjar obtained a proof of the above conjecture (C_2) in the case when X is smooth; in particular he proved Wall's conjecture (C_1) . But his proof relies heavily on the assumption that X is smooth.

The aim of this note is to prove the conjecture (C_2) ; but we need to assume that all the local rings of X have fully-torsion divisor class groups. (In fact a more general result is proved; see our theorem 2.1, and remark 2.2.)

The 'Kempf–Ness theory', as developed by Neeman, is the main ingredient in our proof. We also make use of the Luna slice theorem.

I thank R. V. Gurjar for explaining to me his proof of Wall's conjecture, in particular I make use of his crucial proposition from [G]. I also thank J. N. Damon and J. Wahl for some references, and the Referee for his (her) suggestions to improve the exposition.

1. Notation and preliminaries

By a variety X we shall always mean an algebraic variety $/\mathbb{C}$, and its ring of regular functions is denoted by $\mathbb{C}[X]$. We denote the singular locus of X by Σ_X . Let X be an affine variety on which a reductive linear algebraic group G/\mathbb{C} acts, then by $X//G$ we mean the affine variety $\text{Spec}(\mathbb{C}[X]^G)$, where $\mathbb{C}[X]^G$ denotes the ring of G -invariants in $\mathbb{C}[X]$.

Let us recall the following well known fact about CW complexes (see, e.g., arguments in [LW; Chapter II, Sec. 6]):

(1.1) LEMMA. *Let X be a CW complex, and $Y \subsetneq X$ a (closed) subcomplex. For any $x \in X$, there exists a fundamental system $\{U\}_{U \in \mathcal{U}}$ of (open) neighborhoods of x in X satisfying the following condition:*

Given any $U, V \in \mathcal{U}$, $V \subset U$, the inclusion $V \setminus Y_x \hookrightarrow U \setminus Y_x$ is a homotopy equivalence, where $Y_x := \{x\} \cup Y$. (A)

Now for any neighborhood $W \subset U$ of x ($U \in \mathcal{U}$, but W not necessarily in \mathcal{U}), there of course exists a V in \mathcal{U} such that $V \subset W$. From the condition (A), we easily see that, for any $* \in V \setminus Y_x$, the canonical map

$\pi_1(W \setminus Y_x, *) \rightarrow \pi_1(U \setminus Y_x, *)$ is surjective. (J)

(1.2) DEFINITION. With the notation as in the above lemma, let us further assume that $U \setminus Y_x$ (for some, and hence any $U \in \mathcal{U}$) is connected and non-empty. If this is satisfied, we say that Y does not disconnect X locally at x . In this case, we define the *local fundamental group of X at x with respect to Y* , denoted $\pi_1^{x,Y}(X)$, as the fundamental group $\pi_1(U \setminus Y_x, *)$, for any base point $* \in U \setminus Y_x$ and any $U \in \mathcal{U}$.

Observe that, by the condition (A), for any $V \in \mathcal{U}$ and $*' \in V \setminus Y_x$, $\pi_1(U \setminus Y_x, *)$ is isomorphic with $\pi_1(V \setminus Y_x, *')$, and moreover the isomorphism is unique up to an inner automorphism of $\pi_1(U \setminus Y_x, *)$. In particular, the group $\pi_1^{x,Y}(X)$ is defined only up to an inner automorphism. It is easy to see from (J) that $\pi_1^{x,Y}(X)$ does not depend upon the choice of the fundamental system of neighborhoods \mathcal{U} satisfying (A).

As is well known, for any variety X and a closed subvariety Y , X is a CW complex such that $Y \subset X$ is a subcomplex (see [Gi; §5, Satz 4] or [Lo]). Moreover if X is an irreducible normal variety, then for any closed subvariety $Y \subsetneq X$, Y does not disconnect X locally at any $x \in X$. (This can easily be deduced from [M; page 288, Topological form].) In particular $\pi_1^{x,Y}(X)$ is well defined.

If X is an irreducible normal variety, we will often abbreviate $\pi_1^{\Sigma, x}(X)$ as $\pi_1^x(X)$; and call it the *local fundamental group of X at x* .

2. The main theorem and its proof

Following is our main theorem:

(2.1) **THEOREM.** *Let X be an irreducible normal affine variety, on which a (not necessarily connected) reductive linear algebraic group G/\mathbb{C} acts with quotient $q : X \rightarrow X//G$, such that $\dim X//G \geq 2$. We assume that the following condition (C) is satisfied:*

The union of the codimension-one irreducible components of $q^{-1}(\Sigma_{X//G})$ is locally (in the Zariski topology) set theoretically defined by a single equation. (C)

Assume, in addition, that the local fundamental groups of X at all the points in X are finite. Then the same is true for $X//G$ (i.e. the local fundamental groups of $X//G$ at all the points are finite).

(2.2) **REMARKS.** (a) If all the irreducible components of $q^{-1}(\Sigma_{X//G})$ have $\text{codim} \geq 2$, then of course the condition (C) is vacuously satisfied.

(b) As pointed out by Gurjar; if all the local rings of the variety X (at the closed points) have fully-torsion divisor class groups, then the condition (C) is automatically satisfied for any G action on X .

If X (as in the above theorem) is assumed to be smooth, then all the hypotheses are clearly satisfied. In particular, as a special case of the above theorem, we recover the following main result of [G]:

(2.3) **COROLLARY.** *Let X be an irreducible smooth affine variety, on which a reductive linear algebraic group G acts, such that $\dim X//G \geq 2$. Then $X//G$ has all its local fundamental groups finite.*

(2.4) *Proof of Theorem (2.1).* Set $Y = X//G$, and write $q^{-1}(\Sigma_Y) = D \cup E$; where D (resp. E) is the union of all the irreducible components of $q^{-1}(\Sigma_Y)$ of codim 1 (resp. codim > 1). Then, by the condition (C), $X \setminus D$ is again an affine variety (cf. [N; Corollary 1 on page 52, Chapter V]), and clearly (D being G -stable) $X \setminus D$ is G -stable. Now, by a proposition of Gurjar [G], $(X \setminus D)//G$ is biregular isomorphic with $X//G$. (To prove this, use the fact that the canonical morphism:

$(X \setminus D) // G \rightarrow X // G$ is an isomorphism outside the singular locus and, by assumption X being normal, $(X \setminus D) // G$ as well as $X // G$ are normal.) In particular, we can (and will) replace X by $X \setminus D$ throughout the proof of the theorem; and hence assume that all the irreducible components of $q^{-1}(\Sigma_Y)$ have $\text{codim} \geq 2$.

If $\bar{x} \in Y \setminus \Sigma_Y$, $\pi_1^{\bar{x}}(Y)$ is clearly trivial (since $\dim Y \geq 2$, by assumption). Hence, in what follows, we can assume that $\bar{x} \in \Sigma_Y$.

We first take a G -fixed point $x \in X$ (such that $\bar{x} := q(x) \in \Sigma_Y$), and prove that $\pi_1^{\bar{x}}(Y)$ is finite by crucially using the Kempf–Ness theory:

We fix a maximal compact subgroup $K \subset G$. Then there is a real algebraic K -stable closed subvariety X_c of X and, by Neeman’s deformation theorem [Ne] (also given in [S; §5]), a continuous deformation $\varphi_t : X \rightarrow X$ ($0 \leq t \leq 1$) satisfying the following properties (P_1) – (P_6) :

- (P_1) X_c is contained in the union of all the closed G -orbits of X , and moreover any closed G -orbit intersects X_c in precisely one K -orbit.
- (P_2) The canonical map: $X_c/K \rightarrow X//G$ is a homeomorphism in the Hausdorff topology, where X_c/K denotes the orbit space with the quotient topology coming from the Hausdorff topology on X_c .
- (P_3) φ_0 is the identity map Id .
- (P_4) $\varphi_t|_{X_c} = Id$, for all $0 \leq t \leq 1$.
- (P_5) Image $\varphi_1 \subset X_c$.
- (P_6) $\{\varphi_t(x)\}_{0 \leq t < 1} \subset G \cdot x$, for any $x \in X$. In particular $\varphi_1(x) \in \overline{G \cdot x} \cap X_c$, where $\overline{G \cdot x}$ is the closure in the Hausdorff topology.

Continuing with the proof of our theorem (2.1); from the property (P_6) , it is easy to see that $\varphi_t(X \setminus \Sigma) \subset X \setminus \Sigma$, for any $0 \leq t \leq 1$, where we set $\Sigma := q^{-1}(\Sigma_Y)$. (Even though we do not need, the same is true for any subset $A \subset Y$ instead of Σ_Y .) Further, by the property (P_1) , (x being G -fixed) $x \in X_c$, and by assumption $x \in \Sigma$.

Let W be a (small enough) neighborhood of x in X_c , such that $\pi_1^{x, X_c \cap \Sigma}(X_c) \approx \pi_1(W \setminus \Sigma)$. (It is easy to see, from the above deformation, that $X_c \cap \Sigma$ does not disconnect X_c locally at x .) Since $\varphi_1(x) = x$ (cf. P_4), there exists a (small enough) neighborhood U of x in X such that $\varphi_1(U) \subset W$ (in particular $\varphi_1(U \setminus \Sigma) \subset W \setminus \Sigma$), and moreover $\pi_1^{x, \Sigma}(X) \approx \pi_1(U \setminus \Sigma)$. Since $W \cap U$ is a neighborhood of x in X_c and $\varphi_1|_{W \cap U} = Id$ (cf. P_4), it is easy to see, from (\mathcal{J}) of §1.1, that the induced map

$$\varphi_{1*} : \pi_1^{x, \Sigma}(X) \rightarrow \pi_1^{x, X_c \cap \Sigma}(X_c)$$

is surjective (in fact an isomorphism).

Let q_0 denote the canonical map: $X_c \rightarrow X_c/K$. By virtue of (P_2) , we identify X_c/K with Y . Let us take a (small enough) neighborhood N of \bar{x} in Y (resp. W of x in

X_c), such that $\pi_1^{\bar{x}}(Y) \approx \pi_1(N \setminus \Sigma_Y)$ (resp. $\pi_1^{x, X_c \cap \Sigma}(X_c) \approx \pi_1(W \setminus \Sigma)$). We can assume that $q_0(W) \subset N$, and hence $q_0(W \setminus \Sigma) \subset N \setminus \Sigma_Y$. Since x is a G -fixed (in particular K -fixed) point and K is compact, there exists a fundamental system of neighborhoods of x in X_c , which are all K -stable. We take such a $W' \subset W$. (We can choose W' such that $W' \setminus \Sigma$ is connected.) Then by [B; Chap. II, Theorem 6.2], the induced map $\pi_1(W' \setminus \Sigma) \rightarrow \pi_1((W'/K) \setminus \Sigma_Y)$ (got by the restriction of q_0) has finite cokernel (bounded by the order of K/K^0 , where K^0 is the identity component of K). But q_0 being an open map, W'/K is again a neighborhood of \bar{x} in Y . Hence, by (§) of §1.1, the canonical map $\pi_1((W'/K) \setminus \Sigma_Y) \rightarrow \pi_1(N \setminus \Sigma_Y)$ is surjective. In particular, the induced map

$$q_{0*} : \pi_1^{x, X_c \cap \Sigma}(X_c) \rightarrow \pi_1^{\bar{x}}(Y)$$

has finite cokernel. On composition, we get the map

$$q_{0*} \phi_{1*} : \pi_1^{x, \Sigma}(X) \rightarrow \pi_1^{\bar{x}}(Y),$$

which has finite cokernel. So, to prove the finiteness of $\pi_1^{\bar{x}}(Y)$, it suffices to show that $\pi_1^{x, \Sigma}(X)$ is finite:

Consider the canonical maps α and β as follows:

$$\pi_1^{x, \Sigma}(X) \xleftarrow{\alpha} \pi_1^{x, \Sigma \cup \Sigma_X}(X) \xrightarrow{\beta} \pi_1^x(X).$$

Since $X \setminus \Sigma_X$ is smooth and all the irreducible components of Σ are of codim ≥ 2 (by assumption), the map β is an isomorphism. As is well known, the map α is surjective; but we give an argument (told to me by R. R. Simha) for completeness:

Let U be a non-empty connected open subset (in the Hausdorff topology) of an irreducible normal variety X . Since any subvariety $Y \subsetneq X$ does not disconnect X locally at any point (cf. §1.2), $U \setminus Y$ is connected. Let $p : \tilde{U} \rightarrow U$ be the simply connected cover of U , viewed canonically as a complex analytic variety. Since \tilde{U} is locally homeomorphic to U , $Z := p^{-1}(U \cap Y)$ does not disconnect \tilde{U} locally at any point of \tilde{U} . But then, by a straightforward pointset topological argument, $\tilde{U} \setminus Z$ itself is connected. From this the surjectivity of $\pi_1(U \setminus Y) \rightarrow \pi_1(U)$ follows immediately. This gives the surjectivity of α .

This proves the finiteness of $\pi_1^{x, \Sigma}(X)$ (since, by assumption, $\pi_1^x(X)$ is finite); thereby proving the finiteness of $\pi_1^{\bar{x}}(Y)$, in the case when $G \cdot x = x$.

Now we come to an arbitrary point $\bar{x} \in \Sigma_Y$, and let $G \cdot x$ be the (unique) closed G -orbit lying inside $q^{-1}(\bar{x})$.

By Luna's slice theorem [L; §III], there exists an irreducible affine locally closed subvariety $x \in S \subset X$, which is stable under the reductive subgroup G_x (where

$G_x \subset G$ is the isotropy subgroup at x), and an affine open subset $N \subset Y$, such that the canonical map $\psi : G \times_{G_x} S \rightarrow X$ is étale onto the open subset $q^{-1}(N)$ of X , and moreover the induced map $\bar{\psi} : S//G_x \rightarrow X//G$ is étale onto N . So to prove the finiteness of $\pi_1^{\bar{x}}(X//G) \approx \pi_1^{\bar{x}}(S//G_x)$, since any descending chain of algebraic subgroups of G becomes stationary, it suffices to show that the G_x -variety S satisfies:

- (F₁) S is normal,
- (F₂) The local fundamental groups of S at all the points of S are finite, and
- (F₃) $q_S : S \rightarrow S//G_x$ satisfies the condition (C) of Theorem (2.1).

(F₁) follows trivially, since the map ψ is étale and $G \times_{G_x} S$ fibres over the smooth variety G/G_x with fibre S . Since $\Sigma_{G \times_{G_x} S} = G \times_{G_x} \Sigma_S$ and, by assumption, all the local fundamental groups of X are finite, (F₂) follows.

Observe that $(\bar{\psi})^{-1}(\Sigma_{X//G}) = \Sigma_{S//G_x}$ (since $\bar{\psi}$ is étale). So $q_S^{-1}(\Sigma_{S//G_x}) = q^{-1}(\Sigma_{X//G}) \cap S$, which gives

$$G \times_{G_x} (q_S^{-1}(\Sigma_{S//G_x})) = \psi^{-1}(q^{-1}(\Sigma_{X//G})). \quad (*)$$

The equality (*) clearly shows the validity of (F₃) (since the same is true, by assumption, for the map $q : X \rightarrow X//G$).

This completes the proof of the theorem. \square

(2.5) REMARK (due to R. V. Gurjar). The condition (C) in Theorem (2.1) is not always satisfied. Consider, e.g.,

$$X = \text{Spec}(\mathbb{C}[x_1, x_2, x_3, x_4]/\langle x_1 x_2 - x_3 x_4 \rangle),$$

and $G = \mathbb{C}^*$ acting on X by $t \cdot x_1 = tx_1, t \cdot x_2 = t^{-1}x_2, t \cdot x_3 = tx_3, t \cdot x_4 = t^{-1}x_4$ (for any $t \in \mathbb{C}^*$). Then $\Sigma_{X//\mathbb{C}^*} = \{0\}$, and $q^{-1}(\Sigma_{X//\mathbb{C}^*})$ is the union of two irreducible components (each isomorphic with \mathbb{C}^2); and this does not satisfy the condition (C). Observe however that in this example, $X//\mathbb{C}^*$ has all its local fundamental groups finite.

REFERENCES

- [B] BREDON, G. E., "Introduction to compact transformation groups," Academic Press, New York (1972).
- [Gi] GIESECKE, B., *Simpliziale zerlegung abzählbarer analytischer räume*, Math. Z. 83 (1964), 177–213.
- [G] GURJAR, R. V., *On a conjecture of C. T. C. Wall*, Preprint (1990).
- [Lo] LOJASIEWICZ, S., *Triangulation of semi-analytic sets*. Ann. Scuo. Norm. Sup. Pisa 18 (1964), 449–474.
- [L] LUNA, D., *Slices étales*, Bull. Soc. Math. France 33 (1973), 81–105.

- [LW] LUNDELL, A. T. and WEINGRAM, S., "*The topology of CW complexes*," Van Nostrand Reinhold Company (1969).
- [M] MUMFORD, D., "*The red book of varieties and schemes*," Lecture Notes in Mathematics no. 1358, Springer-Verlag (1988).
- [N] NAGATA, M., "*Lectures on the fourteenth problem of Hilbert*," Tata Institute of Fundamental Research (Bombay) Lecture notes (1965).
- [Ne] NEEMAN, A., *The topology of quotient varieties*. Annals of Math. 122 (1985), 419–459.
- [S] SCHWARZ, G. W., *The topology of algebraic quotients*, In: "*Topological methods in algebraic transformation groups*" (ed. by H. Kraft et al.), Progress in Mathematics Vol. 80, Birkhäuser (1989), 135–151.
- [W] WALL, C. T. C., *Functions on quotient singularities*, Phil. Trans. Royal Soc. London 324 (1987), 1–45.

School of Mathematics
TIFR, Colaba
Bombay 400 005, India

and

University of North Carolina
Chapel Hill
NC 27599-3250, USA

Received November 7, 1990