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## Finiteness of local fundamental groups for quotients of affine varieties under reductive groups

SHRAWAN KUMAR

### 0. Introduction

Let us recall the following conjecture due to C. T. C. Wall:

$(C_1)$  CONJECTURE [W; §1]. *Let  $G$  be a reductive linear algebraic group  $/\mathbb{C}$  acting linearly on an affine space  $\mathbb{C}^n$ . Assume that  $\dim \mathbb{C}^n//G = 2$  (cf. §1). Then the variety  $\mathbb{C}^n//G$  is biregular isomorphic with the variety  $\mathbb{C}^2/\Gamma$ , where  $\Gamma$  is some finite group acting linearly on  $\mathbb{C}^2$ .*

In our attempt to prove the above conjecture, we (together with R. V. Gurjar) were led to the following question (or conjecture) vastly generalizing the above conjecture:

$(C_2)$  CONJECTURE. *Let  $G$  be as above, and assume that  $G$  acts on an irreducible normal affine variety  $X$  over  $\mathbb{C}$ . If the local fundamental groups (cf. §1.2) of  $X$  at all the points of  $X$  are finite, then the same is true for the quotient variety  $X//G$ , provided  $\dim X//G \geq 2$ .*

Recently Gurjar obtained a proof of the above conjecture  $(C_2)$  in the case when  $X$  is smooth; in particular he proved Wall's conjecture  $(C_1)$ . But his proof relies heavily on the assumption that  $X$  is smooth.

The aim of this note is to prove the conjecture  $(C_2)$ ; but we need to assume that all the local rings of  $X$  have fully-torsion divisor class groups. (In fact a more general result is proved; see our theorem 2.1, and remark 2.2.)

The 'Kempf–Ness theory', as developed by Neeman, is the main ingredient in our proof. We also make use of the Luna slice theorem.

I thank R. V. Gurjar for explaining to me his proof of Wall's conjecture, in particular I make use of his crucial proposition from [G]. I also thank J. N. Damon and J. Wahl for some references, and the Referee for his (her) suggestions to improve the exposition.

## 1. Notation and preliminaries

By a variety  $X$  we shall always mean an algebraic variety  $/\mathbb{C}$ , and its ring of regular functions is denoted by  $\mathbb{C}[X]$ . We denote the singular locus of  $X$  by  $\Sigma_X$ . Let  $X$  be an affine variety on which a reductive linear algebraic group  $G/\mathbb{C}$  acts, then by  $X//G$  we mean the affine variety  $\text{Spec}(\mathbb{C}[X]^G)$ , where  $\mathbb{C}[X]^G$  denotes the ring of  $G$ -invariants in  $\mathbb{C}[X]$ .

Let us recall the following well known fact about  $CW$  complexes (see, e.g., arguments in [LW; Chapter II, Sec. 6]):

(1.1) LEMMA. *Let  $X$  be a  $CW$  complex, and  $Y \subsetneq X$  a (closed) subcomplex. For any  $x \in X$ , there exists a fundamental system  $\{U\}_{U \in \mathcal{U}}$  of (open) neighborhoods of  $x$  in  $X$  satisfying the following condition:*

*Given any  $U, V \in \mathcal{U}$ ,  $V \subset U$ , the inclusion  $V \setminus Y_x \hookrightarrow U \setminus Y_x$  is a homotopy equivalence, where  $Y_x := \{x\} \cup Y$ .* (A)

Now for any neighborhood  $W \subset U$  of  $x$  ( $U \in \mathcal{U}$ , but  $W$  not necessarily in  $\mathcal{U}$ ), there of course exists a  $V$  in  $\mathcal{U}$  such that  $V \subset W$ . From the condition (A), we easily see that, for any  $* \in V \setminus Y_x$ , the canonical map

$\pi_1(W \setminus Y_x, *) \rightarrow \pi_1(U \setminus Y_x, *)$  is surjective. (S)

(1.2) DEFINITION. With the notation as in the above lemma, let us further assume that  $U \setminus Y_x$  (for some, and hence any  $U \in \mathcal{U}$ ) is connected and non-empty. If this is satisfied, we say that  $Y$  does not disconnect  $X$  locally at  $x$ . In this case, we define the *local fundamental group of  $X$  at  $x$  with respect to  $Y$* , denoted  $\pi_1^{x,Y}(X)$ , as the fundamental group  $\pi_1(U \setminus Y_x, *)$ , for any base point  $* \in U \setminus Y_x$  and any  $U \in \mathcal{U}$ .

Observe that, by the condition (A), for any  $V \in \mathcal{U}$  and  $*' \in V \setminus Y_x$ ,  $\pi_1(U \setminus Y_x, *)$  is isomorphic with  $\pi_1(V \setminus Y_x, *')$ , and moreover the isomorphism is unique up to an inner automorphism of  $\pi_1(U \setminus Y_x, *)$ . In particular, the group  $\pi_1^{x,Y}(X)$  is defined only up to an inner automorphism. It is easy to see from (S) that  $\pi_1^{x,Y}(X)$  does not depend upon the choice of the fundamental system of neighborhoods  $\mathcal{U}$  satisfying (A).

As is well known, for any variety  $X$  and a closed subvariety  $Y$ ,  $X$  is a  $CW$  complex such that  $Y \subset X$  is a subcomplex (see [Gi; §5, Satz 4] or [Lo]). Moreover if  $X$  is an irreducible normal variety, then for any closed subvariety  $Y \subsetneq X$ ,  $Y$  does not disconnect  $X$  locally at any  $x \in X$ . (This can easily be deduced from [M; page 288, Topological form].) In particular  $\pi_1^{x,Y}(X)$  is well defined.

If  $X$  is an irreducible normal variety, we will often abbreviate  $\pi_1^{x, \Sigma^x}(X)$  as  $\pi_1^x(X)$ ; and call it the *local fundamental group of  $X$  at  $x$* .

## 2. The main theorem and its proof

Following is our main theorem:

(2.1) **THEOREM.** *Let  $X$  be an irreducible normal affine variety, on which a (not necessarily connected) reductive linear algebraic group  $G/\mathbb{C}$  acts with quotient  $q : X \rightarrow X//G$ , such that  $\dim X//G \geq 2$ . We assume that the following condition (C) is satisfied:*

*The union of the codimension-one irreducible components of  $q^{-1}(\Sigma_{X//G})$  is locally (in the Zariski topology) set theoretically defined by a single equation.* (C)

*Assume, in addition, that the local fundamental groups of  $X$  at all the points in  $X$  are finite. Then the same is true for  $X//G$  (i.e. the local fundamental groups of  $X//G$  at all the points are finite).*

(2.2) **REMARKS.** (a) If all the irreducible components of  $q^{-1}(\Sigma_{X//G})$  have  $\text{codim} \geq 2$ , then of course the condition (C) is vacuously satisfied.

(b) As pointed out by Gurjar; if all the local rings of the variety  $X$  (at the closed points) have fully-torsion divisor class groups, then the condition (C) is automatically satisfied for any  $G$  action on  $X$ .

If  $X$  (as in the above theorem) is assumed to be smooth, then all the hypotheses are clearly satisfied. In particular, as a special case of the above theorem, we recover the following main result of [G]:

(2.3) **COROLLARY.** *Let  $X$  be an irreducible smooth affine variety, on which a reductive linear algebraic group  $G$  acts, such that  $\dim X//G \geq 2$ . Then  $X//G$  has all its local fundamental groups finite.*

(2.4) *Proof of Theorem (2.1).* Set  $Y = X//G$ , and write  $q^{-1}(\Sigma_Y) = D \cup E$ ; where  $D$  (resp.  $E$ ) is the union of all the irreducible components of  $q^{-1}(\Sigma_Y)$  of  $\text{codim} 1$  (resp.  $\text{codim} > 1$ ). Then, by the condition (C),  $X \setminus D$  is again an affine variety (cf. [N; Corollary 1 on page 52, Chapter V]), and clearly ( $D$  being  $G$ -stable)  $X \setminus D$  is  $G$ -stable. Now, by a proposition of Gurjar [G],  $(X \setminus D)//G$  is biregular isomorphic with  $X//G$ . (To prove this, use the fact that the canonical morphism:

$(X \setminus D) // G \rightarrow X // G$  is an isomorphism outside the singular locus and, by assumption  $X$  being normal,  $(X \setminus D) // G$  as well as  $X // G$  are normal.) In particular, we can (and will) replace  $X$  by  $X \setminus D$  throughout the proof of the theorem; and hence assume that all the irreducible components of  $q^{-1}(\Sigma_Y)$  have  $\text{codim} \geq 2$ .

If  $\bar{x} \in Y \setminus \Sigma_Y$ ,  $\pi_1^{\bar{x}}(Y)$  is clearly trivial (since  $\dim Y \geq 2$ , by assumption). Hence, in what follows, we can assume that  $\bar{x} \in \Sigma_Y$ .

We first take a  $G$ -fixed point  $x \in X$  (such that  $\bar{x} := q(x) \in \Sigma_Y$ ), and prove that  $\pi_1^{\bar{x}}(Y)$  is finite by crucially using the Kempf–Ness theory:

We fix a maximal compact subgroup  $K \subset G$ . Then there is a real algebraic  $K$ -stable closed subvariety  $X_c$  of  $X$  and, by Neeman’s deformation theorem [Ne] (also given in [S; §5]), a continuous deformation  $\varphi_t : X \rightarrow X$  ( $0 \leq t \leq 1$ ) satisfying the following properties  $(P_1)$ – $(P_6)$ :

- $(P_1)$   $X_c$  is contained in the union of all the closed  $G$ -orbits of  $X$ , and moreover any closed  $G$ -orbit intersects  $X_c$  in precisely one  $K$ -orbit.
- $(P_2)$  The canonical map:  $X_c/K \rightarrow X // G$  is a homeomorphism in the Hausdorff topology, where  $X_c/K$  denotes the orbit space with the quotient topology coming from the Hausdorff topology on  $X_c$ .
- $(P_3)$   $\varphi_0$  is the identity map  $Id$ .
- $(P_4)$   $\varphi_t|_{X_c} = Id$ , for all  $0 \leq t \leq 1$ .
- $(P_5)$  Image  $\varphi_1 \subset X_c$ .
- $(P_6)$   $\{\varphi_t(x)\}_{0 \leq t < 1} \subset G \cdot x$ , for any  $x \in X$ . In particular  $\varphi_1(x) \in \overline{G \cdot x} \cap X_c$ , where  $\overline{G \cdot x}$  is the closure in the Hausdorff topology.

Continuing with the proof of our theorem (2.1); from the property  $(P_6)$ , it is easy to see that  $\varphi_t(X \setminus \Sigma) \subset X \setminus \Sigma$ , for any  $0 \leq t \leq 1$ , where we set  $\Sigma := q^{-1}(\Sigma_Y)$ . (Even though we do not need, the same is true for any subset  $A \subset Y$  instead of  $\Sigma_Y$ .) Further, by the property  $(P_1)$ , ( $x$  being  $G$ -fixed)  $x \in X_c$ , and by assumption  $x \in \Sigma$ .

Let  $W$  be a (small enough) neighborhood of  $x$  in  $X_c$ , such that  $\pi_1^{x, X_c \cap \Sigma}(X_c) \approx \pi_1(W \setminus \Sigma)$ . (It is easy to see, from the above deformation, that  $X_c \cap \Sigma$  does not disconnect  $X_c$  locally at  $x$ .) Since  $\varphi_1(x) = x$  (cf.  $P_4$ ), there exists a (small enough) neighborhood  $U$  of  $x$  in  $X$  such that  $\varphi_1(U) \subset W$  (in particular  $\varphi_1(U \setminus \Sigma) \subset W \setminus \Sigma$ ), and moreover  $\pi_1^{x, \Sigma}(X) \approx \pi_1(U \setminus \Sigma)$ . Since  $W \cap U$  is a neighborhood of  $x$  in  $X_c$  and  $\varphi_1|_{W \cap U} = Id$  (cf.  $P_4$ ), it is easy to see, from  $(\mathcal{S})$  of §1.1, that the induced map

$$\varphi_{1*} : \pi_1^{x, \Sigma}(X) \rightarrow \pi_1^{x, X_c \cap \Sigma}(X_c)$$

is surjective (in fact an isomorphism).

Let  $q_0$  denote the canonical map:  $X_c \rightarrow X_c/K$ . By virtue of  $(P_2)$ , we identify  $X_c/K$  with  $Y$ . Let us take a (small enough) neighborhood  $N$  of  $\bar{x}$  in  $Y$  (resp.  $W$  of  $x$  in

$X_c$ ), such that  $\pi_1^{\bar{x}}(Y) \approx \pi_1(N \setminus \Sigma_Y)$  (resp.  $\pi_1^{x, X_c \cap \Sigma}(X_c) \approx \pi_1(W \setminus \Sigma)$ ). We can assume that  $q_0(W) \subset N$ , and hence  $q_0(W \setminus \Sigma) \subset N \setminus \Sigma_Y$ . Since  $x$  is a  $G$ -fixed (in particular  $K$ -fixed) point and  $K$  is compact, there exists a fundamental system of neighborhoods of  $x$  in  $X_c$ , which are all  $K$ -stable. We take such a  $W' \subset W$ . (We can choose  $W'$  such that  $W' \setminus \Sigma$  is connected.) Then by [B; Chap. II, Theorem 6.2], the induced map  $\pi_1(W' \setminus \Sigma) \rightarrow \pi_1((W'/K) \setminus \Sigma_Y)$  (got by the restriction of  $q_0$ ) has finite cokernel (bounded by the order of  $K/K^0$ , where  $K^0$  is the identity component of  $K$ ). But  $q_0$  being an open map,  $W'/K$  is again a neighborhood of  $\bar{x}$  in  $Y$ . Hence, by  $(\mathcal{S})$  of §1.1, the canonical map  $\pi_1((W'/K) \setminus \Sigma_Y) \rightarrow \pi_1(N \setminus \Sigma_Y)$  is surjective. In particular, the induced map

$$q_{0*} : \pi_1^{x, X_c \cap \Sigma}(X_c) \rightarrow \pi_1^{\bar{x}}(Y)$$

has finite cokernel. On composition, we get the map

$$q_{0*} \phi_{1*} : \pi_1^{x, \Sigma}(X) \rightarrow \pi_1^{\bar{x}}(Y),$$

which has finite cokernel. So, to prove the finiteness of  $\pi_1^{\bar{x}}(Y)$ , it suffices to show that  $\pi_1^{x, \Sigma}(X)$  is finite:

Consider the canonical maps  $\alpha$  and  $\beta$  as follows:

$$\pi_1^{x, \Sigma}(X) \xleftarrow{\alpha} \pi_1^{x, \Sigma \cup \Sigma_x}(X) \xrightarrow{\beta} \pi_1^x(X).$$

Since  $X \setminus \Sigma_x$  is smooth and all the irreducible components of  $\Sigma$  are of  $\text{codim} \geq 2$  (by assumption), the map  $\beta$  is an isomorphism. As is well known, the map  $\alpha$  is surjective; but we give an argument (told to me by R. R. Simha) for completeness:

Let  $U$  be a non-empty connected open subset (in the Hausdorff topology) of an irreducible normal variety  $X$ . Since any subvariety  $Y \subsetneq X$  does not disconnect  $X$  locally at any point (cf. §1.2),  $U \setminus Y$  is connected. Let  $p : \tilde{U} \rightarrow U$  be the simply connected cover of  $U$ , viewed canonically as a complex analytic variety. Since  $\tilde{U}$  is locally homeomorphic to  $U$ ,  $Z := p^{-1}(U \cap Y)$  does not disconnect  $\tilde{U}$  locally at any point of  $\tilde{U}$ . But then, by a straightforward pointset topological argument,  $\tilde{U} \setminus Z$  itself is connected. From this the surjectivity of  $\pi_1(U \setminus Y) \rightarrow \pi_1(U)$  follows immediately. This gives the surjectivity of  $\alpha$ .

This proves the finiteness of  $\pi_1^{x, \Sigma}(X)$  (since, by assumption,  $\pi_1^x(X)$  is finite); thereby proving the finiteness of  $\pi_1^{\bar{x}}(Y)$ , in the case when  $G \cdot x = x$ .

Now we come to an arbitrary point  $\bar{x} \in \Sigma_Y$ , and let  $G \cdot x$  be the (unique) closed  $G$ -orbit lying inside  $q^{-1}(\bar{x})$ .

By Luna's slice theorem [L; §III], there exists an irreducible affine locally closed subvariety  $x \in S \subset X$ , which is stable under the reductive subgroup  $G_x$  (where

$G_x \subset G$  is the isotropy subgroup at  $x$ ), and an affine open subset  $N \subset Y$ , such that the canonical map  $\psi : G \times_{G_x} S \rightarrow X$  is étale onto the open subset  $q^{-1}(N)$  of  $X$ , and moreover the induced map  $\bar{\psi} : S//G_x \rightarrow X//G$  is étale onto  $N$ . So to prove the finiteness of  $\pi_1^{\bar{x}}(X//G) \approx \pi_1^{\bar{x}}(S//G_x)$ , since any descending chain of algebraic subgroups of  $G$  becomes stationary, it suffices to show that the  $G_x$ -variety  $S$  satisfies:

- (F<sub>1</sub>)  $S$  is normal,
- (F<sub>2</sub>) The local fundamental groups of  $S$  at all the points of  $S$  are finite, and
- (F<sub>3</sub>)  $q_S : S \rightarrow S//G_x$  satisfies the condition (C) of Theorem (2.1).

(F<sub>1</sub>) follows trivially, since the map  $\psi$  is étale and  $G \times_{G_x} S$  fibres over the smooth variety  $G/G_x$  with fibre  $S$ . Since  $\Sigma_{G \times_{G_x} S} = G \times_{G_x} \Sigma_S$  and, by assumption, all the local fundamental groups of  $X$  are finite, (F<sub>2</sub>) follows.

Observe that  $(\bar{\psi})^{-1}(\Sigma_{X//G}) = \Sigma_{S//G_x}$  (since  $\bar{\psi}$  is étale). So  $q_S^{-1}(\Sigma_{S//G_x}) = q^{-1}(\Sigma_{X//G}) \cap S$ , which gives

$$G \times_{G_x} (q_S^{-1}(\Sigma_{S//G_x})) = \psi^{-1}(q^{-1}(\Sigma_{X//G})). \quad (*)$$

The equality (\*) clearly shows the validity of (F<sub>3</sub>) (since the same is true, by assumption, for the map  $q : X \rightarrow X//G$ ).

This completes the proof of the theorem. □

(2.5) REMARK (due to R. V. Gurjar). The condition (C) in Theorem (2.1) is not always satisfied. Consider, e.g.,

$$X = \text{Spec}(\mathbb{C}[x_1, x_2, x_3, x_4]/\langle x_1x_2 - x_3x_4 \rangle),$$

and  $G = \mathbb{C}^*$  acting on  $X$  by  $t \cdot x_1 = tx_1, t \cdot x_2 = t^{-1}x_2, t \cdot x_3 = tx_3, t \cdot x_4 = t^{-1}x_4$  (for any  $t \in \mathbb{C}^*$ ). Then  $\Sigma_{X//\mathbb{C}^*} = \{0\}$ , and  $q^{-1}(\Sigma_{X//\mathbb{C}^*})$  is the union of two irreducible components (each isomorphic with  $\mathbb{C}^2$ ); and this does not satisfy the condition (C). Observe however that in this example,  $X//\mathbb{C}^*$  has all its local fundamental groups finite.

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