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Autor: Anderson, Douglas R. / Connolly, F.X.

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Finiteness obstructions and cocompact actions on $S^m \times \mathbb{R}^n$

DOUGLAS R. ANDERSON[1] AND FRANCIS X. CONNOLLY[2]

We recall that an action of the group Γ on the space X is *properly discontinuous* if for every compact set K in X the family $\{\gamma K \mid \gamma \in \Gamma\}$ is locally finite and that the action is *cocompact* if the orbit space X/Γ is compact.

The question of which groups Γ can act freely, properly discontinuously, and cocompactly on $S^m \times \mathbb{R}^n$ has gained interest in recent years. (If the action is not required to be cocompact, this problem is solved for a very large class of groups in [C–P].) This paper studies the following aspect of this question: Suppose Γ acts freely, properly discontinuously, and cocompactly on $S^m \times \mathbb{R}^n$ and that G is a finite subgroup of the group Γ. What restrictions does this impose on G?

It is a classical result of Cartan–Eilenberg [C–E] that in this situation G has periodic cohomology of period d and that $m \equiv -1 \pmod d$. Furthermore, it is natural to expect that any additional restrictions on G will depend on m and n. For example, a well known result of Milnor [Mi] shows that if $n = 0$, G cannot contain a dihedral subgroup; while a recent result of Hambleton–Pedersen [H–P] shows that if $n \geq 2$, then this restriction on G is no longer necessary. So the restrictions on G depend on n. Another such example arises out of the work of Lee [Le] and Madsen–Thomas–Wall [M–T–W]. In [Le], Lee gives examples of finite groups G with periodic cohomology of period d and not containing a dihedral subgroup that do not act freely on $S^{(2r+1)d-1}$ for any $r \geq 0$. On the other hand, the results of [M–T–W] show that such groups G do act freely on S^{2rd-1} for any $r \geq 1$. So the restrictions on the finite subgroups of Γ also depend on m.

The main results of this paper are Theorems I and II below. Since the examples of Lee play a central role in Theorem II, we describe them in more detail. Following Lee [Le; p. 195], we let $Q(2^q, p, 1)$ be the group with presentation

$$\{x, y, z \mid x^2 = (xy)^2 = y^{2^{q-2}},\ z^p = 1, xzx^{-1} = z^{-1}, yzy^{-1} = z^{-1}\}$$

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where p is an odd prime. Notice that since there is a split short exact sequence

$$1 \to C_p \to Q(2^q, p, 1) \to Q(2^q) \to 1,$$

$Q(2^q, p, 1)$ is the semidirect product $C_p \rtimes Q(2^q)$ where $C_p = \langle z \rangle$ is the subgroup generated by z and $Q(2^q)$ is the generalized quaternion group with presentation $\{\bar{x}, \bar{y} \mid \bar{x}^2 = (\bar{x}\bar{y})^2 = \bar{y}^{2^{q-2}}\}$. We embed $Q(2^q)$ in $Q(2^q, p, 1)$ via the homomorphism that sends $\bar{x}$ and $\bar{y}$ respectively, to x and y. This group acts linearly and freely on S^{8r-1} for any $r \geq 1$. On the other hand, Lee [Le] proved the following theorem:

THEOREM (Lee). *If the finite group G contains a copy of $Q(2^q, p, 1)$ with $q \geq 4$, then G cannot act freely on S^{8r+3} for any $r \geq 0$.*

Lee proved this theorem using a semicharacteristic invariant. This result was sharpened in subsequent work by Davis [Da]. He used the Swan–Wall finiteness obstruction to prove the following theorem:

THEOREM (Davis). *Let G be a finite group containing $Q(2^q, p, 1)$ with $q \geq 4$ and p an odd prime with $p \not\equiv -1 \pmod{2^{q-1}}$ and $p \not\equiv 1 \pmod 8$. Then G does not act freely and cellularly on a finite CW complex having the homotopy type of S^{8r+3} for any $r \geq 0$.*

We recall that a *Hadamard manifold* is a simply connected, complete Riemannian manifold of non-positive sectional curvature. For the purposes of this paper, a discrete subgroup $\Gamma \subset \mathrm{Iso}(\mathbb{H})$, where $\mathrm{Iso}(\mathbb{H})$ is the group of isometries of $\mathbb{H}$, is called a *lattice* if the orbit space $\mathbb{H}/\Gamma$ is compact although this is slightly nonstandard terminology. A group Γ is called *lattice-like* if it is virtually torsion free and admits an epimorphism onto a lattice with finite kernel. (Recall a group is *virtually torsion free* if it contains a torsion free subgroup of finite index.) For example, crystallographic groups, discrete, uniform subgroups of a matrix group, and finite groups are lattice-like. Notice that the natural action of a lattice-like group on $\mathbb{H}$ is properly discontinuous.

Unless otherwise stated, in this paper the symbol Γ will denote a lattice-like group and the letters G, H, etc will denote finite groups.

In Section 2 we define an additive category $\mathscr{C}_G(\overline{\mathscr{E}\Gamma}, \Sigma; \mathbb{Z})$ and the transfer homomorphism

$$\mathrm{tr} : \tilde{K}_0(\mathbb{Z}G) \to \tilde{K}_0(\mathscr{C}_G(\overline{\mathscr{E}\Gamma}, \Sigma, \mathbb{Z}))$$

which appears in the following theorem.

THEOREM I. *Suppose the lattice-like group Γ acts freely, cellularly, and cocompactly on the CW complex $\tilde{X}$ and that for every finite subgroup H of Γ, $\tilde{X}/H$ is a finitely dominated CW complex. Let G be a fixed finite subgroup of Γ and $\sigma_G(\tilde{X}/G)$ be its finiteness obstruction. Then* $\operatorname{tr} \sigma_G(\tilde{X}/G) = 0$ *in* $\tilde{K}_0(\mathscr{C}_G(\overline{\mathscr{E}\Gamma}, \Sigma; \mathbb{Z}))$.

A group Γ has *finite virtual cohomological dimension* if it contains a finite index subgroup Γ_0 with finite cohomological dimension. In this case we write $\operatorname{vcd} \Gamma = s$, if Γ_0 has cohomological dimension s. If G is a subgroup of Γ, then $Z_\Gamma(G)$ denotes the centralizer of G in Γ.

The following theorem, which will be derived from the stronger Theorem 5.1 in Section 5, is a generalization of the result of Lee and Davis:

THEOREM II. *Let Γ be a lattice-like group. Suppose that Γ contains a copy of $Q(2^q, p, 1)$ with $q \geq 4$ and for which p is an odd prime p with $p \not\equiv 1 \pmod 8$ and $p \not\equiv -1 \pmod{2^{q-1}}$ and that $Z_\Gamma(C_p)$ is finite. If either*

(1) $3 \leq \operatorname{vcd} Z_\Gamma(Q(2^q))$; *or*

(2) $\operatorname{vcd} \Gamma \leq 3$,

then Γ cannot act freely, cellularly, and cocompactly on a CW complex $\tilde{X}$ homotopy equivalent to S^{8r+3} for some $r \geq 0$. In particular, Γ cannot act freely, properly discontinuously, and cocompactly on $S^{8r+3} \times \mathbb{R}^n$ for any $r \geq 0$.

Notice that if Γ is itself finite, then $Z_\Gamma(C_p)$ must be finite and $\operatorname{vcd} \Gamma = 0$. Hence (2) holds and we recover Davis's theorem. This is not an independent proof of his theorem, however, since we use his proof in the one given here.

Recall that C_p acts freely on $\mathbb{Z}[\zeta_p]$ where ζ_p is a p-th root of unity. To show that Theorem II contains new information, we prove the following corollary:

COROLLARY III. *Let $q \geq 4$ and p be an odd prime with $p \not\equiv 1$ (mod 8) and $p \not\equiv -1$ (mod 2^{q-1}). Let Γ be a crystallographic group of the form $\Gamma = A \rtimes Q(2^q, p, 1)$ where $A = \operatorname{ind}_C^Q(A_0)$, $Q = Q(2^q, p, 1)$, $C = C_p$, and $A_0 = (\mathbb{Z}[\zeta_p])^k$. If $p = 3$, assume that $\operatorname{rank}_{\mathbb{Z}}(A_0) \neq 2$. Then Γ cannot act freely, cellularly, and cocompactly on a CW complex X homotopy equivalent to S^{8r+3} for any $r \geq 0$. In particular, Γ cannot act freely, properly discontinuously, and cocompactly on $S^{8r+3} \times \mathbb{R}^n$ for any $r \geq 0$.*

We remark that the group Γ of Corollary III acts on $A \otimes \mathbb{R} = V$ in such a way that A acts as translations. In addition, $\Gamma/A = Q(2^q, p, 1)$ acts freely and linearly on S^{8r+7} for each $r \geq 0$. Hence Γ acts freely, properly discontinuously and cocompactly on $S^{8r+7} \times V$ via the diagonal action. Thus the "geometric period" of Γ is 8, even though its "algebraic period" in the sense of Farrell cohomology [Fa] is 4.

Section 1 of this paper contains the key geometric result underlying the proof of Theorem I; while Section 2 defines the transfer map mentioned above and gives the proof of Theorem I. Sections 3 and 4 contain the main new technical tools used in this paper. Section 3 introduces the equivariant geometric module theory needed for the proof of Theorem II; while a spectral sequence for calculating this theory is described in Section 4. The proofs of Theorem II and Corollary III are given in Section 5 with the proof of one lemma deferred until Section 6.

The reader who is most interested in Theorem II and its proof should read Sections 1, 2, and 5 first and refer back to Sections 3 and 4 only as need dictates.

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1. The key geometric result

Let Γ be a lattice-like group. Throughout this paper the Hadamard manifold on which Γ acts by isometries will be denoted by $\mathscr{E}\Gamma$. In this section we study the spaces $\mathscr{E}\Gamma$ and $\overline{\mathscr{E}\Gamma}$ that play a basic role in this paper and derive their properties. The notation $\mathscr{E}\Gamma$ for these spaces was introduced in [C–K] where they were studied in a more general context. The main result of this section is Proposition 1.5. It is the key geometric ingredient in the proof of the Theorem I.

LEMMA 1.1. *Let Γ be a lattice-like group. Then Γ contains a normal torsion free subgroup Γ_0 of finite index. Furthermore the left action of Γ on $\mathscr{E}\Gamma$ has the following properties*:

(1) *$\mathscr{E}\Gamma$ admits the structure of a finite dimensional, contractible CW complex on which Γ acts cellularly*;
(2) *For each cell $\sigma \in \mathscr{E}\Gamma$, the stabilizer of σ fixes σ pointwise*;
(3) *For each point $x \in \mathscr{E}\Gamma$, the isotropy subgroup Γ_x is finite*;
(4) *For each finite subgroup G of Γ, $(\mathscr{E}\Gamma)^G$ is contractible*; *and*
(5) *There is a normal subgroup Γ_0 of Γ for which the orbit space $\mathscr{E}\Gamma/\Gamma_0$ is a closed manifold.*

Proof. Let Γ_1 be a torsion free subgroup of finite index in Γ and Γ_0 be the intersection of the (finitely many) conjugates of Γ_1. Then Γ_0 is still torsion free and of finite index but is also normal.

Since Γ acts properly discontinuously on $\mathscr{E}\Gamma$, each isotropy subgroup Γ_x is finite. Thus if Γ_0 is a normal torsion free subgroup of finite index, Γ_0 acts freely, properly discontinuously, and cocompactly on $\mathscr{E}\Gamma$. Hence the orbit space $\mathscr{E}\Gamma/\Gamma_0$ is a closed manifold. Since Γ acts smoothly on $\mathscr{E}\Gamma$, the finite group $G = \Gamma/\Gamma_0$ acts smoothly on $\mathscr{E}\Gamma/\Gamma_0$ and $\mathscr{E}\Gamma/\Gamma_0$ has a an equivariant cell decomposition. The lifting of this decomposition to $\mathscr{E}\Gamma$ endows $\mathscr{E}\Gamma$ with a finite-dimensional CW structure on which Γ acts cellularly. Thus (1), (2), (3) and (5) follow.

We now show that (4) holds. Let $\mathbb{H} = \mathscr{E}\Gamma$ and $G \subset \Gamma$ be a finite subgroup. By a theorem of *E.* Cartan [He; Theorem 13.5, p. 75 (*cf.* also p. 72)], $\mathbb{H}^G \neq \emptyset$. Since Γ acts by isometries, $\mathbb{H}^G$ is totally geodesic. Choose $x \in \mathbb{H}^G$. Then it follows from [Ch–E; Cor. 1.34, p. 37] that $\exp_x : T\mathbb{H}_x \to \mathbb{H}$ is a diffeomorphism which is G-equivariant. Hence $\mathbb{H}^G$ is homeomorphic to $(T\mathbb{H}_x)^G$ which is obviously contractible.

We recall that a group Γ has *finite virtual cohomological dimension* (vcd) if it has a finite index subgroup Γ_0 of finite cohomological dimension and that vcd Γ is defined to be the cohomological dimension of any such subgroup.

LEMMA 1.2. *Let Γ be a lattice-like group. Then* vcd $\Gamma = \dim \mathscr{E}\Gamma$ *and for any finite subgroup G of Γ,* vcd $Z_\Gamma(G) = \dim (\mathscr{E}\Gamma)^G$.

In this lemma, $Z_\Gamma(G)$ is the centralizer of G in Γ; that is $Z_\Gamma(G) = \{\gamma \in \Gamma \mid \gamma g \gamma^{-1} = g \text{ for all } g \in G\}$.

Proof. That vcd $\Gamma = \dim \mathscr{E}\Gamma$ follows trivially from 1.1(5). Let $Z_{\Gamma_0}(G) = Z_\Gamma(G) \cap \Gamma_0$. Then $Z_{\Gamma_0}(G)$ has finite index in $Z_\Gamma(G)$ and acts freely on $(\mathscr{E}\Gamma)^G$. Let $\eta: \Gamma/\Gamma_0$ be the natural homomorphism. By [C–K; Lemma 2.2], $(\mathscr{E}\Gamma)^G/Z_{\Gamma_0}(G)$ is a component of $(\mathscr{E}\Gamma/\Gamma_0)^{\eta(G)}$ under the action of Γ/Γ_0 on the closed manifold $\mathscr{E}\Gamma/\Gamma_0$. Hence it is also a closed manifold and vcd $Z_\Gamma(G) =$ vcd $Z_{\Gamma_0}(G) = \dim (\mathscr{E}\Gamma)^G$ as claimed.

LEMMA 1.3. *The space $\mathscr{E}\Gamma$ has a compactification $\overline{\mathscr{E}\Gamma} = \mathscr{E}\Gamma \cup \Sigma$ with the following properties*:

(1) *The action of Γ on $\mathscr{E}\Gamma$ extends to an action of Γ on $\overline{\mathscr{E}\Gamma}$*;

(2) *For any finite subgroup G of Γ, $\overline{\mathscr{E}\Gamma}$ is a G-homeomorphic to the unit disk in a k-dimensional representation of G*; *and*

(3) *For any compact set $C \subset \mathscr{E}\Gamma$, the sets $\{\gamma C \mid \gamma \in \Gamma\}$ get small near Σ.*

To say that the sets $\{\gamma C \mid \gamma \in \Gamma\}$ *get small near* Σ means that for any point $z \in \Sigma$ and any neighborhood U of z in $\overline{\mathscr{E}\Gamma}$, there is a neighborhood V of z such that if $\gamma C \cap V \neq \emptyset$, then $\gamma C \subset U$.

Proof. The proof given here is essentially that in [F–H; pp. 206–207].

Let $\mathbb{H} = \mathscr{E}\Gamma$ and fix a point $x \in \mathbb{H}$. Choose an orthogonal framing for $T\mathbb{H}_x$. This determines an identification fo $\mathbb{R}^k$ with $T\mathbb{H}_x$ for which the exponential map

$\exp_x: \mathbb{R}^k \to \mathbb{H}$ is a distance non-decreasing diffeomorphism by [Ch–E; Cor. 1.34, p. 37]. Let $\bar{\mathbb{H}}$ be the compactification of $\mathbb{H}$ obtained by adjoining an end point to each geodesic ray in $\mathbb{H}$ emanating from x. Under $\exp_x$ the geodesic rays correspond to linear rays so $\exp_x$ extends to a homeomorphism $\bar{e}_x: D^k \to \bar{\mathbb{H}}$ for which $i \exp_x = \bar{e}_x j$ where $j: \mathbb{R}^k \to D^k$ is a radial embedding. That (1) holds follows from the argument of [F–H; pp. 206–207]. The same argument also shows the following: If Γ_x is the isotropy subgroup of x under the Γ action on $\mathbb{H}$, then Γ_x is finite, $\mathbb{R}^k = T\mathbb{H}_x$ is a linear representation of Γ_x, and $\bar{e}_x : D(T\mathbb{H}_x) \to \bar{\mathbb{H}}$ is a Γ_x-equivariant homeomorphism. Since this observation is independent of the choice of $x \in \mathbb{H}$ and $\mathbb{H}^G \neq \emptyset$ for any finite subgroup of Γ, (2) follows.

Let C be a compact set in $\mathbb{H}$. Since Γ acts by isometries on $\mathbb{H}$ and $\exp_x$ is distance non-decreasing, the sets $\{\exp_x^{-1}(\gamma C)\}$ $(\gamma \in \Gamma)$ have uniformly bounded diameters in $\mathbb{R}^k$. Hence for every $z' \in S^{k-1}$ and every neighborhood U' of z', there is a neighborhood $V' \subset U'$ so that if $\exp_x^{-1}(\gamma C) \cap V' \neq \emptyset$, then $\gamma C' \subset U'$. Since $\bar{e}_x$ is a homeomorphism, (3) now follows.

A *space over* $\mathscr{E}\Gamma$ is a pair (X, p) where X is space and $p : X \to \mathscr{E}\Gamma$ is a continuous map. A map $f : (X, p) \to (Y, q)$ between spaces over $\mathscr{E}\Gamma$ is *continuously controlled at infinity* (or simply, *continuously controlled* or *cc*) if for every $z \in \Sigma$ and every neighborhood U of z in $\overline{\mathscr{E}\Gamma}$ there is a neighborhood V of z so that $fp^{-1}(V) \subset q^{-1}(U)$. If we set $(X, p) \times I = (X \times I, p\pi)$ where π is projection on the first factor, there is an obvious notion of a cc homotopy between cc maps $f, g : (X, p) \to (Y, q)$ and of f being a cc homotopy equivalence. If these maps and these homotopies are also Γ-equivariant, we say that f is a cc Γ-homotopy equivalence.

LEMMA 1.4. *Let (X, p) and (Y, q) be spaces over $\mathscr{E}\Gamma$. Suppose Γ acts on X and Y, that p and q are Γ-equivariant, and that $f : X \to Y$ is a Γ-equivariant map. If X/Γ is compact, then f is cc.*

Proof. Let D be a fundamental domain for $\mathscr{E}\Gamma$ (i.e. D is compact and $\mathscr{E}\Gamma = \bigcup\{\gamma D \mid \gamma \in \Gamma\}$) and consider $E = D \cup qfp^{-1}(D)$. Since p is proper, $p^{-1}(D)$ is compact. Hence so is E. Let $z \in \Sigma$ and U be a neighborhood of z in $\mathscr{E}\Gamma$. By 1.3(3) there is a neighborhood V of z so that if $\gamma E \cap V \neq \emptyset$ for some $\gamma \in \Gamma$, then $\gamma E \subset U$. Suppose now that $x \in p^{-1}(V)$. Since D is a fundamental domain for $\mathscr{E}\Gamma$, $x \in p^{-1}(\gamma D) \subset p^{-1}(\gamma E)$ for some γ. Hence $\gamma E \cap V \neq \emptyset$ and by the choice of V, $\gamma E \subset U$. Suppose $x = \gamma d$ with $d \in p^{-1}(D)$. Then $qf(x) = \gamma qf(d) \in \gamma qfp^{-1}(D) \subset \gamma E \subset U$ since q and f are Γ-equivariant. Thus $f(x) \in q^{-1}(U)$ as required.

A space (X, p) over $\mathscr{E}\Gamma$ for which X is a CW complex is called a *CW complex over* $\mathscr{E}\Gamma$. A CW complex (X, p) over $\mathscr{E}\Gamma$ is *continuously controlled at infinity* (or simply either *continuously controlled* or *cc*) if for every $z \in \Sigma$ and every neighborhood U of z in $\overline{\mathscr{E}\Gamma}$ there is a neighborhood V of z so that if e is a cell in X for which

$p(e) \cap V \neq \varnothing$, then $p(e) \subset U$. All CW complexes over $\mathscr{E}\Gamma$ considered in this paper are assumed to be cc. If p is proper, (X, p) is called a *finite cc CW complex* over $\mathscr{E}\Gamma$.

Let $\tilde{X}$ be any CW complex on which Γ acts freely and cellulary. Then there is an equivariant map $\tilde{J} : \tilde{X} \to \mathscr{E}\Gamma$, unique up to equivariant homotopy, that classifies this action (cf. [C–K]). If $\tilde{X}/\Gamma$ is compact, $(\tilde{X}, \tilde{J})$ is a finite cc CW complex over $\mathscr{E}\Gamma$. Let $\tilde{p}_2 : \tilde{X} \times \mathscr{E}\Gamma \to \mathscr{E}\Gamma$ be projection on the second factor. Then $(\tilde{X} \times \mathscr{E}\Gamma, \tilde{p}_2)$ is a (usually non-finite) cc CW complex over $\mathscr{E}\Gamma$.

PROPOSITION 1.5. *Let $\tilde{X}$ be a CW complex on which Γ acts freely, cellularly, and cocompactly. Suppose that for every finite subgroup $G \subset \Gamma$, $\tilde{X}/G$ is finitely dominated. Then $\tilde{\Psi} = (1, \tilde{J}) : (\tilde{X}, \tilde{J}) \to (\tilde{X} \times \mathscr{E}\Gamma, \tilde{p}_2)$ is a cc Γ-homotopy equivalence.*

Proof. Let Γ_0 be a torsion free subgroup of Γ, as in 1.1, for which there is a short exact sequence $1 \to \Gamma_0 \to \Gamma \xrightarrow{\rho} H \to 1$ with H a finite group and so that $\mathscr{E}\Gamma/\Gamma_0$ is a finite complex. We break the proof into two cases.

Case I. We assume that for every finite subgroup $G \subset \Gamma$, $\tilde{X}/G$ has the homotopy type of a finite complex. In this case consider the diagram

$$\begin{array}{ccc} \tilde{X}/\Gamma_0 & \xrightarrow{\Psi} & \tilde{X} \times_{\Gamma_0} \mathscr{E}\Gamma \\ \downarrow{\scriptstyle J} & & \downarrow{\scriptstyle p_2} \\ \mathscr{E}\Gamma/\Gamma_0 & \xrightarrow{1} & \mathscr{E}\Gamma/\Gamma_0 \end{array}$$

where Ψ, J and p_2, respectively, are induced by $\tilde{\Psi}$, $\tilde{J}$ and $\tilde{p}_2$, respectively. The map $p_2 : \tilde{X} \times_{\Gamma_0} \mathscr{E}\Gamma/\Gamma_0$ is an H-fibration. For any cell $\sigma \in \mathscr{E}\Gamma/\Gamma_0$, let $Y_\sigma = p_2^{-1}(\sigma)$. Then H_σ acts freely on Y_σ, where H_σ is the isotropy subgroup of σ. Let $\tilde{\sigma} \in \mathscr{E}\Gamma$ be a cell in $\mathscr{E}\Gamma$ having $p(\tilde{\sigma}) = \sigma$ where $p : \mathscr{E}\Gamma \to \mathscr{E}\Gamma/\Gamma_0$ is the orbit map and let $\Gamma_{\tilde{\sigma}}$ be its isotropy subgroup. The map ρ restricts to an isomorphism $\Gamma_{\tilde{\sigma}} \to H_\sigma$ and the map $\tilde{X} \times \mathscr{E}\Gamma \to \tilde{X} \times_{\Gamma_0} \mathscr{E}\Gamma$ restricts to a cellular, ρ-equivariant homeomorphism $\tilde{X} \times \tilde{\sigma} \to Y_\sigma$. Since $\tilde{X}/\Gamma_{\tilde{\sigma}}$ has the homotopy type of a finite complex by the assumption above, so does Y_σ/H_σ and Y_σ has the H_σ-homotopy type of a finite complex on which H_σ acts freely and cellularly.

By induction on the skeleta of $\mathscr{E}\Gamma/\Gamma_0$, we can now construct a diagram of H-maps

$$\begin{array}{ccc} \tilde{X} \times_{\Gamma_0} \mathscr{E}\Gamma & \xrightarrow{\Phi} & E \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle q} \\ \mathscr{E}\Gamma/\Gamma_0 & \xrightarrow{1} & \mathscr{E}\Gamma/\Gamma_0 \end{array}$$

with the following properties: E is a finite CW complex on which H acts freely and cellularly; $q : E \to \mathscr{E}\Gamma/\Gamma_0$ is a map so that for each cell $\sigma \in \mathscr{E}\Gamma/\Gamma_0$, the block $E_\sigma = q^{-1}(\sigma)$ is a finite H_σ-complex; $\Phi(Y_\sigma) \subset E_\sigma$ and $\Phi : Y_\sigma \to E_\sigma$ is an H_σ-homotopy equivalence. It follows that $\Phi : \tilde{X} \times_{\Gamma_0} \mathscr{E}\Gamma \to E$ is an H-equivariant block fibration homotopy equivalence. (That is Φ, its homotopy inverse, and the homotopies preserve the blocks Y_σ and E_σ for each σ.) Hence the lift $\tilde{\Phi} : (\tilde{X} \times \mathscr{E}\Gamma, \tilde{p}_2) \to (\tilde{E}, \tilde{q})$ of Φ to Γ_0-covers is a cc homotopy equivalence. Notice that since Φ is H-equivariant, $\tilde{\Phi}$ is Γ-equivariant. In addition, since $\Phi\Psi : \tilde{X}/\Gamma_0 \to E$ is an H-homotopy equivalence between finite CW complexes, $\tilde{\Phi}\tilde{\Psi} : (\tilde{X}, \tilde{J}) \to (\tilde{E}, \tilde{q})$ is also a cc Γ-homotopy equivalence by 1.4. Hence $\tilde{\Psi} : (\tilde{X}, \tilde{J}) \to (\tilde{X} \times \mathscr{E}\Gamma, \tilde{p}_2)$ is a cc Γ-homotopy equivalence as claimed.

Case II (The general case). Let $\tilde{Y} = \tilde{X} \times S^1$, Γ act on $\tilde{Y}$ by acting only on the first factor, and $\tilde{J}_Y = \tilde{J}_X pr$ where $\tilde{J}_X : \tilde{X} \to \mathscr{E}\Gamma$ is the classifying map for the Γ action on $\tilde{X}$ and $pr : \tilde{X} \times S^1 \to \tilde{X}$ is projection on the first factor. Then for each finite subgroup G of Γ, $\tilde{Y}/G = (\tilde{X}/G) \times S^1$ has the homotopy type of a finite complex. Hence $(1, \tilde{J}_Y) : (\tilde{Y}, \tilde{J}_Y) \to (\tilde{Y} \times \mathscr{E}\Gamma, \tilde{p}_2)$ is a cc Γ-homotopy equivalence by Case I. It follows that the induced map on infinite cyclic covers

$$(1, \tilde{J}_X pr) : (\tilde{X} \times \mathbb{R}^1, \tilde{J}_X pr) \to ((\tilde{X} \times \mathbb{R}^1) \times \mathscr{E}\Gamma, \tilde{p}_2)$$

is also a cc Γ-homotopy equivalence. Since the projections maps $(\tilde{X} \times \mathbb{R}^1, \tilde{J}_X pr) \to (\tilde{X}, \tilde{J}_X)$ and $((\tilde{X} \times \mathbb{R}^1) \times \mathscr{E}\Gamma, \tilde{p}_2) \to (\tilde{X} \times \mathscr{E}\Gamma, \tilde{p}_2)$ are obviously cc Γ-homotopy equivalences, it follows that $(1, \tilde{J}_X) : (\tilde{X}, \tilde{J}_X) \to (\tilde{X} \times \mathscr{E}\Gamma, \tilde{p}_2)$ is a cc Γ-homotopy equivalence. This completes the proof of 1.5.

2. Transfers, finiteness obstructions, and the proof of Theorem I

In this section we shall introduce the additive category $\mathscr{C}_G(\overline{\mathscr{E}\Gamma}, \Sigma; \mathbb{Z})$ of equivariant geometric modules over $\overline{\mathscr{E}\Gamma}$ with continuous control at infinity. This category is a variation on the category $\mathscr{C}_{\mathscr{E}\Gamma, G}(\mathbb{Z})$ considered by Hambleton and Pedersen in [H–P] and has the advantage of being a topological invariant of $\overline{\mathscr{E}\Gamma}$. In particular, it does not depend on the metric on $\mathscr{E}\Gamma$. This extra flexibility plays an important role in Section 5 and in particular, in Theorem 5.2. We use this category to define a transfer homomorphism $\mathrm{tr} : \tilde{K}_0(\mathbb{Z}G) \to \tilde{K}_0(\mathscr{C}_G(\overline{\mathscr{E}\Gamma}, \Sigma; \mathbb{Z}))$ and to prove Theorem 1 of the Introduction.

Let Γ be a group, G be a finite subgroup of Γ, and R be a commutative ring with unit. Let $\mathscr{G}_G(\overline{\mathscr{E}\Gamma}, \Sigma; R)$ be the additive category in which an object is a pair (M, p) where M is a left RG-module and $p : M \to \mathscr{P}_f(E)$ is an equivariant map where $\mathscr{P}_f(E)$ is the collection of finite subsets of E. It is required that (M, p) satisfy the following conditions:

(1) For each $x \in \mathscr{E}\Gamma$, $M_x = \{m \in M \mid f(m) \subseteq \{x\}\}$ is a finitely generated free R-submodule of M;
(2) As R-modules $M = \bigoplus M_x$;
(3) $p(m + m') \subseteq p(m) \cup p(m')$ for all $m, m' \in M$;
(4) For each compact set $K \subset E$, $\{x \in K \mid M_x \neq 0\}$ is finite.

A morphism $f : (M, p) \to (N, q)$ is an RG-homomorphism $f : M \to N$ with the following property: If we write f as a family of R-module homomorphisms $\{f_y^x : M_x \to N_y \mid x, y \in \mathscr{E}\Gamma\}$, then for every point $z \in \Sigma$ and every neighborhood U of z, there is a neighborhood V of z with $V \subset U$ and so that if $x \in V$ and $f_y^x \neq 0$, then $y \in U$. We describe this by saying that f is *continuously controlled at infinity*. The operation $(M, p) \oplus (N, q) = (M \oplus N, p \oplus q)$ makes $\mathscr{G}_G(\overline{\mathscr{E}\Gamma}, \Sigma; R)$ into an additive category.

EXAMPLE 2.1. Let $C_n(\mathscr{E}\Gamma; R)$ be the n-th cellular chain group of $\mathscr{E}\Gamma$ with coefficients in R and G be a finite subgroup of Γ. Let $\{e_i \mid i \in I\}$ be a family of representatives for the G-orbits of n-cells in $\mathscr{E}\Gamma$. If $x \in C_n(\mathscr{E}\Gamma; R)$, then $x = \Sigma\, r_{g,i} g e_i$ where the $r_{g,i}$ are elements of R, only finitely many of which are non-zero, and $g \in G$. Define $p : C_n(\mathscr{E}\Gamma; R) \to \mathscr{P}_f(\mathscr{E}\Gamma)$ by setting $p(\Sigma\, r_{g,i} g e_i) = \{g x_i \mid r_{g,i} \neq 0\}$ where x_i is any point of e_i $(i \in I)$. Then $(C_n(\mathscr{E}\Gamma; R), p)$ is an object in $\mathscr{G}_G(\overline{\mathscr{E}\Gamma}, \Sigma; R)$. Although p depends on the choice of the x_i, it follows from 1.3(3) that any two such choices yield isomorphic objects in $\mathscr{G}_G(\overline{\mathscr{E}\Gamma}, \Sigma; R)$.

Let $\mathscr{C}_G(\overline{\mathscr{E}\Gamma}, \Sigma; R)$ be the full subcategory of $\mathscr{G}_G(\overline{\mathscr{E}\Gamma}, \Sigma; R)$ with objects the pairs (M, p) for which M is a free RG-module and M_x is a free RG_x-module for each $x \in \mathscr{E}\Gamma$, where G_x is the isotropy subgroup of x.

In this case when $\mathscr{E}\Gamma$ is a metric space on which G acts by eventually Lipschitz maps, the categories $\mathscr{G}_G(\overline{\mathscr{E}\Gamma}, \Sigma; R)$ and $\mathscr{C}_G(\overline{\mathscr{E}\Gamma}, \Sigma; R)$ are very similar to the categories $\mathscr{G}_{\mathscr{E}\Gamma, G}(R)$ and $\mathscr{C}_{\mathscr{E}\Gamma, G}(R)$ respectively, of [H–P]. In many cases $\mathscr{C}_{\mathscr{E}\Gamma, G}(R)$ is actually a subcategory $\mathscr{C}_G(\overline{\mathscr{E}\Gamma}, \Sigma; R)$ with the only difference being that the hom sets in $\mathscr{C}_G(\overline{\mathscr{E}\Gamma}, \Sigma; R)$ are bigger than those of $\mathscr{C}_{\mathscr{E}\Gamma, G}(R)$.

Let $\mathscr{C}(RG)$ be the category of finitely generated, free RG-modules. The tensor product over R defines a bifunctor

$$\otimes : \mathscr{C}(RG) \times \mathscr{G}_G(\overline{\mathscr{E}\Gamma}, \Sigma; R) \to \mathscr{C}_G(\overline{\mathscr{E}\Gamma}, \Sigma; R).$$

More concretely $\otimes$ sends $(F, (M, p))$ to $(F \otimes M, q)$ where $\otimes = \otimes_R$ and q is such that $(F \otimes M)_x = F \otimes M_x$. Since $F \otimes M_x$ is a finitely generated, free RG-module for every $x \in \mathscr{E}\Gamma$, $(F \otimes M, q)$ is in $\mathscr{C}_G(\overline{\mathscr{E}\Gamma}, \Sigma; R)$. Then $\otimes$ extends to a bifunctor

$$\otimes : \hat{\mathscr{C}}(RG) \times \mathscr{G}_G(\overline{\mathscr{E}\Gamma}, \Sigma; R) \to \hat{\mathscr{C}}_G(\overline{\mathscr{E}\Gamma}, \Sigma; R)$$

where $\hat{\mathscr{A}}$ denotes the idempotent completion of $\mathscr{A}$ (cf. [Fr]). Using this bifunctor, we define a pairing

$$\tilde{K}_0(RG) \otimes K_0(\mathscr{G}_G(\overline{\mathscr{E}\Gamma}, \Sigma; R)) \to \tilde{K}_0(\mathscr{C}_G(\overline{\mathscr{E}\Gamma}, \Sigma; R)),$$

denoted by juxtaposition, by setting $[P][M, p] = [P \otimes M, q]$. Here $P \in \hat{\mathscr{C}}(RG)$ (which is isomorphic to the category of finitely generated projective RG-modules), $(M, p) \in \mathscr{G}_G(\overline{\mathscr{E}\Gamma}, \Sigma; R)$, and for any additive category $\mathscr{A}$, $\tilde{K}_0(\mathscr{A})$ is the cokernel of the obvious homomorphism $K_0(\mathscr{A}) \to K_0(\hat{\mathscr{A}})$. A pairing similar to this one was first investigated in [An].

Let $\chi(\mathscr{E}\Gamma; R) = \Sigma\,(-1)^n[C_n(\mathscr{E}\Gamma; R)] \in K_0(\mathscr{G}_G(\overline{\mathscr{E}\Gamma}, \Sigma; R))$ and define

$$\mathrm{tr} : \tilde{K}_0(RG) \to \tilde{K}_0(\mathscr{C}_G(\overline{\mathscr{E}\Gamma}, \Sigma; R))$$

by setting $\mathrm{tr}\,(y) = y\chi(\mathscr{E}\Gamma; R)$.

We now recall some results of Ranicki [Ran]. Let $\mathscr{F}$ be an additive category, $\mathscr{C}$ be a full subcategory, and $F_* = \{F_n, \partial_n \mid n \geq 0\}$ be a chain complex whose objects are in $\mathscr{F}$. We shall say that F_* is *$\mathscr{C}$-finitely dominated* if there exist a chain complex $C_* = \{C_n, \partial_n\}$ of objects in $\mathscr{C}$, only finitely many of which are nonzero, chain maps $r : F_* \to C_*$ and $i : C_* \to F_*$ in $\mathscr{F}$, and a chain homotopy $h : 1 \simeq ir$ in $\mathscr{F}$. If F_* is $\mathscr{C}$-finitely dominated, Ranicki shows there is chain complex $P_* = \{P_n, \partial_n\}$ in $\hat{\mathscr{C}}$ having $P_n = 0$ for all $n < 0$ and all n sufficiently large and a chain equivalence $P_* \to \hat{F}_*$ in $\hat{\mathscr{F}}$ where $\hat{F}_*$ is the chain complex image of F_* in $\hat{\mathscr{F}}$, i.e. the chain complex $\{(F_n, 1), (\partial_n, 1)\}$. He also shows that $\sigma(F_*) = \Sigma\,(-1)^n[P_n] \in \tilde{K}(\hat{\mathscr{C}})$ is a well defined invariant of F_* that depends only on the chain homotopy type of F_*. This invariant is called the *$\mathscr{C}$-finiteness obstruction* of F_* and vanishes if and only if F_* is $\mathscr{F}$-chain equivalent to a finite chain complex in $\mathscr{C}$.

Let $\mathscr{F}_G(\overline{\mathscr{E}\Gamma}, \Sigma; R)$ be the category containing $\mathscr{C}_G(\overline{\mathscr{E}\Gamma}, \Sigma; R)$ whose objects are pairs (M, p) with M a free RG-module, p a function satisfying conditions (1)–(3) above, and having M_x a free RG_x-submodule for each $x \in \mathscr{E}\Gamma$; and whose morphisms $f = \{f^x_y\}$ are those that are continuously controlled at infinity.

EXAMPLE 2.2. Let (Y, q) be a cc CW complex (respectively, finite cc CW complex) over $\mathscr{E}\Gamma$ on which Γ acts cellularly and freely and for which q is Γ-equivariant. Then for any finite subgroup G of Γ, the cellular chain complex $C_*(Y; R)$ can be regarded as a chain complex in $\mathscr{F}_G(\overline{\mathscr{E}\Gamma}, \Sigma; R)$ (respectively, $\mathscr{C}_G(\overline{\mathscr{E}\Gamma}, \Sigma; R)$). A function $q_n : C_n(Y; R) \to \mathscr{P}_f(\mathscr{E}\Gamma)$ can be defined in a manner similar to that of 2.1. Namely, one chooses a family $\{e_i \mid i \in I\}$ of representatives for the G-orbits of n-cells in Y and a point $x_i \in e_i$. If $x \in C_n(Y; R)$, then $x = \Sigma\, r_{g,i} g e_i$ where the $r_{g,i}$ are elements of R, only finitely many of which are nonzero. Then $q_n(\Sigma\, r_{g,i} g e_i) = \{gq(x_i) \mid n_{g,i} \neq 0\}$.

Proof of Theorem I. By 2.2, $C_*(\tilde{X} \times \mathscr{E}\Gamma; \mathbb{Z})$ is a chain complex of objects in $\mathscr{F}_G(\overline{\mathscr{E}\Gamma}, \Sigma; \mathbb{Z})$. We show that this chain complex is $\mathscr{C}_G(\overline{\mathscr{E}\Gamma}, \Sigma; \mathbb{Z})$ -finitely dominated and evaluate its $\mathscr{C}_G(\overline{\mathscr{E}\Gamma}, \Sigma; \mathbb{Z})$-finiteness obstruction in two ways. Let $C_*(\tilde{X})$ be the cellular chain complex of $\tilde{X}$ regarded as a chain complex of free $\mathbb{Z}G$-modules and $C_*(\mathscr{E}\Gamma; \mathbb{Z})$ be the chain complex of 2.1. Then $C_*(\tilde{X} \times \mathscr{E}\Gamma; \mathbb{Z})$ is $\mathscr{F}_G(\overline{\mathscr{E}\Gamma}, \Sigma; \mathbb{Z})$-chain equivalent to $C_*(\tilde{X}) \otimes C_*(\mathscr{E}\Gamma; \mathbb{Z})$. Since $\tilde{X}/G$ is a finitely dominated CW complex, $C_*(\tilde{X})$ is $\mathbb{Z}G$-finitely dominated, say by C_*. Then $C_*(\tilde{X}) \otimes C_*(\mathscr{E}\Gamma; \mathbb{Z})$ is $\mathscr{C}_G(\overline{\mathscr{E}\Gamma}, \Sigma; \mathbb{Z})$-finitely dominated by $C_* \otimes C_*(\mathscr{E}\Gamma; \mathbb{Z})$. Let $P_* = \{P_n, \partial_n\}$ a chain complex of finite length with P_n in $\hat{\mathscr{C}}(RG)$ for which there is a $\mathbb{Z}G$-chain equivalence $f: P_* \to C_*(\tilde{X})$. Then

$$f \otimes 1 : P_* \otimes C_*(\mathscr{E}\Gamma; \mathbb{Z}) \to C_*(\tilde{X}) \otimes C_*(\mathscr{E}\Gamma; \mathbb{Z})$$

is a $\mathscr{C}_G(\overline{\mathscr{E}\Gamma}, \Sigma; \mathbb{Z})$-chain equivalence. Since $\sigma_G(\tilde{X}/G) = \Sigma\,(-1)^n[P_n]$, a straightforward calculation shows that in $\tilde{K}_0(\mathscr{C}_G(\overline{\mathscr{E}\Gamma}, \Sigma; \mathbb{Z}))$ we have

$$\sigma(C_*(\tilde{X} \times \mathscr{E}\Gamma; \mathbb{Z})) = \sigma(C_*(\tilde{X}) \otimes C_*(\mathscr{E}\Gamma; \mathbb{Z})) = \sigma(P_* \otimes C_*(\mathscr{E}\Gamma; \mathbb{Z})) = \operatorname{tr} \sigma_G(\tilde{X}/G).$$

On the other hand, since $(\tilde{X}, \tilde{J})$ is a finite cc CW complex over $\mathscr{E}\Gamma$ on which Γ acts freely and $\tilde{J}$ is Γ-equivariant, $C_*(\tilde{X}; \mathbb{Z})$ is a $\mathscr{C}_G(\overline{\mathscr{E}\Gamma}, \Sigma; \mathbb{Z})$-chain complex by 2.2. Since $\tilde{\Psi} = (1, \tilde{J}) : (\tilde{X}, \tilde{J}) \to (\tilde{X} \times \mathscr{E}\Gamma, \tilde{p}_2)$ is a cc homotopy equivalence by 1.5, $\tilde{\Psi}$ induces a $\mathscr{C}_G(\overline{\mathscr{E}\Gamma}, \Sigma; \mathbb{Z})$-chain equivalence $C_*(\tilde{X} \times \mathscr{E}\Gamma; \mathbb{Z}) \approx C_*(\tilde{X}; \mathbb{Z})$. Hence

$$\sigma(C_*(\tilde{X} \times \mathscr{E}\Gamma; \mathbb{Z})) = \sigma(C_*(\tilde{X}; \mathbb{Z})) = 0$$

since the latter chain complex is a finite $\mathscr{C}_G(\overline{\mathscr{E}\Gamma}, \Sigma; \mathbb{Z})$-chain complex. By combining this with the equation above, we see that $\operatorname{tr} \sigma_G(\tilde{X}/G) = 0$. This completes the proof of Theorem 1.

3. An equivariant reduced homology theory

The section begins the development of the algebraic tools needed to prove Theorem II. The reader who is most interested in the proof of Theorem II can proceed directly to Section 5, referring back to this section and the next only as the need arises.

The main result in this section is Theorem 3.4, which describes a reduced homology theory defined on the category $\mathscr{CW}_G$ of finite G-CW complexes for G a finite group. We begin by generalizing some of the discussion of Section 2.

Let $\mathscr{E}$ be the category with objects pairs $(\bar{E}, \Sigma)$ with $\bar{E}$ a compact Hausdorff space and Σ a closed subset. Let $E = \bar{E} - \Sigma$. A morphism in $\mathscr{E}$ is a set-theoretic function $f : (\bar{E}_1, \Sigma_1) \to (\bar{E}_2, \Sigma_2)$ which is continuous at every point of Σ_1, has $f^{-1}(\Sigma_2) = \Sigma_1$, and for which $f \mid E_1 \to E_2$ is proper in the sense that if $K \subset E_2$ is compact, then $f^{-1}(K)$ has compact closure in E_1. Here $E_i = \bar{E}_i - \Sigma_i$.

Let G be a group and $\mathscr{E}_G$ be the category with objects pairs $((\bar{E}, \Sigma), \rho)$ where $(\bar{E}, \Sigma)$ is an object in $\mathscr{E}$ and $\rho : G \to \mathrm{Aut}_{\mathscr{E}}(\bar{E}, \Sigma)$ is a homomorphism where $\mathrm{Aut}_{\mathscr{E}}(\bar{E}, \Sigma)$ is the group of automorphisms of $(\bar{E}, \Sigma)$ in $\mathscr{E}$. Thus an object in $\mathscr{E}_G$ is an object of $\mathscr{E}$ equipped with an action of G via automorphisms in $\mathscr{E}$. A morphism in $\mathscr{E}_G$ is a G-equivariant $\mathscr{E}$-morphism $f : (\bar{E}_1, \Sigma_1) \to (\bar{E}_2, \Sigma_2)$.

EXAMPLE 3.1. If G is finite, we let $\mathscr{CW}_G$ be the category of finite CW complexes equipped with a cellular action of G. Taking the closed cone defines a functor $c : \mathscr{CW}_G \to \mathscr{E}_G$ that sends X to (cX, X). Here cX is the quotient space $X \times [0, 1]/X \times \{0\}$ and X is the image of $X \times \{1\}$ in cX. Such pairs (cX, X) give many examples of objects in $\mathscr{E}_G$.

An additive category $\mathscr{C}_G(\bar{E}, \Sigma; R)$ analogous to $\mathscr{C}_G(\overline{\mathscr{E}\Gamma}, \Sigma; R)$ can be defined for any pair $(\bar{E}, \Sigma) \in \mathscr{E}_G$. An object is a pair (M, p) where M is a free RG-module, $p : M \to \mathscr{P}_f(E)$ is a G-equivariant map where $\mathscr{P}_f(E)$ is the set of finite subsets of E, and (M, p) satsifies conditions (1)–(4) of Section 2 and has M_x a free RG_x-module for each $x \in E$. The morphisms are defined as in Section 2 except that $\overline{\mathscr{E}\Gamma}$ is replaced by $\bar{E}$.

REMARK 3.2. If G acts trivially on the pair $(\bar{E}, \Sigma)$ then the functor $J : \mathscr{C}_G(\bar{E}, \Sigma; R) \to \mathscr{C}(\bar{E}, \Sigma; RG)$ that sends (M, p) to $J(M) = \{M_x \mid x \in E\}$ is an isomorphism of categories. Here $\mathscr{C}(\bar{E}, \Sigma; RG)$ is the category of geometric RG-modules on $(\bar{E}, \Sigma)$ with continuous control at infinity introduced in [A–C–F–P]. Notice that if $(\bar{E}, \Sigma) = (c\varnothing, \varnothing)$, then $\mathscr{C}(\bar{E}, \Sigma; RG) = \mathscr{C}(RG)$ is the category of finitely generated free RG-modules.

Following the notation in [A–C–F–P], for any category $\mathscr{A}$, we let $\hat{\mathscr{A}}$ denote its idempotent completion [Fr] and if $\mathscr{A}$ is additive, we let $\mathbb{K}\mathscr{A}$ be the classifying space $B\underline{\underline{A}}^{-1}\underline{\underline{A}}$ where $\underline{\underline{A}} = \mathrm{Iso}\,(\mathscr{A})$ is the category of isomorphisms in $\mathscr{A}$ and $\underline{\underline{A}}^{-1}\underline{\underline{A}}$ is the category of [Gr].

For any object $X \in \mathscr{CW}_G$, let $\mathbb{K}_n^G(X; R) = \Omega\mathbb{K}\hat{\mathscr{C}}_G(cS^{n+1}X, S^{n+1}X; R)$ where S^{n+1} is the $(n+1)^{\text{st}}$ (unreduced) suspension of X ($n \geq 0$). Let $\tilde{\mathbf{K}}^G(X; R) = \{\mathbb{K}_n^G(X; R) \mid n \geq 0\}$.

LEMMA 3.3. *For any space* $X \in \mathscr{CW}_G$, *the sequence of spaces* $\tilde{\mathbf{K}}^G(X; R)$ *has the structure of an* Ω*-spectrum; that is, there are homotopy equivalences* $\varepsilon_n : \mathbb{K}_n^G(X; R) \to \Omega\mathbb{K}_{n+1}^G(X; R)$. *Furthermore the correspondence* $X \mapsto \tilde{\mathbf{K}}^G\ (X; R)$ *defines a functor* $\tilde{\mathbf{K}}^G(\_\,; R)$ *from* $\mathscr{CW}_G$ *to the category of* Ω*-spectra.*

The category of Ω-spectra has objects Ω-spectra and morphisms $f : \{A_n, \alpha_n\} \to \{B_n, \beta_n\}$ families of maps $\{f_n : A_n \to B_n \mid n \geq 0\}$ with $f_{n+1}\alpha_n$ pointed homotopic to $\beta_n f_n$. For any Ω-spectrum $\mathbf{A} = \{A_n, \alpha_n\}$, let $A_* = \pi_*^S \mathbf{A}$ be the stable homotopy of $\mathbb{A}$. The functor A_* takes values in $\mathscr{GA}$ the category of graded abelian groups.

THEOREM 3.4. *The functor* $\tilde{K}_*^G(\_\,; R) = \pi_*^S \tilde{\mathbf{K}}^G(\_\,; R) : \mathscr{CW}_G \to \mathscr{GA}$ *is an equivariant reduced homology theory on the category* $\mathscr{CW}_G$. *That is,*

(i) $\tilde{K}_*^G(\_\,; R)$ *is a homotopy functor.*

(ii) *For any pair* (X, A) *in* $\mathscr{CW}_G$, *there is an exact sequence*

$$\cdots \to \tilde{K}_*^G(A; R) \to \tilde{K}_*^G(X; R) \to \tilde{K}_*^G(X \cup vA; R) \to \tilde{K}_*^G(A; R) \to \cdots.$$

(iii) $\tilde{K}_*^G(pt; R) = 0$.

REMARK 3.5. If G acts trivially on the CW complex X, the functor J of 3.2 induces a natural isomorphism $\imath_* : \tilde{K}_*^G(X; R) \to \tilde{K}_*(X; RG)$ where $\tilde{K}_*(X; RG)$ is the homology theory of [A–C–F–P]. In particular, if $X = \varnothing$, there are isomorphisms $\imath_* : \tilde{K}_*^G(\varnothing; R) \to \tilde{K}_*(\varnothing; RG)$ and $\eta : \tilde{K}_*(\varnothing; RG) \to K_*(RG)$ where η is induced by the isomorphism of categories $J : \mathscr{C}(c\varnothing, \varnothing; RG) \to \mathscr{C}(RG)$ of 3.2.

If (X, A) is a pair in $\mathscr{CW}_G$, let $K_*^G(X, A; R) = \tilde{K}_*^G(X \cup vA; R)$ where the union is over A and vA is a disjoint basepoint if $A = \varnothing$.

COROLLARY 3.6. *The functor* $(X, A) \mapsto K_*^G(X, A; R)$ *is an equivariant homology theory on* $\mathscr{CW}_G$.

Proof. This is an elementary formal consequence of 3.4.

Proofs of 3.3 *and* 3.4. An examination of the proofs of the corresponding statements in [A–C–F–P; Section 4] shows that 3.3 and 3.4 will follow once we have proved the following propositions:

PROPOSITION 3.7. *Let G be a finite group,* $\Sigma \in \mathscr{CW}_G$ and $v\Sigma$ *be the closed cone on* Σ *with vertex* v. *Then* $\mathbb{K}\hat{\mathscr{C}}_G(c(v\Sigma), v\Sigma; R)$ *is contractible.*

PROPOSITION 3.8. *Let X be the pushout of the diagram $X_1 \leftarrow X_0 \rightarrow X_2$ of inclusions of spaces in $\mathscr{C}\mathscr{W}_G$. Then the square*

$$\begin{array}{ccc}
\mathbb{K}\hat{\mathscr{C}}_G(cX_0, X_0; R) & \xrightarrow{i_1} & \mathbb{K}\tilde{\mathscr{C}}_G(cX_1, X_1; R) \\
\Big\downarrow{\scriptstyle i_2} & & \Big\downarrow{\scriptstyle j_1} \\
\mathbb{K}\tilde{\mathscr{C}}_G(cX_2, X_2; R) & \xrightarrow{j_2} & \mathbb{K}\tilde{\mathscr{C}}_G(cX, X; R)
\end{array}$$

is a pullback up to weak homotopy. Hence there is a long exact sequence

$$\cdots \to \pi_{n+1}(\mathbb{K}\mathscr{C}_G) \to \pi_n(\mathbb{K}\mathscr{C}_{G0}) \to \pi_n(\mathbb{K}\mathscr{C}_{G1}) \oplus \pi_n(\mathbb{K}\mathscr{C}_{G2}) \to \pi_n(\mathbb{K}\mathscr{C}_G) \to \cdots$$

($n \geq 0$) where $\mathbb{K}\mathscr{C}_{Gi} = \mathbb{K}\tilde{\mathscr{C}}_G(cX_i, X_i; R)$ ($i = 1, 2, \varnothing$) and $\mathbb{K}\mathscr{C}_{G0} = \mathbb{K}\hat{\mathscr{C}}_G(cX_0, X_0; R)$.

In this proposition, $\tilde{\mathscr{C}}_G(cX_i, X_i; R)$ ($i = 1, 2, \varnothing$) is the idempotent semicompletion of $\mathscr{C}_G(cX_i, X_i; R)$ with respect to $\mathscr{C}_G(cX_0, X_0; R)$. We recall that if $\mathscr{B}$ is a full, additive subcategory of the additive category $\mathscr{A}$, then the *idempotent semicompletion* (or simply, *semicompletion*) *of $\mathscr{A}$ with respect to $\mathscr{B}$* is the full, additive subcategory of $\hat{\mathscr{A}}$ containing those objects (A, p) isomorphic to $(B, q) \oplus (C, 1)$ with $(B, q) \in \hat{\mathscr{B}}$ and $C \in \mathfrak{U}$. The semicompletion is denoted by $\tilde{\mathscr{A}}$.

Proposition 3.8 is a consequence of the following stronger result:

PROPOSITION 3.9. *Let $(\bar{E}_1, \Sigma_1) \xleftarrow{i_1} (\bar{E}_0, \Sigma_0) \xrightarrow{i_2} (\bar{E}_2, \Sigma_2)$ be a diagram of inclusions in $\mathscr{E}_G$ and let $(\bar{E}, \Sigma)$ be its pushout. If Σ_0 is an eventual G-neighborhood retract in $\bar{E}$, then the square*

$$\begin{array}{ccc}
\mathbb{K}\hat{\mathscr{C}}_G(\bar{E}_0, \Sigma_0; \mathfrak{U}) & \xrightarrow{i_1} & \mathbb{K}\tilde{\mathscr{C}}_G(\bar{E}_1, \Sigma_1; \mathfrak{U}) \\
\Big\downarrow{\scriptstyle i_2} & & \Big\downarrow{\scriptstyle j_1} \\
\mathbb{K}\hat{\mathscr{C}}_G(\bar{E}_2, \Sigma_2; \mathfrak{U}) & \xrightarrow{j_2} & \mathbb{K}\tilde{\mathscr{C}}_G(\bar{E}, \Sigma; \mathfrak{U})
\end{array}$$

is a pullback up to weak homotopy. Here the semicompletions are with respect to $\mathscr{C}_G(\bar{E}_0, \Sigma_0; \mathfrak{U})$. Hence there is a long exact sequence

$$\cdots \to \pi_{n+1}(\mathbb{K}\mathscr{C}_G) \to \pi_n(\mathbb{K}\mathscr{C}_{G0}) \to \pi_n(\mathbb{K}\mathscr{C}_{G1}) \oplus \pi_n(\mathbb{K}\mathscr{C}_{G2}) \to \pi_n(\mathbb{K}\mathscr{C}_G) \to \cdots$$

($n \geq 0$) where $\mathbb{K}\mathscr{C}_{Gi} = \mathbb{K}\tilde{\mathscr{C}}_G(\bar{E}_i, \Sigma_i; \mathfrak{U})$ ($i = 1, 2, \varnothing$) and $\mathbb{K}\mathscr{C}_{G0} = \mathbb{K}\hat{\mathscr{C}}_G(\bar{E}_0, \Sigma_0; \mathfrak{U})$.

The requirement that Σ_0 be an *eventual G-neighborhood retract* in $\bar{E}$ means that there is a G-invariant neighborhood N of Σ_0 in $\bar{E}$ and a G-equivariant function $r : (N - \Sigma) \cup \bar{E}_0 \to \bar{E}_0$ with $r^{-1}(\Sigma_0) = \Sigma_0$ and $r \mid \Sigma_0 = \mathrm{id}$ and with the property that if W is a G-invariant open neighborhood of $\Sigma - \Sigma_0$ in $\bar{E} - \bar{E}_0$, $r_W = r \mid$: $((N - \Sigma - W) \cup \bar{E}_0, \Sigma_0) \to (\bar{E}_0, \Sigma_0)$, and $i_W : (\bar{E}_0, \Sigma_0) \to ((N - \Sigma - W) \cup \bar{E}_0, \Sigma_0)$ is the inclusion, then for every point p in Σ_0 and every G-invariant neighborhood U of p in $(N - \Sigma - W) \cup \bar{E}_0$, there is a G-invariant neighborhood V of p in $(N - \Sigma - W) \cup \bar{E}_0$ with $(i_W r_W)^{-1}(V) \subset U$. The map r is called an *eventual G-neighborhood retraction.* An obvious extension of the argument used to prove [A–C–F–P; Lemma 1.6] shows that if Σ_0 is a G-neighborhood retract in Σ, then Σ_0 is an eventual G-neighborhood retract in $c\Sigma$.

Proof of 3.7. The proof is identical with the proof of [A–C–F–P; Corollary 2.5]. The ingredients of that proof are [A–C–F–P; Theorem 2.3] and the fact that $\mathscr{C}_{\mathcal{O}(v\Sigma)}(R)$ is flasque. Here $\mathscr{C}_{\mathcal{O}(v\Sigma)}(R)$ is the category of geometric R-modules over $\mathcal{O}(v\Sigma)$ and bounded morphisms defined in [P–W1] and $\mathcal{O}(v\Sigma) = \Sigma \times [0, \infty)/\Sigma \times \{0\}$ is the large open cone on $v\Sigma$.

Let $X \in \mathscr{C}\mathscr{W}_G$ and observe that the cone action on cX repsects the radial direction. Since the proof of [A–C–F–P; Theorem 2.3] uses only "annular" regions, this observation shows that Theorem 2.3 carries over to the present equivariant setting without change. It is also easy to see that the proof that $\mathscr{C}_{\mathcal{O}(v\Sigma)}(R)$ is flasque given in [P–W2; Lemma 4.1] respects the G-equivariance and shows that the equivariant analogue $\mathscr{C}_{\mathcal{O}(v\Sigma),G}(R)$ of $\mathscr{C}_{\mathcal{O}(v\Sigma)}(R)$ is also flasque. Thus the proof of [A–C–F–P; Corollary 2.5] holds in the present setting and proves 3.7.

Proof of 3.8. This follows from essentially the argument given in [A–C–F–P; Sections 3 and 4]. The only change needed to adapt that argument to the present equivariant setting is to require that the neighborhoods U that appear in the partially ordered set Λ of [A–C–F–P; Lemma 3.3] be G-invariant.

Proof of 3.9. This also follows from the argument of [A–C–F–P; Section 3] modified by the same changes as were needed in the proof of 3.8.

4. A spectral sequence

This section derives a spectral sequence that will be used in the next section. The main result is Theorem 4.2. Before stating it, we recall some definitions and notations.

Let G be a finite group. Let $\mathcal{O}_G$ be the category of canonical orbits of G; i.e. the category of all G-sets of the form G/H where H is a subgroup of G. A *coefficient system* is a covariant functor from $\mathcal{O}_G$ to the category of abelian groups. Such systems are called *simple coefficient systems* in [Br].

Let X be a GW complex. Following [Br], we define the reduced Bredon homology of X with coefficient system F as follows: Let C_p be the discrete G-set of p-cells of X. Let $A_p = \Sigma\, F(G/G_\sigma)$ where the sum runs over $\sigma \in C_p$ and write elements of A_p in the form $\sigma_1 x_1 + \cdots + \sigma_k x_k$ with $x_i \in F(G/G_\sigma)$. Let B_p be the subgroup of A_p generated by all elements of the form $(g\sigma)x - \sigma(g^{-1}x)$. The quotient group $A_p/B_p = C_p^G(X; F)$ is called the *group of equivariant p-chains on X*. The boundary map $\partial_p : C_p^G(X; F) \to C_{p-1}^G(X; F)$ sends the chain σx to $\Sigma\, \tau[\tau : \sigma]F_\tau^\sigma(x)$ where $[\tau : \sigma]$ is the incidence number and $F_\tau^\sigma = F(G/G_\sigma \to G/G_\tau)$. Notice that there is also an augmentation ∂_0: $C_0^G(X; F) \to F(G/G)$ that sends σx to $F(G/G_\sigma \to G/G)(x)$. We set $C_{-1}^G(X; F) = F(G/G)$. The homology of the augmented chain complex $\{C_p^G(X; F); \partial_p\}$ is called the *reduced Bredon homology of X* and is denoted $\tilde{H}_*^G(X; F)$.

REMARK 4.1. If $P > 0$, then $\tilde{H}_P^G(X; F) = H_p^G(X; F)$ is the usual Bredon homology of X. In addition, $\tilde{H}_{-1}^G(X; F) = F(G/G)/\mathrm{im}\,\mathrm{Ind}$ where Ind is the sum of the homomorphisms $F(G/G_\sigma \to G/G)$ over the set of 0-cells of X. In general this group is non-zero.

THEOREM 4.2. *Let $X \in \mathscr{CW}_G$ and $\{G_\sigma \mid \sigma \in X\}$ be the family of isotropy subgroups of the cells σ* in *X. Then there is a spectral sequence $\{E^r, d^r\}$ converging to $\tilde{K}_*^G(X; R)$ with $E_{pq}^2 = \tilde{H}_p^G(X; K_{q-1}(RG_\sigma))$ for all p and q. Here $K_{q-1}(RG_\sigma)$ is coefficient system that sends G/H to $K_{q-1}(RH)$.*

Proof. To simplify notation in this proof, we shall suppress mention of the ring R and will write $\tilde{K}_*^G(X)$ instead of $\tilde{K}_*^G(X; R)$.

Suppose $\dim X = n$ and consider the exact couple obtained by applying $\tilde{K}_*^G$ to the filtration

$$\varnothing \subset X^{(0)} \subset \cdots \subset X^{(p-1)} \subset X^{(p)} \subset \cdots \subset X^{(n)} = X.$$

For $p \geq 0$, this exact couple has $E_{p,q}^1 = \tilde{K}_{p+q}^G(X^{(p)} \cup vX^{(p-1)})$ and $D_{p,q}^1 = \tilde{K}_{p+q}^G(X^{(p)})$. If $p = -1$, $D_{p,q}^1 = \tilde{K}_{q-1}^G(\varnothing)$. Since $\tilde{K}_{q-1}^G(\varnothing) = K_{q-2}(RG) \neq 0$, to get a convergent spectral sequence we redefine this exact couple by setting $E_{-1,q}^1 = \tilde{K}_{q-1}^G(\varnothing) = K_{q-2}(RG)$, $D_{p,q}^1 = 0$ if $p \leq -2$, and letting $D_{p,q}^1 \to E_{p,q}^1$ be the identity if $p \leq -1$. Then $\{E^r, d^r\}$ is the spectral sequence associated with this exact couple.

The careful examination of the homology version of the argument in [Br; p. IV. 6ff] shows that the E^2-term of this spectral sequence is $\tilde{H}_p^G(X; \tilde{K}_q^G)$ where $\tilde{K}_q^G : \mathcal{O}_G \to \mathscr{AB}$ is the coefficient system that sends G/H to $\tilde{K}_q^G(G/H^+)$. To complete the proof of 4.2, it now suffices to identify this coefficient system. That is the content of the next lemma.

LEMMA 4.3. *There is a natural equivalence of functors* $\mu : \tilde{K}_q^G(G/H^+; R) \to K_{q-1}(RH)$.

Proof. The map μ is the composite

$$\tilde{K}_q^G(G/H^+; R) \xrightarrow{(A_*)^{-1}} \tilde{K}_q(S^0; RH) \xrightarrow{\partial} \tilde{K}_{q-1}(\varnothing; RH) \xrightarrow{\eta} K_{q-1}(RH)$$

where η is the isomorphism of 3.5; ∂ is the connecting homomorphism in the exact sequence obtained by applying 3.8 to the diagram $v_- \leftarrow \varnothing \to v_+$ of spaces in $\mathscr{C}\mathscr{W}_G$ and is an isomorphism since the other terms in the exact sequence vanish by 3.7; and $A_* : \tilde{K}_q(S^0; RH) \to \tilde{K}_q^G(G/H^+; R)$ is an isomorphism induced by a map of spectra $A : \tilde{\mathbf{K}}(S^0; RH) \to \tilde{\mathbf{K}}^G(G/H^+; R)$.

Let X be a based H-space and set $Y = G \times_H X/G \times_H \{x_0\}$. If $M = \{M_x \mid x \in cX - X\} \in \mathscr{C}_H(cX, X; R)$, let $\bar{M} = RG \otimes M = \Sigma\, RG \otimes M_x$ where $\otimes$ is tensor over RH. As R-modules, $RG \otimes M_x = \Sigma\, g_j M_x$ where $\{g_j \mid j = 1, \dots, r\}$ is a set of representatives for the cosets of H in G with $g_1 = e$. Define $\bar{p} : \bar{M} \to \mathscr{P}_f(c(Y))$ by setting $\bar{p}(\bar{m}) = \{[g_i, x]\}$ if $\bar{m} \in g_i M_x$ and "extending linearly." That is, if $\bar{m} = \bar{m}_1 + \cdots + \bar{m}_s$ with $0 \neq \bar{m}_i \in RG \otimes M_{x_i}$ $(i = 1, \dots, s)$ and $\bar{m}_i = \bar{m}_{i,j_1} + \cdots + \bar{m}_{i,j_{t(i)}}$ with $0 \neq \bar{m}_{i,j_k} \in g_{j_k} M_{x_i}$ $(k = 1, \dots, t(i))$, then $\bar{p}(\bar{m}) = \{[g_{j_k}, x_i] \mid i = 1, \dots, s;\ k = 1, \dots, t(i)\}$. The conditions that f must satisfy to be cc imply that $\bar{f}$ is cc. It is now easily verified that the correspondence that sends M to $(\bar{M}, \bar{p})$ and f to $\bar{f}$ is a functor $I_H^G : \mathscr{C}_H(cX, X; R) \to \mathscr{C}_G(cY, Y; R)$.

In particular, we may apply this construction to the pointed space $S^n(S^0)$ $(n \geq 0)$ with trivial H-action to obtain a functor $B_n : \mathscr{C}_H(cS^n(S^0), S^n(S^0); R) \to \mathscr{C}_G(cS^n(G/H^+), S^n(G/H^+); R)$. Let J be the isomorphism of 3.2. The sequence of functors

$$A_n = B_n J^{-1} : \mathscr{C}(cS^n(S^0), S^n(S^0); RH) \to \mathscr{C}_G(cS^n(G/H^+), S^n(G/H^+); R)$$

induces a map of Ω-spectra $A : \tilde{\mathbf{K}}(S^0; RH) \to \tilde{\mathbf{K}}^G(G/H^+; R)$. A straightforward adaptation of the "germs of infinity" argument of [A–M]; Lemma 8.7 ff] shows that A_n induces an isomorphism

$$A_{n*} : K_q(\mathscr{C}(cS^n(S^0), S^n(S^0); RH)) \to K_q(\mathscr{C}_G S^n(c(G/H^+), S^n(G/H^+); R) \qquad (4.4)$$

for $q \geq 1$. It then follows directly from (4.4) that $A_* : \tilde{K}_q(S^0; RH) \to \tilde{K}_q^G(G/H^+; R)$ is an isomorphism for all q.

For the reader's convenience, we indicate more completely how to make this adaptation. We make the following interpretations of the categories that appear in the diagram at the beginning of the proof of 8.5 in [A–M; p. 594]: $\mathscr{U}(\mathfrak{U}) = \mathscr{C}(cS^n(S^0), S^n(S^0); RH)$; $\mathscr{B}(\mathfrak{U})$ is the full subcategory of $\mathscr{U}(\mathfrak{U})$ whose objects are

are the geometric G-modules $M = \{M_X \mid$ there is a neighborhood N of $S^n\{+1\}$ in $cS^n(S^0)$ such that $M_x = 0$ for all $x \in U\}$; and $\mathscr{G}_\infty(\mathfrak{U})$ is the full subcategory of $\mathscr{U}(\mathfrak{U})/\mathscr{B}(\mathfrak{U})$ whose objects $M = \{M_X\}$ have $M_x = 0$ if $x \in cS^n\{-1\}$. Similarly, $\mathscr{U}(A) = \mathscr{C}(cS^n(G/H^+),\ S^n(G/H^+); RH)$; $\mathscr{B}(A)$ is the full subcategory of $\mathscr{U}(A)$ whose objects are the geometric G-modules $M = \{M_x \mid$ there is a neighborhood N of $S^n\{+1\}$ in $cS^n(G/H^+)$ such that $M_x = 0$ for all $x \in U\}$; and $\mathscr{G}_\infty(A)$ is the full subcategory of $\mathscr{U}(A)/\mathscr{B}(A)$ whose objects $M = \{M_x\}$ have $M_x = 0$ if $x \in cS^n\{+\}$. With these interpretations, the proofs of [A–M; Lemmas 8.7 and 8.5] carry over without change to prove (4.4).

5. The proofs of Theorem II and Corollary III

This section gives the proofs of Theorem II and Corollary III. Theorem II will be deduced from the sharper Theorem 5.1 below. Let Γ be a group and $\overline{\mathscr{E}\Gamma} = \mathscr{E}\Gamma \cup \Sigma$ be the compactification of $\mathscr{E}\Gamma$ given in 1.3. Then for any finite subgroup G of Γ, by 1.3(3), there is a G-homeomorphism $f: (\overline{\mathscr{E}\Gamma}, \Sigma) \to (c\Sigma, \Sigma)$ where Σ is a G-linear sphere.

THEOREM 5.1. *Let Γ be a lattice-like group. Suppose that Γ acts freely, cellularly, and cocompactly on a CW complex $\tilde{X}$ homotopy equivalent to S^{8r+3} for any $r \geq 0$. If Γ contains a copy of $Q(2^q, p, 1)$ with $q \geq 4$ and p an odd prime satisfying $p \not\equiv 1$ (mod 8) and $p \not\equiv -1$ (mod 2^{q-1}), then either*

(1) *$Z_\Gamma(C_p)$ is infinite; or*

(2) *$H_1^G(\Sigma; K_{-1}(\mathbb{Z}G_\sigma)) \neq 0$ where $H_1^G(\Sigma; K_{-1}(\mathbb{Z}G_\sigma))$ is the Bredon homology of Σ with coefficient system the functor that sends G/H to $K_{-1}(\mathbb{Z}H)$.*

Let $f: G \to Q(2^q)$ be the homomorphism that sends x, y, and z, respectively, to $\bar{x}$, $\bar{y}$, and 1, respectively. Then there is an exact sequence

$$1 \longrightarrow C_p \longrightarrow C \xrightarrow{f} Q \longrightarrow 1$$

where $Q = Q(2^q)$ is the quaternion group. This sequence is split by the monomorphism $j: Q \to G$ that sends $\bar{x}$ to x and $\bar{y}$ to y where we have followed the notation of the introduction. We shall use this notation throughout the next two sections.

Some preliminary results are needed to prove this theorem. Let $f: (\overline{\mathscr{E}\Gamma}, \Sigma) \to (c\Sigma, \Sigma)$ be the G-homeomorphism above. Then f induces an isomorphism of categories $f_*: \mathscr{C}_G(\overline{\mathscr{E}\Gamma}, \Sigma; R) \to \mathscr{C}_G(c\Sigma, \Sigma; R)$.

THEOREM 5.2. *Let Γ be a lattice-like group and G be a finite subgroup of Γ. Suppose there is a G-homeomorphism $f: (\overline{\mathscr{E}\Gamma}, \Sigma) \to (c\Sigma, \Sigma)$. Then there is a commutation diagram with exact row*

$$\begin{array}{ccccc} & & \tilde{K}_0(RG) & \xrightarrow{\mathrm{tr}} & \tilde{K}_0(\mathscr{C}_G(\overline{\mathscr{E}\Gamma}, \Sigma; R)) \\ & & \big\downarrow \eta & & \approx \big\downarrow f_* \\ H_1^G((\Sigma; K_{-1}(RG_\sigma)) & \xrightarrow{d} & \tilde{K}_0(RG)/\mathrm{im\ Ind} & \xrightarrow{i_*} & \tilde{K}_0(c\Sigma, \Sigma; R)) \end{array}$$

Here η *is the natural quotient homomorphism and* Ind *is the sum of the homomorphisms* $\mathrm{Ind}_{G_\sigma}^G : K_{q-1}(RG_\sigma) \to K_{q-1}(RG)$ *over the* 0*-cells* $\sigma \in \Sigma$.

Proof. A standard argument, using 4.1, allows us to extract the exact sequence

$$H_1^G(\Sigma; K_{-1}(RG_\sigma)) \to K_0(RG)/\mathrm{im\ Ind} \xrightarrow{i_*} \tilde{K}_0^G(\Sigma; R) \tag{5.3}$$

from the spectral sequence of 4.2. Let σ be a 0-cell Σ and G_σ be its isotropy group. Note that $K_0(RG)/\mathrm{im\ Ind} = \tilde{K}_0(RG)/\mathrm{im\ Ind}$; while an examination of the definitions shows that $\tilde{K}_0^G(\Sigma; R) = K_0\hat{\mathscr{C}}_G(c\Sigma, \Sigma; R))$. Since there is an exact sequence

$$K_0(\mathscr{C}_G(c\Sigma, \Sigma; R)) \to K_0\hat{\mathscr{C}}_G(c\Sigma, \Sigma; R)) \to \tilde{K}_0\mathscr{C}_G(c\Sigma, \Sigma; R)) \to 0$$

and $K_0(\mathscr{C}_G(c\Sigma, \Sigma; R)) = 0$ by [Pe; Remark 1.6],

$$K_0(\hat{\mathscr{C}}_G(c\Sigma, \Sigma; R)) \approx \tilde{K}_0(\mathscr{C}_G(c\Sigma, \Sigma; R)).$$

Thus the exact sequence (5.3) reduces to the exact row in the diagram. That the square commutes follows by the argument used to prove Theorem 6.3 in [H–P].

When we apply 5.2, $\mathbb{Z}$ will play the role of R.

PROPOSITION 5.4. *Let* $G = Q(2^q, p, 1)$ *with* $q \geq 4$, $p \not\equiv 1$ (*mod* 8) *and* $p \not\equiv -1$ (*mod* 2^{q-1}). *Suppose* G *acts freely and cellularly on the finite dimensional CW complex* $\tilde{X}$ *homotopy equivalent to* S^{8r+3}. *Then* $\sigma_G(\tilde{X}/G) \neq 0$ *in* $\tilde{K}_0(\mathbb{Z}G)/j_*(\tilde{K}_0(\mathbb{Z}Q)$ *where* $\sigma_G(\tilde{X}/G)$ *is the finiteness obstruction of* $\tilde{X}/G$.

Proposition 5.4 is an easy consequence of a result of [Da]. Its proof is given in the next section.

Proof of 5.1. Suppose $Z_\Gamma(C_p)$ is finite. We show that $H_1^G(\Sigma; K_{-1}(\mathbb{Z}G_\sigma)) \neq 0$. Since $i_*\eta(\sigma_G(X/G)) = \mathrm{tr}\,(\sigma_G(\tilde{X}/G)) = 0$ by 5.1 and Theorem I and the row in the diagram of 5.2 is exact, $i_*\eta(\sigma_G(X/G)) \in \mathrm{im}\, d$. On the other hand, by 1.2 $\dim\,(\mathscr{E}\Gamma)^{C_p} = \mathrm{vcd}\, Z_\Gamma(C_p) = 0$. Since $(\overline{\mathscr{E}\Gamma}, \Sigma)$ is G-homeomorphic to $(c\Sigma, \Sigma)$ by 1.3(2), it follows that C_p acts freely on Σ and that every isotropy group of the action of G on Σ is conjugate to a subgroup of Q. Hence the natural homomorphism $\tilde{K}_0(\mathbb{Z}G) \to \tilde{K}_0(\mathbb{Z}G)/j_*\tilde{K}_0(\mathbb{Z}Q)$ factors through $\tilde{K}_0(\mathbb{Z}G)/\mathrm{im\,Ind}$ and $\eta(\sigma_G(X/G)) \neq 0$ in $\tilde{K}_0(\mathbb{Z}G)\mathrm{im\,Ind}$. Hence d is not trivial and $H_1^G(\Sigma; K_{-1}(\mathbb{Z}G_\sigma)) \neq 0$.

Some additional results are needed to prove Theorem II. Although the following result is well known to the experts, we have not found a proof of it in the literature.

LEMMA 5.5. *The group* $K_{-1}(\mathbb{Z}Q(8)) = 0$. *If* $k \geq 4$, *then* $K_{-1}(\mathbb{Z}Q(2^k)) = \mathbb{F}_2$, *the group with* 2 *elements, and* $\mathrm{Ind}_{Q_k}^{Q_{k+1}} : K_{-1}(\mathbb{Z}Q(2^k)) \to K_{-1}(\mathbb{Z}Q(2^{k+1}))$ *is an isomorphism. Here* $Q_k = Q(2^k)$.

Proof. It follows from [Ca; Theorem 3] that we can analyze $K_{-1}(\mathbb{Z}Q)$ by using the following exact sequence

$$0 \longrightarrow K_0(\mathbb{Z}) \longrightarrow K_0(\hat{\mathbb{Z}}_2Q) \oplus K_0(\mathbb{Q}Q) \xrightarrow{\rho} K_0(\hat{\mathbb{Q}}_2Q) \longrightarrow K_{-1}(\mathbb{Z}Q) \longrightarrow 0.$$

It is well known that $K_0(\hat{\mathbb{Z}}_2Q)$ is free abelian of rank 1. The decomposition of $\mathbb{Q}Q$ into simple factors is $(\mathbb{Q})^4 \oplus \Sigma M_2(\mathbb{Q}(\lambda_{2^i})) \oplus \mathbb{Q}(\lambda_{2^{q-1}})\langle -1, -1\rangle$ where the last factor is a quaternionic algebra, $i = 1, \ldots, q-2$, $\lambda_{2^j} = \zeta_{2^j} + \bar{\zeta}_{2^j}$, and ζ_k is a primitive k-th root of unity. Over each of these factors there is a unique simple module and $K_0(\mathbb{Q}Q)$ is free ablian on these simple modules. Since 2 is totally ramified in $\mathbb{Q}(\zeta_{2^j})$ for any j, the decomposition of $\hat{\mathbb{Q}}_2Q$ into simple factors is $(\hat{\mathbb{Q}}_2)^4 \oplus \Sigma M_2(\hat{\mathbb{Q}}_2(\lambda_{2^i})) \oplus \hat{\mathbb{Q}}_2(\lambda_{2^{q-1}})\langle -1, -1\rangle$ and $K_0(\hat{\mathbb{Q}}_2Q)$ is again free abelian on the unique simple modules over the factors. Hence $\mathrm{rk}\, K_0(\hat{\mathbb{Q}}_2Q) = \mathrm{rk}\, K_0(\mathbb{Q}_2Q)$ and $\mathrm{rk}\, K_{-1}(\mathbb{Z}Q) = 0$. Under the homomorphism ρ, except for the last factor, the simple module over a factor of $\mathbb{Q}Q$ maps to the simple module over the corresponding factor of $\hat{\mathbb{Q}}_2Q$. If $k \leq 3$, the same is true of the simple module over the last factors and $K_{-1}(\mathbb{Z}Q) = 0$. If $k \geq 4$ however, the simple module over $\mathbb{Q}(\lambda_{2^{q-1}})\langle -1, -1\rangle$ is the free module of rank 1 while $\hat{\mathbb{Q}}_2(\lambda_{2^{q-1}})\langle -1, -1\rangle$ splits (i.e. is the matrix algebra $M_2(\hat{\mathbb{Q}}_2(\lambda_2^{q-1})))$. Hence ρ maps this generator of $K_0(\mathbb{Q}Q)$ to twice the corresponding generator of $K_0(\hat{\mathbb{Q}}_2Q)$. In this case, then $K_{-1}(\mathbb{Z}Q) = \mathbb{F}_2$. Finally, since $\mathrm{Ind}_{Q_k}^{Q_{k+1}} K_0(\mathbb{Q}Q(2^k)) \to K_0(\mathbb{Q}Q(2^{k+1}))$ preserves the last factors of the above decompositions, this map is the identity if $k \geq 4$.

PROPOSITION 5.6. *Let* $q \geq 4$ *and* p *be an odd prime. Suppose that* $G = Q(2^q, p, 1)$ *acts linearly on the sphere* Σ *such that the cyclic subgroup* C_p *acts freely. Suppose that either* $\dim \Sigma \leq 2$ *or* $\dim \Sigma^Q \geq 2$. *Then* $H_1^G(\Sigma; K_{-1}(\mathbb{Z}G_\sigma)) = 0$.

REMARK 5.7. Proposition 5.6 also holds under the assumption that for all $H \subset Q$, either $L^H = \varnothing$ or $\dim L^H \geq 2$ and $\{H \subset Q \mid \dim \Sigma^H \geq 2\}$ has a unique maximal element. We do not need this result in this paper.

Proof. Suppose $\dim \Sigma \leq 2$. Since C_p acts freely on Σ, $\dim \Sigma \leq 1$. If $\dim \Sigma < 1$, clearly $H_1^G(\Sigma; K_{-1}(\mathbb{Z}G_\sigma)) = 0$. So suppose $\dim \Sigma = 1$. Let $\rho : G \to O(2)$ be the linear representation given by Σ. Since the only finite subgroups of $O(2)$ are cyclic or dihedral and C_p is acting freely, im $\rho = D$ is dihedral of order $2^t p$ for some t and $N = \ker \rho$ is a normal subgroup of order 2^s for some s. Hence $H_1^G(\Sigma; K_{-1}(\mathbb{Z}G_\sigma)) = H_1^D(\Sigma; K_{-1}(\mathbb{Z}G_\sigma))$ and we may compute this group by looking at a D-equivariant cellular decomposition of Σ. This decomposition has a single 1-cell τ. If $N = G_\tau$ is either cyclic or $Q(8)$, then $K_{-1}(\mathbb{Z}G_\tau) = 0$ by [Ba; Theorem 10.6, p. 695] and 5.5. Hence $H_1^G(\Sigma; K_{-1}(\mathbb{Z}G_\sigma)) = 0$. If $N = G_\tau = Q(2^s)$ for some $s \geq 4$, then by 5.5 the coefficient system $K_{-1}(\mathbb{Z}G_\sigma)$ is the constant system with value $\mathbb{F}_2$ and $H_1^D(\Sigma; K_{-1}(\mathbb{Z}G_\sigma)) = H_1^D(\Sigma; \mathbb{F}_2) = H_1(\Sigma/D; \mathbb{F}_2) = 0$ since the orbit space is an interval.

Suppose $\dim \Sigma^Q \geq 2$. Since C_p acts freely on Σ, $Q = G/C_p$ acts on $L = \Sigma/C_p$ and there is an isomorphism

$$H_1^Q(L; K_{-1}(\mathbb{Z}Q_\sigma)) \to H_1^G(\Sigma; K_{-1}(\mathbb{Z}G_\sigma)).$$

Let I be an isotropy subgroup for the Q action on L. Then I is either cyclic of order 2^k with $k \leq q-1$ or quaternionic. If I is cyclic, then $K_{-1}(\mathbb{Z}I) = 0$ by [Ba; Theorem 10.6, p. 695]; while $K_{-1}(\mathbb{Z}Q)$ is given by 5.5. Hence $H_1^Q(L; K_{-1}(\mathbb{Z}Q_\sigma)) = H_1^Q(A; \mathbb{F}_2)$ where A is the union of the subspaces $\{L^H \mid H$ is a quaternionic subgroup of Q of order $>8\}$ and the coefficient system $\mathbb{F}_2$ is constant.

We claim that $H_1^Q(A; \mathbb{F}_2) = 0$. To prove this, let $Q = H_0, H_1, \ldots, H_t$ be a set of conjugacy class representatives for $[H \subset Q \mid H$ is quaternionic of order $>8\}$ ordered so that if H_i is conjugate to a subgroup of H_j, then $j \leq i$. For $k = 0, 1, \ldots, t$ set $L_k = \bigcup\{X^{(H_i)} \mid i \leq k\}$ where $X^{(H)} = \{x \in X \mid (H) \subset (G_x)\}$. Then

$$\varnothing \subset L^Q = L_0 \subset \cdots \subset L_{k-1} \subset L_k \subset \cdots \subset L_t = A$$

is a filtration of A by Q-invariant subspaces. We assert that $H_1^Q(L_0; \mathbb{F}_2) = 0$ and that $H_i^Q(L_k, L_{k-1}; \mathbb{F}_2) = 0$ for $i < 2$ and every $k > 0$. Assuming this, a simple induction argument shows that $H_1^Q(L_k; \mathbb{F}_2) = 0$ for every k. Since $A = L_t$, the claim will follow.

Since $L = \Sigma/C_p$ is a mod 2 homology sphere, each L^H is a mod 2 homology sphere. Since Q acts smoothly on L, each L^H is a manifold. Furthermore $\dim L^H \geq \dim L^Q \geq 2$. Hence each L^H is connected and $H_1(L^H; \mathbb{F}_2) = 0$. In particular $H_1^Q(L_0; \mathbb{F}_2) = H_1(L_0; \mathbb{F}_2) = 0$ as asserted above. Furthermore, since $L_k = L_{k-1} \cup L^{(H_k)}$ and H acts trivially on L^H, there are isomorphisms

$$H_i^{W(H)}(L^H, L^H \cap L_{k-1}; \mathbb{F}_2) \to H_i^{N(H)}(L^H, L^H \cap L_{k-1}; \mathbb{F}_2) \to H_i^Q(L_k, L_{k-1}; \mathbb{F}_2)$$

where $W(H) = N(H)/H$ and the $\mathbb{F}_2$ in the leftmost term is the trivial $\mathbb{F}_2 W(H)$-module. A standard induction argument using Mayer–Vietoris sequences, shows that if

B is any union of the sets $\{L^H \mid H$ is a quaternionic subgroup of Q of order $>8\}$, then B is connected and $H_1(B; \mathbb{F}_2) = 0$. Hence the cellular chain complex $C_*(L^H, L^H \cap L_{k-1}; \mathbb{F}_2)$ is acyclic for $* < 2$. Since this is a chain complex of free $\mathbb{F}_2 W(H)$-modules, the chain complex $C_*(L^H, L^H \cap L_{k-1}; \mathbb{F}_2) \otimes \mathbb{F}_2$ is also acyclic for $* < 2$. Here $\otimes$ is tensor over $\mathbb{F}_2 W(H)$. Hence $H_i^{W(H)}(L^H, L^H \cap L_{k-1}; \mathbb{F}_2) = 0$ for $i < 2$, the second assetion above holds, and the claim is established.

Proof of Theorem II. The proof is by contradiction. So suppose that Γ acts freely, cellularly, and cocompactly on a CW complex $\tilde{X}$ homotopy equivalent to S^{8r+3} for some $r \geq 0$. Since $Z_\Gamma(C_p)$ is finite, $H_1^G(\Sigma; K_{-1}(\mathbb{Z}G_\sigma)) \neq 0$ by 5.1. In addition, since $0 = \operatorname{vcd} Z_\Gamma(C_p) = \dim(\mathscr{E}\Gamma)^{C_p}$ by 1.2, C_p acts freely on Σ. Hence if either $\dim \Sigma = \operatorname{vcd} \Gamma - 1 \leq 2$ or $2 \leq \operatorname{vcd} Z_\Gamma(Q) - 1 = \dim \Sigma^Q$, then $H_1^G(\Sigma; K_{-1}(\mathbb{Z}G_\sigma)) = 0$ by 5.6. This is a contradiction and Theorem II follows.

Proof of Corollary III. Let C_p act on $\mathbb{Z}[\zeta_p]$ by multiplication by a p-th root of unity and $A_0 = (\mathbb{Z}[\zeta_p])^k$ for some $k \geq 1$ (or if $p = 3$, $k \geq 2$). Then $\operatorname{rk} A_0 = (p-1)k$. Let $A = \operatorname{Ind}_C^G A_0$ and Γ be the semidirect product $A \rtimes G$. We show that Γ satisfies the hypotheses of Theorem II. Since Γ is crystallographic, it is lattice-like. A simple Mackey double coset argument shows that C_p acts freely on A. Hence $Z_\Gamma(C_p)$ is contained in G and is finite. Another double coset argument shows that

$$\operatorname{Res}_Q^G A = \operatorname{Res}_Q^G \operatorname{Ind}_C^G A_0 = \operatorname{Ind}_{\{1\}}^Q \mathbb{Z}^{(p-1)k} = (\mathbb{Z}Q)^{(p-1)k}.$$

Since (p, k) either has $p \geq 5$ and $k \geq 1$ or has $p = 3$ and $k \geq 2$, $\operatorname{vcd}(Z_\Gamma(Q)) = \operatorname{rk} A^Q = (p-1)k \geq 4$ and (1) also holds. Corollary III now follows from Theorem II.

6. The Proof of 5.4

After proving some preliminary results, this section gives the proof of Proposition 5.4. The authors want to thank Jim Davis for several discussions about the material in this section that have greatly clarified its presentation.

Let H be any finite group and p be any prime. Let

$$LD'(\hat{\mathbb{Z}}_p H) = \frac{\operatorname{im}\{K_1(\hat{\mathfrak{M}}_p) \to K_1(\hat{\mathbb{Q}}_p H)\}}{\operatorname{im}\{K_1(\hat{\mathbb{Z}}_p H) \to K_1(\hat{\mathbb{Q}}_p H)\}} \tag{6.1}$$

where $\hat{\mathfrak{M}}_p$ is any maximal $\hat{\mathbb{Z}}_p$-order in $\hat{\mathbb{Q}}_p H$. This group is called the *p-local defect group of H*. By choosing $\hat{\mathfrak{M}}_p$ to contain $\hat{\mathbb{Z}}_p H$, we see that the numerator contains the denominator. On the other hand, the numerator is well defined for suppose $\hat{\mathfrak{M}}_{p,1}$ and

$\hat{\mathfrak{M}}_{p,2}$ are two maximal $\hat{\mathbb{Z}}_p$-orders in $\hat{\mathbb{Q}}_p H$. Then by [Re; Theorem 17.3] there is a unit $u \in (\hat{\mathbb{Q}}_p H)^\times$ such that $u\hat{\mathfrak{M}}_{p,1}u^{-1} = \hat{\mathfrak{M}}_{p,2}$. Hence there is a commutative diagram

$$\begin{array}{ccc} K_1(\hat{\mathfrak{M}}_{p,1}) & \longrightarrow & K_1(\hat{Q}_p H) \\ \downarrow{\scriptstyle \phi_{u*}} & & \downarrow{\scriptstyle c_{u*}} \\ K_1(\hat{\mathfrak{M}}_{p,2}) & \longrightarrow & K_1(\hat{\mathbb{Q}}_p H) \end{array}$$

where $c_u : \hat{\mathbb{Q}}_p H \to \hat{\mathbb{Q}}_p H$ is conjugation by u and $\phi_u = c_u \mid \hat{\mathfrak{M}}_{p,1}$. Since c_{u*} is the identity, the numerator is well defined.

In the sequel it is convenient to denote the images in the numerator and denominator of (6.1) by $K_1'(\hat{\mathfrak{M}}_p)$ and $K_1'(\hat{\mathbb{Z}}_p H)$ respectively.

Lemma 6.2. *The correspondence that sends H to $LD'(\hat{\mathbb{Z}}_p H)$ is a functor from finite groups to abelian groups.*

Proof. Let $g : H_1 \to H_2$ be a homomorphism and $g_* : K_1(\hat{\mathbb{Q}}_p H_1) \to K_1(\hat{\mathbb{Q}}_p H_2)$ be the induced homomorphism. Let $\mathfrak{M}_1$ be a maximal $\hat{\mathbb{Z}}_p$-order in $\hat{\mathbb{Q}}_p H_1$. We claim there is a maximal $\hat{\mathbb{Z}}_p$-order $\mathfrak{M}_2$ with $g(\mathfrak{M}_1) \in \mathfrak{M}_2$; for $L = \hat{\mathbb{Z}}_p H_2 \otimes \mathfrak{M}_1$ is a $\hat{\mathbb{Z}}_p$-lattice in $\hat{\mathbb{Q}}_p H_2$ where $\otimes$ is tensor over $\hat{\mathbb{Z}}_p H_1$. Let $\mathfrak{O}_r(L) = \{\lambda \in \hat{\mathbb{Q}}_p H_2 \mid L\lambda < L\}$. It is well known (cf. [Re; p. 109]) that $\mathfrak{O}_r(L)$ is a $\hat{\mathbb{Z}}_p$-order in $\hat{\mathbb{Q}}_p H$ and it is clear that $g(\mathfrak{M}_1) \subset \mathfrak{O}_r(L)$. Let $\mathfrak{M}_2$ be a maximal $\hat{\mathbb{Z}}_p$-order in $\hat{\mathbb{Q}}_p H_2$ containing $\mathfrak{O}_r(L)$. Then $g_* K_1'(\mathfrak{M}_1) \subset K_1'(\mathfrak{M}_2)$. Since it is clear that $g_* K_1'(\hat{\mathbb{Z}}_p H_1) \in K_1'(\hat{\mathbb{Z}}_p H_2)$, g induces a homomorphism $g_* : LD'(\hat{\mathbb{Z}}_p H_1) \to LD'(\hat{\mathbb{Z}}_p H_2)$. The rest of the proof is trivial.

Let $D(H) = \ker\{\tilde{K}_0(\mathbb{Z}H) \to \tilde{K}_0(\mathfrak{M})\}$ where $\mathfrak{M}$ is a maximal $\mathbb{Z}$-order in $\mathbb{Q}H$ containing $\mathbb{Z}H$ and let $K_1'(\mathfrak{M}) = \operatorname{im}\{K_1(\mathfrak{M}) \to K_1(\mathbb{Q}H)\}$.

LEMMA 6.3. *There is an exact sequence*

$$K_1'(\mathfrak{M}) \longrightarrow \sum LD'(\hat{\mathbb{Z}}_p H) \xrightarrow{\partial} D(H) \longrightarrow 0$$

where the sum runs over all primes p dividing $|H|$. Hence ∂ induces an isomorphism

$$\bar{\partial} : \sum LD'(\hat{\mathbb{Z}}_p H)/\operatorname{im} K_1'(\mathfrak{M}) \to D(H).$$

Proof. Consider the diagram

$$\begin{array}{ccccc} \mathbb{Z}H & \longrightarrow & \mathfrak{M} & \longrightarrow & \mathbb{Q}H \\ \downarrow & & \downarrow & & \downarrow \\ \bar{\mathbb{Z}}H & \longrightarrow & \tilde{\mathfrak{M}} & \longrightarrow & \tilde{\mathbb{Q}}H \end{array}$$

in which $\mathfrak{M}$ is a maximal $\mathbb{Z}$ order in $\mathbb{Q}H$ containing $\mathbb{Z}H$, $\tilde{\mathbb{Z}}H = \Pi\, \hat{\mathbb{Z}}_p H$, $\hat{\mathfrak{M}} = \Pi\, \mathfrak{M}_p$ where $\hat{\mathfrak{M}}_p = \hat{\mathbb{Z}}_p \otimes \mathfrak{M}$, $\tilde{\mathbb{Q}}H = \Pi\, \hat{\mathbb{Q}}_p H$, and all products run over all primes p dividing $|H|$. Since the left hand square in the above diagram maps into the outer square, there is an induced map of Mayer–Vietoris sequences. Swan's theorem that projective modules over $\mathbb{Z}H$ are locally free shows that the following commutative diagram has exact rows

$$\begin{tikzcd}
K_1(\mathfrak{M}) \oplus \sum K_1(\hat{\mathbb{Z}}_p H) \arrow[r] \arrow[d] & \sum K_1(\hat{\mathfrak{M}}_p) \arrow[r] \arrow[d] & D(H) \arrow[r] \arrow[d] & 0 \\
K_1(\mathbb{Q}H) \oplus \sum K_1(\hat{\mathbb{Z}}_p H) \arrow[r] & \sum K_1(\hat{\mathbb{Q}}_p H) \arrow[r, "\partial"] & \tilde{K}_0(\mathbb{Z}H) &
\end{tikzcd}$$

Here the sums are again over all primes $p \mid |H|$. Hence there is an exact sequence

$$K_1'(\mathfrak{M}) \oplus \sum K_1'(\hat{\mathbb{Z}}_p H) \longrightarrow \sum K_1'(\hat{\mathfrak{M}}_p) \xrightarrow{\partial} D(H) \longrightarrow 0,$$

where $K_1'(\mathfrak{M}) = \mathrm{im}\,\{K_1(\mathfrak{M}) \to K_1(\mathbb{Q}H)\}$, from which the given exact sequence is easily derived.

REMARK 6.4. If $g : H_1 \to H_2$ is a homomorphism, then the following diagram commutes

$$\begin{tikzcd}
\sum LD'(\hat{\mathbb{Z}}_p H_1) \arrow[r, "\partial"] \arrow[d, "\Sigma g_*"] & D(H_1) \arrow[r] \arrow[d, "g_*"] & 0 \\
\sum LD'(\hat{\mathbb{Z}}_p H_2) \arrow[r, "\partial"] & D(H_2) \arrow[r] & 0
\end{tikzcd}$$

We now specialize the discussion to the case when $G = Q(2^q, p, 1)$. Let $\rho : D(G) \to LD'(\hat{\mathbb{Z}}_p G)/\mathrm{im}\, K_1'(\mathfrak{M})$ be the composite of $\bar{\partial}^{-1}$ and projection on the indicated factor.

THEOREM 6.5 (Davis). *If* $q \geq 4$, $p \not\equiv 1$ *(mod* 8*) and* $p \not\equiv -1$ *(mod* 2^{q-1}*), then* $\rho(\sigma_G(X/G)) \neq 0$.

Proof. Since this result does not appear explicitly in [Da], we indicate how it follows from the results that are there. In particular, by using the decomposition of $\hat{\mathbb{Z}}_p G$ into blocks (i.e. two sided ideals that are direct summands), Davis constructs another projection $\bar{\rho}$ that fits into a commutative diagram

$$
\begin{array}{ccc}
D(G) & \xrightarrow{\partial} & LD'(\hat{\mathbb{Z}}_p G)/\mathrm{im}\, K_1'(\mathfrak{M}) \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \bar{\rho}} \\
D(G) & \xrightarrow{\rho'} & \dfrac{LD'(B_{n-2}) \otimes \mathbb{Z}\left[\dfrac{1}{p}\right]}{\mathrm{im}\, K_1'(\mathfrak{M}_q) \oplus K_1'(\mathfrak{M}_{q_T})}
\end{array}
$$

(The denominator in the lower right term arises since $\mathfrak{M}$ is also decomposed in blocks of which only the indicated pieces map non-trivially into $LD'(B_{n-2})$.) Since [Da; Proof of Theorem 6.1, p. 47] shows that $\rho'(\sigma_G(X/G)) \neq 0$, the stated theorem follows.

Proof of 5.4. We show first that ρ factors through $D(G)/j_* D(Q)$ and then that this group is a subgroup of $\tilde{K}_0(\mathbb{Z}G)/j_* \tilde{K}_0(\mathbb{Z}Q)$. To see the first statement, consider the commutation diagram

$$
\begin{array}{ccc}
LD'(\hat{\mathbb{Z}}_2 Q) & \xrightarrow{\partial} & D(Q) \\
\downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle j_*} \\
K_1'(\mathfrak{M}) \to LD'(\hat{\mathbb{Z}}_2 G) \oplus LD'(\hat{\mathbb{Z}}_p G) & \xrightarrow{\partial} & D(G) \\
\downarrow{\scriptstyle \mathrm{proj}} & & {\scriptstyle \rho}\downarrow{\scriptstyle \eta} \\
\dfrac{LD'(\hat{\mathbb{Z}}_p G)}{\mathrm{im}\, K_1'(\mathfrak{M})} & \longleftarrow & \dfrac{D(G)}{J_* D(Q)}
\end{array}
$$

Since $Q = Q(2^q)$ is a 2-group, the map at the top is onto. Since j_* maps into the first factor and proj is projection on the second factor, proj $j_* = 0$ and the first claim follows. A simple chase of the diagram

$$
\begin{array}{ccc}
D(Q) & \longrightarrow & \tilde{K}_0(\mathbb{Z}Q) \\
{\scriptstyle j_*}\downarrow\uparrow{\scriptstyle f_*} & & {\scriptstyle j_*}\downarrow\uparrow{\scriptstyle f_*} \\
D(G) & \longrightarrow & \tilde{K}_0(\mathbb{Z}G)
\end{array}
$$

using the fact that $f_* j_*$ is the identity shows that $j_* D(Q) = D(G) \cap j_*(\tilde{K}_0(\mathbb{Z}Q)$ establishing the second claim and completing the proof of 5.4.

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Department of Mathematics
Syracuse University
Syracuse, N.Y. 13244

and

Department of Mathematics
University of Notre Dame
Notre Dame, In. 46556

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